AN INTRODUCTION TO VON NEUMANN ALGEBRAS

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Abstract. These notes provide a brief introduction to von Neumann algebras.

Von Neumann algebras were introduced by von Neumann who developed their theory in a series of joint works with Murray in the 1930s-1940s. A von Neumann algebra is a self-adjoint algebra of bounded operators on a Hilbert space which is closed in the weak operator topology. As shown by von Neumann, unital von Neumann algebras admit an entirely algebraic interpretation: they are precisely the commutants of self-adjoint sets of operators. Von Neumann algebras were originally considered in order to formalise quantum mechanics and understand group representations. Unitary representations $\pi : \Gamma \to \mathcal{U}(H)$ naturally give rise to von Neumann algebras: the span of $\pi(\Gamma)$ is a self-adjoint operator algebra and so its weak operator closure is a von Neumann algebra. When Γ is a countable group and π is its left regular representation, this construction retrieves Murray and von Neumann's group von Neumann algebra $L(\Gamma)$. More generally, one can associate a von Neumann algebra to any non-singular measurable action $\Gamma \curvearrowright (X, \mu)$. These constructions, going back to the 1940s, provide connections between von Neumann algebras, group theory and ergodic theory which continue to stimulate research in the area. Over the years, the theory of von Neumann algebras has broadened and diversified in remarkable fashion. It is now organized into three main areas (subfactor theory, free probability, deformation/rigidity theory) and has deep connections to many fields of mathematics and physics, including ergodic theory, geometric group theory, logic (model theory, descriptive set theory), random matrices, tensor categories, quantum field theory and quantum information theory.

1. Basics of von Neumann algebras

This section is devoted to basic notions concerning von Neumann algebras. We introduce the weak and strong operator topologies, define the notion of von Neumann algebras, prove von Neumann's double commutant theorem and present the realization of L^∞-algebras as von Neumann algebras.

Throughout these notes, H will denote a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\xi\| = \sqrt{\langle \xi, \xi \rangle}$. We denote by $\mathbb{B}(H)$ the algebra of all bounded linear operators $T : H \to H$. The *operator norm* of $T \in \mathbb{B}(H)$ is given by

$$\|T\| = \sup_{\|\xi\| \leq 1} \|T\xi\|$$

and its *adjoint* is the unique $T^* \in \mathbb{B}(H)$ such that $\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle, \forall \xi, \eta \in H$.

Exercise 1.1. Let $T, S \in \mathbb{B}(H)$. Prove that $\|TS\| \leq \|T\| \, \|S\|$, $\|T^*\| = \|T\|$ and $\|T^*T\| = \|T\|^2$.

Definition 1.2. An operator $T \in \mathbb{B}(H)$ is called:

- *self-adjoint* (or, *hermitian*) if $T^* = T$.
- a *projection* if $T = T^* = T^2$.

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- a *unitary* if $T^*T = TT^* = 1$.
- an *isometry* if $T^*T = 1$.
- *normal* if $T^*T = TT^*$.
- *positive* (in symbols, $T \geq 0$) if $\langle T\xi, \xi\rangle \geq 0, \forall \xi \in H$.

Remark 1.3. The set of unitary operators $T \in \mathbb{B}(H)$ is a group which is denoted by $\mathcal{U}(H)$.

Remark 1.4. An operator $T \in \mathbb{B}(H)$ is positive if and only if $T = S^*S$, for some $S \in \mathbb{B}(H)$. Given $T_1, T_2 \in \mathbb{B}(H)$, we write $T_1 \leq T_2$ to mean that $T_2 - T_1 \geq 0$.

Definition 1.5. We endow $\mathbb{B}(H)$ with the following three topologies:

- the *norm topology*: $T_i \to T$ if $\|T_i - T\| \to 0$.
- the *strong operator topology* (SOT): $T_i \to T$ if $\|T_i\xi - T\xi\| \to 0$, $\forall \xi \in H$.
- the *weak operator topology* (WOT): $T_i \to T$ if $|\langle T_i\xi, \eta\rangle - \langle T\xi, \eta\rangle| \to 0$, $\forall \xi, \eta \in H$.

The norm topology is stronger than the SOT, which in turn is stronger than the WOT.

Exercise 1.6. Let $(T_i)_{i\in I} \subset \mathbb{B}(H)$ be a net such that $T_i \to T$ (WOT), for some $T \in \mathbb{B}(H)$. For parts (3), (4) and (5) below assume that H is infinite dimensional.

(1) Assume that $(T_i)_{i\in I}$ and T are projections. Prove that $T_i \to T$ (SOT).
(2) Assume that $(T_i)_{i\in I}$ and T are unitaries. Prove that $T_i \to T$ (SOT).
(3) Give an example of a net of projections $(T_i)_{i\in I}$ converging in the WOT but not the SOT.
(4) Give an example of a net of unitaries $(T_i)_{i\in I}$ converging in the WOT but not the SOT.
(5) Prove that the closed unit ball $\{T \in \mathbb{B}(H) \mid \|T\| \leq 1\}$ is WOT but not SOT compact.

While the SOT is strictly stronger than the WOT, we have the following:

Proposition 1.7. *If $C \subset \mathbb{B}(H)$ is a convex set, then $\overline{C}^{\text{SOT}} = \overline{C}^{\text{WOT}}$.*

Proof. Let $y \in \overline{C}^{\text{WOT}}$, $\xi_1, \cdots, \xi_n \in H$ and $\varepsilon > 0$. Then $D = \{(x\xi_1, \cdots, x\xi_n) \mid x \in C\}$ is a convex subset of $H^n = \oplus_{i=1}^n H$. By the Hahn-Banach theorem, the weak and norm closures of D coincide (see, e.g., [KR97, Theorem 1.3.4]). Since $(y\xi_1, \cdots, y\xi_n)$ is in the weak closure of D, it is also in its norm closure. Therefore, we can find $x \in C$ such that $(\sum_{i=1}^n \|x\xi_i - y\xi_i\|^2)^{1/2} < \varepsilon$. This implies that $y \in \overline{C}^{\text{SOT}}$. Since the inclusion $\overline{C}^{\text{SOT}} \subset \overline{C}^{\text{WOT}}$ also holds, we are done. ∎

Definition 1.8. Let H be a complex Hilbert space.

- A subalgebra $A \subset \mathbb{B}(H)$ is called a $*$*-algebra* if $T^* \in A$, $\forall T \in A$.
- A $*$-subalgebra $A \subset \mathbb{B}(H)$ is called a (concrete) C^**-algebra* if it closed in the norm topology.
- A $*$-subalgebra $A \subset \mathbb{B}(H)$ is called a *von Neumann algebra* if it is WOT-closed.

Definition 1.9. A map $\pi : A \to B$ between two C*-algebras A and B is called a $*$*-homomorphism* if it is linear, multiplicative and $*$*-preserving* (i.e., $\pi(a^*) = \pi(a)^*$,$\forall a \in A$). A bijective $*$-homomorphism is called a $*$*-isomorphism.* A $*$-homomorphism $\pi : A \to \mathbb{B}(H)$, for some complex Hilbert space H, is called a *representation* of A.

Exercise 1.10. Let $\mathcal{S} \subset \mathbb{B}(H)$ be a set which is $*$*-closed* (i.e., $T^* \in \mathcal{S}$, $\forall T \in \mathcal{S}$). Prove that the *commutant* of $\mathcal{S}$, defined as $\mathcal{S}' = \{T \in \mathbb{B}(H) \mid TS = ST, \forall S \in \mathcal{S}\}$, is a von Neumann algebra.

Conversely, the next fundamental result shows that every von Neumann algebra arises this way:

Theorem 1.11 (von Neumann's double commutant theorem, [vN30])**.** *If $M \subset \mathbb{B}(H)$ is a unital $*$-subalgebra, then the following three conditions are equivalent:*

(1) M is WOT*-closed.*
(2) M is SOT*-closed.*
(3) $M = M'' := (M')'$.

Here, we say that a subalgebra $M \subset \mathbb{B}(H)$ is unital if it contains the identity operator.

This beautiful result asserts that, for unital $*$-algebras, the analytic condition of being closed in the WOT is equivalent to the algebraic condition of being equal to their double commutant.

Proof. It is clear that (3) $\Rightarrow$ (1) by Exercise 1.10 and that (1) $\Leftrightarrow$ (2) by Proposition 1.7.

To prove that (2) $\Rightarrow$ (3), it suffices to show that if $x \in M''$, $\varepsilon > 0$, and $\xi_1, ..., \xi_n \in H$, then there exists $y \in M$ such that $\|x\xi_i - y\xi_i\| < \varepsilon$, for all $i = 1, ..., n$.

We claim that if p is the orthogonal projection onto an M-invariant closed subspace $K \subset H$, then $p \in M'$. To see this, let $x \in M$. Then $(1-p)xp\xi \in (1-p)(K) = \{0\}$, for all $\xi \in H$. Hence $(1-p)xp = 0$ and so $xp = pxp$. By taking adjoints, we get that $px^* = px^*p$ and hence $px = pxp$, for all $x \in M$. This shows that p commutes with x, as claimed.

Next, assume first that $n = 1$ and let p be the orthogonal projection onto $\overline{M\xi_1} = \overline{\{x\xi_1 \mid x \in M\}}$. Since $\overline{M\xi_1}$ is M-invariant, our claim gives $p \in M'$. Thus, $xp = px$ and $x\xi_1 = xp\xi_1 = px\xi_1 \in \overline{M\xi_1}$. Therefore, there is $y \in M$ such that $\|x\xi_1 - y\xi_1\| < \varepsilon$.

In general, we use a "matrix trick". Let $H^n = \oplus_{i=1}^n H$ and identify $\mathbb{B}(H^n) = \mathbb{M}_n(\mathbb{B}(H))$. Let $\pi : M \to \mathbb{B}(H^n)$ be the "diagonal" $*$-homomorphism given by $\pi(x)(\xi_1 \oplus \cdots \oplus \xi_n) = x\xi_1 \oplus \cdots \oplus x\xi_n$. If $x \in M''$, then Exercise 1.12 below gives that $\pi(x) \in \pi(M)''$. Let $\xi = (\xi_1, \cdots, \xi_n) \in H^n$. By applying the case $n = 1$ we conclude that there is $y \in M$ such that $\|\pi(x)\xi - \pi(y)\xi\| < \varepsilon$. Since $\|\pi(x)\xi - \pi(y)\xi\|^2 = \sum_{i=1}^n \|x\xi_i - y\xi_i\|^2$, we are done. ∎

Exercise 1.12. Prove that $\pi(M'') \subset \mathbb{M}_n(M')'$ and $\pi(M)' \subset \mathbb{M}_n(M')$.

Definition 1.13. A probability space (X, μ) is called *standard* if X is a Polish space and μ is a Borel probability measure on X.

Proposition 1.14. *Let (X, μ) be a standard probability space. Define $\pi : L^\infty(X, \mu) \to \mathbb{B}(L^2(X, \mu))$ by letting $\pi_f(\xi) = f\xi$, for all $f \in L^\infty(X)$ and $\xi \in L^2(X)$. Then $\pi(L^\infty(X))' = \pi(L^\infty(X))$. Therefore, $\pi(L^\infty(X)) \subset \mathbb{B}(L^2(X))$ is a maximal abelian von Neumann subalgebra.*

Proof. Let $T \in \pi(L^\infty(X))'$ and put $g = T(1)$. Then $fg = \pi_f T(1) = T\pi_f(1) = T(f)$ and hence

$$\|fg\|_2 = \|T(f)\|_2 \leqslant \|T\| \, \|f\|_2, \forall f \in L^\infty(X). \tag{1.1}$$

Let $\varepsilon > 0$ and $f = 1_{\{x \in X \, : \, |g(x)| \geqslant \|T\| + \varepsilon\}}$. Then it is clear that $\|fg\|_2 \geqslant (\|T\| + \varepsilon)\|f\|_2$. In combination with inequality (1.1), we get that $(\|T\| + \varepsilon)\|f\|_2 \leqslant \|T\|\|f\|_2$, and so $f = 0$, almost everywhere. Thus, we conclude that $g \in L^\infty(X)$. Since $T(f) = fg = \pi_g(f)$, for all $f \in L^\infty(X)$, and $L^\infty(X)$ is $\|\cdot\|_2$-dense in $L^2(X)$, it follows that $T = \pi_g \in L^\infty(X)$. This proves that $\pi(L^\infty(X))' = \pi(L^\infty(X))$.

If $A \subset \mathbb{B}(L^2(X))$ is an abelian algebra which contains $\pi(L^\infty(X))$, then A commutes with $\pi(L^\infty(X))$. The previous paragraph implies that $A \subset \pi(L^\infty(X))$ which proves that $A = \pi(L^\infty(X))$. ∎

Exercise 1.15. Let I be a set. Define $\pi : \ell^\infty(I) \to \mathbb{B}(\ell^2(I))$ by letting $\pi_f(\xi) = f\xi$, for all $f \in \ell^\infty(I)$ and $\xi \in \ell^2(I)$. Prove that $\pi(\ell^\infty(I))' = \pi(\ell^\infty(I))$. Therefore, $\pi(\ell^\infty(I)) \subset \mathbb{B}(\ell^2(I))$ is a maximal abelian von Neumann subalgebra.

2. The spectral theorem

The spectral theorem for normal matrices $a \in \mathbb{M}_n(C)$ implies that $a = \sum_{z\in\sigma(a)} zp_z$, where p_z is the orthogonal projection onto the eigenspace corresponding an eigenvalue $z \in \sigma(a)$. Given $\Delta \subset \sigma(a)$, put $E(\Delta) = \sum_{z\in\Delta} p_z$. Thus, informally, we have that $a =$ "$\int_{\sigma(a)} z \, \mathrm{d}E(z)$". The spectral theorem proves such a statement for normal operators a on possibly infinite dimensional Hilbert spaces, where E is a so-called spectral measure defined on the spectrum of a.

We start this section by recalling several fundamental facts concerning C*-algebras. We then discuss the expression of representations of abelian C*-algebras in terms of spectral measures, and use this to derive the spectral theorem for normal operators and classify abelian von Neumann algebras.

2.1. **C*-algebras.**

For proofs of the facts presented below, we refer the reader to the introduction to C*-algebras in this volume [Sz22] or [Co99, Chapter 1].

Definition 2.1. An *C*-algebra* is a Banach algebra $(A, \|\cdot\|)$ together with an adjoint operation $* : A \to A$ such that $\forall a, b \in A$ and $\lambda \in \mathbb{C}$ we have

$$(a+b)^* = a^* + b^*,\ (\lambda a)^* = \overline{\lambda} a^*,\ (a^*)^* = a,\ (ab)^* = b^* a^* \ \text{ and } \ \|a^*a\| = \|a\|^2.$$

Such C*-algebras are called *abstract* because, in contrast to *concrete* C*-algebras, they are not a priori represented on a Hilbert space. However, as we will see in Theorem 2.8, any abstract C*-algebra is isomorphic to a concrete one.

If A is a unital C*-algebra and $a \in A$, then the *spectrum* of a is a nonempty compact subset of $\mathbb{C}$ defined by $\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda \cdot 1 - a \text{ is not invertible}\}$.

Theorem 2.2. *Let A and B be unital C*-algebras. Then any unital $*$-homomorphism $\pi : A \to B$ is contractive: $\|\pi(a)\| \le \|a\|, \forall a \in A$. If π is injective, then it is isometric: $\|\pi(a)\| = \|a\|, \forall a \in A$. In particular, any $*$-isomorphism $\pi : A \to B$ is automatically isometric.*

Definition 2.3. Let A a unital C*-algebra. A linear functional $\varphi : A \to \mathbb{C}$ is called *positive* if $\varphi(a^*a) \ge 0, \forall a \in A$. A positive linear functional $\varphi : A \to \mathbb{C}$ is called a *state* if $\varphi(1) = 1$ and *faithful* if $\varphi(a^*a) = 0 \Rightarrow a = 0$.

Exercise 2.4. Let A be a unital C*-algebra and $\varphi : A \to \mathbb{C}$ be a positive linear functional.

(1) (the Cauchy-Schwarz inequality) Prove that $|\varphi(y^*x)|^2 \le \varphi(x^*x)\varphi(y^*y), \forall x, y \in A$.
(2) Prove that φ is bounded and $\|\varphi\| = \varphi(1)$.

Exercise 2.5. Let X be a compact Hausdorff space. Prove that $C(X) = \{f : X \to \mathbb{C} \text{ continuous function}\}$ is an abstract C*-algebra, with the norm $\|f\|_\infty = \sup\limits_{x\in X} |f(x)|$ and adjoint $f^*(x) = \overline{f(x)}$.

The following result shows that every abstract unital abelian C*-algebra A arises this way. We denote by $\widehat{A}$ the set of nonzero homomorphisms $\varphi : A \to \mathbb{C}$. Then $\widehat{A} \subset \{\varphi \in A^* \mid \|\varphi\| = 1\}$ and $\widehat{A}$ is a compact Hausdorff space with respect to the weak*-topology inherited from A^*. Here, A^* denotes the dual Banach space of A, which consists of all bounded linear functionals $\varphi : A \to \mathbb{C}$.

Theorem 2.6 (Gelfand-Naimark)**.** *Let A be a unital abelian C*-algebra. Then the* Gelfand transform $\wedge : A \to C(\widehat{A})$ *given by $\widehat{a}(\varphi) = \varphi(a)$, $\forall a \in A, \varphi \in \widehat{A}$, is a $*$-isomorphism.*

Theorem 2.6 implies the following:

Theorem 2.7 (continuous functional calculus). *Let A be a unital C*-algebra and $a \in A$ be a normal element. Then there exists an isometric unital $*$-homomorphism*

$$C(\sigma(a)) \ni f \mapsto f(a) \in A$$

which maps the identity function on $\sigma(a)$ to a.

By Exercise 1.1, any concrete C*-algebra is an abstract C*-algebra. The converse is also true:

Theorem 2.8 (Gelfand-Naimark-Segal). *Every abstract C*-algebra is $*$-isomorphic to a concrete C*-algebra.*

2.2. **Representations of abelian C*-algebras.** Let $A \subset \mathbb{B}(H)$ be a concrete abelian C*-algebra (e.g., the C*-algebra generated by a normal operator). By Theorem 2.6, A is $*$-isomorphic to $C(\widehat{A})$. This result, however, does not explain how A "acts" on H. The next theorem gives a description of all representations of $C(X)$, where X is a compact Hausdorff space. We denote by $\mathcal{B}$ the σ-algebra of Borel subsets of X, by $B(X)$ the C*-algebra of bounded Borel functions $f : X \to \mathbb{C}$, and by $\mathcal{M}(X)$ the space of complex-valued regular measures on X endowed with the norm $\|\mu\| = \sup\{\int_X f \, \mathrm{d}\mu \mid f \in C(X), \|f\|_\infty \leq 1\}$. In this and the next subsection, we follow the presentation from [Co99, Sections 2.9 and 2.10].

