A NEW CLASS OF GEOMETRICALLY DEFINED HYPERGRAPHS ARISING FROM THE HADWIGER-NELSON PROBLEM*

Sean Fiscus
Duke University
sean.fiscus@duke.edu

Eric Myzelev
University of Pennsylvania
myzelev@sas.upenn.edu

Hongyi Zhang
Haverford College
hzhang3@haverford.edu

Abstract

There is a famous problem in geometric graph theory to find the chromatic number of the unit distance graph on Euclidean space; it remains unsolved. A theorem of Erdős and De-Bruijn simplifies this problem to finding the maximum chromatic number of a finite unit distance graph. Via a construction built on sequential finite graphs obtained from a generalization of this theorem, we have found a class of geometrically defined hypergraphs of arbitrarily large edge cardinality, whose proper colorings exactly coincide with the proper colorings of the unit distance graph on $\mathbb{R}^d$. We also provide partial generalizations of this result to arbitrary real normed vector spaces.

1 Introduction

$\mathbb{Z}^+$ will denote the set of positive integers. A hypergraph is a pair $\mathscr{H} = (V, E)$ in which V is a non-empty set, the set of vertices of $\mathscr{H}$, and $E \subseteq 2^V$ (i.e. E is a set of subsets of V) satisfying $e \in E$ implies $|e| \geq 2$. (In other definitions, singletons may be allowed in E and/or E may be a multi-set.) E is the set of hyperedges, or just edges, of $\mathscr{H}$. A proper coloring of $\mathscr{H}$ is a function $\varphi : V \to C =$ some set of colors, such that

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no $e \in E$ is "monochromatic with respect to φ". This means that for each $e \in E$, $\varphi|_e$ is not constant. The chromatic number of $\mathscr{H}$ is the smallest cardinality $|C|$ such that there is a proper coloring $\varphi : V \to C$.

If $\mathscr{H} = (V,E)$ is a hypergraph and $U \subseteq V$, $|U| \geq 2$, the subhypergraph of $\mathscr{H}$ induced by U is $\mathscr{H}|_U = (U, E \cap 2^U)$. That is, the vertex set of $\mathscr{H}|_U$ is U and the edges of $\mathscr{H}|_U$ are the edges of $\mathscr{H}$ that lie in U. A hypergraph $\mathscr{H} = (V,E)$ in which $|e| = 2$ for every $e \in E$ is a simple graph. The De Bruijn-Erdős theorem [1] concerning the chromatic numbers of infinite graphs is as follows:

If $m \in \mathbb{Z}^+, \mathscr{H} = (V,E)$ is a graph, and $\chi(\mathscr{H}|_F) \leq m$ for every finite set $F \subseteq V$, then $\chi(\mathscr{H}) \leq m$.

To prove our results we shall need a known generalization of this theorem to hypergraphs, which will be provided in the next section.

2 De Bruijn-Erdős for hypergraphs with finite edges

The following theorem generalizes the De Bruijn-Erdős Theorem to hypergraphs. A different proof may be found in [2], Chapter 26.

Theorem 2.1 (D-E)**.** *Suppose that $m \in \mathbb{Z}^+$, $\mathscr{H} = (V,E)$ is a hypergraph with $2 \leq |e| < \infty$ for every $e \in E$, and $\chi(\mathscr{H}|_F) \leq m$ for every finite subset F of V. Then $\chi(\mathscr{H}) \leq m$.*

Remark 2.1.1. Since enlarging the edges of a hypergraph – i.e. putting new vertices in some of the edges to join those that were there to begin with – would seem to make it easier to avoid monochromatic edges when coloring the vertices, we are wondering if we really need the hypothesis, in Theorem D-E, that all edges of $\mathscr{H}$ are finite subsets of V. In the proof to come the hypothesis is actually used, and we do not see a way to avoid that use, except by trading the hypothesis for other, clumsier hypotheses. Is there a mathematical logician in the house?

Proof. Suppose that $\mathscr{H} = (V,E)$, and $m \in \mathbb{Z}^+$ satisfy the hypothesis of Theorem D-E. Let $[m] = \{1,\dots,m\}$. A coloring of V with colors $1,\dots,m$ is an element of $X = [m]^V = \prod_{v \in V}[m]$, the Cartesian product of $[m]$ with itself $|V|$ times. Let $[m]$ have the discrete topology, in which every singleton is an open set. Since $[m]$ is finite, obviously $[m]$ is compact (in this or any other topology). By the Tychonoff theorem [3], X, with the product topology, is compact.

