ON THE CANONICAL BUNDLE FORMULA AND ADJUNCTION FOR GENERALIZED KÄHLER PAIRS

CHRISTOPHER HACON AND MIHAI PĂUN

Abstract. In this article we prove analogs of Kawamata's canonical bundle formula, Kawamata subadjunction and plt/lc inversion of adjunction for generalized pairs on Kähler varieties. We also show that a conjecture of [BDPP13] in dimension $n-1$ implies that the cone theorem holds for any n-dimensional Kähler generalized klt pair $(X, B+\boldsymbol{\beta})$.

Contents

Generalized pairs have been playing an increasingly prominent role in higher dimensional birational geometry (see eg. [Birkar21] and references therein). Their analytic counterparts were introduced in [DHY23] (see Definition 1.1) where it is shown that the minimal model program for compact Kähler generalized klt 3-fold pairs holds. Note that even in the projective case, Definition 1.1 is more general than the usual definition of generalized pairs since the "nef" part is only assumed to be a positive

Christopher Hacon was partially supported by the NSF research grants no: DMS-1952522, DMS-1840190, DMS-2301374.

Mihai Păun gratefully acknowledges support from the DFG.

(1,1) form (instead of a nef divisor). This is extremely useful in the Kähler context as, in many instances, we can replace the use of an arbitrary ample (or big) divisor by a (modified) Kähler class. In particular, using this extra flexibility, [DHY23] shows the finiteness of minimal models for compact Kähler generalized klt 3-fold pairs of general type, and that klt Calabi-Yau Kähler 3-folds are connected by finite sequences of flops. The theory of generalized pairs makes sense in all dimensions and it is hoped that many results from the projective minimal model program will carry through to this setting. In this paper we perform the first steps in this direction. We show that adjunction and inversion of adjunction hold for generalized pairs (both in the plt and lc cases), we prove a canonical bundle formula for generalized klt Kähler pairs and we show that assuming the BDPP conjecture in dimension $n-1$, then the cone theorem for generalized pairs holds in dimension n (and in particular it holds unconditionally in dimension 4). More precisely, we show the following.

Theorem 0.1. *Let $(X, B+\boldsymbol{\beta})$ be a generalized pair and S a component of B of coefficient 1 with normalization $\nu : S^\nu \to S$. Then $(X, B+\boldsymbol{\beta})$ is generalized lc (resp. generalized plt) on a neighborhood of S iff $(S^\nu, B_{S^\nu}+\boldsymbol{\beta}_{S^\nu})$ is generalized lc (resp. S is normal and $(S, B_S+\boldsymbol{\beta}_S)$ is generalized klt).*

Next we turn our attention to the following generalization of Kawamata's adjunction theorem for generalized pairs (cf. [Kawamata98, Theorem 1]).

Theorem 0.2. *Let $(X, B+\boldsymbol{\beta})$ be a generalized log canonical pair such that $(X, B'+\boldsymbol{\beta}')$ is a generalized klt pair and $W \subset X$ is a minimal log canonical center of $(X, B+\boldsymbol{\beta})$. Then W is normal and $(K_X+B+\boldsymbol{\beta})|_W = K_W+B_W+\boldsymbol{\beta}_W$ is a generalized klt pair.*

In order to prove this result, it is necessary to prove the following canonical bundle formula (cf. Theorem 2.3).

Theorem 0.3. *Let $f : X \to Y$ be a projective morphism of compact normal Kähler varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $(X, B+\boldsymbol{\beta})$ is a generalized klt (or generalized lc) pair. If $\gamma \in H^{1,1}_{\mathrm{BC}}(Y)$ is such that $[K_X+B+\boldsymbol{\beta}_X] = f^*\gamma$ then $\gamma = [K_Y+B_Y+\boldsymbol{\beta}_Y]$ where $(Y, B_Y+\boldsymbol{\beta}_Y)$ is a generalized klt (or generalized lc) pair.*

Note that we expect semistable reduction to hold (unconditionally) for morphisms of compact analytic varieties, however the necessary references are not yet available. In the case of projective morphisms, we can deduce semistable reduction from the algebraic case (see [AK00], [Karu99]). It is possible that the results of [BdSB23] are already sufficient for our purposes, but the proof would appear to be more involved and we do not pursue it here. The proof of this result heavily uses Theorem 6.2 (which is a generalization of a result of Guenancia [Gue20]), which roughly speaking, states that $\boldsymbol{\beta}_Y$ is pseudo-effective. By [DHP22, Theorem 2.36], to show that $\boldsymbol{\beta}_Y$ is b-nef, it suffices to show that $\boldsymbol{\beta}_Y|_Z$ is pseudo-effective for any subvariety $Z \subset X$. This can be checked by using semistable reduction and applying Theorem 6.2.

Finally, assuming a key conjecture of Boucksom-Demailly-Paun-Peternell [BDPP13, Conjecture 0.1], we show that the cone theorem for generalized klt pairs holds in arbitrary dimension (and unconditionally for pseudo-effective pairs in dimension ≤ 4).

This provides some evidence that the minimal model program holds in arbitrary dimension for generalized klt pairs. We refer the reader to [DHY23] for the generalized klt minimal model program in dimensions ≤ 3.

Conjecture 0.4. *Let X be a compact Kähler manifold. Then the canonical class K_X is pseudoeffective if and only if X is not uniruled (i.e. not covered by rational curves).*

Note that the above conjecture is known to hold in dimension ≤ 3. Following ideas from [CH20] and using Theorem 0.3 we then prove the following result.