Theorem 2.9. *Let $\pi : C(X) \to \mathbb{B}(H)$ be a $*$-homomorphism. Then there exists a spectral measure $E : \mathcal{B} \to \mathbb{B}(H)$ such that*

$$\pi(f) = \int_X f \, \mathrm{d}E, \ \ \forall f \in C(X).$$

Definition 2.10. A *spectral measure* for $(X, \mathcal{B})$ is a map $E : \mathcal{B} \to \mathbb{B}(H)$ that satisfies the following:

(1) $E(\Delta)$ is a projection, $\forall \Delta \in \mathcal{B}$.
(2) $E(\emptyset) = 0$ and $E(X) = 1$.
(3) $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$, $\forall \Delta_1, \Delta_2 \in \mathcal{B}$.
(4) The map $\mathcal{B} \ni \Delta \mapsto E_{\xi,\eta}(\Delta) := \langle E(\Delta)\xi, \eta\rangle$ belongs to $\mathcal{M}(X)$, $\forall \xi, \eta \in H$.

Lemma 2.11. *Let $E : \mathcal{B} \to \mathbb{B}(H)$ be a spectral measure. If $\xi, \eta \in H$, then $\|E_{\xi,\eta}\| \leqslant \|\xi\| \, \|\eta\|$.*

Proof. Let $\Delta_1, \cdots, \Delta_n \in \mathcal{B}$ be pairwise disjoint sets. Let $\alpha_i \in \mathbb{T}$ such that $|E_{\xi,\eta}(\Delta_i)| = \alpha_i E_{\xi,\eta}(\Delta_i)$. Then $\sum_{i=1}^n |E_{\xi,\eta}(\Delta_i)| = \langle \sum_{i=1}^n \alpha_i E(\Delta_i)\xi, \eta\rangle \leqslant \| \sum_{i=1}^n \alpha_i E(\Delta_i)\xi\| \, \|\eta\|$. Since we also have that

$$\| \sum_{i=1}^n \alpha_i E(\Delta_i)\xi\|^2 = \sum_{i=1}^n \langle E(\Delta_i)\xi, \xi\rangle = \langle E(\cup_{i=1}^n \Delta_i)\xi, \xi\rangle \leqslant \|\xi\|^2,$$

we conclude that $\sum_{i=1}^n |E_{\xi,\eta}(\Delta_i)| \leqslant \|\xi\| \, \|\eta\|$, as desired. ■

Lemma 2.12. *Let $E : \mathcal{B} \to \mathbb{B}(H)$ be a spectral measure. Then for every $f \in B(X)$, there exists an operator $\pi(f) \in \mathbb{B}(H)$ such that $\|\pi(f)\| \leqslant \|f\|_\infty$ and $\langle \pi(f)\xi, \eta\rangle = \int_X f \, \mathrm{d}E_{\xi,\eta}$, for all $\xi, \eta \in H$. Moreover, the map $\pi : B(X) \to \mathbb{B}(H)$ is a $*$-homomorphism.*

Proof. Let $f \in B(X)$. Since the map $H \times H \ni (\xi, \eta) \to \int_X f \, \mathrm{d}E_{\xi,\eta}$ is sesquilinear and satisfies $|\int_X f \, \mathrm{d}E_{\xi,\eta}| \leqslant \|f\|_\infty \|E_{\xi,\eta}\| \leqslant \|f\|_\infty \|\xi\| \|\eta\|$, the existence of $\pi(f)$ is a consequence of Riesz's representation theorem.

Secondly, $\langle \pi(1_\Delta)\xi, \eta\rangle = \int_X 1_\Delta \mathrm{d}E_{\xi,\eta} = E_{\xi,\eta}(\Delta) = \langle E(\Delta)\xi, \eta\rangle$ and so $\pi(1_\Delta) = E(\Delta)$. We get that $\pi(1_{\Delta_1 \cap \Delta_2}) = \pi(1_{\Delta_1})\pi(1_{\Delta_2})$, for every $\Delta_1, \Delta_2 \in \mathcal{B}$. Thus, $\pi(f_1 f_2) = \pi(f_1)\pi(f_2)$, for simple functions $f_1, f_2 \in B(X)$. Since $\|\pi(f)\| \leqslant \|f\|_\infty$, for all $f \in B(X)$, approximating bounded Borel functions by simple functions gives that π is multiplicative. It follows that π is a $*$-homomorphism. ■

Before proving the spectral theorem, we need one additional result.

Lemma 2.13. *Let $\pi : C(X) \to \mathbb{B}(H)$ be a $*$-homomorphism. Then there exists a $*$-homomorphism $\tilde{\pi} : B(X) \to \mathbb{B}(H)$ such that $\tilde{\pi}_{|C(X)} = \pi$. Moreover, if $f \in B(X)$ and $(f_i) \subset B(X)$ is a net such that $\int_X f_i \, \mathrm{d}\mu \to \int_X f \, \mathrm{d}\mu$, for every $\mu \in \mathcal{M}(X)$, then $\tilde{\pi}(f_i) \to \tilde{\pi}(f)$ in the WOT.*

Proof. Let $\xi, \eta \in H$. Note that $C(X) \ni f \to \langle \pi(f)\xi, \eta\rangle \in \mathbb{C}$ is a linear functional such that $|\langle \pi(f)\xi, \eta\rangle| \leqslant \|\pi(f)\| \, \|\xi\| \, \|\eta\| \leqslant \|f\|_\infty \, \|\xi\| \, \|\eta\|$. Riesz's representation theorem implies that there exists $\mu_{\xi,\eta} \in \mathcal{M}(X)$ such that $\int_X f \, \mathrm{d}\mu_{\xi,\eta} = \langle \pi(f)\xi, \eta\rangle$, for all $f \in C(X)$, and $\|\mu_{\xi,\eta}\| \leqslant \|\xi\| \, \|\eta\|$. Note that $\overline{\mu_{\xi,\eta}} = \mu_{\eta,\xi}$, so the map $(\xi, \eta) \to \mu_{\xi,\eta}$ is sesquilinear.

Next, let $f \in B(X)$. Repeating the argument from the proof of Lemma 2.12 shows that there exists an operator $\tilde{\pi}(f) \in \mathbb{B}(H)$ such that $\|\tilde{\pi}(f)\| \leqslant \|f\|_\infty$ and $\langle \tilde{\pi}(f)\xi, \eta\rangle = \int_X f \, \mathrm{d}\mu_{\xi,\eta}$, for all $\xi, \eta \in H$. It is clear that $\tilde{\pi}(f) = \pi(f)$, if $f \in C(X)$. It is also easy to see that $\tilde{\pi}$ is linear and $*$-preserving, so it remains to argue that $\tilde{\pi}$ is multiplicative.

Let $f \in B(X)$ and $g \in C(X)$. Then we can find a net $(f_i) \subset C(X)$ such that $\|f_i\|_\infty \leqslant \|f\|_\infty$, for all i, and $\int_X f_i \, \mathrm{d}\mu \to \int_X f \, \mathrm{d}\mu$, for every $\mu \in \mathcal{M}(X)$ (see [Co99, Lemma 9.7]). Since $\mu_{\xi,\eta} \in \mathcal{M}(X)$, it follows that $\langle \pi(f_i)\xi, \eta\rangle = \int_X f_i \, \mathrm{d}\mu_{\xi,\eta} \to \int_X f \, \mathrm{d}\mu_{\xi,\eta} = \langle \tilde{\pi}(f)\xi, \eta\rangle$, for all $\xi, \eta \in H$. Thus, $\pi(f_i) \to \tilde{\pi}(f)$ in the WOT. Similarly, $\pi(f_i g) \to \tilde{\pi}(fg)$ in the WOT. Since $\pi(f_i g) = \pi(f_i)\pi(g)$, for all i, we deduce that $\tilde{\pi}(fg) = \tilde{\pi}(f)\pi(g)$, for all $f \in B(X)$ and $g \in C(X)$.

Finally, let $f, g \in B(X)$. By approximating g with continuous functions as above and using the last identity, it follows similarly that $\tilde{\pi}(fg) = \tilde{\pi}(f)\tilde{\pi}(g)$. Thus, $\tilde{\pi}$ is multiplicative.

For the moreover assertion, let $f, f_i \in B(X)$ as in the hypothesis. Then for every $\xi, \eta \in H$ we have that $\langle \tilde{\pi}(f_i)\xi, \eta\rangle = \int_X f_i \, \mathrm{d}\mu_{\xi,\eta} \to \int_X f \, \mathrm{d}\mu_{\xi,\eta} = \langle \tilde{\pi}(f)\xi, \eta\rangle$. Therefore, $\tilde{\pi}(f_i) \to \tilde{\pi}(f)$ in the WOT. ■

We are now ready to sketch the proof of the spectral theorem, leaving some details to the reader.

Proof of Theorem 2.9. By Lemma 2.13, π extends to a $*$-homomorphism $\tilde{\pi} : B(X) \to \mathbb{B}(H)$. Define $E : \mathcal{B} \to \mathbb{B}(H)$ by letting $E(\Delta) = \tilde{\pi}(1_\Delta)$. Then one checks that E is a spectral measure. By Lemma 2.12 $\rho : B(X) \to \mathbb{B}(H)$ given by $\rho(f) = \int_X f \, \mathrm{d}E$ is a $*$-homomorphism. Then we have $\rho(1_\Delta) = \int_X 1_\Delta \, \mathrm{d}E = E(\Delta) = \tilde{\pi}(1_\Delta)$, for every $\Delta \in \mathcal{B}$. Consequently, $\rho(f) = \tilde{\pi}(f)$, for every simple function $f \in B(X)$. Since simple functions are $\| \cdot \|_\infty$-dense in $B(X)$ and $\rho, \tilde{\pi}$ are contractive, we get that $\rho = \tilde{\pi}$. In particular, $\pi(f) = \int_X f \, \mathrm{d}E$, for every $f \in C(X)$. This finishes the proof. ■

2.3. The spectral theorem.

Theorem 2.14. *Let $a \in \mathbb{B}(H)$ be a normal operator and $\mathcal{B}$ the σ-algebra of Borel subsets of $\sigma(a)$.*

1. (the spectral theorem) *There is a spectral measure $E : \mathcal{B} \to \mathbb{B}(H)$ such that $a = \int_{\sigma(a)} z \, dE$.*
2. (Borel functional calculus) *The map $B(\sigma(a)) \ni f \to f(a) := \int_{\sigma(a)} f(z) \, dE \in \mathbb{B}(H)$ is a $*$-homomorphism. Moreover, if $f \in B(\sigma(a))$ and $(f_i) \subset B(\sigma(a))$ is a net such that $\int_{\sigma(a)} f_i \, d\mu \to \int_{\sigma(a)} f \, d\mu$, for every $\mu \in \mathcal{M}(\sigma(a))$, then $f_i(a) \to f(a)$ in the* WOT.

Proof. By Theorem 2.7, there exists a $*$-homomorphism $\pi : C(\sigma(a)) \to \mathbb{B}(H)$ such that $\pi(z) = a$. The conclusion now follows directly from Theorem 2.9. □

Corollary 2.15. *Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra.*

1. *If $a \in M$ is normal, then $f(a) \in M$, for every $f \in B(\sigma(a))$.*
2. *M is equal to the norm closure of the linear span of its projections.*

Proof. (1) Let $f \in B(\sigma(a))$. Let $f_i \in C(\sigma(a))$ be a net such that $\|f_i\|_\infty \leqslant \|f\|_\infty$, for all i, and $\int_{\sigma(a)} f_i \, \mathrm{d}\mu \to \int_{\sigma(a)} f \, \mathrm{d}\mu$, for every $\mu \in \mathcal{M}(\sigma(a))$. By Theorem 2.14, we have that $f_i(a) \to f(a)$ in the WOT. Since $f_i(a) \in C^*(a) \subset M$, we conclude that $f(a) \in M$.

(2) If $a \in M$, then we can write $a = b + ic$,where $b, c \in M$ are self-adjoint. So it suffices to show that any self-adjoint $a \in M$ belongs to the norm closure of the linear span of projections of M. To this end, let $\varepsilon > 0$ and write $a = \int_{\sigma(a)} z \, \mathrm{d}E$. Then we can find $\alpha_1, ..., \alpha_n \in \mathbb{R}$ and Borel sets $\Delta_1, ..., \Delta_n \subset \sigma(a)$ such that $\|z - \sum_{i=1}^n \alpha_i 1_{\Delta_i}\|_\infty \leqslant \varepsilon$. It follows that $\|a - \sum_{i=1}^n \alpha_i 1_{\Delta_i}(a)\| \leqslant \varepsilon$. Since the projections $1_{\Delta_i}(a)$ belong to M by part (1), we are done. ■

Exercise 2.16. Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra and $a \in M$ with $a \geqslant 0$. Prove that there exist projections $\{p_n\}_{n=1}^\infty \subset M$ such that $a = \|a\| \sum_{n=1}^\infty 2^{-n} p_n$.

2.4. **Abelian von Neumann algebras.** By Theorem 1.14, $L^\infty(X)$ is a von Neumann algebra, for any standard probability space (X, μ). Conversely, we have:

Theorem 2.17. *Let H be a separable Hilbert space and $M \subset \mathbb{B}(H)$ be an abelian von Neumann algebra. Then M is $*$-isomorphic to $L^\infty(X)$, where (X, μ) is a standard probability space.*

For a von Neumann algebra M, we denote by $(M)_1 = \{x \in M \mid \|x\| \leq 1\}$ its closed *unit ball* and by $M_+ = \{x \in M \mid x \geq 0\}$ the set of its positive elements.

Proof. For simplicity, we only prove this theorem under the additional assumption that there is a vector (called *M-cyclic*) $\xi \in H$ such that $\overline{M\xi} = H$.

Since H is separable, $(\mathbb{B}(H)_1, \mathrm{WOT})$ is a compact metrizable space (see Exercise 1.6(5)) and hence so is $((M)_1, \mathrm{WOT})$. Let $\{a_n\} \subset (M)_1$ be a WOT-dense sequence. Let A be the C*-algebra generated by $\{a_n\}$. Then A is SOT-dense in M and $X = \widehat{A}$ is compact and metrizable. Specifically, we have that $\mathrm{d}(\varphi, \varphi') = \sum_{n=1}^\infty \frac{1}{2^n} |\varphi(a_n) - \varphi'(a_n)|$, for $\varphi, \varphi' \in X$, defines a compatible metric on X.

Let $\pi : C(X) \to A \subset \mathbb{B}(H)$ be the inverse of the Gelfand transform (see Theorem 2.6). By applying Theorem 2.9 we get a spectral measure E on X such that $\pi(f) = \int_X f \, \mathrm{d}E$, for every $f \in C(X)$. Then $\mu(\Delta) = \langle E(\Delta)\xi, \xi \rangle$ defines a measure $\mu \in \mathcal{M}(X)$ such that $\int_X f \, \mathrm{d}\mu = \langle \pi(f)\xi, \xi \rangle$, for every $f \in C(X)$. Thus, for every $f \in C(X)$ we get that

$$\|\pi(f)\xi\|^2 = \langle \pi(f)\xi, \pi(f)\xi \rangle = \langle \pi(f)^*\pi(f)\xi, \xi \rangle = \langle \pi(|f|^2)\xi, \xi \rangle = \int_X |f|^2 \, \mathrm{d}\mu,$$

so $\|\pi(f)\xi\| = \|f\|_{L^2(X)}$. As $\pi(C(X)) = A$ is SOT-dense in M, $\{\pi(f)\xi | f \in C(X)\}$ is dense in $\overline{M\xi} = H$. Since $C(X)$ is dense in $L^2(X)$, we can define a unitary operator $U : L^2(X) \to H$ by

$$U(f) = \pi(f)\xi, \ \ \forall f \in C(X).$$

Let $\rho : L^\infty(X) \to \mathbb{B}(L^2(X))$ be the $*$-homomorphism given by $\rho_f(\eta) = f\eta$. Then for all $f, g \in C(X)$ we have $U\rho_f(g) = U(fg) = \pi(fg)\xi = \pi(f)\pi(g)\xi = \pi(f)U(g)$. As $C(X)$ is dense in $L^2(X)$ we get that $U\rho_f = \pi(f)U$ and thus $\pi(f) = U\rho_f U^*$, for all $f \in C(X)$. Hence, $\pi(C(X)) = U\rho(C(X))U^*$. Since $\rho(C(X))$ is WOT-dense in $L^\infty(X)$, we conclude that $M = UL^\infty(X)U^*$. ■

We recall the isomorphism theorem for standard probability spaces (see [Ke95, Theorem 17.41]). An *isomorphism* between two standard probability spaces (X, μ) and (Y, ν) is a Borel isomorphism $\theta : X \to Y$ (i.e., a bijection such that θ and θ^{-1} are Borel maps) such that $\theta_*\mu = \nu$, where $\theta_*\mu$ is the Borel probability measure on Y given by $\theta_*\mu(Z) = \mu(\theta^{-1}(Z))$, for every Borel set $Z \subset Y$.

Theorem 2.18. *Let (X, μ) be a standard probability space. Assume that μ is non-atomic, i.e., $\mu(\{x\}) = 0$, for every $x \in X$. Then (X, μ) is isomorphic to $([0, 1], \lambda)$, where λ is the Lebesgue measure on $[0, 1]$.*

Definition 2.19. Let M be a von Neumann algebra. A projection $p \in M$ is called *minimal* if every projection $q \in M$ such that $0 \leq q \leq p$ satisfies $q \in \{0, p\}$. A von Neumann algebra M is called *diffuse* if it has no nonzero minimal projections.

Corollary 2.20. *Let H be a separable Hilbert space and $M \subset \mathbb{B}(H)$ be a diffuse abelian von Neumann algebra. Then M is $*$-isomorphic to $L^\infty([0,1],\lambda)$.*

Proof. By Theorem 2.17, M is $*$-isomorphic to $L^\infty(X)$, where (X,μ) is a standard probability space. Since M is diffuse, μ is non-atomic. Otherwise, if $\mu(\{x\}) > 0$, for $x \in X$, then $1_{\{x\}} \in L^\infty(X)$ would be a non-zero minimal projection. Theorem 2.18 thus implies the conclusion. ■

Exercise 2.21. Let M be a von Neumann algebra and $p \in M$ be a projection. Prove that p is minimal if and only if $pMp = \mathbb{C}p$.

Exercise 2.22. Let M be a diffuse von Neumann algebra. Prove that any maximal abelian von Neumann subalgebra $A \subset M$ is diffuse. Hence, deduce that M contains a copy of $L^\infty([0,1],\lambda)$.

Exercise 2.23. Let M be a finite dimensional abelian von Neumann algebra. Prove that M is $*$-isomorphic to $\ell^\infty(\{1,\cdots,n\})$, for some $n \in \mathbb{N}$.

3. Decomposition into types for von Neumann algebras

This section is devoted to the type decomposition for von Neumann algebras. By Corollary 2.15, any von Neumann algebra $M \subset \mathbb{B}(H)$ is generated by projections. To better understand the structure of M, it will be important to "compare" its projections. If H is finite dimensional, then the projections of M can be ordered using the dimension of their range space. A main goal of this section is to define a way to compare projections, in the absence of a suitable notion of dimension.