In the usual definition of the product topology on X, the basic open neighborhoods of a coloring $\varphi \in X$ are the sets
' $N(\psi,F) = \{\psi \in X | \ \psi|_F = \varphi|_F\}$, $F \in \mathscr{F}(V) = \{\text{finite subsets of } V\}$. For $F \in \mathscr{F}(V)$, let $Y_F = \{\varphi \in X | \ \varphi|_F : F \to [m]$ is a proper coloring of $\mathscr{H}|_F\}$. Since $\chi(\mathscr{H}|_F) \leq m, Y_F$ is non-empty. Also, Y_F is closed in X: if $\psi \in X \backslash Y_F$, then $\psi|_F$ is not a proper coloring of H_F, so for some $e \in E \cap 2^F$, $\psi|_e$ is constant (i.e., the vertices in e are all assigned the same $j \in [m]$ by ψ). Any coloring in $N(\psi, e)$ will assign that same j to each vertex of e; therefore, $N(\psi,e) \subseteq X \setminus Y_F$. Thus $X \backslash Y_F$ is open in the product topology, so Y_F is closed.

If $F_1,\dots,F_t \in \mathscr{F}(V)$, then $F_1 \cup \cdots \cup F_t \in \mathscr{F}(V)$, and we have

$$\emptyset \neq Y_{\bigcup_{i=1}^t F_i} \subseteq Y_{F_1} \cap \cdots \cap Y_{F_t}$$

Thus $\{Y_F | F \text{ is in } \mathscr{F}(V)\}$ has the finite intersection property: any intersection of finitely many sets from the family is non-empty.

In a compact topological space, the intersection of all the sets in a collection of closed sets with the finite intersection property is non-empty. Therefore $\bigcap_{F \in \mathscr{F}(V)} Y_F \neq \emptyset$. Suppose $\varphi \in \bigcap_{F \in \mathscr{F}(V)} Y_F$. We claim that $\varphi : V \to [m]$ is a proper coloring of $\mathscr{H}$. Suppose that $e \in E$. Then e is a finite subset of V. Since e is an edge of $\mathscr{H}\mid_e$, $\varphi \in Y_e$ implies that φ assigns at least two colors from [m] to the elements of e. □

3 Hypergraphs Equivalent to the Unit Distance Graph

The motivation for this result is the following question: Does there exist a finite set of triangles S in $\mathbb{R}^d$ such that the number of colors in a coloring of $\mathbb{R}^d$ required to forbid monochromatic copies of triangles in S is the same as the chromatic number of the unit Euclidean distance graph on $\mathbb{R}^d$? The answer is yes. In fact, we prove a stronger result: in addition to generalizing triangles to arbitrary m-point sets, we also show that there is such a set S so that a coloring φ of $\mathbb{R}^d$ forbids congruent copies of m-gons in S if and only if φ forbids unit distance. To make this precise, we introduce a notion of equivalence of hypergraphs.

Definition 3.1. *Let S be a set, and $\mathscr{H} = (S, E_{\mathscr{H}}), \mathscr{G} = (S, E_{\mathscr{G}})$ be hypergraphs where S is the vertex set. We say $\mathscr{H}$ is equivalent to $\mathscr{G}$ if $\chi(\mathscr{H}) = \chi(\mathscr{G})$ and $\varphi : S \to C$, such that $|C| = \chi(\mathscr{H}) = \chi(\mathscr{G})$, is a proper coloring of $\mathscr{H}$ if and only if φ is a proper coloring of $\mathscr{G}$.*

For the purposes of this paper, we define an m-gon as an arbitrary set of cardinality m. This means that some vertices of the m-gon can be collinear, which contradicts the standard geometric definition.

We also define a unit m-gon to be an m-gon in $\mathbb{R}^d$ as defined above with the following additional property: there exists at least one pair of points x, y in m which are Euclidean distance 1 apart.

Theorem 3.1. *Let $M \subset 2^{\mathbb{R}^d}$ be a non-empty finite set of m-gons, for some $m \geq 2$. Define $\mathscr{H}(M)$ as the hypergraph on $\mathbb{R}^d$ with edge set $E = \{X \subset \mathbb{R}^d \mid X \text{ is congruent in } \mathbb{R}^d \text{ to some } T \in M\}$.*

Then there must exist some finite set S of $(m+1)$-gons, such that $\mathscr{H}(M)$ is equivalent to $\mathscr{H}(S)$, where $\mathscr{H}(S)$ is an $(m+1)$-uniform hypergraph with edge set $E = \{Y \in \mathbb{R}^d \mid Y \text{ is congruent in } \mathbb{R}^d \text{ to some element of } S\}$.

Proof. We shall proceed by recursively obtaining sets $S_1, S_2, \ldots; F_1, F_2, \ldots$ satisfying:

1. $S_1 \subseteq S_2 \subseteq \ldots$
2. Each S_j is a finite set of $(m+1)$-gons in $\mathbb{R}^d$, with each $(m+1)$-gon containing some m-gon $X \in M$ as a subset of its points.
3. Defining $E_j :=$ {congruent copies in $\mathbb{R}^d$ of the $(m+1)$-gons in S_j} and $\mathscr{H}_j := (\mathbb{R}^d, E_j)$, we obtain F_j, a finite subset of $\mathbb{R}^d \setminus M$ such that $\chi(\mathscr{H}_j|_{F_j}) = \chi(\mathscr{H}_j)$.
4. $S_{j+1} = S_j \cup \{X \cup \{z\} | z \in F_j, X \in M\}$.

