RECOGNIZING REFERENT UNIT IN FRACTION MULTIPLICATION PROBLEMS: IS THE WHOLE ALWAYS THE SAME?

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Teachers' attention and flexibility on referent unit is important to better understand fractions and fractions operations while it is documented that teachers struggle with it. In this study, we explored teachers' different levels of identifying referent unit in a fraction multiplication problem involving a drawn representation. By analyzing data from five pilot interviews with five middle school mathematics teachers, we found out that teachers attended referent unit differently. Moreover, their different levels of mastering referent unit related to what they view as a whole through their thinking.

Keywords: Referent Unit, Fraction Multiplication/Division, Visual representations, Teacher Knowledge

Backgrounds

Numerous studies have investigated the understanding of fractions among both in-service and pre-service teachers. Given that fraction plays an important role for upper elementary and middle school grades (e.g., CCSSM, 2010), including topics such as using visual models to understand fraction multiplication and division, a robust teaching of this content is necessary. Recent studies have reported that teachers having constrains on comprehension of fraction operations and on facilitating fractions across different representations (Copur-Gencturk & Ölmez, 2022; Izsák et al., 2019; Lee, 2017; Lee et al., 2011; Lo & Luo, 2012; Philipp & Hawthorne, 2015).

Prior Research

Recent research noted about teachers' struggle with referent unit. For example, teachers' attention to referent unit and their flexibility with the referent unit are considered as indicators of understanding of fractions as well as fraction operations. Referent unit is different from unit in a way that referent unit may change according to the situational needs, as Lee (2017) defined: "*referent units* are units that are needed when numbers are embedded in problem situations." (p. 329). Lee et al. (2011) investigated on how twelve teachers' reasoning and understanding of mathematical visual representations with referent unit, and they found out that the participants lack flexibility of keeping track of "unit to which a fraction refers" (p. 204). Consistent with them, Lee (2017) studied 111 pre-service teachers, only 12% of whom showed flexibility of referent unit by providing "appropriate representations" (p. 345). To examine teachers' attention to referent unit, Copur-Gencturk and Ölmez (2022) reported approximately half of their in-service teacher participants (N=246) attended referent unit, by referring "different wholes or unit" where 1/3 could be greater than ½.

While there is limited research particularly examining teacher' understanding of referent unit, making sense of visual representations is often adopted as a way of such kind of exploration, such as length representations or area model representations (Izsák et al., 2019; Lee et al., 2011;

Orrill et al., 2008; Orrill & Brown, 2012). Those studies documented variety level of teachers' struggle with coordinating fractional reasoning in the context of both representations.

Perspectives

Majority research demonstrated that teachers could solve fraction multiplication/division problems by algorithm (e.g., Lo & Luo, 2012), but more importantly, they need to make sense of it both for themselves and thus for their students. Specifically, in the case of referent unit, they need to know how to facilitate the fractions operations in different visual representations, so that they can help with future students' conceptual understanding. According to Shulman (1986), it requires not only teachers' content knowledge (CK) but also their pedagogical content knowledge (PCK), since PCK includes knowledge of students and making sense of different representations to students. To explore teachers' knowledge through this lens, we are driven by this research question: to what extend teachers attend to referent unit when interpreting drawn representation?

Methods

The present study involved conducting interviews with a convenience sample of five middle school mathematics teachers, consisting of two currently practicing teachers (Kevin and Hunter) and three recently retired teachers (Laura, Beth, and Wendy). They are all pseudonymous. Kevin taught at a private school, while the rest of the participants were public school teachers. For two of the retired teachers, they are considered within the context of school settings, because Beth teaches as a long-term substitute and Wendy last taught in the 2019-2020 school year. Among the sample, Beth was the only Black teacher, while the other four participants were white.

Each interview, conducted over Zoom, lasted approximately an hour, during which the teachers were asked to work on tasks on Jamboard. Meanwhile, they were asked to share their screen for both parties to show their pointers moving around or their notes or scratches while they are thinking. Jamboard containing 19 pages of questions related to nine different situations, with one to two questions per screen. The interviews were recorded and transcribed by Zoom, and one of the authors reviewed and edited the transcript to ensure the accuracy of transcribed conversation.

This work focuses on only one area model representation (figure 1) form a situation where there are four different students' work. We first had the teachers looked at four students' work separately, with a purpose of revealing teachers' knowledge about how to make sense of each area model representation from students' work. Before they responded, they were informed to *consider the fraction multiplication problem ¾ × ⅔, and to solve it using an area model, students colored in the model as shown below*. Following the students' drawing, we asked them to solve task one with two separate questions: *Where do you see ¾ in Donald's drawing? How about 2/3?*

Our intention for task one is to capture whether teachers can identify there are two different *wholes* in this representation. Specifically, the first *whole* is the entire rectangular shape, we can consider it as ¾ [of the whole rectangle], or three blue of the whole shape, which represents the blue-shaded parts. However, the second *whole* as we intended now is supposed to be the blue-shaded parts, through which it reflected a thinking of 2/3 [of the blue shaded parts], or 2 parts of 3 blue parts represents the two orange-shaded parts.