3.1. The polar decomposition.

We start this section by discussing the polar decomposition for bounded operators $a \in \mathbb{B}(H)$. This is an analogue of the decomposition of a complex number as the product of a number of absolute value 1 and its absolute value. The *absolute value* of a is given by $|a| = (a^*a)^{\frac{1}{2}}$. Thus, we would like to write $a = v|a|$, where v satisfies $|v| = 1$, i.e., is an isometry. It turns out that this is true, if we allow v to be a *partial* isometry, in the following sense:

Definition 3.1. An operator $v \in \mathbb{B}(H)$ is called a *partial isometry* if $\|v(\xi)\| = \|\xi\|, \forall \xi \in (\ker v)^\perp$. In this case, $(\ker v)^\perp$ and the range $\mathrm{ran}(v) = vH$ are called the *initial* and *final* space of v, respectively.

The *orthogonal complement* of a closed subspace $K \subset H$ is $K^\perp = \{\xi \in H \mid \langle \xi, \eta\rangle = 0, \forall \eta \in K\}$.

Exercise 3.2. Prove that $v \in \mathbb{B}(H)$ is a partial isometry if and only if v^*v is a projection.

Theorem 3.3 (polar decomposition)**.** *If $a \in \mathbb{B}(H)$, then there exists a unique partial isometry $v \in \mathbb{B}(H)$ with initial space $(\ker a)^\perp$ and final space $\overline{\mathrm{ran}(a)}$ such that $a = v|a|$, where $|a| = (a^*a)^{\frac{1}{2}}$.*

Proof. If $\xi \in H$, then $\|a\xi\|^2 = \langle a\xi, a\xi\rangle = \langle a^*a\xi,\xi\rangle = \langle |a|^2\xi,\xi\rangle = \|\,|a|\xi\|^2$. Then the formula $v(|a|\xi) = a\xi$ defines a unitary operator $v : \overline{\mathrm{ran}(|a|)} \to \overline{\mathrm{ran}(a)}$. We extend v to H by letting $v(\eta) = 0$, for all $\eta \in (\overline{\mathrm{ran}(|a|)})^\perp$. Then v is a partial isometry such that $v|a| = a$. By definition the final space of v is $\overline{\mathrm{ran}(a)}$, while the initial space of v is $\overline{\mathrm{ran}(|a|)} = (\ker|a|)^\perp = (\ker a)^\perp$ (where the second equality follows from the first line of the proof). The uniqueness of v is obvious. ■

Exercise 3.4. Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra and $a \in M$. Let v be the partial isometry provided by Theorem 3.3. Define $l(a)$ to be the projection onto $\overline{\mathrm{ran}(a)}$ (the *left support* of a) and $r(a)$ to be the projection onto $(\ker a)^\perp$ (the *right support* of a).

(1) Prove that v commutes with every unitary element $u \in M'$ and deduce that $v \in M$.
(2) Prove that $l(a) = vv^*$ and $r(a) = v^*v$, and use (1) to deduce that $l(a), r(a) \in M$.

3.2. **Projections.** For a von Neumann algebra M, we denote by $\mathcal{P}(M)$ the set of its projections and by $\mathcal{U}(M)$ the group of its unitaries.

Definition 3.5. Let $\{p_i\}_{i\in I} \in \mathbb{B}(H)$ be a family of projections. We denote by

- $\bigvee_{i\in I} p_i$ the smallest projection $p \in \mathbb{B}(H)$ such that $p \geqslant p_i$, $\forall i \in I$.
- $\bigwedge_{i\in I} p_i$ the largest projection $p \in \mathbb{B}(H)$ such that $p \leqslant p_i$, $\forall i \in I$.

Proposition 3.6. *If $p_i \in \mathcal{P}(M)$, $\forall i \in I$, then $\bigvee_{i\in I} p_i, \bigwedge_{i\in I} p_i \in M$.*

Proof. A projection $p \in \mathbb{B}(H)$ belongs to M if and only if p commutes with every $x \in M'$ and if and only if pH is invariant under every $x \in M'$ (see Theorem 1.11 and its proof). ∎

We next use Proposition 3.6 to establish that every von Neumann algebra has a unit.

Corollary 3.7. *Let M be a von Neumann algebra and define $p = \bigvee_{q\in\mathcal{P}(M)} q$. Then $p \in \mathcal{P}(M)$ is a multiplicative unit of M, i.e., $a = pa = ap$, $\forall a \in M$.*

Proof. By Proposition 3.6 we have that $p \in M$. If $a \in M$, then $l(a), r(a) \in \mathcal{P}(M)$ by Exercise 3.1 and thus $l(a), r(a) \leq p$. Since $a = l(a)a = ar(a)$, we get that $a = pa = ap$. ∎

Definition 3.8. Let $M \subset \mathbb{B}(H)$ be a unital von Neumann algebra.

- $\mathcal{Z}(M) = M \cap M'$ is called the *center* of M.
- M is called a *factor* if $\mathcal{Z}(M) = \mathbb{C}1$.
- the *central support* of $p \in \mathcal{P}(M)$ is the smallest projection $z(p) \in \mathcal{Z}(M)$ with $p \leqslant z(p)$.

Lemma 3.9. *$z(p)$ is the orthogonal projection onto $\overline{MpH}$.*

Proof. Let z be the orthogonal projection onto $\overline{MpH}$. Since $pH \subset MpH$, we have that $p \leqslant z$. Since MpH is both M and M' invariant, we get that $z \in M' \cap (M')' = \mathcal{Z}(M)$. Finally, since $p = z(p)p$ we have $MpH = Mz(p)pH = z(p)MpH \subset z(p)H$ and hence $z \leqslant z(p)$. Altogether, $z = z(p)$. ∎

Exercise 3.10. Prove that $z(p) = \bigvee_{u\in\mathcal{U}(M)} upu^*$.

Proposition 3.11. *Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra. Let $p \in \mathcal{P}(M)$ and $p' \in \mathcal{P}(M')$. We denote $pMp = \{pxp \mid x \in M\}$ and $Mp' = \{xp' \mid x \in M\}$ and view them as algebras of operators on the Hilbert spaces pH and $p'H$, respectively. Then we have the following:*

(1) $Mp' \subset \mathbb{B}(p'H)$ is a von Neumann algebra and $(Mp')' = p'M'p'$.
(2) $pMp \subset \mathbb{B}(pH)$ is a von Neumann algebra and $(pMp)' = M'p$.
(3) $\mathcal{Z}(Mp') = \mathcal{Z}(M)p'$ and $\mathcal{Z}(pMp) = \mathcal{Z}(M)p$.

For a proof of this result, see, e.g., [Co99, Proposition 43.8].

3.3. **Equivalence of projections.**

Definition 3.12. Let M be a von Neumann algebra. Two projections $p, q \in M$ are called *equivalent* (in symbols, $p \sim q$) if there exists a partial isometry $v \in M$ such that $p = v^*v$ and $q = vv^*$. We say that p is *dominated* by q (and write $p \prec q$) if $p \sim q'$, for some projection $q' \in M$ with $q' \leqslant q$.

Exercise 3.13. Prove the following:

(1) If $p \sim q$, then $z(p) = z(q)$.
(2) If $p \sim q$ via a partial isometry v, then the map $pMp \ni x \to vxv^* \in qMq$ is a $*$-isomorphism.
(3) If $\{p_i\}_{i\in I}, \{q_i\}_{i\in I}$ are families of mutually orthogonal projections such that $p_i \sim q_i$, $\forall i \in I$, then $\sum_{i\in I} p_i \sim \sum_{i\in I} q_i$. Here, for orthogonal projections $\{p_i\}_{i\in I}$, we let $\sum_{i\in I} p_i = \bigvee_{i\in I} p_i$.

(4) If $p \sim q$ and $z \in \mathcal{Z}(M)$ is a projection, then $zp \sim zq$.

Lemma 3.14. *If M is a von Neumann algebra and $p, q \in \mathcal{P}(M)$, then the following are equivalent:*

(1) $pMq \neq \{0\}$.
(2) there exist nonzero projections $p_1, q_1 \in M$ such that $p_1 \leqslant p, q_1 \leqslant q$ and $p_1 \sim q_1$.
(3) $z(p)z(q) \neq 0$.

Proof. (1) $\Rightarrow$ (2) Let $x \in M$ such that $y = pxq \neq 0$. Let $p_1 = l(y)$ and $q_1 = r(y)$. Then $0 \neq p_1 \leqslant p, 0 \neq q_1 \leqslant q$, $p_1, q_1 \in M$ and $p_1 \sim q_1$ by Exercise 3.4.

(2) $\Rightarrow$ (1) If $v \in M$ is such that $p_1 = vv^*$ and $q_1 = v^*v$, then $0 \neq v = pvq \in pMq$.

(1) $\Rightarrow$ (3) If $z(p)z(q) = 0$, then $pxq = pz(p)xz(q)q = pxz(p)z(q)q = 0$, for all $x \in M$.

(3) $\Rightarrow$ (1) If $pMq = \{0\}$, then $p(xq\xi) = 0$, for all $x \in M$. Since $z(q)$ is the orthogonal projection onto $\overline{MqH}$, we get that $pz(q) = 0$ and thus $p \leqslant 1 - z(q)$. Since the projection $1 - z(q)$ belongs to the center of M, we get that $z(q) \leqslant 1 - z(p)$, hence $z(p)z(q) = 0$. ∎

Exercise 3.15. Let M be a von Neumann algebra and $p \in \mathcal{P}(M)$. Prove that there exist partial isometries $\{v_i\}_{i\in I}$ such that $v_iv_i^* \leq p$ and $\sum_{i\in I} v_i^*v_i = z(p)$.

Theorem 3.16 (the comparison theorem)**.** *Let M be a von Neumann algebra and $p, q \in \mathcal{P}(M)$. Then there exists a projection $z \in \mathcal{Z}(M)$ such that $pz \prec qz$ and $q(1-z) \prec p(1-z)$.*

Proof. By Zorn's lemma, there exist maximal families of mutually orthogonal projections $\{p_i\}_{i\in I}, \{q_i\}_{i\in I}$ such that $p_i \leqslant p, q_i \leqslant q$ and $p_i \sim q_i$, for all $i \in I$. Put $p_1 = \sum_{i\in I} p_i$, and $q_1 = \sum_{i\in I} q_i$. Then $p_1 \sim q_1$. Also, let $p_2 = p - p_1$ and $q_2 = q - q_1$.

Since p_2, q_2 do not have equivalent nonzero subprojections, Lemma 3.14 implies that $z(p_2)z(q_2) = 0$. Thus, if we let $z = z(q_2)$, then $p_2z = 0$ and $q_2(1-z) = 0$. The conclusion now follows since

$$pz = \sum_{i\in I} p_iz + p_2z = \sum_{i\in I} p_iz \sim \sum_{i\in I} q_iz \prec \sum_{i\in I} q_iz + q_2z = qz$$

and similarly $q(1-z) \prec p(1-z)$. ∎

Corollary 3.17. *If M is a factor and $p, q \in \mathcal{P}(M)$, then $p \prec q$ or $q \prec p$.*

Exercise 3.18. Let M be a finite dimensional von Neumann algebra.

(1) Assume that M is a factor. Prove that M has a minimal nonzero projection p. Deduce that there exist pairwise equivalent projections $p_1, \cdots, p_n \in M$ such that $p_1 = p$ and $\sum_{i=1}^n p_i = 1$, for some $n \geq 1$. Use this to conclude that M is $*$-isomorphic to $\mathbb{M}_n(\mathbb{C})$.
(2) Prove that M is $*$-isomorphic to $\bigoplus_{k=1}^K \mathbb{M}_{n_k}(\mathbb{C})$, for some $K, n_1, \cdots, n_K \geq 1$.

3.4. **Classification into types.**

Definition 3.19. Let M be a von Neumann algebra. A projection $p \in M$ is called:

(1) *abelian* if pMp is abelian.
(2) *finite* if whenever $q \in M$ is a projection such that $q \leqslant p$ and $q \sim p$, then $q = p$.

Remark 3.20. Every abelian projection is finite. A subprojection of an abelian (resp. finite) projection is abelian (resp. finite).

Definition 3.21. A unital von Neumann algebra $M \subset \mathbb{B}(H)$ is called

- *finite* if $1 \in M$ is finite.

- of *type I* if any nonzero central projection contains a nonzero abelian subprojection.
- of *type II* if it has no abelian projections and any nonzero central projection contains a nonzero finite subprojection.
- of *type III* if it contains no nonzero finite projection.
- of *type* I_{fin} if it is of type I and finite.
- of *type* I_∞ if it is of type I and not finite.
- of *type* II_1 if it is of type II and finite.
- of *type* II_∞ if it is of type II and not finite.

Remark 3.22. M is finite if and only if any isometry $v \in M$ is a unitary, i.e., $v^*v = 1 \Rightarrow vv^* = 1$.

Theorem 3.23 (decomposition into types)**.** *Let $M \subset \mathbb{B}(H)$ be a unital von Neumann algebra. Then there exist projections $z_1, ..., z_5 \in \mathcal{Z}(M)$ with $\sum_{i=1}^5 z_i = 1$ and $Mz_1, Mz_2, Mz_3, Mz_4, Mz_5$ are von Neumann algebras of type $I_{fin}, I_\infty, II_1, II_\infty$, III, respectively.*

Let $p, q, r \in \mathcal{Z}(M)$ be the maximal projections such that Mp is of type I, Mq is of type II, and r is a finite projection. Then $z_1 = pr, z_2 = p(1-r), z_3 = qr, z_4 = q(1-r)$ and $z_5 = 1-(p+q)$ satisfy the conclusion of Theorem 3.23, see, e.g., [Co99, Theorem 48.16].

Remark 3.24. Any factor M is of one of the types $I_{\text{fin}}, I_\infty, II_1, II_\infty$, or III.

3.5. von Neumann algebras of type I.

Definition 3.25. Let $M \subset \mathbb{B}(H)$ and $N \subset \mathbb{B}(K)$ be von Neumann algebras. For $x \in M, y \in N$ we define $x \otimes y \in \mathbb{B}(H \otimes K)$ by letting $(x \otimes y)(\xi \otimes \eta) = x\xi \otimes y\eta$, for all $\xi \in H, \eta \in K$. The *tensor product von Neumann algebra* $M\overline{\otimes}N \subset \mathbb{B}(H \otimes K)$ is defined as the WOT-closure of the linear span of $\{x \otimes y \mid x \in M, y \in N\}$.

Exercise 3.26. Let K be a Hilbert space and (X, μ) be a standard probability space. Prove that $\mathbb{B}(K)\overline{\otimes}L^\infty(X)$ is a type I_{fin} von Neumann algebra if K is finite dimensional and a type I_∞ von Neumann algebra if K is infinite dimensional.

Any type I von Neumann algebra is isomorphic to $\prod_{i \in I}\Big(\mathbb{B}(K_i)\overline{\otimes}L^\infty(X_i)\Big)$, where $(K_i)_{i\in I}$ are Hilbert spaces and $\{(X_i, \mu_i)\}_{i \in I}$ are standard probability spaces, see [Co99, Section 50] for a proof of this fact. Here, we only prove this fact in the factorial case.

Theorem 3.27. *Any factor M of type I is $*$-isomorphic to $\mathbb{B}(K)$, for some Hilbert space K.*

Proof. Let $p \in M$ be a nonzero abelian projection. Then pMp is both abelian and a factor. Therefore, $pMp = \mathbb{C}p$. Let $\{p_i\}_{i \in I}$ be a maximal family of pairwise orthogonal projections in M that are equivalent to p. Put $q = 1 - \sum_{i \in I} p_i$. We claim that $q = 0$. Indeed, if $q \neq 0$, then by Corollary 3.17 we have that either (1) $p \prec q$ or (2) $q \prec p$. Now, (1) contradicts the maximality of $\{p_i\}_{i\in I}$, while (2) implies that there exists a nonzero projection $q' \leqslant p$ such that $q' \sim q$. Since $pMp = \mathbb{C}p$, it follows that $q' = p$, contradicting again the maximality of $\{p_i\}_{i \in I}$.

We will show that $M \cong \mathbb{B}(\ell^2(I))$. Denote by $\{\delta_i\}_{i\in I}$ the canonical orthonormal basis of $\ell^2(I)$. For $i, j \in I$, we let $e_{i,j} \in \mathbb{B}(\ell^2(I))$ be the "elementary" operator given by $e_{i,j}\delta_k = \delta_{j,k}\delta_i$, for all $k \in I$.

For $i \in I$, let $v_i \in M$ be a partial isometry such that $v_i^*v_i = p$ and $v_iv_i^* = p_i$. Put $v_{i_0} = p$. We define $U : H \to \ell^2(I) \otimes pH$ by letting $U(\xi) = \sum_{i\in I} \delta_i \otimes v_i^*\xi$. Since $\sum_{i\in I} \|v_i^*\xi\|^2 = \sum_{i \in I} \|p_i\xi\|^2 = \|\xi\|^2$, for all $\xi \in H$, it follows that U is a unitary.

We claim that $UMU^* = \mathbb{B}(\ell^2(I))\overline{\otimes}\mathbb{C}p$, so $M \cong \mathbb{B}(\ell^2(I))$. We note that $Uv_iU^* = e_{i,i_0} \otimes p$, $\forall i \in I$. This implies that $U\mathcal{A}U^* = \mathcal{B}\otimes\mathbb{C}p$, where $\mathcal{A} \subset M$ and $\mathcal{B} \subset \mathbb{B}(\ell^2(I))$ are the $*$-algebras generated by

$\{v_i\}_{i\in I}$ and $\{e_{i,i_0}\}_{i\in I}$, respectively. Let $x \in M$ and for $F \subset I$ finite, put $p_F = \sum_{i\in F} p_i$. As $p_F \to 1$, we get that $p_F x p_F \to x$, in the SOT. Since $p_F x p_F = \sum_{i,j\in F} v_i v_i^* x v_j v_j^* = \sum_{i,j\in F} v_i(v_i^* x v_j)v_j^*$ and $v_i^* x v_j \in pMp = \mathbb{C}p$, we get that $p_F x p_F \in \mathcal{A}$. This shows that $\mathcal{A}$ is SOT-dense in M. Similarly, we get that $\mathcal{B}$ is SOT-dense in $\mathbb{B}(\ell^2(I))$. The claim and the theorem are now proven. ■

3.6. **von Neumann algebras of types II and III.** While finding examples of von Neumann algebras of type I is immediate, it is not obvious that type II or III algebras should exist. We next present Murray and von Neumann's group measure space construction [MvN36]. This connects von Neumann algebras with ergodic theory, and leads to examples of factors of types II and III. Moreover, group measure space factors are the subject of intense current research (see, e.g., [Io18]).