Theorem 0.5. *Assume that Conjecture 0.4 holds in dimension n (resp. in dimension $n-1$). Let X be a compact $\mathbb{Q}$-factorial Kähler variety of dimension n such that $(X, B+\boldsymbol{\beta})$ is generalized klt (resp. and $K_X+B+\boldsymbol{\beta}_X$ is pseudo-effective), then there are at most countably many rational curves $\{\Gamma_i\}_{i\in I}$ such that*

$$\overline{\mathrm{NA}}(X) = \overline{\mathrm{NA}}(X)_{K_X+B+\boldsymbol{\beta}_X\geq 0} + \sum_{i\in I}\mathbb{R}^+[\Gamma_i],$$

where $0 < -(K_X+B+\boldsymbol{\beta}_X)\cdot\Gamma_i \leq 2n$. Moreover, if $B+\boldsymbol{\beta}_X$ (or $K_X+B+\boldsymbol{\beta}_X$) is big, then I is finite.

We now turn to a more detailed description of some of the key results in this paper. The most important results in this paper concern versions of the canonical bundle formula (see Theorems 0.3 and 5.2). The typical set up for the canonical bundle formula is an algebraic fiber space $f: X\to Y$ where say X, Y are normal projective varieties, $f_*\mathcal{O}_X=\mathcal{O}_Y$, and a log canonical pair (X,B) such that $K_X+B\sim_{\mathbb{Q},Y}0$. We can then write

$$K_X+B\sim_{\mathbb{Q}} f^*(K_Y+B_Y+M_Y)$$

where B_Y is the boundary part that measures the singularities of f, and the moduli part M_Y is a $\mathbb{Q}$-divisor class which measures the variation of the fibers of f. For example if f is an elliptic fbration, then $M_Y = j^*\mathcal{O}_{\mathbb{P}^1}(\frac{1}{12})$ where j denotes the j function, and if X_P is a smooth fiber of multiplicity m over a codimension 1 point $P\in Y$ and $B=0$, then the coefficient of B_Y along P is $1-\frac{1}{m}$.

The canonical bundle formula, roughly speaking, states that if the morphism f is sufficiently well prepared (eg. $B=0$ and all fibers have simple normal crossings), then (Y, B_Y) is log canonical and the the moduli part M_Y is a nef $\mathbb{Q}$-divisor. In particular (Y, B_Y) is a generalized log canonical pair (and in fact this is the key motivation for introducing generalized pairs [BZ16]).

Results along this line are established in [Kawamata98], see [Kollar07] for a detailed discussion. The positivity of the nef part is deduced from general positivity properties of pushforwards of the canonical bundle (see [Kollar86]). The intuition here is that after performing several reductions, we can in fact assume that the moduli part coincides with $f_*\omega_{X/Y}$.

While the canonical bundle formula has a large number of extremely important applications (eg. to sub-adjunction [Kawamata98]), it is clear that in order to run proofs by induction on the dimension, it is important to establish versions of the canonical bundle formula that work for generalized pairs of the form $(X, B+M)$. This is achieved in [Fil20]. Note that on projective varieties, nef classes are limits of ample divisors and hence one hopes the result for generalized pairs [Fil20] follow as a limit of the result for the usual pairs [Kawamata98].

In the Kähler context it is more practical to work with generalized pairs $(X, B+\beta)$ where $\beta \in H^{1,1}_{\mathrm{BC}}(X)$ is a nef class (see Definition 1.1) and $f: X \to Y$ is a holomorphic map of Kähler manifolds such that $K_X + B + \beta = f^*\gamma$ for some $\gamma \in H^{1,1}_{\mathrm{BC}}(Y)$. One can then define the boundary B_Y and moduli parts $\beta_Y \in H^{1,1}_{\mathrm{BC}}(Y)$ just as in the projective case. Unluckily, it does not follow that β is a limit of $\mathbb{Q}$-divisors and hence the arguments from the projective case do not apply in the Kähler case.

Our strategy is to first prove an analog of the positivity of $f_*\omega_{X/Y}$. We show that if $K_X + B + \beta = f^*\gamma$ and $(X, B+\beta)$ is generalized log canonical, then $K_{X/Y} + B + \beta$ is pseudo-effective (the precise statement is contained in Theorem 2.2, which is an easy consequence of Theorem 6.2 that generalizes [Gue20]). Once the morphism $f : X \to Y$ is sufficiently prepared (and B is re-chosen appropriately), we have $K_{X/Y} + B + \beta = f^*\beta_Y$ and hence β_Y is also pseudo-effective. By [DHP22], it is known that to show that β_Y is nef, it suffices to show that $\beta_Y|_W$ is nef where W is (the normalization of) any subvariety of Y. To verify this we consider $f_W : X_W \to W$ (where $(\ldots)_W$ denotes restriction over W) and then apply Theorem 2.2 to the induced pair $(X_W, B_W + \beta_W)$. For the details of the proof see Theorem 2.3. Note that for technical reasons we have to assume that f is a projective morphism, but we expect the result will hold without this assumption.

Therefore, the technical heart of this paper is Theorem 6.2 (which is the key ingredient in the proof of Theorem 2.2). To gain some intuition, note that in the setting of Theorem 6.2 we consider $K_{X/Y} + B + \beta \equiv f^*\gamma + L$ where L is a $\mathbb{Q}$-line bundle such that $\kappa(L|_{X_y}) \geq 0$ for general $y \in Y$. Restricting over open subsets $U \subset Y$ we may trivialize γ and treat $\beta|_{X_U}$ as a $\mathbb{Q}$-line bundle $F_U = \mathcal{O}_{X_U}(L - K_{X/Y} - B)$. Following [PT18], we construct a positive current $\Theta_U \geq 0$ fiber-wise from the m-th root of the sections of $mL|_{X_y}$ (for $m > 0$ sufficiently big and divisible). These currents then glue together to give a positive current $\Theta \equiv K_{X/Y} + B + \beta$.

