Rose-Hulman Undergraduate Mathematics Journal

Volume 24
Issue 1

Article 5

Constructing Spanning Sets of Affine Algebraic Curvature Tensors

Stephen J. Kelly
University of Chicago, sjk7777@comcast.net

Constructing Spanning Sets of Affine Algebraic Curvature Tensors

Cover Page Footnote

I would like to thank Dr. Corey Dunn for his excellent mentorship during this project. I would also like to thank Dr. Rolland Trapp for his support. This research was generously made possible by California State University, San Bernardino and NSF grant 2050894.

ℝose-Hulman Undergraduate Mathematics Journal
Volume 24, Issue 1, 2023

Constructing Spanning Sets of Affine Algebraic Curvature Tensors

By *Stephen Kelly*

Abstract. In this paper, we construct two spanning sets for the affine algebraic curvature tensors. We then prove that every 2-dimensional affine algebraic curvature tensor can be represented by a single element from either of the two spanning sets. This paper provides a means to study affine algebraic curvature tensors in a geometric and algebraic manner similar to previous studies of canonical algebraic curvature tensors.

1 Introduction

When studying the geometry of a pseudo-Riemannian manifold (M, g), it is natural to investigate the geometry of its curvature tensor. This tensor has three symmetries that enable it to be studied algebraically as an algebraic curvature tensor (ACT). We define an ACT below.

Definition 1.1. Let V be a n-dimensional real vector space. We define an *algebraic curvature tensor* to be an $R \in \bigotimes^4(V^*)$ such that R satisfies the following algebraic properties of a Riemannian curvature tensor:

1. $R(x, y, z, w) = -R(y, x, z, w)$,
2. $R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0$,
3. $R(x, y, z, w) = R(z, w, x, y)$.

One studies algebraic curvature tensors because they allow algebraic methods to be used to understand geometric properties of manifolds. The current body of work on ACTs has mostly focused on linear dependence and decompositions of ACTs [2, 5, 6, 10], but there has also been some investigation of ACTs in specific model spaces [8].

The commonality among all of these investigations has been the existence of some set which spans the vector space of ACTs; we refer to the elements of any such set as *canonical ACTs*

Mathematics Subject Classification. 53B20
Keywords. affine algebraic curvature tensors, curvature tensors, spanning sets

While many manifolds have curvature tensors that can be represented by ACTs, this is not the case for every manifold. As will be discussed in Section 3, the third symmetry of ACTs is based upon an assumption about the interaction between a manifold's connection and the metric. But, this assumption does not necessarily hold for all manifolds of interest. The limits of this assumption motivate us to consider affine algebraic curvature tensors (AACT) which describes the curvature tensors of manifolds where the interaction assumption does not hold. This enables us to study the geometry of a more general set of manifolds through algebraic investigations of its curvature tensor. Below we define an *affine algebraic curvature tensor*.

Definition 1.2. Using V as above, an AACT is an $R \in \bigotimes^4(V^*)$ such that R satisfies the following two properties:

1. $R(x, y, z, w) = -R(y, x, z, w)$,
2. $R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0$.

Note that an AACT is just an ACT without (3) in Definition 1.

The goal of this paper is to find two spanning sets of the affine algebraic curvature tensors. This is significant because having these sets will allow researchers to generalize the results previously only proved for ACTs like the results from [5, 6, 8, 10]. Moreover, as will be explained in Section 3, the study of ACTs places each ACT on a manifold with a metric and relies upon assumptions about the interaction of the manifold's connection with the metric. Readers familiar with differential geometry will know this structure as the Levi-Civita connection. When studying AACTs, we do not assume we have this structure which lets us work with a broader set of geometric objects. As such, mathematicians will be able to analyze the broader subject of affine geometry through the algebraic perspective of AACTs.

The organization of this paper is as follows. In Section 2 we will review the necessary differential geometry to understand the differences between affine and classical differential geometry. This will set the stage for the following sections and highlight why the generalization of ACTs to AACTs is not immediate.

In Section 3 we will use geometric methods to determine a reasonable spanning set which will provide us with the first notion of a canonical AACT. Specifically, we will highlight how the canonical ACTs require the Levi-Civita connection, while the canonical AACTs do not. This serves to highlight the geometric nature of AACTs and clarify how the algebraic proofs in Sections 4 and 6 have their roots in geometry. In Section 4 we demonstrate this set is indeed a spanning set.

In Section 5 we will produce a second spanning set, but we will do so without any geometric intuition. We do this by adjusting the canonical AACTs found in Section 4 in an analogous way to how the canonical symmetric ACTs are adjusted to produce the canonical anti-symmetric ACTs [3].

Then, in Section 6 we will prove that every AACT on a two dimensional vector space can be represented by a single canonical AACT in both the symmetric and anti-symmetric build. These final sections will act as a jumping off point for further study of the canonical representations of AACTs.

2 Preliminaries

The goal of this section is to explain all of the necessary differential geometry to define the Levi-Civita connection and the curvature tensor of a manifold. This section will follow the standard submanifold and connection theory as laid out in [4]. This section assumes knowledge of undergraduate analysis and some manifold theory. Those unfamiliar with either of these can see [7] and Chapter 2 of [4] respectively.

With that being said, one of the fundamental objects of study for this paper are connections on manifolds. We will be using the definition below.

Definition 2.1. Let $C^\infty(M)$ be the set of smooth real valued functions on the manifold M. Moreover, let $\pi : E \to M$ be a vector bundle on the manifold M, $\mathscr{T}(M)$ be the tangent bundle of M, and $\mathscr{E}(M)$ be the space of smooth sections of E. A connection in E is a map

$$\nabla : \mathscr{T}(M) \times \mathscr{E}(M) \to \mathscr{E}(M),$$

written as $(X, Y) \to \nabla_X Y$ that has the following properties:

1. $\nabla_X Y$ is $C^\infty(M)$-linear over X. That is for $f, g \in C^\infty(M)$ and X_1, X_2 vector fields in $\mathscr{T}(M)$ we have
$$\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y$$
2. $\nabla_X Y$ is $\mathbb{R}$-linear in Y. That is for $a, b \in \mathbb{R}$ and Y_1, Y_2 vector fields in $\mathscr{E}(M)$ we have
$$\nabla_X a_1 Y_1 + a_2 Y_2 = a_1 \nabla_X Y_1 + a_2 \nabla_X Y_2.$$
3. $\nabla_X Y$ satisfies this product rule for $f \in C^\infty(M)$:
$$\nabla_X (fY) = f\nabla_X Y + (Xf)Y.$$

We call $\nabla_X Y$ the covariant derivative of Y along X. It can be thought of as the derivative of Y in the direction of X. Next, we define a linear connection on M.

Definition 2.2. An linear connection is a connection in the tangent bundle TM. That is

$$\nabla : \mathscr{T}(M) \times \mathscr{T}(M) \to \mathscr{T}(M).$$

A well known and important result is that every manifold admits a linear connection. See page 52 of [4] for more details on the proof of this fact. We will use this fact in Section 3.

Linear connections are fundamental objects in differential geometry, and so naturally we would like to know how they act with regard to the tangent and normal bundles of a manifold. Definition 5 lays that out for us.