Let Γ be a countable group and (X, μ) a σ-finite standard measure space. We say that an action $\Gamma \curvearrowright (X, \mu)$ is *nonsingular* if for every $g \in \Gamma$ and measurable set $Y \subset X$, the set gY is measurable and $\mu(Y) = 0 \Rightarrow \mu(gY) = 0$. We denote by $g_*\mu$ the measure on X given by $g_*\mu(Y) = \mu(g^{-1}Y)$. Since $g_*\mu \prec \mu$, we have a Radon-Nykodym derivative $\frac{\mathrm{d}g_*\mu}{\mathrm{d}\mu} \in L^1(X, \mu)_+$ such that

$$\int_X f \frac{\mathrm{d}g_*\mu}{\mathrm{d}\mu}\,\mathrm{d}\mu = \int_X f\,\mathrm{d}g_*\mu = \int_X f \circ g\,\mathrm{d}\mu, \ \ \forall f \in L^\infty(X)_+.$$

This equation implies that the formula $\sigma_g(f)(x) = \big(\frac{\mathrm{d}g_*\mu}{\mathrm{d}\mu}(x)\big)^{\frac{1}{2}} f(g^{-1}x)$, for $x \in X$, $f \in L^2(X)$, defines a unitary operator σ_g on $L^2(X)$. Let $\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ be the left regular representation given by $\lambda(g)(\delta_h) = \delta_{gh}$. Denote $H = L^2(X)\bar{\otimes}\ell^2\Gamma$ and define a unitary representation $u : \Gamma \to \mathcal{U}(H)$ be letting $u_g = \sigma_g \otimes \lambda(g)$. We also define a $*$-homomorphism $\pi : L^\infty(X) \to \mathbb{B}(H)$ by letting $\pi(f)(\xi \otimes \delta_g) = f\xi \otimes \delta_g$, and view $L^\infty(X) \subset \mathbb{B}(H)$, via π. Then

$$u_g f u_g^* = \sigma_g(f), \ \ \forall f \in L^\infty(X), g \in \Gamma.$$

Definition 3.28. The *group measure space von Neumann algebra* $L^\infty(X) \rtimes \Gamma \subset \mathbb{B}(H)$ is defined as the *WOT*-closure of the linear span of $\{fu_g \mid f \in L^\infty(X), g \in \Gamma\}$.

Definition 3.29. A nonsingular action $\Gamma \curvearrowright (X, \mu)$ is called:

- *ergodic* if every Γ-invariant measurable set $Y \subset X$ satisfies $\mu(Y) \in \{0, 1\}$.
- *(essentially) free* if $\mu(\{x \in X \mid gx = x\}) = 0$, for every $g \in \Gamma \setminus \{e\}$.

Theorem 3.30. *Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic nonsingular action of a countable group Γ. Then $L^\infty(X) \rtimes \Gamma$ is a factor of*

(1) type I if μ has atoms.
(2) type II$_1$ if μ is non-atomic and there is a finite Γ-invariant measure ν such that $\nu \sim \mu$.
(3) type II$_\infty$ if μ is non-atomic and there is an infinite Γ-invariant measure ν such that $\nu \sim \mu$.
(4) type III if there is no (finite or infinite) Γ-invariant measure ν such that $\nu \sim \mu$.

For a proof of this theorem, see [Ta79, Theorem 7.12, Chapter V]. We will prove item (2) of this result as part of Proposition 5.15.

Example 3.31. Let Γ be a countable discrete group and μ be its counting measure. Then the left translation $\Gamma \curvearrowright (\Gamma, \mu)$ is clearly free ergodic and measure preserving. The group measure space factor $\ell^\infty(\Gamma) \rtimes \Gamma$ is of type I, and is in fact $*$-isomorphic to $\mathbb{B}(\ell^2(\Gamma))$.

Example 3.32. Let G be a non-discrete second countable locally compact group, m_G be a left Haar measure of G and $\Gamma < G$ a countable dense subgroup. For instance, we can take the inclusion $\Gamma < G$ to be $\mathbb{Z} \equiv \{\exp(2\pi i n\alpha) \mid n \in \mathbb{Z}\} < \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, or $\mathbb{Q} < \mathbb{R}$. Then the left translation action $\Gamma \curvearrowright (G, m_G)$ is free, ergodic and measure preserving. The group measure space factor $L^\infty(G) \rtimes \Gamma$ is of type II$_1$ if G is compact and of type II$_\infty$ if G is non-compact.

Example 3.33. In the setting of Example 3.32, let θ be a topological automorphism of G. Assume that G is non-compact and connected, $\theta(\Gamma) = \Gamma$ and $\theta_* m_G \neq m_G$. View $\theta_{|\Gamma}$ as an automorphism of Γ and define the semidirect product group $\widetilde{\Gamma} = \Gamma \rtimes_\theta \mathbb{Z}$. The action $\widetilde{\Gamma} \curvearrowright (G, m_G)$ given by $(g, n) \cdot x = g\theta^n(x)$, for $g \in \Gamma, n \in \mathbb{Z}, x \in G$, is free, ergodic and nonsingular. Then there is no $\widetilde{\Gamma}$-invariant measure ν on G with $\nu \sim m_G$. Otherwise, ergodicity of $\Gamma \curvearrowright (G, m_G)$ would imply that $\nu = \lambda m_G$, for a $\lambda > 0$, contradicting that $\theta_* m_G \neq m_G$. Thus, $L^\infty(G) \rtimes \widetilde{\Gamma}$ is a factor of type III. For a concrete example, one can take $\Gamma = \mathbb{Q}$, $G = \mathbb{R}$ and $\theta(x) = rx$, for any $r \in \mathbb{Q} \setminus \{0, \pm 1\}$.

4. Tracial von Neumann algebras

In the rest of these notes, we will focus on the study of von Neumann subalgebras of II_1 factors. These are exactly the von Neumann algebras which admit a trace. The existence of a trace is an extremely useful property in particular because it allows to define an equivalence-invariant notion of dimension for projections. For a comprehensive reference on tracial von Neumann algebras, we refer the reader to the book in preparation [AP22].

4.1. **Tracial von Neumann algebras.** Let M, N be von Neumann algebras. A map $\Phi : M \to N$ is called *positive* if $\Phi(M_+) \subset N_+$. A positive map $\Phi : M \to N$ is called *normal* if $\Phi(x_i) \to \Phi(x)$, for any increasing net $(x_i) \subset M_+$ such that $x_i \to x$ (SOT). A positive linear functional $\varphi : M \to \mathbb{C}$ is called *tracial* if $\varphi(xy) = \varphi(yx), \forall x, y \in M$, and *faithful* if $\varphi(x) = 0 \Rightarrow x = 0, \forall x \in M_+$.

Remark 4.1. A positive linear functional $\varphi : M \to \mathbb{C}$ is normal if and only if it is *completely additive*: $\varphi(\sum_{i \in I} p_i) = \sum_{i \in I} \varphi(p_i)$, for any family $(p_i)_{i \in I} \subset M$ of mutually orthogonal projections. For a proof of this result, see [AP22, Theorem 2.5.5].

Definition 4.2. A von Neumann algebra M is called *tracial* if it admits a *trace*, i.e., a faithful normal tracial state $\tau : M \to \mathbb{C}$. (In short, we call the pair (M, τ) a tracial von Neumann algebra.)

The following exercise provides a standard source of faithful states.

Exercise 4.3. Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra and $\xi \in H$ such that $\overline{M'\xi} = H$ (ξ is called an *M'-cyclic* vector). Prove that the positive linear functional $\varphi : M \to \mathbb{C}$ given by $\varphi(x) = \langle x\xi, \xi \rangle$ is faithful.

Examples 4.4. (of tracial von Neumann algebras)

(1) $L^\infty(X)$ is a tracial von Neumann algebra with the trace given by $\tau(f) = \int_X f \, \mathrm{d}\mu$.
(2) $\mathbb{M}_n(\mathbb{C})$ is a tracial von Neumann algebra with the *normalized* trace $\tau([a_{i,j}]) = \frac{1}{n} \sum_{i=1}^n a_{i,i}$.
(3) More generally, $\mathbb{M}_n(L^\infty(X))$ is a tracial von Neumann algebra, where $\tau([f_{i,j}]) = \frac{1}{n} \sum_{i=1}^n \int_X f_{i,i} \, \mathrm{d}\mu$.

Remark 4.5. Any tracial von Neumann algebra M is finite. If $v \in M$ satisfies $v^*v = 1$, then vv^* is a projection, so $1 - vv^*$ is a projection. As $\tau(1 - vv^*) = \tau(v^*v - vv^*) = 0$ and τ is faithful, $vv^* = 1$.

Theorem 4.6 (existence of the trace)**.** *Any finite von Neumann algebra M on a separable Hilbert space H is tracial. Any II_1 factor is a tracial von Neumann algebra.*

Remark 4.7. Any finite von Neumann algebra $M \subset \mathbb{B}(H)$ admits a *normal center-valued trace* $\Psi : M \to \mathcal{Z}(M)$ (see [KR97, Chapter 8], for a constructive proof, and [Co99, Section 55], for a proof based on the Ryll-Nardzewski fixed point theorem). In particular, any II_1 factor M is tracial. If H is separable, then $\mathcal{Z}(M)$ is isomorphic to $L^\infty(X)$, for a standard probability space (X, μ), by Theorem 2.17. Then $\tau(T) = \int_X \Psi(T) \, \mathrm{d}\mu$ defines a trace on M.

Exercise 4.8. Let M be a II_1 factor with a faithful normal tracial state τ. Prove that two projections $p, q \in M$ are equivalent if and only if $\tau(p) = \tau(q)$.

Exercise 4.9. Let (M,τ) be a diffuse tracial von Neumann algebra. Prove that for every $t\in[0,1]$, there exists a projection $p\in M$ such that $\tau(p)=t$.

Exercise 4.10. (uniqueness of the trace) Let M be a II_1 factor with a faithful normal tracial state τ. Prove that any tracial state $\tau':M\to\mathbb{C}$ must be equal to τ.

4.2. **The standard representation.** A von Neumann algebra can sit in many ways inside $\mathbb{B}(H)$. In this section, we show that any tracial von Neumann algebra (M,τ) has a canonical Hilbert space representation. This is a particular case of the GNS construction (see [Sz22] or [Co99, Chapter 1]).

Endow M with the scalar product $\langle x,y\rangle=\tau(y^*x)$. Define $L^2(M)$ as the closure of M with respect to the norm $\|x\|_2=\sqrt{\tau(x^*x)}$. Let $M\ni x\to\widehat{x}\in L^2(M)$ be the canonical embedding. Then

$$\|xy\|_2^2=\tau(y^*x^*xy)\leqslant\|x^*x\|\tau(y^*y)=\|x\|^2\|y\|_2^2,\forall x,y\in M.$$

Thus, letting $\pi(x)(\widehat{y})=\widehat{xy}$, for all $x,y\in M$, defines a $*$-homomorphism $\pi:M\to\mathbb{B}(L^2(M))$ called the *standard representation* of M. Then π is isometric and thus $\pi(M)$ is a C*-algebra. Moreover, $\pi(M)$ is a von Neumann algebra (see [AP22, Theorem 2.6.1]). Hereafter, we view $M\subset\mathbb{B}(L^2(M))$ by identifying M with $\pi(M)$.

We next show that the commutant of M in the standard representation is anti-isomorphic to M. Define $J:L^2(M)\to L^2(M)$ by letting $J(\widehat{x})=\widehat{x^*}$. Then J is a conjugate linear unitary involution: $J(\alpha\widehat{x}+\beta\widehat{y})=\overline{\alpha}J(\widehat{x})+\overline{\beta}J(\widehat{y})$, $\langle J(\widehat{x}),J(\widehat{y})\rangle=\langle\widehat{y},\widehat{x}\rangle$, $\forall\alpha,\beta\in\mathbb{C}$, $x,y\in M$, and $J^2=I$.

Theorem 4.11. *$M'=JMJ$.*

Proof. Denote $H=L^2(M)$. Notice that $\{x\widehat{1}\mid x\in M\}$ is dense in H and $J(x\widehat{1})=x^*\widehat{1}$, for all $x\in M$. Using these properties for every $x,y,z\in M$ we get that

$$JxJy(z\widehat{1})=JxJ(yz\widehat{1})=Jx(z^*y^*\widehat{1})=J(xz^*y^*\widehat{1})=yzx^*\widehat{1}=yJ(xz^*\widehat{1})=yJxJ(z\widehat{1}).$$

Thus $JMJ\subset M'$ and hence $\{x'\widehat{1}\mid x'\in M'\}\supset\{JxJ\widehat{1}\mid x\in M\}=\{\widehat{x^*}\mid x\in M\}=\{\widehat{x}\mid x\in M\}$. This implies that $\{x'\widehat{1}\mid x'\in M'\}$ is dense in H. Further, if $x'\in M'$, then for all $y\in M$ we have

$$\langle Jx\widehat{1},y\widehat{1}\rangle=\langle Jy\widehat{1},x\widehat{1}\rangle=\langle x^*y^*\widehat{1},\widehat{1}\rangle=\langle y^*x^*\widehat{1},\widehat{1}\rangle=\langle x^*\widehat{1},y\widehat{1}\rangle.$$

This shows that $Jx\widehat{1}=x^*\widehat{1}$, for all $x\in M'$. Altogether, have shown that the two properties satisfied by M are also satisfied by M'. Thus, we deduce that $JM'J\subset M''=M$ and hence $JMJ=M'$. ∎

Exercise 4.12. Let (M,τ) be a tracial von Neumann algebra. Let $\xi\in L^2(M)$ and $C>0$ such that $\|x\xi\|_2\leq C\|x\|_2,\forall x\in M$. Prove that $\xi=\widehat{y}$, for some $y\in M$.

4.3. **Hilbert modules.** Next, we address the following question: on what Hilbert spaces H other than $L^2(M)$ can a tracial von Neumann algebra M be represented? The answer, showing that H is isomorphic to a direct sum of specific Hilbert subspaces of $L^2(M)$, is important as it allows us to define a notion of dimension for H as an M-module. This is crucial in applications such as defining L^2-Betti numbers for groups and manifolds.

Definition 4.13. Let M be a von Neumann algebra. A *left Hilbert M-module* is a Hilbert space H together with a unital normal $*$-homomorphism $\pi:M\to\mathbb{B}(H)$. (Note that defining $x\xi:=\pi(x)(\xi)$ makes H a left M-module.)

Example 4.14. If $p\in M$ is a projection, then $L^2(M)p:=JpJ(L^2(M))$ is a left Hilbert M-module.

Exercise 4.15. Let $x\in M$ and denote by $p\in M$ the right support projection of x. Prove that the left Hilbert M-module $\overline{M\widehat{x}}\subset L^2(M)$ is isomorphic to $L^2(M)p$.

Theorem 4.16. *If H is a left Hilbert M-module, there exists a family of projections $\{p_i\}_{i\in I}$ in M such that $H \cong \oplus_{i\in I} L^2(M)p_i$. More precisely, there exists a unitary operator $U : H \to \oplus_{i\in I} L^2(M)p_i$ such that $U(x\xi) = xU(\xi)$, $\forall x \in M, \xi \in H$. The* dimension *of H is defined by letting*

$$\dim_M(H) := \sum_{i\in I} \tau(p_i).$$

The proof of Theorem 4.16 uses the next lemma and the exercise following it.

Lemma 4.17 (Radon-Nikodym)**.** *Let $\varphi : M \to \mathbb{C}$ be a linear functional with $0 \leqslant \varphi(x) \leqslant \tau(x)$, $\forall x \in M_+$. Then there is $y \in M$ such that $0 \leqslant y \leqslant 1$ and $\varphi(x) = \tau(xy)$, for all $x \in M$.*

Proof. The Cauchy-Schwarz inequality (see Exercise 2.4(1)) gives that

$$|\varphi(y^*x)|^2 \leqslant \varphi(x^*x)\varphi(y^*y) \leqslant \tau(x^*x)\tau(y^*y) = \|x\|_2^2\|y\|_2^2, \forall x, y \in M.$$

In particular, $|\varphi(x)| \leqslant \|x\|_2 = \|\widehat{x}\|_2$, for all $x \in M$. By Riesz's representation theorem we find $\xi \in L^2(M)$ such that $\varphi(x) = \langle \widehat{x}, \xi\rangle$, for all $x \in M$. Next, for $y \in M$, we get that

$$\|y\xi\|_2 = \sup_{x\in M, \|x\|_2\leqslant 1} |\langle \widehat{x}, y\xi\rangle| = \sup_{x\in M, \|x\|_2\leqslant 1} |\langle \widehat{y^*x}, \xi\rangle| = \sup_{x\in M, \|x\|_2\leqslant 1} |\varphi(y^*x)| \leqslant \|y\|_2.$$

By Exercise 4.12 we can find $y \in M$ such that $\xi = \widehat{y^*}$. Thus, $\varphi(x) = \tau(xy)$, for all $x \in M$. It is left as an exercise to show that $0 \leqslant y \leqslant 1$. ∎

Exercise 4.18. Let M be a von Neumann algebra and $\varphi, \psi : M \to \mathbb{C}$ be normal positive linear functionals such that $\varphi(1) < \psi(1)$. Prove that there exists a nonzero projection $q \in M$ such that $\varphi(x) \leq \psi(x), \forall x \in (qMq)_+$. (Let $(r_i) \subset M$ be a maximal family of mutually orthogonal projections such that $\varphi(r_i) \geq \psi(r_i)$, for every i. Then the projection $q = 1 - \sum_i r_i$ has the desired property.)

Proof of Theorem 4.16. The proof relies on the following claim:

Claim. Let $\xi \in H \setminus \{0\}$. Then we can find nonzero projections $q, p \in M$ such that $\overline{Mq\xi} \cong L^2(M)p$.

Proof of the claim. Define $\varphi : M \to \mathbb{C}$ by letting $\varphi(x) = \langle x\xi, \xi\rangle$, for $x \in M$. Let $c > 0$ such that $\varphi(1) < c\tau(1)$. Since φ and $c\tau$ are normal positive linear functionals on M, by Exercise 4.18 we can find a projection $q \in M$ such that $\varphi(x) \leqslant c\tau(x)$, for all $x \in (qMq)_+$.

By applying Lemma 4.17, we can find $y \in qMq$ such that $0 \leqslant y \leqslant c$ and $\varphi(x) = \tau(xy)$, for all $x \in qMq$. Let $z \in (qMq)_+$ such that $z^2 = y$. If $x \in M$, then since $qx^*xq \in qMq$, we get that

$$\|x(q\xi)\|^2 = \varphi(qx^*xq) = \tau(qx^*xqy) = \tau(x^*xy) = \tau(x^*xz^2) = \|x\widehat{z}\|_2^2,$$

and thus $\|x(q\xi)\| = \|x\widehat{z}\|_2$. Thus, $\theta : \overline{Mq\xi} \to \overline{M\widehat{z}} \subset L^2(M)$ given by $\theta(x(q\xi)) = x\widehat{z}$ extends to a unitary operator. It follows that $\overline{Mq\xi} \cong \overline{M\widehat{z}}$. The claim now follows from Exercise 4.15.