Before giving the recursion, let us note that if the $S_j, F_j, \mathscr{H}_j = (\mathbb{R}^d, E_j)$ are as above then we have the following observations.

(i) Since, for each $j = 1,2,\ldots$ and $e \in E_j$, e contains a copy of some $X \in M$, $\chi(\mathscr{H}_j) \leq \chi(\mathscr{H}(M))$.

(ii) In view of the definition of E_j, $S_1 \subseteq S_2 \subseteq \ldots$ implies that $E_1 \subseteq E_2 \subseteq \ldots$, and thus $\chi(\mathscr{H}_1) \leq \chi(\mathscr{H}_2) \leq \ldots$

(iii) $\chi(\mathscr{H}(M))$ is finite. To see this, observe that in $\mathbb{R}^d$ with the Euclidean norm $|| \; ||$, the sets $\{x,y\}$ congruent to a two-set $\{u,v\}$ are just the sets satisfying $||x-y|| = ||u-v||$. Therefore, a coloring of $\mathbb{R}^d$ forbidding congruent copies of $\{u,v\}$ is, in other jargon, the same as a coloring which forbids the distance $||u-v||$. For every positive distance, the smallest number of colors needed to forbid that distance is $\chi(\mathbb{R}^d, 1)$, the chromatic number of the Euclidean unit distance graph on $\mathbb{R}^d$.
Suppose that $|M| = n$ (recall that M is finite). From each $T \in M$ select a 2-set D; let the 2-sets selected be
$D_1, D_2, ..., D_n$ and the distances determined by the 2 points in these sets be $d_1, .., d_n$. For each $i = 1, ..., n$, let φ_i be a coloring of $\mathbb{R}^n$ with $\chi(\mathbb{R}^d, 1)$ colors that forbids the distance d_i. Now color $\mathbb{R}^d$ by an assignment ψ of n-tuples: $\psi(r) = (\varphi_1(r), ..., \varphi_n(r))$. We now have colored $\mathbb{R}^d$ with $\chi(R^d, 1)^n < \infty$ colors, and it is easy to see that ψ is a proper coloring of $\mathscr{H}(M)$: For any $T' \subseteq \mathbb{R}^d$ congruent to some $T \in M$, $\mathscr{T}'$ will contain a doubleton congruent to one of the D_i; to the two vectors in that doubleton, φ_i will assign different colors, which means that ψ will assign different n-tuples to the 2 vectors, which means that T' is not monochromatic.

(iv) By (i), (iii), and Theorem D-E, it follows that for each j there is a finite set $F_j \in \mathbb{R}^d$ such that $\chi(\mathscr{H}_j|_{F_j}) = \chi(\mathscr{H}_j)$.

(v) Suppose ρ is an isometry of $\mathbb{R}^d$ and F is a finite non-empty subset of $\mathbb{R}^d$. Because ρ maps each $e \in E_j$ to an edge $\rho(e) \in E_j$, $\chi(\mathscr{H}_j|_F) = \chi(\mathscr{H}_j|_{\rho(F)})$.

The recursion:

Let $S_1 = \{X \cup \{a\} \mid X \in M\}$, for some $a \in \mathbb{R}^d \backslash \bigcup_{T \in M} T$. Let $F_1 \subseteq \mathbb{R}^d$ be a finite set such that $\chi(\mathscr{H}_1|_{F_1}) = \chi(\mathscr{H}_1)$; as explained in (iv) above, Theorem D-E guarantees the existence of such an F_1, and by (v), we can assume $F_1 \cap (\bigcup_{T \in M} T) = \emptyset$; if $F_1 \cap (\bigcup_{T \in M} T) \neq \emptyset$, replace F_1 by a translate of itself.

From there, the recursion is dictated in 3 and 4: From S_j we get E_j; the existence of F_j is guaranteed. Then we define S_{j+1} by 4, above, and roll on.

Since the integer sequence $(\chi(\mathscr{H}_j))_j$ is non-decreasing and bounded above by $\chi(\mathscr{H}(M))$, clearly it will be eventually constant. If that eventual constant value were $\chi(\mathscr{H}(M))$, we would be almost done, except for showing that every proper coloring of $\mathscr{H}_k$ also properly colors $\mathscr{H}(M)$. All will be accomplished by the following.