Definition 2.3. Let X and Y be vector fields in the tangent bundle of a manifold, M, M be a submanifold of N, and N have the connection $\overline{\nabla}$. Let N have the metric $g(\cdot,\cdot)$. Finally, let $\mathcal{N}(\mathrm{M})$ be the normal bundle of M. We define $\overline{\nabla}^{\top}$ and $\overline{\nabla}^{\perp}$ to be the tangential and perpendicular components of ∇, respectively. That is to say that

$$\overline{\nabla}^{\top} : \mathcal{T}(\mathrm{M}) \times \mathcal{T}(\mathrm{M}) \to \mathcal{T}(\mathrm{M})$$

and

$$\overline{\nabla}^{\perp} : \mathcal{T}(\mathrm{M}) \times \mathcal{T}(\mathrm{M}) \to \mathcal{N}(\mathrm{M}).$$

We call $(\overline{\nabla}_{\mathrm{X}}\mathrm{Y})^{\perp}$ the second fundamental form, and will denote it as $\eta(\mathrm{X},\mathrm{Y})$.

Because $\overline{\nabla}^{\top}$ describes the $\overline{\nabla}$'s behavior on $\mathcal{T}(\mathrm{M})$ and $\overline{\nabla}^{\perp}$ describes the $\overline{\nabla}$'s behavior on $\mathcal{N}(\mathrm{M})$ it stands to reason that $\overline{\nabla}^{\top}$ and $\overline{\nabla}^{\perp}$ can combine to make $\overline{\nabla}$. Proposition 1 formalizes this notion.

Proposition 2.4. *Let* M *be a submanifold of* N*. Let* N *have the connection* $\overline{\nabla}$ *and metric* $g(\cdot,\cdot)$*. If* $\mathrm{X},\mathrm{Y} \in \mathcal{T}(\mathrm{M})$ *are extended arbitrarily to vector fields on the ambient space, the following formula holds along* M*:*

$$\overline{\nabla}_{\mathrm{X}}\mathrm{Y} = (\overline{\nabla}_{\mathrm{X}}\mathrm{Y})^{\top} + \eta(\mathrm{X},\mathrm{Y}).$$

Details of this proof can be found on page 135 of [4]. Moreover, details for a proof that the choice of extension of X and Y does not change the result can be found on page 50 of [4].

We can now move on to defining the Lie bracket of vector fields. This is the last step required to define the Levi-Civita connection. Note that the Lie Bracket does not require the manifold to have a metric.

Definition 2.5. Let X and Y be vector fields on a manifold, M, and let $f \in C^{\infty}(\mathrm{M})$. Then, we define Lie bracket, $[\mathrm{X},\mathrm{Y}](f)$, to be

$$[\mathrm{X},\mathrm{Y}](f) := \mathrm{X}(\mathrm{Y}(f)) - \mathrm{Y}(\mathrm{X}(f)).$$

When referring to the Lie bracket as an operator we write $[\mathrm{X},\mathrm{Y}]$.

Up until this point, there have been no differences between affine differential geometry and classical differential geometry. The Levi-Civita connection (as defined below) is the first split in these fields. In classical differential geometry the Levi-Civita connection is frequently used, but in in affine differential geometry we use a connection that may not be the Levi-Civita connection.

Definition 2.6. Let M have the metric $g(\cdot,\cdot)$ and let X, Y, and Z be vector fields on M. The Levi-Civita connection (or Riemannian connection), denoted as ∇^{LC}, is the unique linear connection on M such that

1. $\nabla^{LC}_X Y - \nabla^{LC}_Y X = [X, Y]$,
2. $X(g(Y,Z)) = g(\nabla^{LC}_X Y, Z) + g(Y, \nabla^{LC}_X Z)$.

We say ∇^{LC} is (1) torsion-free and (2) metric compatible, respectively.

The fundamental fact about the Levi-Civita connection is that it is the unique connection on M that has both of these properties. Details of a proof of this fact can be found on page 69 of [4]. We can apply these properties to derive the Weingarten equation.

Proposition 2.7. *Let* M *be a submanifold of* N *with metric* $g(\cdot,\cdot)$*, and let* N *have a metric,* $\overline{g}(\cdot,\cdot)$*, and the Levi-Civita connection,* $\overline{\nabla}^{LC}$*. Suppose* $X, Y \in \mathscr{T}(M)$ *and* $Z \in \mathscr{N}(M)$*. When* X, Y, Z *are extended arbitrarily to the ambient space, the following equation holds at points of* M*:*

$$g(\overline{\nabla}^{LC}_X Z, Y) = -g(Z, \eta(X, Y)).$$

We call this the Weingarten equation.

The Weingarten equation is critical in deriving the canonical ACTs, but it requires $\overline{\nabla}$ to be torsion-free and metric compatible. By uniqueness, that means it requires $\overline{\nabla}$ to be the Levi-Civita connection. Since we do not assume $\overline{\nabla}$ is metric compatible in affine geometry we cannot use this equation. As we will show later in Section 3, this is the primary reason that the canonical ACTs are not the complete set of canonical AACTs.

Now that we have completed our first goal of defining the Levi-Civita connection, we can move on to defining the curvature tensor. To do that we must first define the curvature operator, and we will also define some important properties of connections. Note that the curvature operator does not require a metric, but the curvature tensor does.

Definition 2.8. Let X,Y, and Z be vector fields on M endowed with the connection ∇. Then we define the curvature operator on M with respect to ∇ to be

$$\mathscr{R}(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

∇ is said to be flat if

$$\mathscr{R} \equiv 0.$$

∇ is said to be symmetric or torsion-free if

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Also, we denote the curvature operator on $\mathbb{R}^n$ as $\overline{\mathscr{R}}(X,Y)Z$. Note that when R^n is equipped with a flat, symmetric connection $\overline{\nabla}$, the curvature operator $\overline{\mathscr{R}} = 0$.

Now we can finally use the curvature operator to define the curvature tensor.

Definition 2.9. Let X,Y, Z, and W be vector fields on (M, g). Then we define the curvature tensor on M with respect to ∇ to be

$$R(X,Y,Z,W) := g(\mathscr{R}(X,Y)Z,W).$$

Notice that since connections determine the curvature operator they ultimately determine the curvature tensor too.

3 Geometric Derivation of the Spanning Set

As was mentioned in Section 1, the point of this section is to provide the reader with a geometric justification for our choice of spanning set.

For this section we will make three standing assumptions: (1) M^{n-1} is a manifold of dimension $n-1$ embedded into $\mathbb{R}^n$, (2) $g(\cdot,\cdot)$ is a metric on $\mathbb{R}^n$, and (3) $\mathbb{R}^n$ has a flat, symmetric connection $\overline{\nabla}$.

These assumptions allow us to conclude the next proposition whose proof is similar to the one on page 135 in [4].

Proposition 3.1. $(\overline{\nabla})^\top$ *is a torsion-free connection on* M.

Proof. First we will prove $(\overline{\nabla})^\top$ is a connection on M and then that it is torsion-free.

1. $(\overline{\nabla}_X Y)^\top$ is $C^\infty(M)$-linear over X because

$$\begin{aligned}(\overline{\nabla}_{fX_1+hX_2}Y)^\top &= (f\overline{\nabla}_{X_1}Y)^\top + (h\overline{\nabla}_{X_2}Y)^\top \\ &= f(\overline{\nabla}_{X_1}Y)^\top + h\overline{\nabla}_{X_2}Y)^\top\end{aligned}$$

   since $\overline{\nabla}$ is a connection.

2. $(\overline{\nabla}_X Y)^\top$ is $\mathbb{R}$-linear in Y because for $a_1, a_2 \in \mathbb{R}$ we have

$$\begin{aligned}(\overline{\nabla}_X a_1 Y_1 + a_2 Y_2)^\top &= (\overline{\nabla}_X a_1 Y_1)^\top + (\overline{\nabla}_X a_2 Y_2)^\top \\ &= a_1(\overline{\nabla}_X Y_1)^\top + a_2(\overline{\nabla}_X Y_2)^\top.\end{aligned}$$

   This too is because $\overline{\nabla}$ is a connection.