Data was analyzed qualitatively to determine whether each teacher successfully identified the two different *wholes* as we indented to, if not, what *whole* did they refer to when they responded to the two separate interview questions.

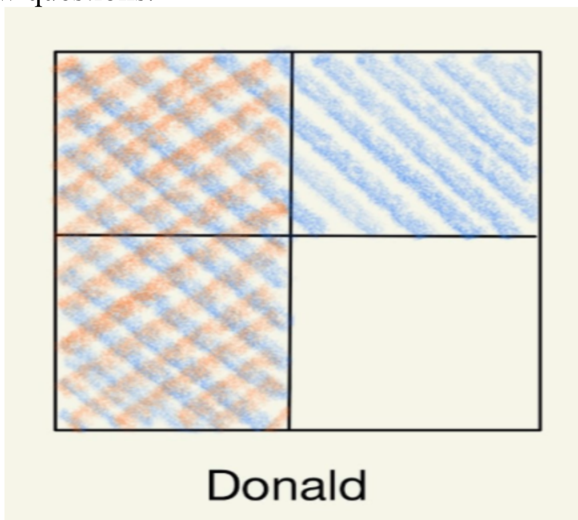

Figure 1: A student's work of modeling $^{3}/_{4} \times {}^{2}/_{3}$

Results

Where Did the Teachers See 3/4

All five teachers identified ¾ as the blue-shaded parts quickly. They either marked the three blue shaded parts on Jamboard or mentioned it as the "blue lines." Interestingly, when they talked, they did not explicitly refer to the relationship of the blue shaded parts to the whole rectangle. Only one teacher, Kevin, explicitly attended to the relationship of the blue section to the whole by typing a note to himself on Jamboard saying, "Blue=3/4 of whole."

Regardless of their ways of indication (verbal or written), we agreed that all of these five teachers were aware of where the 3/4 could be seen in this representation, with recognizing the whole as the entire rectangular shape.

Where Did the Teachers See 2/3

Among five teachers, two teachers (Kevin and Laura) successfully identified 2/3 as two orange parts of three blue parts. Kevin typed his notes on Jamboard saying, "Red [Orange] = 2/3 of blue shaded". Laura explained, "…so the two-thirds are the orange lines going this way, and the two-thirds doesn't include this one [the top right blue part], so it is 2/3 of the blue lines."

The other three teachers explicitly expressed confusion when they trying to identify 2/3. For example, Wendy mentioned that the denominator "is confusing," because she thought the student did not realize "[the whole rectangular shape] is not partitioned into thirds." Similarly, Hunter said, "I honestly don't see the two-thirds… I would like to cut it into three pieces, but I don't see where the two-thirds are in his drawing." Further extending the idea of partitioning that Hunter introduced, Beth said, "…where is [2/3]? Um…In order to multiply this [2/3], I would have done it [the whole rectangle] into twelfths, because I could see it a little better: 6/12."

All three of these teachers (Wendy, Hunter, and Beth) were unable to identify where the 2/3 appears in the representation and they also all seemed to be searching for 2/3 of the original rectangle, rather than 2/3 of the 3/4. Wendy and Hunter wanted to partition the entire rectangle into fourths or into thirds. Thus, we interpreted that they were not shifting to think about a different whole when they were looking for 2/3. Even though Beth gave an alternative way of partitioning the entire rectangle into a common denominator (twelfths), she was trying to find both ¾ and 2/3 in the original rectangle, rather than attending to the whole to which each fraction referred. In another words, she was thinking of the same whole.

In contrast, Kevin and Laura were smoothly and explicitly, transitioning from the-entire-rectangle-as-a-whole into the blue-shaded-part-as-a-whole. Thus, we concluded that, two of the five teachers showed some flexibility of referent unit as they successfully identified what the unit is based on the situational needs.

Discussion

The present study is consistent with prior research about teachers struggle with referent unit (e.g., Izsák, 2008) and provide more evidence on the difficulties teachers have on identifying referent unit when they deal with the area model for fraction multiplication situations. Our findings revealed that it was not anomalous that the given area (rectangle) was conceived as the whole by teachers even when the discussion should have been about the blue section of the rectangle. This is the same phenomenon described by Izsák (2008) who drew from Steffe's (1994; 2003) work with children to explain the issue as the teacher not attending to two levels of units: the whole rectangle and the new "whole" comprised of the three blue boxes.

Interestingly, as one of four different student representations in this pilot task, the researchers had anticipated that Donald's drawing would be relatively clear for teachers. However, it seemed that the other representations, which were all area models partitioned into more equal pieces were easier overall. Additional analysis needs to be completed to determine the extent to which the teachers attended to the shifting unit in the other drawn representations. We ponder whether this task, with its four unique area models, might differentiate teachers' understanding of referent units.

By illustrating the importance of looking at whether teachers attended to referent units in a drawn representation situated in fraction multiplication problem, we hope to contribute on the existing research about teachers' attention as well as flexibility of referent unit.

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