Clearly $\chi(\mathscr{H}_1) \geq 2 > 1$ whereas $\chi(\mathscr{H}_k) \leq \chi(\mathscr{H}(M)) \leq k$ for $k \geq \chi(\mathscr{H}(M))$. Therefore there is a first value of $k \in \mathbb{Z}^+$ such that $\chi(\mathscr{H}_k) \leq k$.

We have $k > 1$ and

$$k - 1 < \chi(\mathscr{H}_{k-1}) \leq \chi(\mathscr{H}_k) \leq k$$

whence $\chi(\mathscr{H}_{k-1}) = \chi(\mathscr{H}_k) = k \leq \chi(\mathscr{H}(M))$.

Let $\varphi : \mathbb{R}^d \to \{1, \ldots, k\}$ be a proper coloring of $\mathscr{H}_k$, and since $S_{k-1} \subset S_k$, φ is also a proper coloring of $\mathscr{H}_{k-1}$. If φ is a proper coloring of $\mathscr{H}(M)$, then $\chi(\mathscr{H}(M)) = k$, and, since φ is an arbitrarily chosen proper coloring of $\mathscr{H}_k$, the claim of this theorem will be affirmed, with $S = S_k$.

Suppose, on the contrary, that φ does not properly color $\mathscr{H}(M)$, implying that for some $X' = \{a_1, ..., a_m\}$ a congruent copy of some $X \in M$, $\varphi(a_1) = \varphi(a_2) = ... = \varphi(a_m)$. We can, without loss of generality, convene that $\varphi(a_1) = ... = \varphi(a_m) = k$. Let $\rho : \mathbb{R}^d \to \mathbb{R}^d$ be an isometry such that $\rho(X) = X'$. Consider any point $\rho(z) \in \rho(F_{k-1})$, and note that if it was colored with color k then $X' \cup \{\rho(z)\}$ would be monochromatic and congruent to $X \cup \{z\}$, an edge in $\mathscr{H}_{k-1}$, which is impossible since φ properly colors $\mathscr{H}_{k-1}$. Thus φ is a proper coloring of $\mathscr{H}_{k-1} |_{F_{k-1}}$ with $k-1$ colors. This means that $\chi(\mathscr{H}_{k-1}) = \chi(\mathscr{H}_{k-1} |_{F_{k-1}}) \leq k - 1$, and contradicts the fact that $\chi(\mathscr{H}_{k-1}) = k$.
□ □

Corollary 3.1.1. *For all integers $m \geq 2$, there exists a finite set S of unit m-gons, such that $\mathscr{H}(S))$ is equivalent to $(\mathbb{R}^d, 1)$, where $\mathscr{H}(S) = \mathscr{H}(\mathbb{R}^d, E)$ and $E = \{X \subset \mathbb{R}^d \mid X$ is congruent in $\mathbb{R}^d$ to some element of $S\}$ and $(\mathbb{R}^d, 1)$ is the Euclidean unit distance graph on $\mathbb{R}^d$.*

Proof. We prove the statement by induction. When $m = 2$, the corollary is trivially true.

Now consider $m > 2$ and suppose that the corollary holds true for $m - 1$. Then there exists a finite set S_{m-1} of $m-1$-gons as asserted in the Corollary. By the Theorem, there exists a finite set S_m of m-gons such that $\chi(\mathscr{H}(S_m)) = \chi(\mathscr{H}(S_{m-1}))$, and any proper coloring of $\mathscr{H}(S_m)$ with $\chi(\mathscr{H}(S_m)$ colors is also a proper coloring of $\mathscr{H}(S_{m-1})$.

According to the inductive hypothesis,
$\chi(\mathscr{H}(S_{m-1})) = \chi(\mathbb{R}^d, 1)$, and any proper coloring of $\mathscr{H}(S_{m-1})$ with $\chi(\mathscr{H}(S_{m-1})$ colors is also a proper coloring of the Euclidean unit distance graph on $\mathbb{R}^d$. Thus, $\chi(\mathscr{H}(S_m)) = \chi(\mathbb{R}^d, 1)$, and any proper coloring of $\mathscr{H}(S_m)$ with $\chi(\mathscr{H}(S_m))$ colors is also a proper coloring of the Euclidean unit distance graph on $\mathbb{R}^d$.

Also note that because there are two points distance 1 apart in all of the $m-1$-gons in S_{m-1}, for every m-gon in S_m, there must also be two points distance 1 apart. That is, S_m is a finite set of unit m-gons. □

4 Generalizing to Non-Euclidean Norms on $\mathbb{R}^d$

In the preceding sections, distance in $\mathbb{R}^d$ was provided by the Euclidean norm, hereinafter to be denoted as $||\cdot||_2$. Some, but not all, of Theorem 3.1 and its corollary survives generalization to the setting of a finite-dimensional normed vector space over

$\mathbb{R}$. Without loss of generality, the vector space will be $\mathbb{R}^d$ and the norm will be denoted as $||\cdot||$.