Finally, we remark that it is possible to give direct arguments with analytic techniques to prove stronger versions of the canonical bundle formula Theorem 2.3. In fact Theorem 5.2 shows that if we further assume that β contains a smooth positive representative, then we can conclude the stronger fact that β_Y is a closed positive current with zero Lelong numbers.

Acknowledgment. The authors would like to thank J. Cao, O. Das and M. Temkin for useful communications. Thanks go equally to S. Boucksom and C.-M. Pan for interesting discussions about Lelong numbers.

1. Preliminaries

Here we recall some definitions and results from [DHY23]. We will say that S is relatively compact if $S \subset S'$ is an open subset whose closure is compact. Similarly $\pi : X \to S$ is a proper morphism to a relatively compact space S if there is a proper morphism $\pi' : X' \to S'$ where $S \subset S'$ is an open subset whose closure is compact and $X = \pi'^{-1}(S)$.

Definition 1.1. *Let $\pi : X \to S$ be a proper morphism of normal Kähler varieties such that S is relatively compact and X is a normal compact Kähler variety, $\nu : X' \to X$ a resolution of singularities, B' an $\mathbb{R}$-divisor on X' with simple normal crossings support, and $\beta' \in H^{1,1}_{\mathrm{BC}}(X')$, such that*

(1) $B := \nu_* B' \geq 0$,
(2) $[\beta'] \in H^{1,1}_{\mathrm{BC}}(X')$ is nef over S, and
(3) $[K_{X'} + B' + \beta'] = \nu^*\gamma$, where $\gamma \in H^{1,1}_{\mathrm{BC}}(X)$.

Then we let $\beta = \nu_*\beta'$ and we say that $\nu : (X', B' + \beta') \to (X, B + \beta)$ is a generalized pair (over S). We will often abuse notation and say that $(X/S, B+\beta)$ (or $(X, B+\beta)$) is a generalized pair (over S) and $\nu : (X', B' + \beta') \to (X, B + \beta)$ is a resolution. If moreover $\mathrm{Ex}(\nu)$ *is a divisor such that* $\mathrm{Ex}(\nu) + B'$ *has simple normal crossings, then we say that ν is a log resolution and if* $\mathrm{Ex}(\nu)$ *supports a relatively ample divisor, then ν is projective. We will often assume that $X = S$ and omit $\pi : X \to S$.*

Remark 1.2. *Note that we can define the corresponding nef b-(1,1) form $\boldsymbol{\beta} := \overline{\beta'}$ as follows. For any bi-meromorphic morphism $p : X'' \to X'$ we define $\boldsymbol{\beta}_{X''} = p^*\beta'$ and for any bi-meromorphic morphism $q : X'' \to X'''$ we let $\boldsymbol{\beta}_{X'''} = q_*\boldsymbol{\beta}_{X''}$. Using the projection formula, one easily checks that $q_*\boldsymbol{\beta}_{X''}$ is well defined (i.e. $\boldsymbol{\beta}_{X'''}$ does not depend on the choice of the common resolution X'' of X' and X''') and that for any bi-meromorphic morphism $r : X_1 \to X_2$ of birational models of X, we have $r_*\boldsymbol{\beta}_{X_1} = \boldsymbol{\beta}_{X_2}$. We say that $\overline{\beta'}$ descends to X'. Note that for any bi-meromorphic morphism $p : X'' \to X'$, we also have $\overline{\beta'} = \overline{\beta''}$ where $\beta'' = \boldsymbol{\beta}_{X''}$, and so $\boldsymbol{\beta}$ also descends to X''.*

Similarly, if $\nu : Y \to X'$ is a proper morphism, then write $K_Y + B_Y = \nu^*(K_{X'} + B')$. For any proper morphism $\mu : Y \to Y'$ we have $B_{Y'} = \mu_* B_Y$. In this way we have defined a b-divisor $\mathbf{B}$ (whose trace $\mathbf{B}_Y$ on Y is B_Y). Since the b-divisor $\mathbf{K} + \mathbf{B} = \overline{K_{X'} + B_{X'}}$ and the b-$(1,1)$-form $\boldsymbol{\beta} = \overline{\beta_{X'}}$ descend to X', we say that the generalized pair $(X, B + \beta)$ descends to X'.

We will often denote the generalized pair $\nu : (X', B'+\beta') \to (X, B+\beta)$ by $(X, B+\boldsymbol{\beta})$ where $\boldsymbol{\beta} = \overline{\beta'}$. Note that then $\beta' = \boldsymbol{\beta}_{X'}$ and $B' = \nu^*(K_X + B + \beta) - (K_{X'} + \beta')$.

We define the *generalized discrepancies* $a(P; X, B+\boldsymbol{\beta}) = -\mathrm{mult}_P(\mathbf{B}_Y)$, where P is a prime divisor on a bimeromorphic model Y of X. We say that $(X, B+\boldsymbol{\beta})$ is *generalized klt* or generalized Kawamata log terminal (resp. *generalized lc* or generalized log canonical) if for any log resolution $\nu : X' \to X$, we have $\lfloor \mathbf{B}_{X'} \rfloor \leq 0$, i.e. $a(P; X, B + \boldsymbol{\beta}) > -1$ for all prime divisors P over X (resp. $a(P; X, B + \boldsymbol{\beta}) \geq -1$ for all prime divisors P over X). This can be checked on a single given log resolution. We say that $(X, B+\boldsymbol{\beta})$ is *generalized dlt* (divisorially log terminal) if there is an open subset $U \subset X$ such that $(U, (B + \boldsymbol{\beta})|_U)$ is a log resolution (of itself) and $-1 \leq a(P; X, B + \boldsymbol{\beta}) \leq 0$ for any prime divisor P on U and $-1 < a(P; X, B + \boldsymbol{\beta})$ for any prime divisor P over X with center contained in $X \setminus U$.

