Forbidding Monochromatic Translates of Scaled Three-Point Sets

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This research was supported by NSF (DMS) grant no. 1950563.

Take two distinct points in the plane, u and v. Let's introduce Cartesian coordinates so that the plane becomes $\mathbb{R}^2$ and $u = (0,0)$, $v = (1,0)$. Let $S = \{u, v\}$.

Is it possible to color the plane with finitely many colors so that no subset S' of the plane congruent to S is monochromatic (i.e., the two points in S' bear different colors, for all such S')? If so, what is the smallest number of colors necessary to accomplish this feat?

As readers of Geombinatorics well know, the answer to the first question is yes, and the answer to the second, denoted $\chi(\mathbb{R}^2)$, is not known exactly, but it is one of 5, 6, 7 [Gre18]. Now we ask: what is the smallest number of colors needed to color $\mathbb{R}^2$ so that no translate of S in $\mathbb{R}^2$ is monochromatic? The answer is 2. There are uncountably many such colorings of $\mathbb{R}^2$ with red and blue, but here is one, an easily described "strip" coloring: color $(x,y) \in \mathbb{R}^2$ red if $\lfloor x \rfloor$ is even and blue if $\lfloor x \rfloor$ is odd.

Another question: is it possible to color the plane with finitely many colors so that no set similar to S is monochromatic? Clearly not, because every 2-subset of $\mathbb{R}^2$ is similar to S. What about coloring $\mathbb{R}^2$ so no set

$$S' = \{(a,b),(a+c,b) : a,b,c \in \mathbb{R}, c \neq 0\},$$

a translate of a nonzero scalar times S, is monochromatic? Again the answer is no, because in any such coloring, any two points on a horizontal line would have to be colored differently.

For any set $F \subseteq \mathbb{R}^2$ with $|F| > 2$ we can ask questions analogous to those above with S replaced by F. We are far from the first to consider such questions. For instance, in [Erd+75] it is noted that if F is the set of 3 vertices of an equilateral triangle then the plane can be 2-colored with a strip coloring so that no set congruent to F is monochromatic, and it is conjectured that such 3-subsets of $\mathbb{R}^2$ are the only ones such that $\mathbb{R}^2$ can be 2-colored so as to forbid monochromatic sets congruent to the given set. In [Ben+13] it is shown that for many $F \subseteq \mathbb{R}^2$, with $|F| = 3$, $\mathbb{R}^2$ can be 4-colored so as to forbid monochromatic sets congruent to F, and for "fat triangles" –

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three point sets such that the triangle determined by them is "close" to being equilateral, only 3 colors are required.

We use the following theorem proven by Van der Waerden in order to prove a few results about forbidding monochromatic translates of multiples of three point sets when coloring the plane.

Theorem 1. *(Van der Waerden, 1927 [Wae27]). Given $k,m \in \mathbb{Z}^+$, there exists some $N(k,m) \in \mathbb{Z}^+$ such that if the points $\{x \in \mathbb{Z} : 1 \leq x \leq N(k,m)\}$ are colored with k colors, there must be a monochromatic arithmetic progression of length m.*

Theorem 2. *Let $T \subseteq \mathbb{R}^2$ be a set of three points. Then the plane $\mathbb{R}^2$ cannot be colored with a finite number of colors so that monochromatic triples that are translates of multiples of T are forbidden.*

Proof. We assume for a contradiction that we can color $\mathbb{R}^2$ with a finite number of colors so that monochromatic triples that are translates of multiples of T are forbidden, and fix such a coloring. Let k denote the number of colors used in the coloring, noting that we clearly must have $k > 1$.

We first consider the case where the three points are not collinear. We only consider triples where one point is $(0,0)$, the second point is $(1,0)$, and the third point is in the first quadrant with x-coordinate between 0 and 1 since we can move and rotate the coordinate axes and then scale T so that we have this form for any T. Let (Δ_x, Δ_y) denote the third vertex, where $0 \leq \Delta_x \leq 1$ and $\Delta_y > 0$.

We recursively define a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$, using the N from Van der Waerden's theorem, by

- $f(1) = 3k$
- $f(x) = N(k, 2 \cdot f(x-1))$.