Two sets $X, Y \subseteq \mathbb{R}^d$ are <u>congruent</u> <u>copies</u> of each other in $(\mathbb{R}^d, ||\cdot||)$ if and only if one of them is the image of the other under a composition, in either order, of a surjective linear isometry of $(\mathbb{R}^d, ||\cdot||)$ and a translation. With this definition of congruence, we lose one of the support beams to our geometric intuition that may seem essential to the proof of Theorem 3.1: there can exist $u, v, x, y \in \mathbb{R}^d$ such that $||u-v|| = ||x-y|| > 0$ and yet $\{u, v\}$ and $\{x, y\}$ are not congruent.

However, note: if $\{u, v\}$ and $\{x, y\}$ are congruent, then $||u-v|| = ||x-y||$. Consequently, if φ is a coloring of $\mathbb{R}^d, ||\cdot||)$ which forbids a distance $a > 0$, then $|\varphi(e)| > 1$ if $e \subseteq \mathbb{R}^d$ contains two points a distance a apart.

For $a > 0$ let $\chi((\mathbb{R}^d, ||\cdot||), a)$ denote the smallest $|C|$ such that some coloring $\varphi: \mathbb{R}^d \to C$ forbids the distance a. By the properties of norms, it is clear that $\chi((\mathbb{R}^d, ||\cdot||), a) = \chi((\mathbb{R}^d, ||\cdot||), 1)$ for all $a > 0$. Also very important for our generalizations:

Lemma 4.1. *For all* $||\cdot||$, $\chi((\mathbb{R}^d, ||\cdot||), 1) < \infty$.

This is well known, but we will provide a proof outline in an Appendix.

From Lemma 4.1 and the definition of congruence we obtain, as in section 3, and by the same argument, the following.

Lemma 4.2. *Suppose that* $\S$ *is a finite collection of subsets of* $\mathbb{R}^d$*, each subset with at least 2 elements, and* $\mathscr{H}(\S) = (\mathbb{R}^d, E(\S))$ *is defined by* $E(\S) = \{T \subseteq \mathbb{R}^d | T$ *is congruent to some* $S \in \S\}$*. Then* $\chi(\mathscr{H}(\S)) < \infty$.

Theorem 4.3. *For any norm* $||\cdot||$ *on* $\mathbb{R}^d$*, Theorem 3.1 holds with the Euclidean norm replaced by* $||\cdot||$*, provided the phrase "congruent in* $\mathbb{R}^d$*" is replaced by "congruent in* $(\mathbb{R}^d, ||\cdot||)$*."*

In view of Lemmas 4.1 and 4.2, the proof is straightforward; follow the path of argument in section 3.

However, generalizing Corollary 3.1.1 looks to us like a lost cause. In the case $||\cdot|| = ||\cdot||_2$, for any two unit vectors u, v, there is a linear isometry of $(\mathbb{R}^d, ||\cdot||_2)$ that takes u into v, whence $\{\underline{0}, u\}$ and $\{\underline{0}, v\}$ are congruent. Thus the graph $((\mathbb{R}^d, ||\cdot||), 1)$ is the hypergraph $\mathscr{H}(\{\underline{0}, u\})$, which makes the base of the induction proof of the Corollary, when $m = 2$, "trivial".

Given a non-Euclidean norm $||\cdot||$ on $\mathbb{R}^d$, we can, with reference to Theorem 4.3, find $\S_2 \subseteq \S_3 \subseteq \dots$ such that $\S_m$ is a finite set of unit m-gons in $(\mathbb{R}^d, ||\cdot||)$, $m = 2, 3, \dots$, and the hypergraphs $\mathscr{H}(\S_m)$ are all equivalent, but we can only be sure that $\chi(\mathscr{H}(\S_m)) \le \chi((\mathbb{R}^d, ||\cdot||), 1)$, and even if we happen to have equality, we see no way of assuring that every proper coloring of $\chi(\S_m)$ with number of colors equal to the chromatic number of graph will also properly color the unit distance graph on $(\mathbb{R}^d, ||\cdot||)$.

On the other hand, we have no proof that there is no generalization of Corollary 3.1.1 for some or all non-Euclidean norms $||\cdot||$ on $\mathbb{R}^d$, $d > 1$. Perhaps the question is worth investigating. (Yes, we are aware that Corollary 3.1.1 will hold for every norm arising from an inner product on $\mathbb{R}^d$; $\mathbb{R}^d$ with such a norm is linearly and isometrically isomorphic to $(\mathbb{R}^d, ||\cdot||_2)$.)

We can obtain m-uniform hypergraphs in $(\mathbb{R}^d, ||\cdot||)$ with the same chromatic number as $((\mathbb{R}^d, ||\cdot||), 1)$ if we abandon the practice of defining edge sets as the sets congruent to one of a finite set of m-gons. However, our method does not provably produce hypergraphs equivalent to the unit distance graph on $(\mathbb{R}^d, ||\cdot||)$.