3. Finally, $(\overline{\nabla}_X Y)^\top$ follows the product rule. We see that for $f \in C^\infty(M)$

$$(\overline{\nabla}_X fY)^\top = (f\overline{\nabla}_X Y + (Xf)Y)^\top = f(\overline{\nabla}_X Y)^\top + ((Xf)Y)^\top.$$

So, $(\overline{\nabla}_X Y)^\top$ is a connection. Next, it is torsion-free because

$$(\overline{\nabla}_X Y)^\top - (\overline{\nabla}_Y X)^\top = (\overline{\nabla}_X Y - \overline{\nabla}_Y X)^\top = [X,Y]^\top = [X,Y]$$

where we use the fact that if $X, Y \in \mathscr{T}(M)$ and M is embedded in $\mathbb{R}^n$, then $[X,Y] \in \mathscr{T}(M)$, which implies that $[X,Y]^\top = [X,Y]$. □

For simplicity we will now denote $\overline{\nabla}^\top$ simply as ∇. More importantly, this proposition is critical to expanding the curvature in a meaningful way that enables us to find our canonical tensors. Performing this expansion will also need one last property of η.

Proposition 3.2. $\eta(A,B)$ *is a symmetric 2-tensor. We denote this as*

$$\eta \in S^2(V^*).$$

A proof of Proposition 4 can be found in Lemma 8.1 of [4].

Having explained all of this background we can now begin to focus on demonstrating the geometric roots of our first spanning set for the AACTs.

First of all, notice that η maps two vector fields on M into $\mathscr{N}(M)$. Let k be a basis vector for $\mathscr{N}(M)$. Then $\eta(A,B) = h(A,B)k$ for some symmetric bilinear form $h \in S^2(V^*)$.

Let us look at $\overline{R}$, the curvature tensor of $\mathbb{R}^n$ with respect to the flat connection $\overline{\nabla}$. Let X, Y, Z and W be vector fields on $\mathbb{R}^n$ that are tangent to M. We see that by expanding $\overline{\nabla}$ into $\nabla + \eta$ we get that

$$\begin{aligned}
0 = \overline{R}(X,Y,Z,W) &= g(\overline{\mathscr{R}}(X,Y)Z,W)\\
&= g(\overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z, W)\\
&= g(\overline{\nabla}_X(\nabla_Y Z + \eta(Y,Z)) - \overline{\nabla}_Y(\nabla_X Z + \eta(X,Z)) - \overline{\nabla}_{[X,Y]}Z, W)\\
&= g(\overline{\nabla}_X(\nabla_Y Z) + \overline{\nabla}_X(\eta(Y,Z)) - \overline{\nabla}_Y(\nabla_X Z) - \overline{\nabla}_Y(\eta(X,Z)) - \overline{\nabla}_{[X,Y]}Z, W)\\
&= g(\nabla_X\nabla_Y Z + \eta(X,\nabla_Y Z), W) - g(\nabla_Y\nabla_X Z + \eta(Y,\nabla_X Z), W)\\
&\quad - g(\nabla_{[X,Y]}Z + \eta([X,Y],Z), W) + g(\overline{\nabla}_X(\eta(Y,Z)), W) - g(\overline{\nabla}_Y(\eta(X,Z)), W).
\end{aligned}$$

Recall that W is a vector field in the tangent bundle of M, and that η exclusively maps into the normal bundle. So, we can eliminate their inner products to get

$$\begin{aligned}
0 = \overline{R}(X,Y,Z,W) = g(\nabla_X\nabla_Y Z, W) - g(\nabla_Y\nabla_X Z, W) - g(\nabla_{[X,Y]}Z, W)\\
+ g(\overline{\nabla}_X(\eta(Y,Z)), W) - g(\overline{\nabla}_Y(\eta(X,Z)), W).
\end{aligned}$$

Then, by the definition of the curvature tensor this simplifies to

$$0 = \overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(\overline{\nabla}_X(\eta(Y,Z)), W) \\ - g(\overline{\nabla}_Y(\eta(X,Z)), W).$$

At this point in the derivation for ACTs one normally uses the Weingarten equation here to simplify the expression further [4]. However, we do not do that because we do not necessarily have metric compatibility. So, instead we will substitute η with $h(A,B)k$. It is this difference that will ultimately lead us to canonical AACTs that are distinct from the canonical ACTs. When we make our substitution we get

$$0 = \overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(\overline{\nabla}_X(h(Y,Z)k), W) \\ - g(\overline{\nabla}_Y(h(X,Z)k), W).$$

Using the product rule of connections gives us

$$0 = \overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + g((h(Y,Z)\overline{\nabla}_X k) + X(h(Y,Z))k, W) \\ - g(h(X,Z)\overline{\nabla}_Y k + X(h(X,Z))k, W).$$

Then, since $X(h(X,Z))k$ is in the normal bundle of M its inner product with W is 0. Hence we can simplify it down to

$$0 = \overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + g((h(Y,Z)\overline{\nabla}_X k), W) \\ - g(h(X,Z)\overline{\nabla}_Y k, W).$$

Simply letting $\alpha(X,W) := -g(\overline{\nabla}_X k, W)$ gives us

$$0 = \overline{R}(X,Y,Z,W) = R(X,Y,Z,W) - h(Y,Z)\alpha(X,W) + h(X,Z)\alpha(Y,W).$$

But, $\overline{R} \equiv 0$ because $\overline{\nabla}$ is flat. So we have the result

$$R(X,Y,Z,W) = \alpha(X,W)h(Y,Z) - \alpha(Y,W)h(X,Z).$$

This is strikingly similar to the tensors in the spanning set of the ACTs, so we suspect that these functions span the AACTs.

But, before we prove that, we return to the vector space setting and verify that these functions are AACTs to begin with.

Proposition 3.3. $R(X,Y,Z,W) = \alpha(X,W)h(Y,Z) - \alpha(Y,W)h(X,Z)$ *is an AACT if* $\alpha \in \bigotimes^2(V^*)$ *and* $h \in S^2(V^*)$.

Proof. We first check the anti-symmetry in the first two spots. We get that

$$\begin{aligned} R(X,Y,Z,W) &= \alpha(X,W)h(Y,Z) - \alpha(Y,W)h(X,Z) \\ &= -\left(\alpha(Y,W)h(X,Z) - \alpha(X,W)h(Y,Z)\right) = -(R(Y,X,Z,W)). \end{aligned}$$

Then, checking the Bianchi identity gives us that

$$\begin{aligned} R(X,Y,Z,W) &+ R(Y,Z,X,W) + R(Z,X,Y,W) \\ &= \alpha(X,W)h(Y,Z) - \alpha(Y,W)h(X,Z) \\ &+ \alpha(Y,W)h(Z,X) - \alpha(Z,W)h(Y,X) \\ &+ \alpha(Z,W)h(X,Y) - \alpha(X,W)h(Z,Y) \\ &= 0. \end{aligned}$$

□

Definition 3.4. We define

$$R_{\alpha,h}(X,Y,Z,W) := \alpha(X,W)h(Y,Z) - \alpha(Y,W)h(X,Z),$$

where $\alpha \in \bigotimes^2(V^*)$ and $h \in S^2(V^*)$.