Let $y_1 = 0$. By Van der Waerden's theorem, coloring the points

$$\{(x, y_1) \in \mathbb{Z}^2 : 0 \leq x \leq f(k+2)\}$$

with k colors guarantees that there is a monochromatic arithmetic progression of length $f(k+1)$. Call this color C_1, and let s_1 denote the difference between consecutive elements of the arithmetic progression. Define

$$X_1 = \{x \in \mathbb{Z} : (x, y_1) \text{ is in the arithmetic progression of color } C_1\},$$

noting that $|X_1| = f(k+1)$.

Now for $1 < n \leq k+1$, continue the sequence of y-coordinates for which there exist monochromatic arithmetic progressions $\{y_i\}$, the sequence of colors of the arithmetic progressions $\{C_i\}$, the sequence of differences between consecutive elements of the arithmetic progressions $\{s_i\}$, and the sequence of sets comprising the arithmetic progressions $\{X_i\}$ as follows.

Let $y_n = y_{n-1} + s_{n-1} \cdot \Delta_y$. We have that $|X_{n-1}| = f(k+2-(n-1))$ by how we defined X_1 and continue to define the X_i. Thus by Van der Waerden's theorem, coloring the points

$$\{(x, y_n) \in \mathbb{R}^2 : x - \Delta_x \in X_{n-1}\}$$

with k colors guarantees that there is a monochromatic arithmetic progression (in the set of points to be considered) of length $2 \cdot f(k+2-n)$. Take the $f(k+2-n)$ points of this arithmetic progression with the smallest x-coordinates as our next arithmetic progression. Note that since the x-coordinates of the points form an arithmetic progression, this is still an arithmetic progression in $\mathbb{R}^2$. Call the color of the arithmetic progression C_n (which is not yet known to be distinct from any of the C_i), and let s_n denote the difference between consecutive elements of the arithmetic progression, noting that s_{n-1} divides s_n. Define

$$X_n = \{x \in X_{n-1} + \Delta_x : (x, y_n) \text{ is in the arithmetic progression of color } C_n\},$$

noting that $|X_n| = f(k+2-n)$.

By this point we have defined the sequences of length $k+1$ $\{y_i\}$, $\{C_i\}$, $\{s_i\}$, and $\{X_i\}$. Let $y_{k+2} = y_{k+1} + s_{k+1}$.

We prove that all of the C_i are distinct by proving that for all n such that $1 \le n \le k+1$, no points

$$\left\{(x + c\Delta_x s_n, y_n + c\Delta_y s_n) : x \in X_n, c \in \mathbb{Z}^+, y_n + c\Delta_y s_n \le y_{k+2}\right\}$$

can be a color C_i, $1 \le i \le n$, by strong induction.

We prove the claim for n, assuming that no points

$$\left\{(x + c\Delta_x s_{n-1}, y_{n-1} + c\Delta_y s_{n-1}) : x \in X_{n-1}, c \in \mathbb{Z}^+, y_{n-1} + c\Delta_y s_{n-1} \le y_{k+2}\right\}$$

can be a color C_i, $1 \le i \le n-1$.

Since the arithmetic progression that the points in X_n come from has $f(k+2-n)$ points with s_n as the distance between any two consecutive points, and since there are $f(k+2-n)$ points that continue this arithmetic progression to the right, we have that any point $(x, y_n) \in X_n$ is distance cs_n to the left of another point of the same color on $y = y_n$ for $1 \le c \le f(k+2-n)$. Thus we have that no point in

$$\{(x + c\Delta_x s_n, y_n + c\Delta_y s_n) : x \in X_n, 1 \le c \le f(k+2-n)\}$$

can be color C_n since otherwise this would create a monochromatic triple of color C_n that is a translate of a multiple of T. The largest value of $y_n + c\Delta_y s_n$ that is allowed by this is

$$y_n + \Delta_y s_n \cdot f(k+2-n) = \Delta_y \cdot \sum_{i=1}^{n-1} s_i + \Delta_y s_n \cdot f(k+2-n).$$

We show that this value is greater than or equal to y_{k+2}.