Finally, let $\{H_i\}_{i\in I}$ be a maximal family of mutually orthogonal left Hilbert M-sub-modules of H such that for every $i \in I$, there exists a projection $p_i \in \mathcal{M}$ with $H_i \cong L^2(M)p_i$. Then the above claim implies that $H = \oplus_{i\in I} H_i$, which finishes the proof. ∎

4.4. **Hilbert bimodules.** In the early 1980s, Connes discovered that Hilbert bimodules give an appropriate representation theory for tracial von Neumann algebras (see [Co82, Po86] and [AP22, Chapter 13]).

Definition 4.19. Let (M, τ) be a tracial von Neumann algebra. A *Hilbert M-bimodule* is a Hilbert space H equipped with commuting normal $*$-homomorphisms $\pi : M \to \mathbb{B}(H)$, $\rho : M^{\text{op}} \to \mathbb{B}(H)$, where M^{op} is the opposite von Neumann algebra of M. We write $x\xi y = \pi(x)\rho(y^{\text{op}})\xi$.

Examples 4.20. (of Hilbert bimodules)

(1) The *trivial bimodule* $L^2(M)$ with $x\xi y = xJy^*J\xi$.
(2) The *coarse bimodule* $L^2(M)\bar{\otimes}L^2(M)$ with $x(\xi\otimes\eta)y = x\xi\otimes\eta y$.
(3) $L^2(\widetilde{M})$ with $x\xi y = \alpha(x)J\beta(y)^*J\xi$, where $(\widetilde{M},\widetilde{\tau})$ is a tracial von Neumann algebra and $\alpha,\beta : M\to\widetilde{M}$ are $*$-homomorphisms such that $\widetilde{\tau}\circ\alpha = \widetilde{\tau}\circ\beta = \tau$.

4.5. **Jones' basic construction.** Let (M,τ) be a tracial von Neumann algebra in its standard representation and $B\subset M$ be a von Neumann subalgebra. Theorem 4.11 implies that

$$B\subset M = JM'J\subset JB'J\subset\mathbb{B}(L^2(M)).$$

The von Neumann algebra $JB'J$ is called the basic construction associated to $B\subset M$. As shown in Proposition 4.24, this is generated by M and the orthogonal projection from $L^2(M)$ onto $L^2(B)$. The basic construction was used by Jones (via an iteration argument) to prove his famous index theorem for subfactors [Jo83]. It is now a key tool in the study of II_1 factors.

Definition 4.21. Let M be a von Neumann algebra and $B\subset M$ be a von Neumann subalgebra. A positive linear map $E : M\to B$ is called a *conditional expectation* if it satisfies the following:

(1) $E(b) = b$, $\forall b\in B$.
(2) $E(b_1xb_2) = b_1E(x)b_2$, $\forall b_1,b_2\in B, x\in M$.

Proposition 4.22. *Let (M,τ) be a tracial von Neumann algebra and $B\subset M$ be a von Neumann subalgebra. Then there exists a unique conditional expectation $E : M\to B$ such that $\tau\circ E = \tau$.*

Proof. Let $e_B : L^2(M)\to L^2(B)$ be the orthogonal projection, where $L^2(B)$ denotes the $\|\cdot\|_2$-closure of $\{\hat{b}\mid b\in B\}$. If $x\in M$ and $b\in B$, then $be_B(\hat{x}) = e_B(\widehat{bx})$ and hence

$$\|be_B(\hat{x})\|_2 = \|e_B(\widehat{bx})\|_2\leqslant\|\widehat{bx}\|_2 = \|bx\|_2\leqslant\|x\|\,\|b\|_2 = \|x\|\,\|\hat{b}\|_2.$$

Thus, there is $T\in\mathbb{B}(L^2(B))$ such that $T(\hat{b}) = be_B(\hat{x})$. Since $T\in B'$, we get that $T\in JBJ$, and thus $e_B(\hat{x})\in\hat{B}$. One checks that $E_B : M\to B$ given by $\widehat{E_B(x)} = e_B(\hat{x})$ satisfies the conclusion. ∎

Definition 4.23. Let (M,τ) be a tracial von Neumann algebra and $B\subset M$ be a von Neumann subalgebra. The *basic construction* $\langle M,e_B\rangle$ is the von Neumann subalgebra of $\mathbb{B}(L^2(M))$ generated by M and the orthogonal projection e_B from $L^2(M)$ onto $L^2(B)$.

Proposition 4.24. *We have the following:*

(1) $Je_B = e_BJ$, $be_B = e_Bb$ and $e_Bxe_B = E_B(x)e_B$, $\forall b\in B, x\in M$.
(2) $\langle M,e_B\rangle = (JBJ)' = JB'J$.
(3) $e_B\in\langle M,e_B\rangle$ has central support 1.
(4) The linear span of Me_BM is an SOT*-dense $*$-subalgebra of $\langle M,e_B\rangle$.*
(5) There exists a semifinite faithful normal trace $\mathrm{Tr} : \langle M,e_B\rangle\to\mathbb{C}$ such that

$$\mathrm{Tr}(xe_By) = \tau(xy),\forall x,y\in M.$$

(6) If $p\in\langle M,e_B\rangle$ is a projection, then $pL^2(M)$ is a right Hilbert B-module and

$$\dim(pL^2(M)_B) = \mathrm{Tr}(p).$$

Proof. (1) The proof of this assertion is left as an exercise.

(2) Since $\langle M,e_B\rangle' = M'\cap\{e_B\}' = JMJ\cap\{e_B\}' = J(M\cap\{e_B\}')J = JBJ$, the double commutant theorem implies that $\langle M,e_B\rangle = (JBJ)'$.

(3) Since $\langle M,e_B\rangle e_BL^2(M)\supset\langle M,e_B\rangle e_B\hat{1} = \langle M,e_B\rangle\hat{1}\supset M\hat{1}$, by Lemma 3.9 we deduce that $e_B\in\langle M,e_B\rangle$ has central support 1.

(4) Let $\mathcal{M}$ be the SOT-closure of the linear span of Me_BM. Then $\mathcal{M}$ is a von Neumann algebra and a two sided ideal of $\langle M, e_B\rangle$. In particular, $ue_Bu^* \in \mathcal{M}$, for every $u \in \mathcal{U}(\langle M, e_B\rangle)$. Thus, by (3) and Exercise 3.10 we get that $1 = \bigvee_{u\in\mathcal{U}(\langle M,e_B\rangle)} ue_Bu^* \in \mathcal{M}$. This implies that $\mathcal{M} = \langle M, e_B\rangle$.

(5) Since the central support of e_B in $\langle M, e_B\rangle$ is 1, there are partial isometries $(v_i) \subset \langle M, e_B\rangle$ such that $v_i^*v_i \leq e_B$ and $\sum_i v_iv_i^* = 1$ (see Exercise 3.15).

We define a normal weight $\mathrm{Tr} : \langle M, e_B\rangle_+ \to [0, +\infty]$ by letting

$$\mathrm{Tr}(T) = \sum_i \langle Tv_i\widehat{1}, v_i\widehat{1}\rangle.$$

Let $T \in \langle M, e_B\rangle$ with $\mathrm{Tr}(T^*T) = 0$. Then $Tv_i\widehat{1} = 0$ and thus $Tv_i\widehat{b} = Tv_iJb^*J\widehat{1} = Jb^*JTv_i\widehat{1} = 0$, for every $b \in B$. Hence, $Tv_iL^2(B) = \{0\}$ for every i. Since $v_i^*v_i \leq e_B$, we have $Tv_iv_i^*L^2(M) \subset Tv_iL^2(B)$ and thus $Tv_iv_i^* = 0$ for every i. Since $\sum_i v_iv_i^* = 1$, we get that $T = 0$, so Tr is faithful.

To show that Tr is a trace, note that since $\langle M, e_B\rangle e_B = Me_B$, we can find $w_i \in M$ such that $v_i = v_ie_B = w_ie_B$. Let $x, y \in M$. By appying the identity $\sum_i w_ie_Bw_i^* = \sum_i v_iv_i^* = 1$ to $\widehat{x} \in L^2(M)$, we derive that $\sum_i w_iE_B(w_i^*x) = x$. Also, if $a, b, c \in M$, then $\langle ae_Bbe_B\widehat{1}, ce_B\widehat{1}\rangle = \tau(E_B(c^*a)b)$. By combining these identities we get that

$$\mathrm{Tr}(xe_By) = \sum_i \langle xe_Byw_ie_B\widehat{1}, w_ie_B\widehat{1}\rangle = \sum_i \tau(E_B(w_i^*x)yw_i) = \sum_i \tau(w_iE_B(w_i^*x)y) = \tau(xy). \tag{4.1}$$

Thus, if $T = xe_By$ and $T' = x'e_By'$, for some $x, y, x', y' \in M$, then

$$\mathrm{Tr}(TT') = \tau(xE_B(yx')y') = \tau(E_B(y'x)E_B(yx')) = \tau(x'E_B(y'x)y) = \mathrm{Tr}(T'T).$$

This proves that Tr is a trace.

(6) Equation (4.1) shows that Tr does not depend on the choice on $\{v_i\}_{i\in I}$. Thus, we may assume that there is a subset $J \subset I$ such that $p = \sum_{i\in J} v_iv_i^* = \sum_{i\in J} w_ie_Bw_i^*$. Thus, $\mathrm{Tr}(p) = \sum_{i\in J} \tau(w_iw_i^*)$. Since $v_i^*v_i = E_B(w_i^*w_i)e_B$ is a projection, $E_B(w_i^*w_i)$ is a projection. We leave it as an exercise to check that $w_ie_Bw_i^*(L^2(M))$ is isomorphic to $E_B(w_i^*w_i)L^2(B)$, as a right Hilbert B-module. Thus, $pL^2(M) \cong \bigoplus_{i\in J} E_B(w_i^*w_i)L^2(B)$, as right Hilbert B-modules, hence

$$\dim(pL^2(M)_B) = \sum_{i\in I} \tau(E_B(w_i^*w_i)) = \sum_{i\in J} \tau(w_i^*w_i) = \sum_{i\in J} \tau(w_iw_i^*) = \mathrm{Tr}(p),$$

which finishes the proof. ■

Remark 4.25. Let $\mathcal{M}$ be a von Neumann algebra endowed with a normal, faithful, semi-finite trace Tr. For $1 \leq p < \infty$, define $\|x\| = \mathrm{Tr}(|x|^p)^{\frac{1}{p}}$, for every $x \in \mathcal{M}$. The Banach space $\mathrm{L}^p(\mathcal{M})$ is defined as the closure of the set $\{x \in \mathcal{M} \mid \|x\|_p < \infty\}$ with respect to $\|\cdot\|_p$. If $\mathcal{M} = \langle M, e_B\rangle$, then $\mathrm{L}^p(\mathcal{M})$ is equal to the closure of the span of Me_BM with respect to $\|\cdot\|_p$.

4.6. **Popa's intertwining-by-bimodules technique.** Given subalgebras A, B of a von Neumann algebra M, it is a natural question whether $uAu^* \subset B$, for some $u \in \mathcal{U}(M)$. To address this question, Popa developed a technique, called *intertwining-by-bimodules.* This technique has been instrumental in the progress made in the classification of II_1 factors via Popa's deformation/rigidity theory (see the surveys [Po06, Va10, Io18]) and is now a fundamental tool in the study of II_1 factors.

Theorem 4.26 (Popa, [Po03])**.** *Let A, B be von Neumann subalgebras of a tracial von Neumann algebra (M, τ). Then the following conditions are equivalent:*

(1) There is no net $(u_i) \subset \mathcal{U}(A)$ such that $\|E_B(xu_iy)\|_2 \to 0, \forall x, y \in M$.
(2) There is a nonzero projection $e \in A' \cap \langle M, e_B\rangle$ such that $\mathrm{Tr}(e) < +\infty$.

(3) *There are nonzero projections $p \in A, q \in B$, a nonzero partial isometry $v \in M$ and a $*$-homomorphism $\theta : pAp \to qBq$ such that $v^*v \leq p, vv^* \leq q$ and $\theta(x)v = vx, \forall x \in pAp$.*
(4) *There exists an A-B-subbimodule $\mathcal{H}$ of $\mathrm{L}^2(M)$ such that $\dim(\mathcal{H}_B) < +\infty$.*

If conditions (1)-(4) hold, we write $A \prec_M B$ and say that a corner of A embeds into B inside M*.*

Next, we mention two cases when $A \prec_M B$ implies the existence of $u \in \mathcal{U}(M)$ such that $uAu^* \subset B$.

Definition 4.27. Let (M, τ) be a tracial von Neumann algebra. We say that a von Neumann subalgebra $A \subset M$ is a *Cartan subalgebra* if it is maximal abelian and the normalising group $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}$ satisfies $\mathcal{N}_M(A)'' = M$.

Theorem 4.28 (Popa, [Po01])**.** *Let A, B be Cartan subalgebras of a II_1 factor M. Then $A \prec_M B$ if and only if there exists $u \in \mathcal{U}(M)$ such that $uAu^* = B$.*

Remark 4.29. Let (M, τ) be a tracial von Neumann algebra and $A, B \subset M$ be von Neumann subalgebras. Assume that $A' \cap M = \mathbb{C}1$ and the inclusion $B \subset M$ is *mixing*: $\|E_B(xb_ny)\|_2 \to 0$, for every $x, y \in M$ with $E_B(x) = 0$ and any sequence $b_n \subset (B)_1$ such that $b_n \to 0$ in the WOT. Then $A \prec_M B$ if and only if there exists $u \in \mathcal{U}(M)$ such that $uAu^* \subset B$ [Po03].

5. Examples of tracial von Neumann algebras

5.1. **The hyperfinite II_1 factor.** For $n \geq 1$, let $A_n = \mathbb{M}_{2^n}(\mathbb{C})$ and $\tau_n : A_n \to \mathbb{C}$ be the normalized trace. Consider the diagonal embedding $A_n \subset A_{n+1}$ given by

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

Define $A = \cup_{n \geq 1} A_n$ and notice that A is a $*$-algebra which is equipped with a norm $\|\cdot\|$ which satisfies $\|x^*x\| = \|x\|^2$, for all $x \in A$. Moreover, $\tau : A \to \mathbb{C}$ defined by $\tau(x) = \tau_n(x)$, if $x \in A_n$, is a faithful tracial linear functional which satisfies $|\tau(x)| \leq \|x\|$, for all $x \in A$.

We denote by H the closure of A with respect to the norm $\|x\|_2 = \sqrt{\tau(x^*x)}$, and consider the GNS $*$-homomorphism $\pi : A \to \mathbb{B}(H)$ given by $\pi(x)(\widehat{y}) = \widehat{xy}$, for all $x, y \in A$.

Theorem 5.1. *$R := \overline{\pi(A)}^{\mathrm{WOT}}$ is a II_1 factor and the map $\varphi : R \to \mathbb{C}$ given by $\varphi(x) = \langle x\widehat{1}, \widehat{1} \rangle$ is a normal faithful tracial state such that $\varphi \circ \pi = \tau$.*

Proof. Showing that φ is tracial on R is equivalent to proving that $\langle y\hat{1}, x^*\hat{1} \rangle = \langle x\hat{1}, y^*\hat{1} \rangle$, for all $x, y \in R$. Since this holds for all $x, y \in \pi(A)$ ($\langle \pi(y)\hat{1}, \pi(x)^*\hat{1} \rangle = \langle \hat{y}, \hat{x^*} \rangle = \tau(xy)$, for all $x, y \in A$) and $\pi(A)$ is SOT-dense in R, we deduce that φ is tracial on R.

Given $z \in A$, we have $\|yz\|_2^2 = \tau(z^*y^*yz) = \tau(yzz^*y^*) \leq \|zz^*\|\tau(yy^*) = \|z\|^2\|y\|_2^2$, for all $y \in A$. This implies the existence of $\rho(z) \in \mathbb{B}(H)$ such that $\rho(z)(\hat{y}) = \widehat{yz}$, for all $y \in A$. Since $\rho(z) \in \pi(A)'$, we get that $\rho(z) \in R'$. Thus, $R'\hat{1} \supset \{\rho(z)\hat{1} \mid z \in A\} = \{\hat{z} \mid z \in A\}$ and since $\{\hat{z} \mid z \in A\}$ is $\|\cdot\|_2$-dense in H, Exercise 4.3 gives that φ is faithful.

Since φ is a normal state, the second assertion of the theorem is proven.

Finally, let us show that R is a factor. To this end, let $x \in \mathcal{Z}(R)$ and put $x_0 = x - \varphi(x) \cdot 1$. For $n \geq 1$, let $R_n = \pi(A_n) \subset R$ and $E_n : R \to R_n$ be the unique φ-preserving conditional expectation. Then $E_n(x) \in \mathcal{Z}(R_n)$. Since $R_n \cong \mathbb{M}_{2^n}(\mathbb{C})$ is factor and E_n is φ-preserving, we get that $E_n(x) = \varphi(E_n(x)) \cdot 1 = \varphi(x) \cdot 1$ or equivalently $E_n(x_0) = 0$. Thus, for every $n \geq 1$ and $y \in R_n$ we have that $\varphi(x_0y) = \varphi(E_n(x_0y)) = \varphi(E_n(x_0)y) = 0$. Hence, $\varphi(x_0y) = 0$, for all $y \in \pi(A)$. Since $\pi(A)$ is SOT-dense in R, we conclude that this equality holds for every $y \in R$. In particular, we have that $\varphi(x_0x_0^*) = 0$. Since φ is faithful we conclude that $x_0 = 0$ and thus $x = \varphi(x)1 \in \mathbb{C}1$. ■

Definition 5.2. A von Neumann algebra M is called *hyperfinite* if it admits an increasing sequence $(M_n)_{n\geq 1}$ of finite dimensional $*$-subalgebras such that $\cup_{n\geq 1} M_n$ is SOT-dense in M.

The II_1 factor R from Theorem 5.1 is hyperfinite by definition. Murray and von Neumann [MvN43] proved that any hyperfinite II_1 factor is isomorphic to R, which justifies the following:

Definition 5.3. The II_1 factor R is called **the** *hyperfinite II_1 factor.*

As it turns out, R is the smallest II_1 factor:

Exercise 5.4. Let M be a II_1 factor and $\tau : M \to \mathbb{C}$ be a faithful normal tracial state.

(1) By Exercise 4.9, there exists a projection $p \in M$ such that $\tau(p) = \frac{1}{2}$. Use this fact to prove that there exists an injective unital $*$-homomorphism $\rho : \mathbb{M}_2(\mathbb{C}) \to M$.
(2) Prove that there exists an injective unital $*$-homomorphism $\pi : R \to M$.

5.2. **Group von Neumann algebras.** Let Γ be a countable group. The left and right regular representations $\lambda, \rho : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ are given by $\lambda(g)(\delta_h) = \delta_{gh}$ and $\rho(g)(\delta_h) = \delta_{hg^{-1}}$. The *group von Neumann algebra* $L(\Gamma) \subset \mathbb{B}(\ell^2(\Gamma))$ is the WOT-closure of the linear span of $\{\lambda(g) \mid g \in \Gamma\}$ [MvN43]. We denote by $R(\Gamma) \subset \mathbb{B}(\ell^2(\Gamma))$ the WOT-closure of the linear span of $\{\rho(g) \mid g \in \Gamma\}$.