Lemma 1.3. *Let $(X, B + \boldsymbol{\beta})$ be a generalized klt (resp. generalized dlt) variety. If $K_X + B$ is $\mathbb{Q}$-Cartier, then (X, B) is klt (resp. dlt).*

Proof. Let $f: X' \to X$ be a log resolution, $K_{X'} + B' + \boldsymbol{\beta}_{X'} = f^*(K_X + B + \boldsymbol{\beta}_X)$, where $\lfloor B' \rfloor \leq 0$, as $(X, B + \boldsymbol{\beta})$ is generalized klt. Let $K_{X'} + B^\sharp = f^*(K_X + B)$, then

$$E := K_{X'} + B' - f^*(K_X + B) \equiv f^*\boldsymbol{\beta}_X - \boldsymbol{\beta}_{X'}$$

and so E is exceptional and $-E$ is nef over X. By the negativity lemma $E \geq 0$. But then

$$B' = E + f^*(K_X + B) - K_{X'} = B^\sharp + E$$

and so $\lfloor B^\sharp \rfloor \leq 0$, i.e. (X, B) is klt. The statement about dlt singularities follows similarly. $\square$

In dimension 2, the situation is particularly simple as shown by the following lemma.

Lemma 1.4. *If $(X, B + \boldsymbol{\beta})$ is a generalized klt, dlt, lc surface, then $K_X + B$ is $\mathbb{R}$-Cartier, (X, B) is klt, dlt, lc and $\boldsymbol{\beta}_X$ is nef.*

Proof. See [DHY23]. $\square$

The next result shows that, working locally over X, generalized klt pairs behave similarly to the usual klt pairs, and in particular they have rational singularities.

Theorem 1.5. *Let $(X, B + \boldsymbol{\beta})$ be a generalized klt pair, then X has rational singularities and if we replace X by a relatively compact Stein open subset, then the following hold:*

(1) there exists a small bimeromorphic morphism $\mu: X^\sharp \to X$ such that $X^\sharp$ is $\mathbb{Q}$-factorial,
(2) if $K_{X^\sharp} + B^\sharp + \boldsymbol{\beta}_{X^\sharp} = \mu^*(K_X + B + \boldsymbol{\beta}_X)$, then $\beta_{X^\sharp} \equiv_X \Delta^\sharp$ so that $(X^\sharp, B^\sharp + \Delta^\sharp)$ is klt, and
(3) if $\Delta = \mu_*\Delta^\sharp$, then $(X, B + \Delta)$ is klt.

Proof. This follows from [DHY23] but we include a proof for the convenience of the reader. Note that rational singularities is a local property and hence follows from (3) and [Fuj22, Theorem 3.12].

(1-2) Let $\nu: X' \to X$ be a projective log resolution of $(X, B + \beta)$ and write $K_{X'} + B' + \beta' = \nu^*(K_X + B + \beta)$ so that $\boldsymbol{\beta} = \overline{\beta'}$. Let E be the reduced exceptional divisor and for $0 < \epsilon \ll 1$, let $B^* = (B')^{>0} + \epsilon E$ and $F = (B')^{<0} + \epsilon E$, then $K_{X'} + B^* + \beta' = \nu^*(K_X + B + \beta) + F$ where the support of F equals the set of all exceptional divisors, and $(X', B^* + \beta')$ is generalized klt. In particular $\beta' \equiv_X F - (K_{X'} + B^*)$ where $F - (K_{X'} + B^*)$ is an $\mathbb{R}$-divisor, nef over X. As ν is projective and X is Stein, we may assume that $F - (K_{X'} + B^*)$ is big. But then $\beta' \equiv_X \Delta'$, where $\Delta' > 0$ is an effective $\mathbb{R}$-divisor such that $(X', B^* + \Delta')$ is klt. We may therefore run the relative $K_{X'} + B^* + \Delta'$ mmp ([Fuj22] and [DHP22]) and hence we may assume that we have a birational map $\psi: X' \dashrightarrow X^\sharp$ such that if $F^\sharp = \psi_*F$, $B^\sharp = \psi_*B^*$, $\beta^\sharp = \psi_*\beta'$ and $\Delta^\sharp = \psi_*\Delta'$, then

$$F^\sharp \equiv_X K_{X^\sharp} + B^\sharp + \beta^\sharp \equiv_X K_{X^\sharp} + B^\sharp + \Delta^\sharp$$

is nef over X so that $F^\sharp = 0$ by the negativity lemma. Therefore $\mu: X^\sharp \to X$ is a small bimeromorphic morphism, $B^\sharp = \mu_*^{-1}B$ and $X^\sharp$ is $\mathbb{Q}$-factorial. Clearly $(X^\sharp, B^\sharp + \Delta^\sharp)$

is klt. Note that each step of the above mmp preserves the numerical equivalence $\beta^\sharp \equiv_X \Delta^\sharp$, and in particular $K_{X^\sharp} + B^\sharp + \beta^\sharp = \mu^*(K_X + B + \beta)$.

(3) By the base point free theorem [Nak87, Theorem 4.8], we have $K_{X^\sharp} + B^\sharp + \Delta^\sharp \sim_{\mathbb{Q},X} 0$ and the claim follows. □

The following result is a technical result that is useful in many situations, especially when proving results by induction on the dimension. It shows that up to replacing X by a higher model, the locus of non-klt singularities is contained inside the reduced boundary of a carefully chosen strongly $\mathbb{Q}$-factorial dlt pair. Recall that a $\mathbb{Q}$-line bundle is a reflexive rank 1 sheaf such that there exists a positive integer m such that the reflexive hull $L^{[m]} := (L^{\otimes m_0})^{\vee\vee}$ is a line bundle. A variety X is strongly $\mathbb{Q}$-factorial if every reflexive rank 1 sheaf is a $\mathbb{Q}$-line bundle.