Let Q be a collection of finite subsets of $\mathbb{R}^d$, each with at least 2 elements, and let

$$E_0 = \{e \subseteq \mathbb{R}^d \mid \text{for some } f \in Q, e \text{ and } f \text{ are congruent}\}$$

For each $t \in \mathbb{Z}^+$, let

$$B_t = \{f \cup T \mid f \in Q, T \subseteq \mathbb{R}^d, f \cap T = \emptyset, \text{ and } |T| = t\}$$

$$E_t = \{e \subseteq \mathbb{R}^d \mid \text{ for some } g \in B_t, e \text{ and } g \text{ are congruent}\}$$

It will be important to notice that because, for each $f \in Q$, we form infinitely many sets in B_t by taking the union of f with each and every $t-$subset of $\mathbb{R}^d \setminus f$, it follows that if $e \in E_0$, $T' \subseteq \mathbb{R}^d \setminus e$, and $|T'| = t$, then $e \cup T' \in E_t$.

Theorem 4.4. *With $t \in \mathbb{Z}^+, Q, B_t, E_0$, and E_t as above, let $\mathscr{H}_0 = (\mathbb{R}^d, E_0)$ and $\mathscr{H}_t = (\mathbb{R}^d, E_t)$. If $\chi(\mathscr{H}_0) < \infty$ then $\chi(\mathscr{H}_t) = \chi(\mathscr{H}_0)$.*

Proof. Since each $e \in E_t$ contains an $e' \in E_0$, it follows that a proper coloring of $\mathscr{H}_0$ will also serve as a proper coloring of $\mathscr{H}_t$. Thus,

$$\chi(\mathscr{H}_t) \leq \chi(\mathscr{H}_0) < \infty$$

By the same argument,

$$\chi(\mathscr{H}_t) \leq \chi(\mathscr{H}_{t-1}) \leq \chi(\mathscr{H}_0)$$

It suffices to show that $\chi(\mathscr{H}_t) = \chi(\mathscr{H}_{t-1})$ for each $t \in \mathbb{Z}^+$.

Let $k = \chi(\mathscr{H}_t)$ and suppose that $k < \chi(\mathscr{H}_{t-1})$. Let $\varphi : \mathbb{R}^d \to \{1, ..., k\}$ be such that no edge $g \in E_t$ is monochromatic, with reference to φ.

Since $k < \chi(\mathscr{H}_{t-1})$, there must exist some $B \in E_{t-1}$ such that φ assigns the same color to every element of B. Without loss of generality, we can assume that this color is k.

For every $u \in \mathbb{R}^d \setminus B$, $B \cup \{u\} \in E_t$. Therefore $\varphi(u) \in \{1, ..., k-1\}$; otherwise, $B \cup \{u\}$ would be monochromatic under the coloring φ. Thus, φ restricted to $\mathbb{R}^d \setminus B$ is a proper coloring of $\mathscr{H}_t|_{\mathbb{R}^d \setminus B}$ with $k-1$ colors.

We shall now use Theorem $D-E$ to prove the existence of a proper coloring of $\mathscr{H}_t$ with colors $\{1, ..., k-1\}$ which will contradict $k = \chi(\mathscr{H}_t)$. Since this contradiction descends from the assumption that $k = \chi(\mathscr{H}_t) < \chi(\mathscr{H}_{t-1})$ it will follow that $\chi(\mathscr{H}_t) = \chi(\mathscr{H}_{t-1})$ and the theorem will be proven.

Suppose $F \subset \mathbb{R}^d$ is finite and $\chi(\mathscr{H}_t|_F) = \chi(\mathscr{H}_t)$. We aim to show that $\mathscr{H}_t|_F$ can be properly colored with no more than $k-1$ colors, which will imply that

$$\chi(\mathscr{H}_t) = \chi(\mathscr{H}_t|_F) \leq k-1 < k = \chi(\mathscr{H}_t).$$

Let $v \in \mathbb{R}^d$ be such that $(v+F) \cap B = \emptyset$. Then φ colors $v+F$ with no more than

$k-1$ colors so that for each $\alpha \in 2^{v+F} \cap E_t$, φ assigns more than one color to the elements of α. Now color F as follows: color $f \in F$ with $\varphi(v+f)$. Since every translate of every $\alpha \in E_t$ is in E_t, and no $\alpha \in 2^{v+F} \cap E_t$ is monochromatic under coloring by φ, we have what we wanted, a proper coloring of $\mathscr{H}_t|_F$ with colors from $\{1,...,k-1\}$. □