Convention. Following the tradition in the subject, we denote $u_g := \lambda(g)$, for $g \in \Gamma$.

Proposition 5.5. *$\tau : L(\Gamma) \to \mathbb{C}$ given by $\tau(x) = \langle x\delta_e, \delta_e\rangle$ is a faithful normal tracial state. Moreover, $L(\Gamma)' = R(\Gamma)$.*

Proof. Since $\tau(1) = 1$ and $\tau(x^*x) = \|x\delta_e\|^2 \geqslant 0$, for all $x \in M$, we get that τ is a normal state. Since $\tau(u_g u_h) = \tau(u_{gh}) = \delta_{gh,e} = \delta_{hg,e} = \tau(u_{hg}) = \tau(u_h u_g)$, we get that τ is a trace. If $\tau(x^*x) = 0$, then the first line of the proof implies that $x\delta_e = 0$. If $g \in \Gamma$, then since $\rho(g^{-1}) \in L(\Gamma)'$, we get that $x\delta_g = x(\rho(g^{-1})\delta_e) = \rho(g^{-1})(x\delta_e) = 0$. This implies that $x = 0$, hence τ is faithful.

We identify $L^2(L(\Gamma))$ with $\ell^2\Gamma$ via the unitary $u_g \to \delta_g$. Under this identification, the involution J becomes $J(\delta_g) = \delta_{g^{-1}}$. Now, if $g, h \in \Gamma$, then $Ju_gJ(\delta_h) = Ju_g\delta_{h^{-1}} = J\delta_{gh^{-1}} = \delta_{hg^{-1}} = \rho(g)(\delta_h)$. This shows that $Ju_gJ = \rho(g)$, for all $g \in \Gamma$, hence $L(\Gamma)' = JL(\Gamma)J = R(\Gamma)$. ■

Notation 5.6. For $x \in L(\Gamma)$, we write $x\delta_e = \sum_{g\in\Gamma} x_g\delta_g \in \ell^2\Gamma$. Observe that in the above identification $L^2(L(\Gamma)) = \ell^2(\Gamma)$, we have that $\widehat{x} = x\delta_e$. The complex coefficients $\{x_g\}_{g\in\Gamma}$ are called the *Fourier coefficients* of x and can be calculated as $x_g = \langle x\delta_e, \delta_g\rangle = \tau(xu_g^*)$. We will write $x = \sum_{g\in\Gamma} x_g u_g$, where the convergence holds in the $\|\cdot\|_2$ (but not necessarily the WOT!).

Exercise 5.7. Let $x, y \in L(\Gamma)$ and let $x = \sum_{g\in\Gamma} x_g u_g$, $y = \sum_{g\in\Gamma} y_g u_g$ be their Fourier expansions. Prove that $x^* = \sum_{g\in\Gamma} \overline{x_{g^{-1}}} u_g$ and $xy = \sum_{g\in\Gamma}(\sum_{h\in\Gamma} x_h y_{h^{-1}g})u_g$.

Remark 5.8. Let Γ be a countable abelian group. The group of all homomorphisms $\eta : \Gamma \to \mathbb{T}$ is a compact abelian group, called the *dual* of Γ and denoted $\widehat{\Gamma}$. Let μ be the Haar measure of $\widehat{\Gamma}$. For $g \in \Gamma$, let $\hat{g} \in L^2(\widehat{\Gamma})$ be given by $\hat{g}(\eta) = \eta(g)$. Then the map $U : \ell^2(\Gamma) \to L^2(\widehat{\Gamma})$ given by $U(\delta_g) = \hat{g}$ extends to a unitary such that $UL(\Gamma)U^* = L^\infty(\widehat{\Gamma})$. In particular, $L(\Gamma)$ is $*$-isomorphic to $L^\infty(\widehat{\Gamma})$. If Γ is infinite, then μ has no atoms and so $L(\Gamma)$ is $*$-isomorphic to $L^\infty([0,1], \lambda)$ by Corollary 2.20.

The next result clarifies when $L(\Gamma)$ is a II_1 factor.

Proposition 5.9. *Let Γ be a countable group. Then $L(\Gamma)$ is a factor if and only if Γ has infinite conjugacy classes (or, is* icc*): the conjugacy class $\{hgh^{-1} \mid h \in \Gamma\}$ is infinite, for every $g \in \Gamma \setminus \{e\}$.*

Proof. ($\Rightarrow$) Assume that $C = \{hgh^{-1} \mid h \in \Gamma\}$ is finite, for some $g \neq e$. Then $x = \sum_{k\in C} u_k$ belongs to the center of $L(\Gamma)$ and $x \notin \mathbb{C}1$.

($\Leftarrow$) Assume that Γ is icc and let x be an element in the center of $L(\Gamma)$. Let $x = \sum_{g\in\Gamma} x_g u_g$ be the Fourier expansion of x and $h \in \Gamma$. Let $y = \sum_{g\in\Gamma} y_g u_g$ for the Fourier expansion of $y = u_h x u_h^*$. Then $y_g = \tau(yu_g^*) = \tau(u_h x u_h^* u_g^*) = \tau(xu_{hgh^{-1}}^*) = x_{hgh^{-1}}$. Since x commutes with u_h, we get that $y = x$, and hence $x_{hgh^{-1}} = x_g$, for all $g, h \in \Gamma$. Since $\sum_{g\in\Gamma} |x_g|^2 = \|x\|_2^2 < \infty$, and Γ is icc, we conclude that $x_g = 0$, for all $g \in \Gamma \setminus \{e\}$. Thus, $x \in \mathbb{C}1$. ■

Exercise 5.10. Prove that the following countable groups are icc:

(1) the group S_∞ of bijections $\pi : \mathbb{N} \to \mathbb{N}$ such that $\{n \in \mathbb{N} \mid \pi(n) \neq n\}$ is finite.
(2) the free product group $\Gamma = \Gamma_1 * \Gamma_2$, where Γ_1, Γ_2 are any groups with $|\Gamma_1| > 1$ and $|\Gamma_2| > 2$. In particular, the free group $\mathbb{F}_n$ on $n \geq 2$ generators is icc.
(3) $\mathrm{SL}_n(\mathbb{Z}) := \{A \in \mathbb{M}_n(\mathbb{Z}) \mid \det(A) = 1\}$, for every odd $n \geqslant 3$.

5.3. **Group measure space von Neumann algebras.** In this section, we study group measure space von Neumann algebras arising from pmp actions. Let Γ be a countable group and (X, μ) a standard probability space. An action $\Gamma \curvearrowright (X, \mu)$ is called *probability measure preserving* (*pmp*) if for every $g \in \Gamma$ and measurable set $Y \subset X$, the set gY is measurable and $\mu(gY) = \mu(Y)$.

Recall that the group measure space von Neumann algebra $L^\infty(X) \rtimes \Gamma$ is defined as the closure, in the WOT, of the linear span of $\{fu_g \mid f \in L^\infty(X), g \in \Gamma\} \subset \mathbb{B}(H)$. Here, $H = L^2(X) \otimes \ell^2\Gamma$, we define a unitary representation $u : \Gamma \to \mathcal{U}(H)$ and view $L^\infty(X) \subset \mathbb{B}(H)$ by

$$u_g(\xi \otimes \delta_h) = \sigma_g(\xi) \otimes \delta_{gh} \quad \text{and} \quad f(\xi \otimes \delta_h) = f\xi \otimes \delta_h, \qquad \forall f \in L^\infty(X), \xi \in L^2(X), g, h \in \Gamma,$$

where $\sigma_g(\xi)(x) = \xi(g^{-1}x)$. Also, we recall that $u_g f u_g^* = \sigma_g(f)$, for every $f \in L^\infty(X), g \in \Gamma$.

Proposition 5.11. *$\tau : L^\infty(X) \rtimes \Gamma \to \mathbb{C}$ given by $\tau(x) = \langle x(1 \otimes \delta_e), 1 \otimes \delta_e\rangle$ is a faithful normal tracial state.*

Proof. For all $f \in L^\infty(X)$ and $g \in \Gamma$ we have that

$$\tau(fu_g) = \langle fu_g(1 \otimes \delta_e), 1 \otimes \delta_e\rangle = \langle f \otimes \delta_g, 1 \otimes \delta_e\rangle = \delta_{g,e} \int_X f \, \mathrm{d}\mu.$$

If $f_1, f_2 \in L^\infty(X)$ and $g_1, g_2 \in \Gamma$, then $f_1 u_{g_1} f_2 u_{g_2} = f_1 \sigma_{g_1}(f_2) u_{g_1 g_2}$ and $f_2 u_{g_2} f_1 u_{g_2} = f_2 \sigma_{g_2}(f_1) u_{g_2 g_1}$. Since $\tau(\sigma_g(f)) = \tau(f)$, for all $f \in L^\infty(X)$ and $g \in \Gamma$, we get that $\tau(f_1 u_{g_1} f_2 u_{g_2}) = \tau(f_2 u_{g_2} f_1 u_{g_2})$. This implies that τ is a trace. We leave the rest of the proof as an exercise. ■

Proposition 5.12. *Let $\Gamma \curvearrowright (X, \mu)$ be a pmp action. Denote $M = L^\infty(X) \rtimes \Gamma$ and $A = L^\infty(X)$. Every $a \in M$ has a unique* Fourier expansion *of the form $a = \sum_{g\in\Gamma} a_g u_g$, where $a_g = E_A(au_g^*) \in A$, for every $g \in \Gamma$, and the series converges in $\|\cdot\|_2$. Moreover, we have the following:*

- $a^* = \sum_{g\in\Gamma} \sigma_{g^{-1}}(a_g^*) u_{g^{-1}}$.
- $\|a\|_2^2 = \sum_{g\in\Gamma} \|a_g\|_2^2$.
- $ab = \sum_{g\in\Gamma} (\sum_{h\in\Gamma} a_h \sigma_h(b_{h^{-1}g})) u_g$.

Proof. The formula $U(fu_g) = f \otimes \delta_g$ defines a unitary operator $U : L^2(M) \to L^2(X) \otimes \ell^2\Gamma$. Thus, every $a \in M$ can be written as $a = \sum_{g\in\Gamma} a_g u_g$, where $a_g \in L^2(X)$ satisfy $\sum_{g\in\Gamma} \|a_g\|_2^2 = \|a\|_2^2$. Moreover, we have that $\hat{a}_e = e_A(\hat{a})$ and thus $a_e = E_A(a)$. Since $au_h^* = \sum_{g\in\Gamma} a_{gh} u_g$, we get that $a_h = E_A(au_h^*)$, for every $h \in \Gamma$. We leave the rest of the proof as an exercise. ■

Lemma 5.13. *A pmp action $\Gamma \curvearrowright (X, \mu)$ is ergodic if and only if any function $f \in L^2(X)$ which satisfies that $\sigma_g(f) = f$, for every $g \in \Gamma$, is essentially constant.*

Proof. ($\Leftarrow$) If Y is a Γ-invariant set, then $f = 1_Y \in L^2(X)$ is a Γ-invariant function. Thus, there is $c \in \mathbb{C}$ such that $f = c$. As $f^2 = f$, we get that $c \in \{0, 1\}$, hence $\mu(Y) = \int_X f \, \mathrm{d}\mu = c \in \{0, 1\}$.

($\Rightarrow$) Let $f \in L^2(X)$ be a $\sigma(\Gamma)$-invariant function. If f is not constant, then it admits at least two distinct essential values $z, w \in \mathbb{C}$. Let $\delta = |z - w|/2$. Then $Y = \{x \in X | \, |f(x) - z| < \delta\}$ and $Z = \{x \in X | \, |f(x) - w| < \delta\}$ are disjoint, Γ-invariant, measurable sets. Since $\mu(Y) > 0$ and $\mu(Z) > 0$, we get a contradiction with the ergodicity of the action. ■

Exercise 5.14. Let Γ be an infinite group and (Y, ν) be a non-trivial standard probability space. Define $(X, \mu) = (Y^\Gamma, \nu^{\otimes \Gamma})$. Consider the *Bernoulli action* $\Gamma \curvearrowright (X, \mu)$ given by $gx = (x_{g^{-1}h})_{h \in \Gamma}$, for every $g \in \Gamma$ and $x = (x_h)_{h \in \Gamma} \in X$. Prove that this action is pmp, essentially free and ergodic. Moreover, prove that this action is *mixing*: $\lim_{g \to \infty} \mu(gY \cap Z) = \mu(Y)\mu(Z)$, $\forall Y, Z \subset X$ measurable.

Proposition 5.15. *Let $\Gamma \curvearrowright (X, \mu)$ be a pmp action. Denote $M = L^\infty(X) \rtimes \Gamma$ and $A = L^\infty(X)$.*

(1) The action $\Gamma \curvearrowright (X, \mu)$ is free if and only if $A \subset M$ is maximal abelian, i.e., $A' \cap M = A$.
(2) Assume that the action $\Gamma \curvearrowright (X, \mu)$ is free. Then M is a factor if and only if the action $\Gamma \curvearrowright (X, \mu)$ is ergodic.

Proof. (1) Assume that $A' \cap M = A$. Let $g \in \Gamma \setminus \{e\}$ and put $Y = \{x \in X \mid gx = x\}$. Since $1_Y \sigma_g(f) = 1_Y f$, for all $f \in A$, we get that $a = 1_Y u_g \in A' \cap M$. Hence $a \in A$ and thus $a = E_A(a) = 0$, showing that $\mu(Y) = 0$. This implies that the action is free.

Conversely, assume that the action is free. Let $a \in A' \cap M$ and $a = \sum_{g \in \Gamma} a_g u_g$ be its Fourier decomposition. If $b \in A$, then $\sum_{g \in \Gamma} b a_g u_g = ba = ab = \sum_{g \in \Gamma} a_g \sigma_g(b) u_g$, thus $b a_g = \sigma_g(b) a_g$, for all $g \in \Gamma$. Let $g \in \Gamma \setminus \{e\}$ and put $Y_g = \{x \in X \mid a_g(x) \neq 0\}$. The last equality gives that $b(g^{-1}x) = b(x)$, for almost every $x \in Y_g$. Since (X, μ) is a standard probability space, we can find a sequence of measurable sets $(X_n) \subset X$ which separate points in X. By applying the last identity to $b = 1_{X_n}$, for all $n \geq 1$, we deduce that $g^{-1}x = x$, for almost every $x \in Y_g$. Since the action is free, we get that $\mu(Y_g) = 0$, hence $a_g = 0$. As this holds for all $g \in \Gamma \setminus \{e\}$, we conclude that $a \in A$.

(2) Since the action is free, (1) implies that $\mathcal{Z}(M) = A \cap M' = \{a \in A \mid \sigma_g(a) = a, \forall g \in \Gamma\}$. By Lemma 5.13, the conclusion follows. ■

Exercise 5.16. Let Γ be an icc group and $\Gamma \curvearrowright (X, \mu)$ be a pmp action. Prove that $L^\infty(X) \rtimes \Gamma$ is a II_1 factor if and only if the action $\Gamma \curvearrowright (X, \mu)$ is ergodic.

5.4. **Cartan subalgebras and orbit equivalence.** By Proposition 5.15, $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$ is a Cartan subalgebra, for every free pmp action $\Gamma \curvearrowright (X, \mu)$. It is a fundamental observation of Singer [Si55] (see also Feldman and Moore's work [FM75]) that the isomorphism class of the inclusion $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$ captures exactly the orbit equivalence class of the action $\Gamma \curvearrowright (X, \mu)$.

Proposition 5.17 (Singer, [Si55])**.** *If $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are free pmp actions, then the following conditions are equivalent:*

(1) There exists a $*$-isomorphism $\pi : L^\infty(X) \rtimes \Gamma \to L^\infty(Y) \rtimes \Lambda$ such that $\pi(L^\infty(X)) = L^\infty(Y)$.
(2) The actions are orbit equivalent*, i.e., there exists an isomorphism $\theta : (X, \mu) \to (Y, \nu)$ (called an* orbit equivalence *between the actions) such that $\theta(\Gamma x) = \Lambda \theta(x)$, for a.e. $x \in X$.*

Both implications of Proposition 5.17 are important. Thus, (1) $\Rightarrow$ (2) reduces the classification of group measure space factors to the classification of actions up to orbit equivalence, provided that the Cartan subalgebras can be shown to be unique. Conversely, the implication (2) $\Rightarrow$ (1) provides a von Neumann algebraic approach to the study of orbit equivalence of actions.

Proof. Denote $A = L^\infty(X), B = L^\infty(Y), M = L^\infty(X) \rtimes \Gamma$ and $N = L^\infty(Y) \rtimes \Lambda$.

(1) $\Rightarrow$ (2) Since $\pi_{|A} : A \to B$ is a $*$-isomorphism, we can find an isomorphism $\theta : (X,\mu) \to (Y,\nu)$ such that $\pi(a) = a \circ \theta^{-1}$, for all $a \in A$ (see [AP22, Theorem 3.3.4]). We will prove that θ is the desired orbit equivalence. To this end, fix $g \in \Gamma$ and denote $v = \pi(u_g)$. Then v normalises B and thus we can find an isomorphism $\alpha : (Y,\nu) \to (Y,\nu)$ such that $vbv^* = b \circ \alpha$, for all $b \in B$.

Claim. $\alpha(y) \in \Lambda y$, for almost every $y \in Y$.

Consider the Fourier expansion $v = \sum_{h\in\Lambda} v_h u_h$, where $v_h \in B$ for all $h \in \Lambda$. Since $vb = (b \circ \alpha)v$, we deduce that $v_h(b \circ h^{-1}) = v_h(b \circ \alpha)$, for all $h \in \Lambda$ and $b \in B$. If we let $Y_h = \{y \in Y \mid v_h(y) \neq 0\}$, then the same argument as in the proof of Proposition 5.15 shows that $\alpha(y) = h^{-1}y$, for almost every $y \in Y_h$ and all $h \in \Lambda$. Now, if we let $Z = Y \setminus (\cup_{h\in\Lambda} Y_h)$, then $1_Z v_h = 0$, for all $h \in \Lambda$ and thus $1_Z v = \sum_{h\in\Lambda}(1_Z v_h)u_h = 0$. Hence $\nu(Z)^{1/2} = \|1_Z\|_2 = \|1_Z v\|_2 = 0$, which implies that the set $\cup_{h\in\Lambda} Y_h$ is co-null in Y. This clearly implies the claim.

If $a \in A$, then $a \circ g^{-1} \circ \theta^{-1} = \pi(u_g a u_g^*) = v\pi(a)v^* = \pi(a) \circ \alpha = a \circ \theta^{-1} \circ \alpha$. Thus, $g^{-1} \circ \theta^{-1} = \theta^{-1} \circ \alpha$ hence $\theta \circ g^{-1} = \alpha \circ \theta$. Together with the claim, this implies that $\theta(g^{-1}x) = \alpha(\theta(x)) \in \Lambda\theta(x)$, for almost every $x \in X$. Since $g \in \Gamma$ is arbitrary, we conclude that $\theta(\Gamma x) \subset \Lambda\theta(x)$, for almost every $x \in X$. Since the reverse inclusion can be proved similarly, it follows that θ is an orbit equivalence.