Theorem 1.6 (DLT models)**.** *Let $(X, B + \boldsymbol{\beta})$ be a generalized pair, where X is relatively compact Stein. Then there exists a projective birational morphism $f^{\mathrm{m}} : X^{\mathrm{m}} \to X$ such that X^{m} is strongly $\mathbb{Q}$-factorial, all exceptional divisors P have discrepancy $a(X, B + \boldsymbol{\beta}, P) \le -1$ and $(X^{\mathrm{m}}, (f^{\mathrm{m}})^{-1}_* B + \mathrm{Ex}(f^{\mathrm{m}}))$ is generalized dlt.*

Proof. The proof is similar to the proof in the case of the usual generalized pairs (see [Fil20, Theorem 3.2]), which in turn is based on ideas of Hacon (see [KK10]). We include the argument for the convenience of the reader. Recall that X is strongly $\mathbb{Q}$-factorial if for every reflexive rank 1 sheaf F there exists an integer $m > 0$ such that $(F^{\otimes m})^{**}$ is locally free (see [DH20, Definition 2.2]).

Let $f : X' \to X$ be a log resolution of the generalized pair $(X, B + \boldsymbol{\beta})$ and write $K_{X'} + B' + \beta' = f^*(K_X + B + \beta)$ where $\beta = \boldsymbol{\beta}_X$ and $\beta' = \boldsymbol{\beta}_{X'}$. We may assume that f is defined by a sequence of blow ups over centers of codimension ≥ 2 in X, and hence f is a projective morphism and so we have $C \ge 0$ an f-exceptional divisor such that $-C$ is relatively ample. We write $B' = f^{-1}_*\{B\} + E^+ + F - G$ where E^+, F, G are supported on the divisors of discrepancy $a \le -1,\ -1 < a < 0,\ a > 0$ respectively and we let $E = \mathrm{red}(E^+)$ be the reduced divisor with the same support as E. For any $0 < \epsilon, \mu, \nu < 1$, we have

$$E + (1+\nu)F - \mu C + \beta' = (1 - \epsilon\mu)E + (1+\nu)F + \mu(\epsilon E - C) + \beta'.$$

Note that $-\mu C + \beta' \equiv_X -\mu C - (K_{X'} + B')$ is an ample $\mathbb{R}$-divisor (over X) and hence for $0 < \epsilon \ll 1$, $\mu(\epsilon E - C) + \beta'$ is also numerically equivalent to an ample $\mathbb{R}$-divisor (over X). Since X is Stein, we may write

$$-\mu C + \beta' \equiv_X H_1 \qquad \text{and} \qquad \mu(\epsilon E - C) + \beta' \equiv_X H_2$$

where $B' + H_1 + H_2$ has simple normal crossing support and $\lfloor H_1 \rfloor = \lfloor H_2 \rfloor = 0$. Let

$$\Delta_{\epsilon,\mu,\nu} := f^{-1}_*\{B\} + (1 - \epsilon\mu)E + (1+\nu)F + H_2 \equiv_X f^{-1}_*\{B\} + E + (1+\nu)F + H_1,$$

then $(X', \Delta_{\epsilon,\mu,\nu})$ is klt for $0 < \nu \ll 1$, and by [DHP22] there is a $\mathbb{Q}$-factorial minimal model $\psi : X' \dashrightarrow X^{\mathrm{m}}_{\epsilon,\mu,\nu}$ over X so that $K_{X^{\mathrm{m}}_{\epsilon,\mu,\nu}} + \Delta^{\mathrm{m}}_{\epsilon,\mu,\nu}$ is nef over X. By the equations above, this is also a minimal model for the dlt pair $(X', f^{-1}_*\{B\} + E + (1+\nu)F + H_1)$. Let $B^{\mathrm{m}}_{\epsilon,\mu,\nu} = \psi_*(f^{-1}_*\{B\} + E + F)$, then $\Delta^{\mathrm{m}}_{\epsilon,\mu,\nu} = B^{\mathrm{m}}_{\epsilon,\mu,\nu} + \psi_*(\nu F + H_1)$, and $(X^{\mathrm{m}}_{\epsilon,\mu,\nu}, B^{\mathrm{m}}_{\epsilon,\mu,\nu})$ is dlt. Define

$$N := K_{X^{\mathrm{m}}_{\epsilon,\mu,\nu}} + B^{\mathrm{m}}_{\epsilon,\mu,\nu} + \nu F^{\mathrm{m}}_{\epsilon,\mu,\nu} + H^{\mathrm{m}}_{1,\epsilon,\mu,\nu} \equiv_X K_{X^{\mathrm{m}}_{\epsilon,\mu,\nu}} + \Delta^{\mathrm{m}}_{\epsilon,\mu,\nu}$$

$$T := K_{X^{\mathrm{m}}_{\epsilon,\mu,\nu}} + B^{\mathrm{m}}_{\epsilon,\mu,\nu} + (E^+ - E)^{\mathrm{m}}_{\epsilon,\mu,\nu} - G^{\mathrm{m}}_{\epsilon,\mu,\nu} + \beta^{\mathrm{m}}_{\epsilon,\mu,\nu} \equiv_X 0.$$