So, we have that the $R_{\alpha,h}$'s are AACTs, and that they can be found in a similar way to how one finds the canonical ACTs in the metric compatible setting. In the next section, we will prove our conjecture that the $R_{\alpha,h}$'s really are a spanning set.

4 The Symmetric Build

In this section, we build on the geometric arguments of Section 3 and prove that the $R_{\alpha,h}$'s are a spanning set of the AACTs. This point also marks a shift in our methods. While Sections 2 and 3 mostly used geometric and analytic methods, Sections 4, 5, and 6 will be algebraic in nature. Importantly, this shift emphasizes why Theorem 1 matters. It ties these fundamentally geometric $R_{\alpha,h}$'s to the algebraic set of AACTs. Thus, it enables us to look at affine geometry in a new algebraic light and view the algebraic AACTs from a geometric perspective. That said, due to the change in methods we must now introduce two new maps that will be crucial to proving Theorem 1 below.

Remark 4.1. Let V be a real vector space of dimension n, and let $(e_1, e_2, \ldots, e_n)$ be a basis for V. Then, the map $e^i \otimes e^j \otimes e^k \otimes e^l : V^4 \to \mathbb{R}$ maps the input $\alpha(e_i, e_j, e_k, e_l)$ to α for $\alpha \in \mathbb{R}$ and all other combinations of basis vectors to 0.

More advanced readers will know this as a tensor product of the dual basis of V^*, but for our purposes it suffices to know how this map acts on these basis vectors of V. Understanding it in this way allows us to make the next definition, inspired by Gilkey's approach on page 42 of [3].

Definition 4.2. Let $(e_1, e_2, ..., e_n)$ be a basis for V. We define the map $T_{ijkl} : V^4 \to \mathbb{R}$ to be

$$T_{ijkl} := e^i \otimes e^j \otimes e^k \otimes e^l - e^j \otimes e^i \otimes e^k \otimes e^l.$$

In particular, T_{ijkl} maps (e_i, e_j, e_k, e_l) to 1, (e_j, e_i, e_k, e_l) to -1, and all other combinations of basis vectors to 0.

Having said that we can begin Theorem 1 by formally defining the span of $R_{\alpha,h}$'s.

Definition 4.3. We define B to be the set $\text{span}(\{R_{\alpha,h} : \alpha \in \bigotimes^2(V^*), h \in S^2(V^*)\})$.

This leads us to the main result.

Theorem 4.4. *The set of affine algebraic curvature tensors on* V *equals* B.

Proof. Let $(e_1, e_2, ..., e_n)$ be a basis for V, and let each c_{ijkl} be a real number. Furthermore, let W be an arbitrary affine algebraic curvature tensor. Then, W can be expressed as follows:

$$\begin{aligned} W = &\sum_{i,j \text{ distinct}} c_{ijij} T_{ijij} + c_{ijii} T_{ijii} \\ &+ \sum_{i,j,k \text{ distinct}} c_{ijki} T_{ijki} + c_{ijik} T_{ijik} + c_{ijkk} T_{ijkk} \\ &+ \sum_{i,j,k,l \text{ distinct}} c_{ijkl} T_{ijkl} \end{aligned} \tag{1}$$

where all indices go from 1 to n. This expression of W is possible because each T_{ijkl} essentially encodes how W acts on (e_i, e_j, e_k, e_l). As such, adding together all possible combinations of the T_{ijkl}'s encodes how W acts on every set of basis vectors. Therefore, when this sum is scaled properly by c_{ijkl}'s this sum is equivalent to W itself. We can see that proving each of these summands is in B will prove $W \in B$. Following this logic, we will break this argument into cases dealing with each of these sums.

Case 1: T_{ijij}

Fix i and j. If there is a linear combination of $R_{\alpha,h}$'s that equals T_{ijij} on the basis vectors, then $T_{ijij} \in B$. If we can do this for all i and j, then we will have that the linear combination of the T_{ijij}'s will be in B as well.

Now, for any arbitrary $\alpha \in \bigotimes^2(V)$, $h \in S^2(V)$ we have that

$$R_{\alpha,h}(e_i, e_j, e_i, e_j) = \alpha(e_i, e_j) h(e_j, e_i) - \alpha(e_j, e_j) h(e_i, e_i).$$

If we pick α and h so that $\alpha(e_j, e_j) = -1, h(e_i, e_i) = 1$, and they map all other basis vectors to 0, then $T_{ijij}(e_i, e_j, e_i, e_j) = R_{\alpha,h}(e_i, e_j, e_i, e_j) = 1$. This in turn means that

$T_{ijij}(e_j, e_i, e_i, e_j) = R_{\alpha,h}(e_j, e_i, e_i, e_j) = -1$. We also know that T_{ijij} sends all other combinations of basis vectors to 0, so we need to show that $R_{\alpha,h}$ does as well.

In order for an input to be non-zero for $R_{\alpha,h}$, we have to have e_i in the third position and e_j in the fourth position. Also, e_i and e_j must occupy either the first or second positions. Hence, the only non-zero inputs are (e_i, e_j, e_i, e_j) and (e_j, e_i, e_i, e_j) which is what was desired. Therefore $T_{ijij} = R_{\alpha,h} \in B$. And, since this process did not rely on specific values for i and j, we know that $\sum_{i,j \text{ distinct}} c_{ijij} T_{ijij} \in B$.

Case 2: T_{ijii}

Similarly to the last case, we want to find an $R_{\alpha,h}$ that equals T_{ijii}. Again, for an arbitrary $\alpha \in \bigotimes^2(V^*)$, $h \in S^2(V^*)$ we have that

$$R_{\alpha,h}(e_i, e_j, e_i, e_i) = \alpha(e_i, e_i)h(e_j, e_i) - \alpha(e_j, e_i)h(e_i, e_i).$$

So, if we pick α and h so that $\alpha(e_j, e_i) = -1, h(e_i, e_i) = 1$, and they map all other basis vectors to 0, then we have that $T_{ijii} = R_{\alpha,h}$.

Case 3: T_{ijik}

Following the same logic as the previous cases, we see that for an arbitrary α and h we have that

$$R_{\alpha,h}(e_i, e_j, e_i, e_k) = \alpha(e_i, e_k)h(e_j, e_i) - \alpha(e_j, e_k)h(e_i, e_i).$$

So, picking α and h such that $\alpha(e_j, e_k) = -1, h(e_i, e_i) = 1$ and all other combinations of basis vectors are mapped to zero gives us that $T_{ijik} = R_{\alpha,h}$.

Case 4: T_{ijki} and T_{ijkk}

We start off in a similar manner to the last three cases, and see that

$$R_{\alpha,h}(e_i, e_j, e_k, e_i) = \alpha(e_i, e_i)h(e_j, e_k) - \alpha(e_j, e_i)h(e_i, e_k).$$

Pick α and h such that $\alpha(e_j, e_i) = -1, h(e_i, e_k) = 1$, and all other basis vectors get mapped to 0. Unlike in previous cases where we could ignore the symmetry of h by defining its only nonzero term to be (e_i, e_i), in this case we also have that $h(e_j, e_k) = h(e_k, e_j) = 1$. This means that

$$\begin{cases} R_{\alpha,h}(e_i, e_j, e_k, e_i) = 1, \\ R_{\alpha,h}(e_j, e_i, e_k, e_i) = -1, \\ R_{\alpha,h}(e_k, e_j, e_i, e_i) = 1, \\ R_{\alpha,h}(e_j, e_k, e_i, e_i) = -1, \end{cases}$$

and all other inputs are 0. So unlike in the previous cases $R_{\alpha,h} \neq T_{ijki}$, although, $R_{\alpha,h} = T_{ijki} + T_{kjii}$.