Since the number of points in one of the arithmetic progressions is at least one less than the number of points in the next arithmetic progression times the distance between the points in this next arithmetic progression (relative to points in the first arithmetic progression being one unit apart), we have that for any $1 \leq i \leq k$,

$$f(k+2-i) \geq \frac{s_{i+1}}{s_i} \cdot (f(k+2-(i+1))-1).$$

Therefore

$$\begin{aligned}
f(k+2-n) &\geq \frac{s_{n+1}}{s_n} \cdot (f(k+2-(n+1))-1) \\
&= \frac{s_{n+1}}{s_n} \cdot f(k+2-(n+1)) - \frac{s_{n+1}}{s_n} \\
&\geq f(1) \cdot \prod_{i=n}^{k} \frac{s_{i+1}}{s_i} - \sum_{i=n}^{k} \prod_{j=n}^{i} \frac{s_{j+1}}{s_j} \\
&= f(1) \cdot \frac{s_{k+1}}{s_n} - \sum_{i=n}^{k} \frac{s_{i+1}}{s_n} \\
&\geq 3k \cdot \frac{s_{k+1}}{s_n} - k \cdot \frac{s_{k+1}}{s_n} \\
&\geq 2k \cdot \frac{s_{k+1}}{s_n}.
\end{aligned}$$

This means that

$$\begin{aligned}
\Delta_y \cdot \sum_{i=1}^{n-1} s_i + \Delta_y s_n \cdot f(k+2-n) &\geq \Delta_y \cdot \sum_{i=1}^{n-1} s_i + \Delta_y s_n \cdot 2k \cdot \frac{s_{k+1}}{s_n} \\
&= \Delta_y \cdot \sum_{i=1}^{n-1} s_i + \Delta_y \cdot 2k \cdot s_{k+1} \\
&\geq \Delta_y \cdot \sum_{i=1}^{n-1} s_i + \Delta_y \cdot \sum_{i=n}^{k+1} s_i \\
&= \Delta_y \cdot \sum_{i=1}^{k+1} s_i \\
&= y_{k+2}.
\end{aligned}$$

Therefore we have that no point in

$$\left\{(x+c\Delta_x s_n, y_n + c\Delta_y s_n) : x \in X_n, c \in \mathbb{Z}^+, y_n + c\Delta_y s_n \leq y_{k+2}\right\}$$

can be color C_n.

We now consider the colors C_i, $1 \leq i \leq n-1$ if there are any i in that range (so when $n \neq 1$). Since we know that s_{n-1} divides s_n and both are positive integers, we may fix $l \in \mathbb{Z}^+$ such that $s_n = ls_{n-1}$. Then for any $c \in \mathbb{Z}^+$ we have that

$$y_n + c\Delta_y s_n = (y_{n-1} + s_{n-1} \cdot \Delta_y) + c\Delta_y (ls_{n-1}) = y_{n-1} + (1+cl)\Delta_y s_{n-1}$$

so since $X_n \subseteq X_{n-1} + \Delta_x$ and $1 + cl > 1$, we have that

$$\begin{gathered}\left\{(x + c\Delta_x s_n, y_n + c\Delta_y s_n) : x \in X_n, c \in \mathbb{Z}^+, y_n + c\Delta_y s_n \leq y_{k+2}\right\} \subseteq \\ \left\{(x + c\Delta_x s_n, y_{n-1} + c\Delta_y s_{n-1}) : x \in X_{n-1}, c \in \mathbb{Z}^+, y_{n-1} + c\Delta_y s_{n-1} \leq y_{k+2}\right\}\end{gathered}$$

and so no points

$$\left\{(x + c\Delta_x s_n, y_n + c\Delta_y s_n) : x \in X_n, c \in \mathbb{Z}^+, y_n + c\Delta_y s_n \leq y_{k+2}\right\}$$

can be a color C_i, $1 \leq i \leq n$, by our inductive hypothesis together with the fact that have already considered the color C_n. This proves our induction and so each of the C_i, $1 \leq i \leq k+1$, are distinct colors.

This is a contradiction since we only have k distinct colors that are used in our coloring, so we conclude that such a coloring cannot exist. Since the number of colors and the coloring were arbitrary, this proves that $\mathbb{R}^2$ cannot be colored with a finite number of colors so that translates of multiples of T are forbidden.

We now consider the case where the three points of T are collinear. This reduces to showing that we cannot color the line $\mathbb{R}$ with a finite number of colors such that translates of multiples of some triple $T^{'}$ are not monochromatic. Without loss of generality we may assume that the first point is 0, the second is 1, and we call the third point $p > 0$.

If p is rational, then we can fix some $a, b \in \mathbb{Z}^+$ such that $p = \frac{a}{b}$. Then by Van der Waerden's theorem we may fix a monochromatic arithmetic progression within the integer points of length $a+b$. Call the distance between consecutive elements of this arithmetic progression d. The first point in this arithmetic progression is a distance of db from the b-th point, which is a distance of da from the final point. Thus we have a translate of db times $T^{'}$ that is monochromatic, a contradiction.