We then have

$$T - N \equiv_X \mu C^{\mathrm{m}} + (E^+ - E)^{\mathrm{m}}_{\epsilon,\mu,\nu} - G^{\mathrm{m}}_{\epsilon,\mu,\nu} - \nu F^{\mathrm{m}}_{\epsilon,\mu,\nu} =: D^{\mathrm{m}}_{\epsilon,\mu,\nu},$$

where $-D^{\mathrm{m}}_{\epsilon,\mu,\nu}$ is nef and the pushforward of $D^{\mathrm{m}}_{\epsilon,\mu,\nu}$ to X is effective. By the negativity lemma, $D^{\mathrm{m}}_{\epsilon,\mu,\nu} \geq 0$. The divisors C, $E^+ - E$, F and G are independent of ϵ, μ, ν, thus if $0 < \mu \ll \nu \ll 1$, then $G^{\mathrm{m}}_{\epsilon,\mu,\nu} = \nu F^{\mathrm{m}}_{\epsilon,\mu,\nu} = 0$. Then let $X^{\mathrm{m}} := X^{\mathrm{m}}_{\epsilon,\mu,\nu}$, then X^{m} is strongly $\mathbb{Q}$-factorial as it is the output of a minimal model program (see [DH20, Lemma 2.5]) and $(X^{\mathrm{m}}, (f^{\mathrm{m}})^{-1}_* B + \mathrm{Ex}(f^{\mathrm{m}}))$ is generalized dlt. $\square$

1.1. **Boundary and moduli parts.** Throughout this section we will assume that $\pi : Z \to S$ is a proper morphism of relatively compact normal analytic varieties, $f : X \to Z$ is a proper morphism of normal Kähler varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Z$ and $(X/S, B + \boldsymbol{\beta})$ a generalized pair which is generalized log canonical over an open subset of Z. Recall that by assumption there is a log resolution $\nu : X' \to X$ of $(X, B + \boldsymbol{\beta})$, i.e.

(1) $\boldsymbol{\beta} = \overline{\boldsymbol{\beta}_{X'}}$ (that is $\boldsymbol{\beta}$ descends to X'),
(2) $\boldsymbol{\beta}_{X'}$ is nef over S,
(3) $\mathrm{Ex}(\nu)$ is a divisor and $\nu^{-1}(B) \cup \mathrm{Ex}(\nu)$ has simple normal crossings.

We let $K_{X'} + B_{X'} + \boldsymbol{\beta}_{X'} = \nu^*(K_X + B + \boldsymbol{\beta}_X)$ and we say that $K_{X'} + B_{X'} + \boldsymbol{\beta}_{X'}$ is the log-crepant pull-back of $K_X + B + \boldsymbol{\beta}_X$.

Definition 1.7. *For any prime divisor Q on Z, let*

$$a_Q = a_Q(X, B + \boldsymbol{\beta}) = \sup\{t \in \mathbb{R} | (X, B + tf^*Q + \boldsymbol{\beta}) \text{ is glc over } \eta_Q\}.$$

In this definition "glc" means that there is an analytic open subset $Z^0 \subset Z$ intersecting Q such that $(X, B + a_Q f^*Q + \boldsymbol{\beta})$ is glc but not gklt over Z^0. Note that as Z is normal, Z_{sing} has codimension at least 2 and so Q is Cartier in codimension 1. Since $(X, B+\boldsymbol{\beta})$ is a generalized log canonical pair, f is proper, and Z is relatively compact, it is easy to see that $a_Q = 1$ for all but finitely many divisors $Q \subset Z$. We can then define the boundary divisor $B_Z = B(X/Z, B + \boldsymbol{\beta}) = \sum(1 - a_Q)Q$.

Remark 1.8. *If $\eta : X' \to X$ and $\mu : Z' \to Z$ are proper bimeromorphic maps of normal varieties and $f' : X' \to Z'$ is holomorphic, $K_{X'} + B' + \boldsymbol{\beta}_{X'} = \eta^*(K_X + B + \boldsymbol{\beta}_X)$, then we say that $(X', B' + \boldsymbol{\beta})$ is the induced generalized pair. If $Q' = \mu^{-1}_* Q$, then it is easy to see that $a_Q(X, B + \boldsymbol{\beta}) = a_{Q'}(X', B' + \boldsymbol{\beta})$. In particular, if $B_{Z'}$ is the boundary divisor for $(X', B' + \boldsymbol{\beta})$, then $\mu_* B_{Z'} = B_Z$, i.e. the boundary divisor defined by the above formula is in fact a b-divisor which we denote by $\mathbf{B}^Z$ (so that $B_Z = \mathbf{B}^Z_Z$ and $B_{Z'} = \mathbf{B}^Z_{Z'}$).*

Definition 1.9. *If $K_X + B + \boldsymbol{\beta}_X \equiv f^*\gamma$ for some $\bar{\partial}, \partial$ closed form γ, then we define the moduli part $\beta_Z := \gamma - (K_Z + B_Z)$ of $(X/Z, B_X + \boldsymbol{\beta}_X)$. If $K_Y + B_Y + \boldsymbol{\beta}_Y$ is the log-crepant pull-back of $K_X + B + \boldsymbol{\beta}_X$ and $\beta_{Z'}$ is the moduli part of $(X'/Z', B_{X'} + \boldsymbol{\beta}_{X'})$, then it is easy to see that $\mu_* \beta_{Z'} = \beta_Z$ and so we have a b-(1,1) form $\boldsymbol{\beta}^Z$ such that $\beta_Z = \boldsymbol{\beta}^Z_Z$.*

Definition 1.10. *The pair $(X/Z, B+\boldsymbol{\beta})$ is said to be BP stable (over Z) if $\mathbf{K}+\mathbf{B} = \overline{(K_Z + B_Z)}$ i.e. if $K_Z + B_Z$ is $\mathbb{R}$-Cartier and for any contraction $f' : X' \to Z'$ which is birationally equivalent to f and such that the induced maps $\mu : Z' \to Z$ and $\eta : X' \to X$ are projective birational morphisms, then $K_{Z'} + \mathbf{B}^Z_{Z'} = \mu^*(K_Z + \mathbf{B}^Z_Z)$. Note that if $K_X + B + \boldsymbol{\beta}_X \equiv f^*\gamma$, then the moduli part also descends to Z i.e. $\boldsymbol{\beta}^Z = \overline{\beta_Z}$*