Now we will examine T_{kjii}. Consider another arbitrary $\tilde{\alpha} \in \bigotimes^2(V^*)$ and $\tilde{h} \in S^2(V^*)$. Then, we have that

$$R_{\tilde{\alpha},\tilde{h}}(e_k, e_j, e_i, e_i) = \tilde{\alpha}(e_k, e_i)\tilde{h}(e_j, e_i) - \tilde{\alpha}(e_j, e_i)\tilde{h}(e_k, e_i).$$

Pick $\tilde{\alpha}$ and $\tilde{h}$ so that $\tilde{\alpha}(e_k,e_i)=1$ and $\tilde{h}(e_j,e_i)=\tilde{h}(e_i,e_j)=1$. That means that

$$\begin{cases} R_{\tilde{\alpha},\tilde{h}}(e_k,e_j,e_i,e_i)=1,\\ R_{\tilde{\alpha},\tilde{h}}(e_j,e_k,e_i,e_i)=-1,\\ R_{\tilde{\alpha},\tilde{h}}(e_k,e_i,e_j,e_i)=1,\\ R_{\tilde{\alpha},\tilde{h}}(e_i,e_k,e_j,e_i)=-1,\end{cases}$$

and all other inputs are 0. Thus, $R_{\tilde{\alpha},\tilde{h}}=T_{kjii}+T_{kiji}$.

Let $W(e_i,e_j,e_k,e_l)=c_{ijkl}$ for any $i,j,k,l\in\{1,\dots,n\}$. Since W satisfies the Bianchi identity, we can pick these constants so that $c_{ijkl}+c_{jkil}+c_{kijl}=0$. We claim that all terms in the T_{ijki} and T_{kjii} sums can be expressed by a linear combination of $R_{\alpha,h}$ and $R_{\tilde{\alpha},\tilde{h}}$. We need

$$\begin{cases} xR_{\alpha,h}(e_i,e_j,e_k,e_i)+yR_{\tilde{\alpha},\tilde{h}}(e_i,e_j,e_k,e_i)=c_{ijki},\\ xR_{\alpha,h}(e_j,e_k,e_i,e_i)+yR_{\tilde{\alpha},\tilde{h}}(e_j,e_k,e_i,e_i)=c_{jkii},\\ xR_{\alpha,h}(e_k,e_i,e_j,e_i)+yR_{\tilde{\alpha},\tilde{h}}(e_k,e_i,e_j,e_i)=c_{kiji}.\end{cases}$$

If we let $x=c_{ijki}$ and $y=c_{kiji}$ we can check that we get

$$c_{ijki}R_{\alpha,h}(e_i,e_j,e_k,e_i)+c_{kiji}R_{\tilde{\alpha},\tilde{h}}(e_i,e_j,e_k,e_i)=c_{ijki}\cdot 1+c_{kiji}\cdot 0=c_{ijki}.$$

Similarly, for the input (e_k,e_i,e_j,e_i) this linear combination gives us

$$c_{ijki}R_{\alpha,h}(e_k,e_i,e_j,e_i)+c_{kiji}R_{\tilde{\alpha},\tilde{h}}(e_k,e_i,e_j,e_i)=c_{ijki}\cdot 0+c_{kiji}\cdot 1=c_{kiji}.$$

The final non-zero, independent input is (e_j,e_k,e_i,e_i). We have that

$$c_{ijki}R_{\alpha,h}(e_j,e_k,e_i,e_i)+c_{kiji}R_{\tilde{\alpha},\tilde{h}}(e_j,e_k,e_i,e_i)=c_{ijki}\cdot -1+c_{kiji}\cdot -1=c_{jkii}.$$

And, since

$$c_{ijki}R_{\alpha,h}+c_{kiji}R_{\tilde{\alpha},\tilde{h}}=c_{ijki}(T_{ijki}+T_{kjii})+c_{kiji}(T_{kjii}+T_{kiji})$$

we see that all other combinations of basis vectors are mapped to 0.

Hence, $R_{\alpha,h}+R_{\tilde{\alpha},\tilde{h}}$ covers the T_{ijki} and T_{jkii} terms, and thus the $\sum c_{ijki}T_{ijki}$ and $\sum c_{ijkk}T_{ijkk}$ summations can be completely replicated through the summation of $(R_{\alpha,h}+R_{\tilde{\alpha},\tilde{h}})$'s. So, the sums are in B.

Case 5: T_{ijkl} The final case is very similar Case 4. We start by considering an arbitrary $R_{\alpha,h}$ for the input (e_i,e_j,e_k,e_l). This is

$$R_{\alpha,h}(e_i,e_j,e_k,e_l)=\alpha(e_i,e_l)h(e_j,e_k)-\alpha(e_j,e_l)h(e_i,e_k).$$

Let $\alpha(e_i,e_l)=1$ and $h(e_j,e_k)=h(e_k,e_j)=1$. The non-zero, independent inputs are (e_i,e_j,e_k,e_l) and (e_i,e_k,e_j,e_l) both of which map to 1. Thus, $R_{\alpha,h}=T_{ijkl}+T_{ikjl}$.

We now shift to consider another arbitrary $R_{\tilde{\alpha},\tilde{h}}$ for the input (e_i, e_k, e_j, e_l). That gives us

$$R_{\tilde{\alpha},\tilde{h}}(e_i, e_k, e_j, e_l) = \tilde{\alpha}(e_i, e_l)\tilde{h}(e_k, e_j) - \tilde{\alpha}(e_k, e_l)\tilde{h}(e_i, e_j).$$

Let $\tilde{\alpha}(e_k, e_l) = -1$ and $\tilde{h}(e_i, e_j) = \tilde{h}(e_j, e_i) = 1$. Then the non-zero, independent inputs of $R_{\tilde{\alpha},\tilde{h}}$ are (e_i, e_k, e_j, e_l) and (e_j, e_k, e_i, e_l). Thus, $R_{\tilde{\alpha},\tilde{h}} = T_{ikjl} + T_{jkil}$.

As in the previous case, let $W(e_i, e_j, e_k, e_l) = c_{ijkl}$ for any $i, j, k, l \in \{1, ..., n\}$. Again, we can pick these constants so that $c_{ijkl} + c_{jkil} + c_{kijl} = 0$. Then, we can consider a linear combination $xR_{\alpha,h} + yR_{\tilde{\alpha},\tilde{h}}$. We need

$$\begin{cases} xR_{\alpha,h}(e_i, e_j, e_k, e_l) + yR_{\tilde{\alpha},\tilde{h}}(e_i, e_j, e_k, e_l) = c_{ijkl}, \\ xR_{\alpha,h}(e_j, e_k, e_i, e_l) + yR_{\tilde{\alpha},\tilde{h}}(e_j, e_k, e_i, e_l) = c_{jkil}, \\ xR_{\alpha,h}(e_k, e_i, e_j, e_l) + yR_{\tilde{\alpha},\tilde{h}}(e_k, e_i, e_j, e_l) = c_{kijl}. \end{cases}$$