Now if p is irrational, we recursively define a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$, using the N from Van der Waerden's theorem, by

- $f(1) = 3k$
- $f(x) = N(k, 1 + f(x-1))$.

By Van der Warden's theorem, coloring the points $\mathbb{Z}^+$ with k colors guarantees that there is a monochromatic arithmetic progression of length $1 + f(k+1)$. Call this color C_1, and let s_1

denote the difference between consecutive elements of the arithmetic progression. Define

$$X_1 = \{j \in \mathbb{Z}^+ : j \text{ is in the arithmetic progression of color } C_1, \\ \text{excluding the single rightmost point}\} \subseteq \mathbb{R},$$

noting that $|X_1| = f(k+1)$. Let b_1 denote the smallest element of the arithmetic progression.

Now for $1 < n \leq k+1$, continue the sequence of colors of the arithmetic progressions $\{C_i\}$, the sequence of differences between consecutive elements of the arithmetic progressions $\{s_i\}$, and the sequence of sets comprising the arithmetic progressions $\{X_i\}$ as follows.

We have that $|X_{n-1}| = f(k+2-(n-1)$ by how we defined X_1 and continue to define the X_i. Thus by Van der Warden's theorem, coloring the points

$$\{x \in \mathbb{R} : x - (1+p)s_{n-1} \in X_{n-1}\}$$

with k colors guarantees that there is a monochromatic arithmetic progression (in the set of points to be considered) of length $1 + f(k+2-n)$. Call the color of the arithmetic progression C_n (which is not yet known to be distinct from any of the C_i) and let s_n denote the difference between consecutive elements of the arithmetic progression, noting that s_{n-1} divides s_n. Define

$$X_n = \{x \in X_{n-1} + s_{n-1}(1+p) : x \text{ is in the arithmetic progression of color } C_n, \\ \text{excluding the rightmost point}\},$$

noting that $|X_n| = f(k+2-n)$. Let b_n denote the smallest element of the arithmetic progression.

By this point we have defined the sequences of length $k+1$ $\{C_i\}$, $\{s_i\}$, $\{X_i\}$, and $\{b_i\}$. We note that all the X_i are disjoint since each point in X_i is the sum of an integer and $\sum_{j=1}^{i-1} s_j$ times the irrational p.

We prove that all of the C_i are distinct by proving that for all n such that $1 \leq n \leq k$, no points

$$W_n = \left\{ b_n + j(p+1)s_n + cs_n : 1 \leq j \leq \sum_{i=n}^{k} \frac{s_i}{s_n}, 0 \leq c < f(k+2-n) - j \right\}$$

can be a color C_i, $1 \leq i \leq n$, by strong induction. This will be sufficient because $X_{n+1} \subseteq W_n$ for $1 \leq n \leq k$ (consider $j = 1$).

We prove the claim for n, assuming that no points

$$\left\{ b_{n-1} + j(p+1)s_{n-1} + cs_{n-1} : 1 \leq j \leq \sum_{i=n-1}^{k} \frac{s_i}{s_{n-1}}, 0 \leq c < f(k+2-(n-1)) - j \right\}$$

can be a color C_i, $1 \leq i \leq n-1$.

Since X_n is an arithmetic progression with $f(k+2-n)$ points with s_n as the distance between any two consecutive points, we have that, for $j \in \mathbb{Z}^+$, any point $x \in X_n$ excepting the rightmost j

points is distance $j \cdot s_n$ to the left of another point of the same color. Thus we have that no point in
$$\left\{ b_n + j(p+1)s_n + cs_n : 1 \le j \le \sum_{i=n}^{k} \frac{s_i}{s_n}, 0 \le c < f(k+2-n) - j \right\}$$
can be color C_n since otherwise this would create a monochromatic triple of color C_n that is scaled to have the distance between the first two points be $j \cdot s_n$ and is translated onto the arithmetic progression.

We now consider the colors C_i, $1 \le i \le n-1$ if there are any i in that range (so when $n \neq 1$). Since we know that s_{n-1} divides s_n and both are positive integers, we may fix $l \in \mathbb{Z}^+$ such that $s_n = ls_{n-1}$.