(2) $\Rightarrow$ (1) Let $\theta : (X,\mu) \to (Y,\nu)$ be an orbit equivalence. Define a $*$-isomorphism $\pi : A \to B$ by letting $\pi(a) = a \circ \theta^{-1}$. Our goal is to show that π extends to a $*$-isomorphism $\pi : M \to N$.

To this end, fix $g \in \Gamma$. Then $(\theta \circ g^{-1} \circ \theta^{-1})(y) \in \Lambda \cdot y$, for almost every $y \in Y$. For $h \in \Lambda$, put $Y_{g,h} = \{y \in Y \mid (\theta \circ g^{-1} \circ \theta^{-1})(y) = h^{-1}y\}$. Then $\{Y_{g,h}\}_{h\in\Lambda}$ is a measurable partition of Y. Since $h^{-1}Y_{g,h} = \{y \in Y \mid (\theta \circ g \circ \theta^{-1})(y) = hy\}$, we also have that $\{h^{-1}Y_{g,h}\}_{h\in\Lambda}$ is a measurable partition of Y. Using the last two facts, one checks that the formula $\pi(u_g) = \sum_{h\in\Lambda} 1_{Y_{g,h}} u_h$ defines a unitary in N such that $\pi(u_g)\pi(a)\pi(u_g)^* = \pi(a \circ g^{-1})$, for all $a \in A$. This entails that π extends to a $*$-homomorphism from the linear span of $\{au_g \mid a \in A, g \in \Gamma\}$ to N. Moreover, π is trace preserving. We leave it as an exercise to show that π extends to a $*$-isomorphism $\pi : M \to N$. ■

5.5. **Tensor product von Neumann algebras.** We next establish that the class of tracial von Neumann algebras is closed under tensor products.

Proposition 5.18. *Let (M_1,τ_1) and (M_2,τ_2) be tracial von Neumann algebras. Then $M_1\overline{\otimes}M_2$ is a tracial von Neumann algebra. Moreover, if M_1 and M_2 are II_1 factors, then so is $M_1\overline{\otimes}M_2$.*

The proof of the moreover assertion of Proposition 5.18 relies on the following exercise:

Exercise 5.19. Let (M,τ) be a tracial von Neumann algebra. For $x \in M$, let $K_x \subset L^2(M)$ be the $\|\cdot\|_2$-closure of the convex hull of the set $\{\widehat{uxu^*} \mid u \in \mathcal{U}(M)\}$. Then we have

(1) Assume that M is a II_1 factor and let $x \in M$. Prove that $\tau(x)\hat{1}$ is the unique element of minimal $\|\cdot\|_2$ of K_x. Deduce in particular that $\tau(x)\hat{1} \in K_x$.
(2) Assume that the linear span of the set of $x \in M$ such that $\tau(x)\hat{1} \in K_x$ is $\|\cdot\|_2$-dense in M. Prove that M is II_1 factor.

Proof of Proposition 5.18. Using that $M_1\overline{\otimes}M_2 \subset \mathbb{B}(L^2(M_1) \otimes L^2(M_2))$, define $\tau : M_1\overline{\otimes}M_2 \to \mathbb{C}$ by $\tau(x) = \langle x(\hat{1} \otimes \hat{1}), \hat{1} \otimes \hat{1}\rangle$. Then $\tau(x_1 \otimes x_2) = \tau_1(x_1)\tau_2(x_2)$ for all $x_1 \in M_1, x_2 \in M_2$, and τ is a trace. Since $M_1'\overline{\otimes}M_2' \subset (M_1\overline{\otimes}M_2)'$ and $\hat{1} \otimes \hat{1}$ is $M_1'\overline{\otimes}M_2'$-cyclic, Exercise 4.3 implies that τ is faithful.

For the moreover assertion, assume that M_1 and M_2 are II_1 factors. If $x_1 \in M_1$ and $x_2 \in M_2$, then by Exercise 5.19(1) we get that $\tau_1(x_1)\hat{1} \in K_{x_1}$ and $\tau_2(x_2)\hat{1} \in K_{x_2}$. This easily implies that $\tau(x_1 \otimes x_2)(\hat{1} \otimes \hat{1}) = \tau_1(x_1)\hat{1} \otimes \tau_2(x_2)\hat{1} \in K_{x_1 \otimes x_2}$. Since the linear span of $\{x_1 \otimes x_2 \mid x_1 \in M_1, x_2 \in M_2\}$ is SOT-dense and so $\|\cdot\|_2$-dense in $M_1\overline{\otimes}M_2$, Exercise 5.19(2) gives that $M_1\overline{\otimes}M_2$ is a II_1 factor. ■

Exercise 5.20. Let (X_1, μ_1) and (X_2, μ_2) be standard probability spaces and consider the product probability space $(X_1 \times X_2, \mu_1 \otimes \mu_2)$. Prove that $L^\infty(X_1)\overline{\otimes}L^\infty(X_2) \cong L^\infty(X_1 \times X_2)$.

Exercise 5.21. Let Γ_1 and Γ_2 be countable groups. Prove that $L(\Gamma_1)\overline{\otimes}L(\Gamma_2) \cong L(\Gamma_1 \times \Gamma_2)$.

5.6. **Free product von Neumann algebras.** We next recall the definition of the of two tracial von Neumann algebras (M_1, τ_1) and (M_2, τ_2).

Denote $H_i = L^2(M_i) \ominus \mathbb{C}\widehat{1}$, for $i \in \{1, 2\}$, and define the Hilbert space

$$H = \mathbb{C}\widehat{1} \oplus \Big(\bigoplus_{n\geq 1} \bigoplus_{i_1\neq i_2 \neq \cdots \neq i_n} H_{i_1} \otimes H_{i_2} \otimes \cdots \otimes H_{i_n}\Big).$$

Also, for $i \in \{1, 2\}$, define the Hilbert space

$$H_i = \mathbb{C}\widehat{1} \oplus \Big(\bigoplus_{n\geq 1} \bigoplus_{i\neq i_1\neq i_2 \neq \cdots \neq i_n} H_{i_1} \otimes H_{i_2} \otimes \cdots \otimes H_{i_n}\Big).$$

Then we have a natural unitary identification $H = L^2(M_i)\otimes H_i$ which allows us to view $M_i \subset \mathbb{B}(H)$.

Definition 5.22. The free product von Neumann algebra $M_1 * M_2$ is defined as the von Neumann algebra generated by $M_1, M_2 \subset \mathbb{B}(H)$.

Proposition 5.23. *Let (M_1, τ_1) and (M_2, τ_2) be tracial von Neumann algebras. Then $M_1 * M_2$ is a tracial von Neumann algebra.*

Proof. Let $\tau : M \to \mathbb{C}$ be the normal state given by $\tau(x) = \langle x\hat{1}, \hat{1}\rangle$. Let $x = x_1x_2 \cdots x_n$, where $n \geq 1$, $x_j \in M_{i_j}$ and $\tau(x_j) = 0$, for every $1 \leq j \leq n$, and $i_j \neq i_{j+1}$, for every $1 \leq j \leq n - 1$. Since $\widehat{x1} = \widehat{x}_{i_1} \otimes \widehat{x}_{i_2} \otimes \cdots \otimes \widehat{x}_{i_n} \in H_{i_1} \otimes H_{i_2} \otimes \cdots \otimes H_{i_n}$ is orthogonal to $\hat{1}$ we get that $\tau(x) = 0$.

Let $y = y_m \cdots y_2y_1$, where $m \geq 1$, $y_k \in M_{l_k}$ and $\tau(y_k) = 0$, for every $1 \leq k \leq m$, and $l_k \neq l_{k+1}$, for every $1 \leq k \leq m - 1$. By using the previous paragraph, it follows that $\tau(xy) = \tau(yx) = 0$, unless $n = m$ and $i_j = l_j$, for every $1 \leq j \leq n$, in which case we have that $\tau(xy) = \tau(yx) = \prod_{j=1} \tau_{i_j}(x_jy_j)$. In either case, we get that $\tau(xy) = \tau(yx)$ and since the linear span of x (respectively, y) of the form above is SOT-dense in $\{z \in M \mid \tau(z) = 0\}$, we deduce that τ is tracial. ∎

Exercise 5.24. Let (M_1, τ_1) and (M_2, τ_2) be tracial von Neumann algebras such that M_1 is diffuse and $M_2 \neq \mathbb{C}1$. Let $(u_k) \in \mathcal{U}(M_1)$ be a sequence such that $u_k \to 0$, weakly. Prove that $u_kxu_k^* \to 0$, weakly, for every $x \in M_1 * M_2$ with $E_{M_1}(x) = 0$. Use this to prove that $M_1 * M_2$ is a II_1 factor.

Exercise 5.25. Let Γ_1 and Γ_2 be countable groups. Prove that $L(\Gamma_1) * L(\Gamma_2) \cong L(\Gamma_1 * \Gamma_2)$.

5.7. **Ultraproduct von Neumann algebras.** We end this section by defining ultraproducts of tracial von Neumann algebras. We start by reviewing the notion of free ultrafilters on $\mathbb{N}$.

Definition 5.26. The *Stone-Čech compactification* of $\mathbb{N}$, denoted by $\beta\mathbb{N}$, is defined as the Gelfand dual of the abelian C*-algebra $\ell^\infty(\mathbb{N})$. An *ultrafilter* of $\mathbb{N}$ is an element of $\beta\mathbb{N}$, i.e., a non-zero homomorphism $\omega : \ell^\infty(\mathbb{N}) \to \mathbb{C}$. For $n \in \mathbb{N}$, we denote by $e_n \in \beta\mathbb{N}$ the evaluation at n, i.e., $e_n(f) = f(n)$. An ultrafilter $\omega \in \beta\mathbb{N}$ is *free* if it does not belong to $\mathbb{N} \equiv \{e_n\}_{n\in\mathbb{N}}$.

Remark 5.27. We have that $\beta\mathbb{N} \setminus \mathbb{N} \neq \emptyset$. To see this, let $K_n \subset \beta\mathbb{N}$ be the weak*-closure of $\{e_k \mid k > n\}$. Then K_n is weak*-compact by Alaoglu's theorem and $K_{n+1} \subset K_n$, for all n. Thus, $\cap_n K_n \neq \emptyset$. If $\omega \in \cap_n K_n$, then $\omega \in K_n$ and thus $\omega(\delta_n) = 0$, for all $n \in \mathbb{N}$. This shows that $\omega \notin \mathbb{N}$.

Exercise 5.28. If $\omega \in \beta\mathbb{N}$, we denote $\lim_{n\to\omega} x_n := \omega((x_n)_n)$, for every $(x_n)_n \in \ell^\infty(\mathbb{N})$. Prove that if $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\omega} x_n = x$.

Warning. The notation $\lim\limits_{n\to\omega}$ will be used to denote the limit of a sequence of complex numbers along an ultrafilter ω and should not be confused with the limit as n approaches an ordinal ω.

Definition 5.29. Let $\omega \in \beta\mathbb{N}\setminus\mathbb{N}$ and (M_n, τ_n) be a sequence of tracial von Neumann algebras. Let $\prod_{n\in\mathbb{N}} M_n$ be the C^*-algebra of sequences (x_n) with $x_n \in M_n, \forall n \in \mathbb{N}$, and $\|(x_n)\| := \sup \|x_n\| < \infty$. Let $\mathcal{I}_\omega \subset \prod_{n\in\mathbb{N}} M_n$ be the two-sided norm closed ideal consisting of sequences $(x_n) \in \prod_{n\in\mathbb{N}} M_n$ such that $\lim_{n\to\omega} \|x_n\|_2 = 0$. The *ultraproduct* $\prod_{n\in\omega} M_n$ is defined as the quotient $\prod_{n\in\mathbb{N}} M_n/\mathcal{I}_\omega$.

Then $\prod_{n\in\omega} M_n$ is a tracial von Neumann algebra which has a canonical trace $\tau_\omega((x_n)) = \lim\limits_{n\to\omega} \tau_n(x_n)$ (see [BO08, Appendix 4.A] or [AP22, Proposition 5.4.1] for a proof of this fact).

If $M_n = M$, $\forall n \in \mathbb{N}$, we denote $\prod_{n\in\omega} M_n$ by M^ω and call it the *ultrapower* of M.

6. Properties of von Neumann algebras

In the first two parts of this section we present two fundamental representation-theoretic properties (amenability and property (T)) of groups and von Neumann algebras. We end this section by briefly discussing two asymptotic properties of II_1 factors (property Gamma and McDuff's property).

6.1. Amenability.

Definition 6.1. A countable group Γ is called *amenable* if there exists a state $\varphi : \ell^\infty(\Gamma) \to \mathbb{C}$ which is invariant under the left translation action: $\varphi(g \cdot f) = \varphi(f)$, for all $g \in \Gamma$ and $f \in \ell^\infty(\Gamma)$. Here, $g \cdot f \in \ell^\infty(\Gamma)$ is defined as $(g \cdot f)(h) = f(g^{-1}h)$.

Theorem 6.2. *Let Γ be a countable group. Then the following conditions are equivalent:*

(1) Γ is amenable.
(2) Γ satisfies the Reiter condition*: there exists a sequence of non-negative functions $f_n \in \ell^1(\Gamma)$ such that $\|f_n\|_1 = 1$, for all n, and $\lim_{n\to\infty} \|g \cdot f_n - f_n\|_1 = 0$, for all $g \in \Gamma$.*
(3) Γ satisfies the Følner condition*: there exists a sequence of finite subsets $F_n \subset \Gamma$ such that $\lim_{n\to\infty} |gF_n \setminus F_n|/|F_n| = 0$, for all $g \in \Gamma$.*
(4) the left regular representation of Γ has almost invariant vectors*: there exists a sequence $\xi_n \in \ell^2(\Gamma)$ such that $\|\xi_n\|_2 = 1$, for all n, and $\lim_{n\to\infty} \|\lambda(g)\xi_n - \xi_n\|_2 = 0$, for all $g \in \Gamma$.*

Proof. The proof relies on two very useful tricks, due to Day (the proof of (1) $\Rightarrow$ (2)) and Namioka (the proof of (2) $\Rightarrow$ (3)). Enumerate $\Gamma = \{g_n\}_{n\geq 1}$.

(1) $\Rightarrow$ (2) Fix $n \geq 1$ and consider the convex subset

$$C := \{(g_1 \cdot f - f, g_2 \cdot f - f, \cdots, g_n \cdot f - f) \mid f \in \ell^1(\Gamma), f \geq 0, \|f\|_1 = 1\}$$

of the Banach space $\ell^1(\Gamma)^{\oplus_n}$ with the norm $\|(f_1, f_2, \cdots, f_n)\| = \sum_{i=1}^n \|f_i\|_1$.

We claim that $\mathbf{0} = (0, 0, \cdots, 0) \in \overline{C}^{\|\cdot\|}$. Assuming this claim, we can find $f_n \in \ell^1(\Gamma)$ such that $f_n \geq 0, \|f_n\|_1 = 1$ and $\sum_{i=1}^n \|g_i \cdot f_n - f_n\|_1 \leq 1/n$. This clearly implies (2).

If the claim were false, then since $\overline{C}^{\|\cdot\|} \subset \ell^1(\Gamma)^{\oplus_n}$ is a closed convex set and $(\ell^1(\Gamma)^{\oplus_n})^* = \ell^\infty(\Gamma)^{\oplus_n}$, the Hahn-Banach separation theorem implies the existence of $F_1, F_2, \cdots, F_n \in \ell^\infty(\Gamma)$ and $\alpha > 0$ such that $\sum_{i=1}^n \Re\langle g_i \cdot f - f, F_i\rangle \geq \alpha$, for any $f \in \ell^1(\Gamma)$ with $f \geq 0$ and $\|f\|_1 = 1$.

If we put $F = \sum_{i=1}^n \Re(g_i^{-1} \cdot F_i - F_i)$, then the last inequality rewrites as $\langle f, F\rangle \geq \alpha$, for any $f \in \ell^1(\Gamma)$ with $f \geq 0$ and $\|f\|_1 = 1$. For $f = \delta_g$, this implies that $F(g) \geq \alpha$, for all $g \in \Gamma$. Thus, we get that $\varphi(F) \geq \varphi(\alpha 1) = \alpha > 0$. On the other hand, $\varphi(F) = \sum_{i=1}^n (\varphi(\Re(g_i^{-1} \cdot F_i)) - \varphi(\Re F_i)) = 0$. This gives the desired contradiction.

(2) $\Rightarrow$ (3) If $f_1, f_2 \in \ell^1(\Gamma)$ and $f_1, f_2 \geq 0$, then Fubini's theorem implies that

$$\|f_1 - f_2\|_1 = \int_0^\infty \|1_{\{f_1>t\}} - 1_{\{f_2>t\}}\|_1 \mathrm{d}t \quad \text{and} \quad \|f_1\|_1 = \int_0^\infty \|1_{\{f_1>t\}}\|_1 \mathrm{d}t. \tag{6.1}$$

By (2), for any $n \geq 1$ we can find $f \in \ell^1(\Gamma)$ such that $f \geq 0$, $\|f\|_1 = 1$ and $\sum_{i=1}^n \|g_i \cdot f - f\|_1 < 1/n$. For $t > 0$, let $K_t = \{f > t\}$. Since $f \in \ell^1(\Gamma)$, we get that K_t is a finite subset of Γ. Also, note that $\{g \cdot f > t\} = gK_t$ and thus $\|1_{\{g\cdot f>t\}} - 1_{\{f>t\}}\|_1 = |gK_t \triangle K_t|$, for all $g \in \Gamma$. By combining the last inequality with (6.1), we derive that

$$\int_0^\infty \sum_{i=1}^n |g_i K_t - K_t| \mathrm{d}t < 1/n = 1/n\|f\|_1 = \int_0^\infty (|K_t|/n)\mathrm{d}t.$$

Hence, there is $t_n > 0$ such that $F_n := K_{t_n}$ satisfies $\sum_{i=1}^n |g_i F_n \triangle F_n| < |F_n|/n$. This proves (3).

(3) $\Rightarrow$ (4) Let $\xi_n := 1_{F_n}/\sqrt{|F_n|}$. Then $\|\xi_n\|_2 = 1$ and $\|\lambda(g)\xi_n - \xi_n\|_2 = \sqrt{|gF_n \triangle F_n|/|F_n|}$, for all $n \geq 1$ and $g \in \Gamma$, which clearly implies (4).

(4) $\Rightarrow$ (1) Let ω be a free ultrafilter on $\mathbb{N}$. Define $\varphi : \ell^\infty(\Gamma) \to \mathbb{C}$ by letting $\varphi(f) = \lim_{n\to\omega}\langle f\xi_n, \xi_n\rangle$. Then φ is a state and $\varphi(g \cdot f) = \lim_{n\to\omega}\langle f(g^{-1} \cdot \xi_n), g^{-1} \cdot \xi_n\rangle = \varphi(f)$, for all $f \in \ell^\infty(\Gamma)$, $g \in \Gamma$. ∎

Exercise 6.3. Let Γ be a countable group. Assume that (a) any finitely generated subgroup of Γ is amenable or (b) Γ is abelian. Prove that Γ is amenable.