Suppose now that $\eta : X' \to X$ is a proper generically finite morphism. We define the pull-back $\boldsymbol{\beta}' := \eta^*\boldsymbol{\beta}$ as follows. Let $\nu : Y \to X$ be a log resolution of $(X, B+\boldsymbol{\beta})$ and Y' be a resolution of the normalization of the main component of $Y \times_X X'$ so that $\rho : Y' \to Y$ is a generically finite holomorphic map and $\nu' : Y' \to X'$ is a bimeromorphic map. Then let $\boldsymbol{\beta}' = \overline{(\rho^*\boldsymbol{\beta}_Y)}$ and $K_{Y'} + B_{Y'} = \rho^*(K_Y + B_Y)$. Since $\boldsymbol{\beta}'_{Y'} = \rho^*\boldsymbol{\beta}_Y$ is nef over S, $(Y', B_{Y'} + \boldsymbol{\beta}'/S)$ defines a generalized pair. Now let $B' = \nu'_* B_{Y'}$, then

$$K_{X'} + B' + \boldsymbol{\beta}'_{X'} = \nu'_*(\rho^*(\nu^*(K_X + B_X + \boldsymbol{\beta}_X))) = \eta^*(K_X + B_X + \boldsymbol{\beta}_X)$$

by the projection formula. We will say that $(X', B' + \boldsymbol{\beta}')$ is the *log crepant pull-back* of $(X, B+\boldsymbol{\beta})$.

Lemma 1.11. *If $\boldsymbol{\beta}$ descends to X, then $\boldsymbol{\beta}'$ descends to X'.*

Proof. By assumption $\nu^*\boldsymbol{\beta}_X = \boldsymbol{\beta}_Y$. Then

$$\nu'^*(\eta^*\boldsymbol{\beta}_X) = \rho^*\nu^*\boldsymbol{\beta}_X = \rho^*\boldsymbol{\beta}_Y = \boldsymbol{\beta}'_{Y'}.$$

Thus $\boldsymbol{\beta}' = \overline{\eta^*\boldsymbol{\beta}_X}$, i.e. $\boldsymbol{\beta}'$ descends to X'. □

Given a generically finite map $\mu : Z' \to Z$ and $h : X' \to X \times_Z Z'$ a proper bimeromorphic map from a normal variety to the main component, then we obtain a base change diagram

$$\begin{array}{ccc} X' & \xrightarrow{\eta} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow{\mu} & Z \end{array} \tag{1}$$

Lemma 1.12. *Let $B_Z = B(X/Z, B+\boldsymbol{\beta})$ be the boundary divisor for $f : (X, B+\boldsymbol{\beta}) \to Z$ and $B_{Z'} = B(X'/Z', B_{X'} + \boldsymbol{\beta}')$ be the boundary divisor for $f' : (X', B_{X'} + \boldsymbol{\beta}') \to Z'$ where $\boldsymbol{\beta}' = \eta^*\boldsymbol{\beta}$ and $K_{X'} + B_{X'} + \boldsymbol{\beta}_{X'}$ is the log-crepant pull-back of $K_X + B + \boldsymbol{\beta}_X$. If μ is finite, then $K_{Z'} + B_{Z'} = \mu^*(K_Z + B_Z)$.*

Proof. See [Amb99, Theorem 3.2]. □

Definition 1.13. *A morphism $f : X \to Z$ is weakly semistable* [AK00, Definition 0.1] *if*

(1) X and Z admit toroidal structures $U_X \subset X$ and $U_Z \subset Z$, with $U_X = f^{-1}(U_Z)$;
(2) with this structure, the morphism f is toroidal;
(3) the morphism f is equidimensional;
(4) all the fibers of the morphism f are reduced; and
(5) Z is nonsingular.

See [AK00, Section 1] *for a discussion on toroidal morphisms. If (X,B) is a pair, then we say that (X,B) has good horizontal divisors if for every $x\in X$ there exists a local model $X_\sigma = X_{\sigma'}\times \mathbb{A}^l$ where the horizontal divisors (i.e. the divisors dominating Z) are exactly the pull-backs of the coordinate hyperplanes in $\mathbb{A}^l$* [Karu99, Definition 9.1].

We recall the following facts.

Theorem 1.14. *(1) Weak semistability is preserved by base change (see* [Karu99, Lemma 8.3]*).*

(2) Let $f : X \to Z$ be a projective morphism of quasi-projective varieties and $W \subset X$ a closed subset a proper subscheme, then there exists a diagram

$$\begin{array}{ccc} U_{X'}\subset X' & \xrightarrow{\mu_X} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ U_{Z'}\subset Z' & \xrightarrow{\mu_Z} & Z \end{array} \tag{2}$$

such that X', Z' are nonsingular, μ_Z and μ_X are birational, f' is toroidal and $f^{-1}(W)\subset X'\setminus U_{X'}$ is a snc divisor [AK00, Theorem 2.1].

(3) There exists a finite surjective morphism $Z''\to Z'$ such that denoting by X'' the normalization of $X\times_{Z'} Z''$, we have that $U_{X''}\subset X''$ and $U_{Z''}\subset Z''$ are toroidal embeddings, the projection $f'' : X''\to Z''$ is an equidimensional toroidal morphism with reduced fibers, and $(f'')^{-1}(U_{Z''}) = U_{X''}$. If (X,B) is a pair, then we may assume $(X'',B'')\to Z''$ is weakly semistable with good horizontal divisors where B'' is the support of the inverse image of B and the $X''\to X$ exceptional divisors see [AK00, Proposition 5.1] *and* [Karu99, Theorem 9.5].