Since the smallest point of X_n plus $s_n f(k+2-n)$ gives the largest point of the arithmetic progression that contains X_n (so X_n with one more point), and since this largest point must be $(p+1)s_{n-1}$ more than a point of X_{n-1}, the largest that b_n can be is when the point from X_{n-1} that it is $(p+1)s_{n-1}$ larger than is the largest point of X_{n-1}, so
$$b_n \le b_{n-1} + s_{n-1} f(k+2-(n-1)) - s_n f(k+2-n).$$

The smallest that b_n can be is $(p+1)s_{n-1}$ greater than b_{n-1}, so there is some $d \in \mathbb{N}$ with $0 \le d \le f(k+2-(n-1)) - 1 - lf(k+2-n)$ such that $b_n = b_{n-1} + (p+1)s_{n-1} + ds_{n-1}$.

Thus
$$\begin{aligned}
&\left\{ b_n + j(p+1)s_n + cs_n : 1 \le j \le \sum_{i=n}^{k} \frac{s_i}{s_n}, 0 \le c < f(k+2-n) - j \right\} \\
&= \left\{ b_{n-1} + (p+1)s_{n-1} + ds_{n-1} + jl(p+1)s_{n-1} + cls_{n-1} : 1 \le j \le \sum_{i=n}^{k} \frac{s_i}{s_n}, 0 \le c < f(k+2-n) - j \right\} \\
&= \left\{ b_{n-1} + (jl+1)(p+1)s_{n-1} + (cl+d)s_{n-1} : 1 \le j \le \sum_{i=n}^{k} \frac{s_i}{s_n}, 0 \le c < f(k+2-n) - j \right\}.
\end{aligned}$$

Considering the bounds on $(jl+1)$ and $(cl+d)$ in order to rewrite this as a subset of the set we made the inductive hypothesis about,
$$\begin{aligned}
0 \le cl + d &< lf(k+2-n) - jl + f(k+2-(n-1)) - 1 - lf(k+2-n) \\
&= f(k+2-(n-1)) - (jl+1).
\end{aligned}$$

Since the largest value of $jl+1$ allowed is
$$1 + l\sum_{i=n}^{k} \frac{s_i}{s_n} = 1 + \sum_{i=n}^{k} \frac{s_i}{s_{n-1}} = \sum_{i=n-1}^{k} \frac{s_i}{s_{n-1}},$$

we have that

$$\left\{ b_n + j(p+1)s_n + cs_n : 1 \leq j \leq \sum_{i=n}^{k} \frac{s_i}{s_n}, 0 \leq c < f(k+2-n) - j \right\}$$

$$\subseteq \left\{ b_{n-1} + j(p+1)s_{n-1} + cs_{n-1} : 1 \leq j \leq \sum_{i=n-1}^{k} \frac{s_i}{s_{n-1}}, 0 \leq c < f(k+2-(n-1)) - j \right\}$$

and so no points

$$\left\{ b_n + j(p+1)s_n + cs_n : 1 \leq j \leq \sum_{i=n}^{k} \frac{s_i}{s_n}, 0 \leq c < f(k+2-n) - j \right\}$$

can be a color C_i, $1 \leq i \leq n$, by our inductive hypothesis together with the fact that have already considered the color C_n. This proves our induction and so each of the C_i, $1 \leq i \leq k+1$, are distinct colors.

This is a contradiction since we only have k distinct colors that are used in our coloring, so we conclude that such a coloring cannot exist. Since the number of colors and the coloring were arbitrary, this proves that $\mathbb{R}^2$ cannot be colored with a finite number of colors so that translates of multiples of T are forbidden.

Therefore the theorem is true whether the points are collinear or not.

□

Upon inspecting the points that are used in the previous proof, we see that, even with possibly needing to re-coordinatize the plane, if the original triple T was on integer points then if we scale every point in the proof by some constant factor (so that we initially assume that the second point of T is just on the x-axis rather than at $(1,0)$ so that the third point is on an integer point in the original axes, which has no effect on the validity of the proof), they will all be on integer points. In the case where the three points are collinear, we simply need to find an arithmetic progression of length equal to the distance between the two extreme points, which is easily done by Van der Waerden's theorem. Thus we arrive at the following result.

Corollary 1. *Let $T \subseteq \mathbb{Z}^2$ be a set of three points. Then the plane $\mathbb{Z}^2$ cannot be colored with a finite number of colors so that monochromatic triples that are translates of multiples of T are forbidden.*

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