So, we find $x = c_{ijkl}$ and $y = c_{jkil}$. Plugging these inputs into the linear combination shows us that

$$c_{ijkl}R_{\alpha,h}(e_i, e_j, e_k, e_l) + c_{jkil}R_{\tilde{\alpha},\tilde{h}}(e_i, e_j, e_k, e_l) = c_{ijkl} \cdot 1 + c_{jkil} \cdot 0 = c_{ijkl},$$
$$c_{ijkl}R_{\alpha,h}(e_j, e_k, e_i, e_l) + c_{jkil}R_{\tilde{\alpha},\tilde{h}}(e_j, e_k, e_i, e_l) = c_{ijkl} \cdot 0 + c_{jkil} \cdot 1 = c_{jkil},$$

and

$$\begin{aligned} c_{ijkl}R_{\alpha,h}(e_k, e_i, e, j, e_l) + c_{jkil}R_{\tilde{\alpha},\tilde{h}}(e_k, e_i, e_j, e_l) &= c_{ijkl} \cdot -1 + c_{jkil} \cdot -1 \\ &= -(c_{ijkl} + c_{jkil}) = c_{kjil}. \end{aligned}$$

Notice that the last equality follows from the Bianchi identity. Finally, we know that

$$c_{ijkl}R_{\alpha,h} + c_{kijl}R_{\tilde{\alpha},\tilde{h}} = c_{ijkl}(T_{ijkl} + T_{ikjl}) + c_{kijl}(T_{ikjl} + T_{jkil}).$$

So, we can see that the $c_{ijkl}R_{\alpha,h} + c_{kijl}R_{\tilde{\alpha},\tilde{h}}$ is zero for all other inputs. As a result, a summation of $(R_{\alpha,h} + R_{\tilde{\alpha},\tilde{h}})$'s can give the same output as $\sum c_{ijkl}T_{ijkl}$. So, $\sum c_{ijkl}T_{ijkl} \in B$.

All of the sums in (1) are in the span of B so we can conclude that $W \in B$. □

5 The Anti-Symmetric Build

The definition of the $R_{\alpha,h}$ from the previous sections was inspired by geometric considerations. However, for the anti-symmetric case we obtain our spanning set through analogy to the canonical anti-symmetric ACTs. See page 2 of [3].

Definition 5.1. We define

$$R_{\alpha,p}(X, Y, Z, W) = \alpha(X, W)p(Y, Z) - \alpha(Y, W)p(X, Z) - 2\alpha(Z, W)p(X, Y)$$

where $\alpha \in \bigotimes^2(V^*)$ and $p \in \Lambda^2(V^*)$, where $\Lambda^2(V^*)$ is the set of anti-symmetric 2-tensors on V.

From here we must first check that our $R_{\alpha,p}$'s are AACTs.

Proposition 5.2. $R_{\alpha,p}$ *is an AACT.*

Proof. Again, we check the anti-symmetry in the first two spots. We get that

$$\begin{aligned}R_{\alpha,p}(X,Y,Z,W) &= \alpha(X,W)p(Y,Z)-\alpha(Y,W)p(X,Z)-2\alpha(Z,W)p(X,Y)\\ &= -(\alpha(Y,W)p(X,Z)-\alpha(X,W)p(Y,Z)-2\alpha(Z,W)p(Y,X)\ )\\ &= -R_{\alpha,p}(Y,X,Z,W).\end{aligned}$$

Then, checking the Bianchi identity gives us that

$$\begin{aligned}&R_{\alpha,p}(X,Y,Z,W)+R_{\alpha,p}(Y,Z,X,W)+R_{\alpha,p}(Z,X,Y,W)\\ &\quad= \alpha(X,W)p(Y,Z)-\alpha(Y,W)p(X,Z)-2\alpha(Z,W)p(X,Y)\\ &\quad+\alpha(Y,W)p(Z,X)-\alpha(Z,W)p(Y,X)-2\alpha(X,W)p(Y,Z)\\ &\quad+\alpha(Z,W)p(X,Y)-\alpha(X,W)p(Z,Y)-2\alpha(Y,W)p(Z,X)\\ &\quad= 0.\end{aligned}$$

□

With that we formally define the span of the set of $R_{\alpha,p}$'s.

Definition 5.3. We define Q to be the set $\operatorname{span}(\{R_{\alpha,p}:\alpha\in\bigotimes^2(V^*),p\in\Lambda^2(V)\})$.

Much like in Section 4, this leads us to the main theorem of Section 5.

Theorem 5.4. *The set of affine algebraic curvature tensors on* V *equals* Q.

Proof. This proof follows very similarly to the proof of Theorem 1 in the previous section. In fact, we define T_{ijkl} the same as in the previous theorem, and we break up an arbitrary AACT, W, in the same way. We again have

$$\begin{aligned}W = &\sum_{i,j\text{ distinct}} c_{ijij}T_{ijij}+c_{ijii}T_{ijii}\\ &+\sum_{i,j,k\text{ distinct}} c_{ijki}T_{ijki}+c_{ijik}T_{ijik}+c_{ijkk}T_{ijkk}\\ &+\sum_{i,j,k,l\text{ distinct}} c_{ijkl}T_{ijkl}\end{aligned}\tag{2}$$

Also, as we had done in the previous proof let $W(e_i,e_j,e_k,e_l)=c_{ijkl}$ for any $i,j,k,l\in\{1,\ldots,n\}$, and pick these c's so that they satisfy the Bianchi identity.
Case 1: T_{ijij}

We again consider an arbitrary $R_{\alpha,h}$ with the input (e_i, e_j, e_i, e_j). We see that

$$R_{\alpha,p}(e_i, e_j, e_i, e_j) = \alpha(e_i, e_j)p(e_j, e_i) - \alpha(e_j, e_j)p(e_i, e_i) - 2\alpha(e_i, e_j)p(e_i, e_j).$$

Let $\alpha(e_i, e_j) = 1$, $p(e_j, e_i) = 1$, and all other independent combinations of basis vectors be mapped to 0. Then, we have that $R_{\alpha,p}(e_i, e_j, e_i, e_j) = 3$ and $R_{\alpha,p}(e_j, e_i, e_i, e_j) = -3$. As such, we need to show that $R_{\alpha,p}$ is zero on all other independent combinations of basis vectors.

In order for $R_{\alpha,p}$ to be non-zero, its input must have an e_j in its fourth slot. But, then all other combinations of basis vectors will be (e_i, e_j, e_i, e_j), (e_j, e_i, e_i, e_j), or (e_i, e_i, e_j, e_j). The last of these is 0 for all AACTs due to anti-symmetry in the first two positions. So, we have found an $R_{\alpha,p}$ such that $\frac{1}{3}R_{\alpha,p} = T_{ijij} \in Q$.

Case 2: T_{ijii}

Similarly, computing $R_{\alpha,p}$ with the input (e_i, e_j, e_i, e_i) gives us that

$$R_{\alpha,p}(e_i, e_j, e_i, e_i) = \alpha(e_i, e_i)p(e_j, e_i) - \alpha(e_j, e_i)p(e_i, e_i) - 2\alpha(e_i, e_i)p(e_i, e_j).$$

Let $\alpha(e_i, e_i) = 1$ $p(e_j, e_i) = 1$, and all other combinations of basis vectors be mapped to 0. Then the only non-zero, independent input is (e_i, e_j, e_i, e_i) which is mapped to 3. So, $\frac{1}{3}R_{\alpha,p} = T_{ijii} \in Q$.

Case 3: T_{ijik} We start in the same way as before by computing $R_{\alpha,p}(e_i, e_j, e_i, e_k)$. If we let $\alpha(e_i, e_k) = 1$, $p(e_j, e_i) = 1$, and all other combinations of basis vectors be mapped to 0 we get that

$$R_{\alpha,p}(e_i, e_j, e_i, e_k) = \alpha(e_i, e_k)p(e_j, e_i) - \alpha(e_j, e_k)p(e_i, e_i) - 2\alpha(e_i, e_k)p(e_i, e_j) = 3.$$

As in the first two cases this is the only non-zero, independent input, and thus we get that $\frac{1}{3}R_{\alpha,p} = T_{ijik} \in Q$.