Corollary 4.4.1. *Let $||\cdot||$ be a norm on $\mathbb{R}^d$. For each $m \in \mathbb{Z}^+$ such that $m > 2$, define $E_m = \{T \subseteq \mathbb{R}^d \mid |T| = m$ and T contains 2 points $||\cdot||$-distance 1 apart$\}$. Let $\mathscr{G}_m = (\mathbb{R}^d, E_m)$. Then $\chi(\mathscr{G}_m) = \chi((\mathbb{R}^d, ||\cdot||), 1)$ for all m.*

Proof. If $Q = \{\{\underline{0}, u\} | u \in \mathbb{R}^d$ and $||u|| = 1\}$, then, in terms used in the Theorem's statement, $\mathscr{H}_0 = ((\mathbb{R}^d, ||\cdot||), 1)$, the unit distance graph on $(\mathbb{R}^d, ||\cdot||)$ and, for each $m > 2$, $\mathscr{H}_{t-2} = \mathscr{G}_m$. The conclusion follows from the Theorem. □

5 Explicit Construction of Hypergraph

In this section, we are in $\mathbb{R}^d$ with the usual Euclidean norm. We will give a "constructive" proof of a weaker version of Theorem 3.1.

By applying Theorem D-E for hypergraphs to Theorem 3.1, it can be shown that for every finite m-uniform hypergraph in $\mathbb{R}^d$, there must exist a finite $(m+1)$-uniform hypergraph of equal chromatic number. However, this deduction is non-constructive because all known proofs of Theorem D-E use the axiom of choice, and thus we have no control over what the finite hypergraph might look like. In this section, we show how to take a finite m-uniform hypergraph and construct a finite $(m+1)$-uniform hypergraph of equal chromatic number. This allows us to "construct", for any $m \in \mathbb{Z}^+$, $m > 2$, a finite m-uniform hypergraph in $\mathbb{R}^d$ with chromatic number equal to $\chi(\mathbb{R}^d, 1)$. We use "construct" in quotations, however, because this hypergraph construction will use a finite unit distance graph G with chromatic number $\chi(\mathbb{R}^d, 1)$.

Theorem 5.1. *Let $\mathscr{H}$ be a finite, m-uniform hypergraph with vertices in $\mathbb{R}^d$. There exists a finite, $(m+1)$-uniform hypergraph $\mathscr{H}'$ with vertices in $\mathbb{R}^d$ such that $\chi(\mathscr{H}') = \chi(\mathscr{H})$.*

Proof. Let $k = \chi(\mathscr{H})$, and F be the vertex set $V(\mathscr{H})$. Note that for all translates $F + v \subset \mathbb{R}^d$ of F where $v \in \mathbb{R}^d$, the chromatic number of corresponding hypergraph $\mathscr{H} + v$, with vertex set $V(\mathscr{H} + v) = F + v$ and $E(\mathscr{H} + v) = \{e + v \mid e \in E(\mathscr{H})\}$, must also be k.

We now construct an $(m+1)$-uniform hypergraph $\mathscr{H}'$ in the following way:

1. Let the vertex set of $\mathscr{H}'$ be the union of k disjoint translates of $F = V(\mathscr{H})$, which we will call $F_1, F_2, ..., F_k$. Denote by $\mathscr{H}_i$ the translate of $\mathscr{H}$ onto F_i.
2. Define the edge set of $\mathscr{H}'$ by

$$E(\mathscr{H}') = \{\{e \cup v\} \mid e \in \mathscr{H}_i,\ v \in F_j, \text{ where } j > i\}$$

   In other words, $E(\mathscr{H}')$ consists of all $(m+1)$-gons such that m points are from an edge in $\mathscr{H}_i$ and the remaining point is from some F_j with $j > i$.

By construction, $\mathscr{H}'$ is finite and $(m+1)$-uniform. Also, any proper coloring φ of $\mathscr{H}$ can be extended to a proper coloring of $\mathscr{H}'$ by coloring each copy F_i of $V(\mathscr{H})$ in $V(\mathscr{H}')$ by the same colors assigned to $V(\mathscr{H})$. Doing so, all $e \in E(\mathscr{H}_i)$ will be properly colored, implying that all $e \in E(\mathscr{H}')$ must be properly colored by construction. Thus, we have shown that $k \geq \chi(H')$, and need only to confirm that $k \not> \chi(H')$. To show this, we demonstrate that any coloring of $\mathscr{H}'$ with fewer than k colors must yield a monochromatic edge.

Consider a coloring $\varphi : \mathbb{R}^d \to \{1,...,k-1\}$. It is clear φ must not properly color $\mathscr{H}_i$ for each $\mathscr{H}_i \in \mathscr{H}_1,...,\mathscr{H}_k$. Thus for each $\mathscr{H}_i$, there must be some edge $e = \{a_1,...,a_m\} \in E(\mathscr{H}_i)$ such that $\varphi(a_1) = ... = \varphi(a_m)$.