Proposition 6.4. *$\mathbb{F}_2$ is not amenable.*

Proof. Assume by contradiction that there exists a left translation invariant state $\varphi : \ell^\infty(\mathbb{F}_2) \to \mathbb{C}$. Define $m : \mathcal{P}(\mathbb{F}_2) \to [0, 1]$ by $m(A) = \varphi(1_A)$. Then m is finitely additive ($m(A\cup B) = m(A)+m(B)$, for every disjoint $A, B \subset \mathbb{F}_2$) and left invariant ($m(gA) = m(A)$, for every $g \in \mathbb{F}_2$ and $A \subset \mathbb{F}_2$).

Let a and b be the free generators of $\mathbb{F}_2$. Let S be the set of elements of $\mathbb{F}_2$ whose reduced form begins with a non-zero power of a, and put $T = \mathbb{F}_2 \setminus S$. Then $aT \subset S$, $bS\cup b^2S \subset T$ and $bS\cap b^2S = \emptyset$. Thus, we get $m(S) \geq m(aT) = m(T) \geq m(bS \cup b^2S) = m(bS) + m(b^2S) = 2m(S)$. This implies that $m(S) = m(T) = 0$. Since $m(S) + m(T) = m(\mathbb{F}_2) = 1$, this provides a contradiction. ∎

Exercise 6.5. Let Γ_1 and Γ_2 be any countable groups such that $|\Gamma_1| > 1$ and $|\Gamma_2| > 2$. Prove that the free product group $\Gamma = \Gamma_1 * \Gamma_2$ is not amenable.

Definition 6.6. A tracial von Neumann algebra (M, τ) is called *amenable* if there exists a state $\Phi : \mathbb{B}(L^2(M)) \to \mathbb{C}$ such that $\Phi_{|M} = \tau$ and $\Phi(Tx) = \Phi(xT)$, for all $x \in M$ and $T \in \mathbb{B}(L^2(M))$.

Theorem 6.7. *Let Γ be a countable group. Then Γ is amenable if and only if $L(\Gamma)$ is amenable.*

Proof. Assume that Γ is amenable and let $\varphi : \ell^\infty(\Gamma) \to \mathbb{C}$ be a left translation invariant state. Define a state $\Phi : \mathbb{B}(\ell^2(\Gamma)) \to \mathbb{C}$ by letting

$$\Phi(T) := \varphi(g \mapsto \langle T\delta_g, \delta_g\rangle).$$

If $T \in L(\Gamma)$, then for all $g \in \Gamma$ we have

$$\langle T\delta_g, \delta_g\rangle = \langle T\rho(g)\delta_e, \rho(g)\delta_e\rangle = \langle \rho(g)^*T\rho(g)\delta_e, \delta_e\rangle = \langle T\delta_e, \delta_e\rangle = \tau(T),$$

and thus $\Phi(T) = \tau(T)$. If $T \in \mathbb{B}(\ell^2(\Gamma))$ and $h \in \Gamma$, then the left invariance of φ gives that

$$\Phi(\lambda(h)T\lambda(h)^*) = \varphi(g \mapsto \langle \lambda(h)T\lambda(h)^*\delta_g, \delta_g\rangle) = \varphi(g \mapsto \langle T\delta_{h^{-1}g}, \delta_{h^{-1}g}\rangle) = \Phi(T).$$

Thus, if $\mathcal{C} := \{x \in L(\Gamma) \mid \Phi(Tx) = \Phi(xT), \text{ for all } T \in \mathbb{B}(\ell^2(\Gamma))\}$, then $\lambda(g) \in \mathcal{C}$, for all $g \in \Gamma$. By Cauchy-Schwarz, we have that $|\Phi(Tx)|^2 \leq \Phi(TT^*)\Phi(x^*x) \leq \|T\|^2\Phi(x^*x) = \|T\|^2\|x\|_2^2$ and similarly

$|\Phi(xT)|^2 \leq \|T\|^2\|x\|_2^2$, for all $x \in L(\Gamma)$ and $T \in \mathbb{B}(\ell^2(\Gamma))$. This implies that $\mathcal{C}$ is $\|\cdot\|_2$-closed. Since $\mathcal{C}$ contains the linear span of $\lambda(\Gamma)$, we conclude that $\mathcal{C} = L(\Gamma)$. This shows that $L(\Gamma)$ is amenable.

Conversely, assume that $L(\Gamma)$ is amenable. Let Φ be a state on $\mathbb{B}(\ell^2(\Gamma))$ such that $\Phi(Tx) = \Phi(xT)$, for all $x \in L(\Gamma)$ and $T \in \mathbb{B}(\ell^2(\Gamma))$. Consider the natural embedding $\ell^\infty(\Gamma) \subset \mathbb{B}(\ell^2(\Gamma))$ and notice that $\lambda(g)f\lambda(g)^* = f \circ g^{-1} = g \cdot f$, for all $f \in \ell^\infty(\Gamma)$ and $g \in \Gamma$. Thus, for all $f \in \ell^\infty(\Gamma)$ and $g \in \Gamma$, we have that $\Phi(g \cdot f) = \Phi(\lambda(g)f\lambda(g)^*) = \Phi(f)$. This implies that Γ is amenable. ■

Exercise 6.8. Prove that any hyperfinite tracial von Neumann algebra (M, τ) is amenable.

It is a remarkable fact, proved by Connes in his celebrated work [Co75], that the converse is true. A tracial von Neumann algebra (M, τ) is called *separable* if the Hilbert space $L^2(M)$ is separable.

Theorem 6.9 (Connes, [Co75])**.** *Let (M, τ) be a separable tracial von Neumann algebra. Then the following are equivalent:*

1. *M is amenable.*
2. *M is* injective*: there exists a conditional expectation $E : \mathbb{B}(L^2(M)) \to M$.*
3. *there exists a sequence $\xi_n \in L^2(M)\bar{\otimes}L^2(M)$ such that $\langle x\xi_n, \xi_n\rangle = \tau(x)$ for all $n \geq 1$ and $\lim_{n\to\infty} \|x\xi_n - \xi_n x\| = 0$, for all $x \in M$.*
4. *M is hyperfinite.*

Consequently, any separable amenable II_1 factor M is isomorphic to R. In particular, $L(\Gamma)$ is isomorphic to R, for any icc amenable group Γ.

For a proof of this result, see [AP22, Chapters 11 and 13].

6.2. **Property (T).**

Definition 6.10. [Ka67] A countable group Γ has *Kazhdan's property (T)* if any unitary representation $\pi : \Gamma \to \mathcal{U}(H)$ which has almost invariant vectors admits a nonzero invariant vector.

Examples of countable groups with property (T) include $\mathrm{SL}_n(\mathbb{Z})$, for $n \geq 3$, and, more generally, any lattice in a simple Lie group of rank at least 2 [Ka67]. For more on property (T), see [BdHV08].

Exercise 6.11. Let Γ be a countable group and $(\Gamma_n)_{n\in\Gamma}$ be an increasing sequence of subgroups with $\cup_{n\in\mathbb{N}}\Gamma_n = \Gamma$. Endow $X = \sqcup_{n\in\mathbb{N}}\Gamma/\Gamma_n$ with the left multiplication action of Γ and denote by $\pi : \Gamma \to \mathcal{U}(\ell^2(X))$ the associated unitary representation. Prove that

1. π has almost invariant vectors.
2. π has nonzero invariant vectors if and only if $\Gamma_n = \Gamma$ for some $n \in \mathbb{N}$.

Deduce that if Γ has property (T), then it is finitely generated.

Exercise 6.12. Prove that any countable amenable group Γ which has property (T) must be finite.

We next recall Connes and Jones' notion of property (T) for II_1 factors.

Definition 6.13. [CJ85] A II_1 factor M has property (T) if there exists $F \subset M$ finite and $\delta > 0$ such that whenever H is a Hilbert M-bimodule and $\xi \in H$ is a unit vector with $\max_{x\in F} \|x\xi - \xi x\| < \delta$, there exists a nonzero vector $\eta \in H$ such that $x\eta = \eta x, \forall x \in M$.

Exercise 6.14. Prove that any amenable tracial factor M which has property (T) must be finite dimensional.

Proposition 6.15 (Connes and Jones, [CJ85])**.** *Let Γ be a countable icc group. Then Γ has property (T) if and only if $L(\Gamma)$ has property (T).*

Proof. We prove the "only if" assertion and refer the reader to [CJ85] for the proof of the "if" assertion. Assume that Γ has property (T) and enumerate $\Gamma = \{g_n\}_{n\in\mathbb{N}}$. Then there are $S \subset \Gamma$ finite and $\delta > 0$ such that if $\pi : \Gamma \to \mathcal{U}(H)$ is a unitary representation and $\xi \in H$ is a unit vector with $\max_{g\in S} \|\pi(g)\xi - \xi\| < \delta$, then H has a nonzero $\pi(\Gamma)$-invariant vector. Otherwise, for any $n \in \mathbb{N}$ we find a unitary representation $\pi_n : \Gamma \to \mathcal{U}(H_n)$ without nonzero invariant vectors and a unit vector $\xi_n \in H_n$ such that $\max_{1\leq i\leq n} \|\pi_n(g_i)\xi_n - \xi_n\| \leq \frac{1}{n}$. Then the representation $\pi = \bigoplus_{n\in\mathbb{N}} \pi_n$ has almost invariant vectors but no nonzero invariant vectors.

Let H be a Hilbert $L(\Gamma)$-bimodule which has a unit vector ξ such that $\max_{g\in S} \|u_g\xi - \xi u_g\| < \delta$. Define a unitary representation $\pi : \Gamma \to \mathcal{U}(H)$ by letting $\pi(g)\eta = u_g\eta u_g^*$, for every $g \in \Gamma, \eta \in H$. Then $\|\pi(g)\xi - \xi\| = \|u_g\xi - \xi u_g\| < \delta$, for every $g \in S$. By the previous paragraph, we can find a nonzero vector η such that $\pi(g)\eta = \eta$ and thus $u_g\eta = \eta u_g$, for all $g \in \Gamma$. Since the linear span of $\{u_g \mid g \in \Gamma\}$ is SOT-dense in $L(\Gamma)$ we conclude that $x\eta = \eta x$, for all $x \in L(\Gamma)$. ■

Definition 6.16. Let M be a II_1 factor. We denote by $\mathrm{Aut}(M)$ be group of automorphisms of M and by $\mathrm{Inn}(M) = \{\mathrm{Ad}(u) \mid u \in \mathcal{U}(M)\}$ the subgroup of inner automorphisms of M. We endow $\mathrm{Aut}(M)$ with the pointwise $\|\cdot\|_2$-topology: $\theta_i \to \theta \iff \|\theta_i(x) - \theta(x)\|_2 \to 0, \forall x \in M$.

The *outer automorphism group* of M is defined as the quotient group

$$\mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Inn}(M).$$

The *fundamental group* of M is defined by

$$\mathcal{F}(M) = \{\frac{\tau(p)}{\tau(q)} \mid p, q \in M \text{ nonzero projections such that } pMp \cong qMq\}.$$

The fundamental group was introduced by Murray and von Neumann in [MvN43] who showed that it is a multiplicative subgroup of $(0, +\infty)$, and that $\mathcal{F}(R) = (0, +\infty)$. The outer automorphism group of R is also a very large group that contains every second countable locally compact group. In contrast, II_1 factors with property (T) have "small" (countable) symmetry groups:

Proposition 6.17. *Let M be a II_1 factor with property (T). Then* $\mathrm{Inn}(M)$ *is an open subgroup of* $\mathrm{Aut}(M)$. *Thus,* $\mathrm{Out}(M)$ *is countable.*

Proof. Let $F \subset M, \delta > 0$ as in Definition 6.13. To prove that $\mathrm{Inn}(M) < \mathrm{Aut}(M)$ is an open subgroup it suffices to show that a neighborhood of Id_M in $\mathrm{Aut}(M)$ is contained in $\mathrm{Inn}(M)$. Let $\theta \in \mathrm{Aut}(M)$ such that $\max_{x\in F} \|\theta(x) - x\|_2 < \delta$. Consider the Hilbert space $H = L^2(M)$ with the Hilbert M-bimodule structure given by $x\xi y = \theta(x)Jy^*J\xi$. Then for every $x \in F$ we have

$$\|x\widehat{1} - \widehat{1}x\|_2 = \|\theta(x) - x\|_2 < \delta.$$

Thus, we can find a unit vector $\eta \in L^2(M)$ such that $\theta(x)\eta = Jx^*J\eta, \forall x \in M$. Then $\varphi : M \to \mathbb{C}$ given by $\varphi(x) = \langle x\eta, \eta\rangle$ is a tracial state. Exercise 4.10 gives that $\varphi = \tau$. So $\|x\eta\|_2 = \|x\|_2, \forall x \in M$, and Exercise 4.12 implies that $\eta = \widehat{u}$, for some $u \in \mathcal{U}(M)$. Therefore, $\theta = \mathrm{Ad}(u)$ belongs to $\mathrm{Inn}(M)$.

Since $\mathrm{Inn}(M) < \mathrm{Aut}(M)$ is an open subgroup, it is also closed. Moreover, the quotient group $\mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Inn}(M)$ is both discrete and separable, and thus must be countable. ■

This result was proved by Connes in [Co80] when $M = L(\Gamma)$, for an icc property (T) group Γ. Connes moreover established that $\mathcal{F}(M)$ is countable.

In a major breakthrough in [Po01], Popa gave the first examples of II_1 factors M with trivial fundamental group, $\mathcal{F}(M) = \{1\}$. The existence of II_1 factors M with trivial outer automorphism group, $\mathrm{Out}(M) = \{e\}$, was proved later on in [IPP05]. Only very recently, the first examples of II_1 factors with property (T) that have trivial fundamental group and, respectively, trivial outer automorphism group were found in [CDHK20] and [CIOS21].

6.3. Property Gamma and McDuff's property.

Definition 6.18. Let M be a II_1 factor. A sequence $(x_n) \subset M$ is called *uniformly bounded* if $\sup_n \|x_n\| < \infty$ and *almost central* if $\|x_n y - y x_n\|_2 \to 0$, $\forall y \in M$. We say that M has *property Gamma* if it admits a uniformly bounded central sequence (x_n) such that $\inf_n \|x_n - \tau(x_n)1\|_2 > 0$.

Property Gamma was introduced by Murray and von Neumann in [MvN43] who used it to show that $R \ncong L(\mathbb{F}_2)$. Specifically, they proved that R has property Gamma while $L(\mathbb{F}_2)$ does not.

Let M be a II_1 factor with property Gamma and $(x_n) \subset M$ be as in Definition 6.18. Consider the "diagonal" embedding $M \subset M^\omega$ given by $M \ni y \mapsto (y_n = y) \in M^\omega$ and view $x = (x_n) \in M^\omega$. Then x belongs to $M' \cap M^\omega = \{a \in M^\omega \mid ab = ba, \forall b \in M\}$. Since $\|x - \tau(x)1\|_2 = \lim\limits_{n \to \omega} \|x_n - \tau(x_n)1\|_2 > 0$, we get that $M' \cap M^\omega \neq \mathbb{C}1$. Conversely, if $M' \cap M^\omega \neq \mathbb{C}1$, then M has property Gamma.

Moreover, the following holds:

Theorem 6.19. *Let M be a separable II_1 factor. Then the following are equivalent:*

1. *M has property Gamma.*
2. *$M' \cap M^\omega \neq \mathbb{C}1$, where ω is a free ultrafilter on $\mathbb{N}$.*
3. $\mathrm{Inn}(M)$ *is not a closed subgroup of* $\mathrm{Aut}(M)$.
4. *There exists a sequence of unit vectors $\xi_n \in L^2(M) \ominus \mathbb{C}\hat{1}$ such that $\|x\xi_n - \xi_n x\|_2 \to 0, \forall x \in M$.*

For a proof of this result, we refer the reader to [AP22, Chapter 15]. The above equivalences were obtained as follows: (1) $\Leftrightarrow$ (2) in [Co74], (1) $\Leftrightarrow$ (3) in [Co74, Sa74], and (1) $\Leftrightarrow$ (4) in [Co75].

Exercise 6.20. A group Γ is called *inner amenable* if the representation $\pi : \Gamma \to \mathcal{U}(\ell^2(\Gamma \setminus \{e\}))$ given by $\pi(g)(\delta_h) = \delta_{ghg^{-1}}$ has almost invariant vectors. Assume that $\Gamma \curvearrowright (X, \mu)$ is a pmp action of a non-inner amenable group Γ and let $M = L^\infty(X) \rtimes \Gamma$. Prove that $M' \cap M^\omega \subset L^\infty(X)^\omega$. Deduce, by taking X to consist of one point, that $L(\Gamma)$ is a II_1 factor without property Gamma.

Exercise 6.21. Let Γ be a non-amenable group such that the centralizer $\{h \in \Gamma \mid gh = hg\}$ is amenable, $\forall g \in \Gamma \setminus \{e\}$. Prove that Γ is not inner amenable. Deduce that $\mathbb{F}_2$ is not inner amenable.

Definition 6.22. Let M be a II_1 factor. We say that M is *McDuff* if it admits two uniformly bounded central sequences $(x_n), (y_n)$ such that $\inf_n \|x_n y_n - y_n x_n\|_2 > 0$.

The following is the main result of [Mc70]:

Theorem 6.23. *Let M be a separable II_1 factor. Then the following are equivalent:*

1. *M is McDuff.*
2. *$M' \cap M^\omega$ is not abelian, where ω is a free ultrafilter on $\mathbb{N}$.*
3. *M is isomorphic to $M \overline{\otimes} R$.*

If M is McDuff then it has property Gamma. The converse is false, as the next example shows.

Example 6.24. Let $\mathbb{Z} \curvearrowright^\sigma (X, \mu)$ be a free ergodic pmp action and $\delta : \mathbb{F}_2 \to \mathbb{Z}$ be an onto homomorphism. Consider the ergodic pmp action $\mathbb{F}_2 \curvearrowright^{\sigma \circ \delta} (X, \mu)$. Since $\mathbb{F}_2$ is not inner amenable, Exercise 6.20 gives that $M = L^\infty(X) \rtimes_{\sigma \circ \delta} \mathbb{F}_2$ is a II_1 factor which satisfies $M' \cap M^\omega \subset L^\infty(X)^\omega$. Since σ is not strongly ergodic (equivalently, $(L^\infty(X) \rtimes \mathbb{Z})' \cap L^\infty(X)^\omega \neq \mathbb{C}1$) , $\sigma \circ \delta$ is not strongly ergodic and thus $M' \cap L^\infty(X)^\omega \neq \mathbb{C}1$. Therefore, $M' \cap M^\omega$ is nontrivial and abelian. In other words, M has property Gamma but is not McDuff.

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