Case 4: T_{ijki} and T_{jkii}

We let $\alpha(e_j, e_i) = 1$, $p(e_i, e_k) = -1$, and compute that $R_{\alpha,p}(e_i, e_j, e_k, e_i) = 1$. This $R_{\alpha,p}$ has two other distinct inputs, namely (e_j, e_k, e_i, e_i) which is also mapped to 1 and (e_k, e_i, e_j, e_i) which is mapped to -2. So, $R_{\alpha,p} = T_{ijki} + T_{jkii} - 2T_{kiji}$.

We now look at the T_{jkii} case. Let $\tilde{\alpha}(e_k, e_i) = 1$, $\tilde{p}(e_j, e_i) = -1$, and compute that

$$R_{\tilde{\alpha},\tilde{p}}(e_j, e_k, e_i, e_i) = \tilde{\alpha}(e_j, e_i)\tilde{p}(e_k, e_i) - \tilde{\alpha}(e_k, e_i)\tilde{p}(e_j, e_i) - 2\tilde{\alpha}(e_i, e_i)\tilde{p}(e_j, e_k) = 1.$$

We also see that $R_{\tilde{\alpha},\tilde{p}}$ maps (e_k, e_i, e_j, e_i) to 1 and (e_i, e_j, e_k, e_i) to -2. So, $R_{\tilde{\alpha},\tilde{p}} = T_{jkii} + T_{kiji} - 2T_{ijki}$.

Much like in case 4 of the symmetric build's proof, we solve the following systems of equations:

$$\begin{cases} xR_{\alpha,p}(e_i, e_j, e_k, e_i) + yR_{\tilde{\alpha},\tilde{p}}(e_i, e_j, e_k, e_i) = c_{ijki}, \\ xR_{\alpha,p}(e_j, e_k, e_i, e_i) + yR_{\tilde{\alpha},\tilde{p}}(e_j, e_k, e_i, e_i) = c_{jkii}, \\ xR_{\alpha,p}(e_k, e_i, e_j, e_i) + yR_{\tilde{\alpha},\tilde{p}}(e_k, e_i, e_j, e_i) = c_{kiji}. \end{cases}$$

Solving this ultimately gives us that $x = \frac{2c_{jkii}+c_{ijki}}{3}$ and $y = \frac{c_{jkii}-c_{ijki}}{3}$. Finally, we see that if we plug in x and y into the above linear combination and evaluate sum at (e_i, e_j, e_k, e_i), (e_j, e_k, e_i, e_i), or (e_k, e_i, e_j, e_i) we get the desired result. Moreover, since

$$\begin{aligned}\frac{2c_{jkii}+c_{ijki}}{3}\mathrm{R}_{\alpha,p} + \frac{c_{jkii}-c_{ijki}}{3}\mathrm{R}_{\tilde{\alpha},\tilde{p}} = \frac{2c_{jkii}+c_{ijki}}{3}&(\mathrm{T}_{jkii}+\mathrm{T}_{ijki}-2\mathrm{T}_{kiji}) \\ &+\frac{c_{jkii}-c_{ijki}}{3}(\mathrm{T}_{jkii}+\mathrm{T}_{kiji}-2\mathrm{T}_{ijki})\end{aligned}$$

these are the only independent inputs that do not map to 0. Hence, we have that both $\sum c_{ijki}\mathrm{T}_{ijki}$ and $\sum c_{jkii}\mathrm{T}_{jkii}$ can be expressed as a sum of $\mathrm{R}_{\alpha,p}$'s.

Case 5: T_{ijkl}

As in the last few cases, we let $\alpha(e_i, e_l) = 1$, $p(e_j, e_k) = 1$, and all other combinations of basis vectors be mapped to 0. Then, we get that

$$\mathrm{R}_{\alpha,p}(e_i, e_j, e_k, e_l) = \alpha(e_i, e_l)p(e_j, e_k) - \alpha(e_j, e_l)p(e_i, e_k) - 2\alpha(e_k, e_l)p(e_i, e_j) = 1.$$

This has two other non-zero, independent inputs: (e_k, e_i, e_j, e_l) which is mapped to 1 and (e_j, e_k, e_i, e_l) which is mapped to -2. So, $\mathrm{R}_{\alpha,p} = \mathrm{T}_{ijkl} + \mathrm{T}_{kijl} - 2\mathrm{T}_{jkil}$.

Now let $\tilde{\alpha}(e_k, e_l) = 1$, $\tilde{p}(e_i, e_j) = 1$, and compute

$$\mathrm{R}_{\tilde{\alpha},\tilde{p}}(e_k, e_i, e_j, e_l) = \tilde{\alpha}(e_k, e_l)\tilde{p}(e_i, e_j) - \tilde{\alpha}(e_i, e_l)\tilde{p}(e_k, e_j) - 2\tilde{\alpha}(e_j, e_l)\tilde{p}(e_k, e_i) = 1.$$

This also has two other non-zero, independent inputs outside its kernel, (e_j, e_k, e_i, e_l) which maps to 1 and (e_i, e_j, e_k, e_l) which maps to -2. So, $\mathrm{R}_{\tilde{\alpha},\tilde{p}} = \mathrm{T}_{kijl} + \mathrm{T}_{jkil} - 2\mathrm{T}_{ijkl}$.

Much like in Case 4, we now solve the following system of equations:

$$\begin{cases} x\mathrm{R}_{\alpha,p}(e_i, e_j, e_k, e_l) + y\mathrm{R}_{\tilde{\alpha},\tilde{p}}(e_i, e_j, e_k, e_l) = c_{ijkl}, \\ x\mathrm{R}_{\alpha,p}(e_k, e_i, e_j, e_l) + y\mathrm{R}_{\tilde{\alpha},\tilde{p}}(e_k, e_i, e_j, e_l) = c_{kijl}, \\ x\mathrm{R}_{\alpha,p}(e_j, e_k, e_i, e_l) + y\mathrm{R}_{\tilde{\alpha},\tilde{p}}(e_j, e_k, e_i, e_l) = c_{jkil}. \end{cases}$$

We get that $x = \frac{2c_{kijl}+c_{ijkl}}{3}$ and $y = \frac{c_{kijl}-c_{ijkl}}{3}$. So, we can express $\sum c_{ijkl}\mathrm{T}_{ijkl}$ as a sum of $(x\mathrm{R}_{\alpha,p} + y\mathrm{R}_{\tilde{\alpha},\tilde{p}})$'s. Hence, $\sum c_{ijkl}\mathrm{T}_{ijkl} \in \mathrm{Q}$.

As such, all of the sums in (2) are a linear combination of $\mathrm{R}_{\alpha,p}$'s which implies that $\mathrm{W} \in \mathrm{Q}$. Therefore Q spans the AACTs. □

6 Representation by Canonical AACTs

Now that we have created two spanning sets for the AACTs researchers can begin to ask questions about their representations. This section acts as a starting point for this research by proving that the maximum number of canonical AACTs are required to represent an arbitrary AACT in a 2-dimensional real vector space is 1. We formally define this notion, which is the title of this section, in Definitions 14 and 15.