We now show that there must be an edge $e \in \mathscr{H}'$ which φ monochromatically colors.

Let e_1 be a monochromatic edge in $E(\mathscr{H}_1)$. Without loss of generality suppose that $\varphi(e_1) = \{k-1\}$. If any vertex $v \in F_2 \cup \, ... \, \cup F_k$ were colored by $k-1$, we would have found a monochromatic edge in $\mathscr{H}'$, by our construction of the edges in $\mathscr{H}'$. Thus, we can assume that $\varphi(v) \in \{1,...,k-2\}$ for $v \in F_2 \cup \, ... \, \cup F_k$.

Likewise, there must be a monochromatic edge e_2 in $E(\mathscr{H}_2)$, because F_2 is colored now with even fewer colors than F_1 was. Because $\varphi(F_2) \in \{1,...,k-2\}$, we can without loss of generality suppose that $\varphi(e_2) = \{k-2\}$.

Continuing in this fashion until reaching F_k, we see that for each color $\alpha \in \{1,...,k-1\}$, there must appear a monochromatic edge in some $\mathscr{H}_i$ for $i \in \{1,...,k-1\}$. Now consider $\varphi(v)$ for $v \in F_k$. No matter which color is chosen for v, there will appear a monochromatic edge in $\mathscr{H}'$ of the form $e_i \cup v$ for some $e_i \in \mathscr{H}_i$. We see that it is impossible to properly color $\mathscr{H}'$ with fewer than k colors, implying that $\chi(\mathscr{H}') = k$.

□

Corollary 5.1.1. *For all integers $m \geq 2$, there exists a finite, m-uniform hypergraph $\mathscr{H}'$ with vertices in $\mathbb{R}^d$ such that $\chi(\mathscr{H}) = \chi(\mathbb{R}^d, 1)$.*

Proof. We prove the statement by induction. When $m = 2$, the result follows directly from the D-E Theorem.

Now consider $m > 2$ and suppose the theorem holds true for $m-1$. Then there exists a finite, $(m-1)$-uniform hypergraph $\mathscr{H}$ with vertices in $\mathbb{R}^d$ such that $\chi(\mathscr{H}) = \chi(\mathbb{R}^d, 1)$. By the previous theorem, there must exist an m-uniform hypergraph $\mathscr{H}'$ with vertices in $\mathbb{R}^d$ such that $\chi(\mathscr{H}') = \chi(\mathscr{H}) = \chi(\mathbb{R}^d, 1)$. □

□

Remark 5.1.1. When $m = 2$, the hypergraph is not obtained constructively, but with this method, we still have control over the large-scale geometric structure of the m-uniform hypergraph based on how we position the finite hypergraphs when $m > 2$, and hence the term “construct”.

Appendix

Summary of the proof that for any norm $|| \cdot ||$ on $\mathbb{R}^d$, $\chi(\mathbb{R}^d, 1) < \infty$.

Let $||\cdot||$ be a norm on $\mathbb{R}^d$ and let $||\cdot||_\infty$ be the norm on $\mathbb{R}^d$ defined by $||(a_1,...,a_d)||_\infty = \max[|a_i|; 1 \leq i \leq d]$.

The proof of this well-known result will rely on the even better-known fact that any two norms on $\mathbb{R}^d$ are equivalent. The equivalence of $||\cdot||$ and $||\cdot||_\infty$ means that there exist $c, C > 0$ such that for all $u \in \mathbb{R}^d$,

$$c||u||_\infty \leq ||u|| \leq C||u||_\infty.$$

The rightmost inequality can be used to prove the existence of $\varepsilon > 0$ such that the $||\cdot||$-diameter of the d-dimensional cube $[0,\varepsilon]^d$ is < 1. [Just take $0 < \varepsilon < \frac{1}{c}$.] Then the leftmost inequality above implies the existence of an integer $m > 0$ such that for each of the unit coordinate vectors $e_j = (S_{1j},...,S_{dj})$ (in which S_{ij} is the Kronecker delta), $||(m-1)e_j|| > 1$.

Next we partition the cube $Q = [0, m\varepsilon)^d$ into m^d little cubes, $\sum_{t=0}^{d-1}[j_t\varepsilon, (j_t+1)\varepsilon)$, $(j_0,...,j_{d-1}) \in \{0,1,...,m-1\}^d$, and color Q with m^d colors; one to each little cube. Finally we color $\mathbb{R}^d$ with those same colors by tiling $\mathbb{R}^d$ with translates (with colors attached) of Q.

This will be a proper coloring of $((\mathbb{R}^d, ||\cdot||), 1)$ because any two points bearing the same color are either within the same little cube— no two points of which are $||\cdot||$-distance ≥ 1 apart,— or in two different little cubes, in which case the distance between them is > 1.

Thus $\chi(\mathbb{R}^d, ||\cdot||) \leq m^d < \infty$.

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