Definition 6.1. Let V be an n-dimensional vector space. We define

$$\sigma(\mathrm{R}) = \min\left\{k : \mathrm{R} = \sum_{i=1}^{k} \mathrm{R}_{\alpha_i, h_i}\right\},$$

for R an AACT. Then, we define

$$\sigma(n) = \max_{\mathrm{R} \in \mathrm{AACT}} \sigma(\mathrm{R}).$$

Note that $\max_{\mathrm{R}\in\mathrm{AACT}} \sigma(\mathrm{R})$ exists because the dimension of the space of AACTs is $\frac{n^2(n^2-1)}{3}$ [9], and thus each spanning set needs no more than $\frac{n^2(n^2-1)}{3}$ tensors to linearly combine to any AACT. That lets us express the following theorem.

Theorem 6.2. $\sigma(2) = 1$.

Proof. Let A be an arbitrary AACT in a 2-dimensional real vector space. A's behavior for any (x, y, z, w) is defined by its behavior on a set of basis vectors of M. Let $\{e_1, e_2\}$ be basis vectors for M. Then, by evaluating A on all the independent sets of basis vectors we can show that there is an $\mathrm{R}_{\alpha,h} = \mathrm{A}$. If there was such an $\mathrm{R}_{\alpha,h}$ then we would need

$$\begin{cases} \mathrm{A}_{1211} = \alpha(e_1, e_1)h(e_2, e_1) - \alpha(e_2, e_1)h(e_1, e_1), \\ \mathrm{A}_{1212} = \alpha(e_1, e_2)h(e_2, e_1) - \alpha(e_2, e_2)h(e_1, e_1), \\ \mathrm{A}_{1221} = \alpha(e_1, e_1)h(e_2, e_2) - \alpha(e_2, e_1)h(e_1, e_2), \\ \mathrm{A}_{1222} = \alpha(e_1, e_2)h(e_2, e_2) - \alpha(e_2, e_2)h(e_1, e_2). \end{cases}$$

Then, simply letting $h(e_1, e_1) = 1$, $h(e_2, e_2) = 1$, $-\alpha(e_2, e_1) = \mathrm{A}_{1211}$, $-\alpha(e_2, e_2) = \mathrm{A}_{1212}$, $\alpha(e_1, e_1) = \mathrm{A}_{1221}$, and $\alpha(e_1, e_2) = \mathrm{A}_{1222}$ gives us an $\mathrm{R}_{\alpha,h} = \mathrm{A}$. □

In the same vein of thought we define the maximum number of anti-symmetric AACTs required to express an arbitrary AACT in n-dimensions.

Definition 6.3. Let V be an n-dimensional vector space. We define

$$\kappa(\mathrm{R}) = \min\left\{k : \mathrm{R} = \sum_{i=1}^{k} \mathrm{R}_{\alpha_i, p_i}\right\},$$

for R an AACT. Then, we define

$$\kappa(n) = \max_{\mathrm{R} \in \mathrm{AACT}} \kappa(\mathrm{R}).$$

Note that $\max_{\mathrm{R}\in\mathrm{AACT}} \kappa(\mathrm{R})$ exists for the same reason that $\max_{\mathrm{R}\in\mathrm{AACT}} \sigma(\mathrm{R})$ exists. This gives us a similar result to Theorem 3.

Theorem 6.4. $\kappa(2) = 1$.

Proof. By the same logic as in the previous proof, we consider if there was an $R_{\alpha,p} = A$. Then we would need

$$\begin{cases} A_{1211} = \alpha(e_1,e_1)p(e_2,e_1) - \alpha(e_2,e_1)p(e_1,e_1) - 2\alpha(e_1,e_1)p(e_1,e_2), \\ A_{1212} = \alpha(e_1,e_2)p(e_2,e_1) - \alpha(e_2,e_2)p(e_1,e_1) - 2\alpha(e_1,e_2)p(e_1,e_2), \\ A_{1221} = \alpha(e_1,e_1)p(e_2,e_2) - \alpha(e_2,e_1)p(e_1,e_2) - 2\alpha(e_2,e_1)p(e_1,e_2), \\ A_{1222} = \alpha(e_1,e_2)p(e_2,e_2) - \alpha(e_2,e_2)p(e_1,e_2) - 2\alpha(e_2,e_2)p(e_1,e_2). \end{cases}$$

So, much like in Theorem 3, letting $p(e_1,e_1) = 1$, $p_(e_2,e_2) = 1$, $-\alpha(e_2,e_1) = A_{1211}$, $-\alpha(e_2,e_2) = A_{1212}$, $\alpha(e_1,e_1) = A_{1221}$, and $\alpha(e_1,e_2) = A_{1222}$ gives us an $R_{\alpha,p} = A$. □

7 Conclusion and Open Problems

Thus far we have constructed two spanning sets for the AACTs and discovered $\sigma(2)$ and $\kappa(2)$. As was mentioned in the introduction, this is only the starting point for questions concerning AACTs. Here are just a few interesting open problems:

1. What are upper bounds for $\sigma(n)$ and $\kappa(n)$? Are these bounds sharp?

2. Under what conditions can linear independence of multiple canonical AACTs occur?

3. Is there a geometric proof that the symmetric build spans the AACTs? There is such a proof for the ACTs, but this proof has not been adapted for AACTs. Adapting it to the AACTs would require that each $R_{\alpha,h}$ be geometrically realized on a manifold, M, with a connection, ∇. M would then have to be embedded into $\mathbb{R}^n$ in a way such that $\overline{\nabla}$ is flat in $\mathbb{R}^n$ and $\nabla = (\overline{\nabla})^\top$. It is unknown if such an embedding is always possible.

4. In [1] the authors mention an object similar to our $R_{\alpha,h}$'s, but instead of $h \in S^2(V^*)$ they use the manifold's metric, g. They then go on to prove that this is an AACT. Is the set $\{R_{\alpha,g} | \alpha \in \bigotimes^2(V^*)\}$ a spanning set for the AACTs?

8 Acknowledgements

I would like to thank Dr. Corey Dunn for his excellent mentorship during this project. I would also like to thank Dr. Rolland Trapp for his support. This research was generously made possible by California State San Bernardino and NSF grant 2050894.

References

[1] M. Brozos-Vázquez, P. B. Gilkey and S. Nikčević, *Geometric realizations of curvature*, ICP Advanced Texts in Mathematics, 6, Imp. Coll. Press, London, 2012. MR2964268

[2] J. Díaz-Ramos and E. García-Río, A note on the structure of algebraic curvature tensors, Linear Algebra Appl. **382** (2004), 271–277. MR2050112

[3] P. B. Gilkey, *Geometric properties of natural operators defined by the Riemann curvature tensor*, World Sci. Publishing, Inc., River Edge, NJ, 2001. MR1877530

[4] J. M. Lee, *Riemannian manifolds*, Graduate Texts in Mathematics, 176, Springer, New York, 1997. MR1468735

[5] G.López, An Upper Bound on $\eta(n)$, Unpublished manuscript. (2018).

[6] K. Ragosta, Canonical Expressions of Algebraic Curvature Tensors, Unpublished manuscript. (2019).

[7] W. Rudin, *Principles of mathematical analysis*, McGraw-Hill, Inc., New York, 1953. MR0055409

[8] R.Shapiro, Algebraic Curvature Tensors of Einstein and Weakly Einstein Model Spaces, Unpublished manuscript. (2017).

[9] R. S. Strichartz, Linear algebra of curvature tensors and their covariant derivatives, Canad. J. Math. **40** (1988), no. 5, 1105–1143. MR0973512

[10] C.Tripp, Linear Independence of Algebraic Curvature Tensors in the Higher-Signature Setting, Unpublished manuscript. (2019).

Stephen Kelly
University of Chicago
sjk7777@comcast.net