Lemma 1.15. *Suppose that $(X/Z, \mathrm{Supp}(B))$ is weakly semistable with good horizontal divisors, $\boldsymbol{\beta}$ descends to X and (Z,Σ) is a simple normal crossings pair such that $f(B^v)\subset\Sigma$. Then, the corresponding boundary divisor B_Z descends to Z, i.e. $(X/Z,B)$ is BP-stable.*

Proof. Since $\boldsymbol{\beta}$ descends to X, we can disregard $\boldsymbol{\beta}$, and the claim follows from the usual argument for pairs, see eg. [Fil20, 4.13]. $\square$

Proposition 1.16. *Let $f : (X, B+\boldsymbol{\beta})\to Z$ be a locally projective morphism of relatively compact normal analytic varieties such that $(X,B+\boldsymbol{\beta})$ is a pair which is glc over an open subset of Z. Then $\mathbf{B}^Z$ descends to some model Z' i.e. if $f' : X'\to Z'$ is bimeromorphic to $f : X\to Z$ and $(X', B'+\boldsymbol{\beta})$ is the induced pair, $\boldsymbol{\beta} = \overline{\boldsymbol{\beta}_{X'}}$ then $(X'/Z', B')$ is BP-stable.*

Proof. We may assume that $\boldsymbol{\beta}$ descends to X and hence we may disregard $\boldsymbol{\beta}$ in what follows. The question is local over Z and hence we may assume that f is projective and Z is relatively compact and Stein. By GAGA (see [AT19, Apendix B, C] for the details) we may assume that $f : X\to Z$ is a projective morphism of quasi-projective varieties.

Let $f' : X'\to Z'$ and $f'' : X''\to Z''$ be the morphisms defined by Theorem 1.14. We may assume that $\pi_X^{-1}(\mathrm{Supp}(B)) + \mathrm{Ex}(\pi_X)$ has good horizontal divisors (over Z'') where $\pi_X : X'' \to X$. Let $K_{X''} + B'' = \pi_X^*(K_X + B)$, then $(X''/Z'', B'')$ is BP-stable by Lemma 1.15. We must show that if $\nu_{Z'} : Z'_1 \to Z'$ is any birational map, then $K_{Z'_1} + B_{Z'_1} = \nu_{Z'}^*(K_{Z'} + B_{Z'})$ where $B_{Z'_1}$ is the corresponding boundary divisor. Since $Z'' \to Z'$ is finite, then $K_{Z''} + B_{Z''} = \mu_{Z'}^*(K_{Z'} + B_{Z'})$ by Lemma 1.12 (where $\mu_{Z'} : Z'' \to Z'$ and $B_{Z''}$ is the corresponding boundary divisor). Let Z''_1 be the normalization of $Z'' \times_{Z'} Z'_1$, then $\mu_{Z'_1} : Z''_1 \to Z'_1$ is finite and hence $\mu_{Z'_1}^*(K_{Z'_1} + B_{Z'_1}) = K_{Z''_1} + B_{Z''_1}$. Let $\nu_{Z''} : Z''_1 \to Z''$. Since $(X''/Z'', B'')$ is BP-stable and $\mu_{Z'} \circ \nu_{Z''} = \nu_{Z'} \circ \mu_{Z'_1}$, then

$$\mu_{Z'_1}^*(K_{Z'_1}+B_{Z'_1}) = K_{Z''_1}+B_{Z''_1} = \nu_{Z''}^*(K_{Z''}+B_{Z''}) = \nu_{Z''}^*(\mu_{Z'}^*(K_{Z'}+B_{Z'})) = \mu_{Z'_1}^*(\nu_{Z'}^*(K_{Z'}+B_{Z'})).$$

Pushing forward, it follows that $K_{Z'_1} + B_{Z'_1} = \nu_{Z'}^*(K_{Z'} + B_{Z'})$.

□

Theorem 1.17. *Let (X, B) be a pair and $f : X \to Z$ be a projective morphism of compact complex manifolds with connected fibers, then there exists a birational morphism $Z' \to Z$ and a finite morphism $Z'' \to Z'$ and a morphism $f'' : X'' \to Z''$ birational to $X \times_Z Z''$ such that $(X''/Z'', B'')$ is weakly semistable with good horizontal divisors where B'' is the support of the inverse image of B and the $X'' \to X$ exceptional divisors.*

Proof. Let $\phi : (\mathcal{X}, \mathcal{B}) \to \mathcal{Z}$ the corresponding component of the Hilbert scheme. Replacing $\mathcal{Z}$ by a resolution of the image of Z and replacing Z by a higher model, we may assume that $Z \to \mathcal{Z}$ is surjective, generically finite of smooth manifolds, and $(X, B) = (\mathcal{X}, \mathcal{B}) \times_{\mathcal{Z}} Z$ over an open subset of Z. By Theorem 1.14, there are

(1) birational morphisms $\mu_{\mathcal{Z}} : \mathcal{Z}' \to \mathcal{Z}$ and $\mu_{\mathcal{X}} : \mathcal{X}' \to \mathcal{X}$ such that the induced map $\phi' : \mathcal{X}' \to \mathcal{Z}'$ is a toroidal morphism of smooth varieties,
(2) a finite surjective morphism $\mathcal{Z}'' \to \mathcal{Z}'$ such that denoting by $\mathcal{X}''$ the normalization of $\mathcal{X}'' \times_{\mathcal{Z}'} \mathcal{Z}''$ then $\phi'' : \mathcal{X}'' \to \mathcal{Z}''$ is an equidimensional toroidal morphism with reduced fibers and $(\mathcal{X}'', \mathcal{B}'') \to \mathcal{Z}''$ is weakly semistable with good horizontal divisors where $\mathcal{B}''$ is the support of the inverse image of $\mathcal{B}$ and the $\mathcal{X}'' \to \mathcal{X}$ exceptional divisors.

We let $Z' = Z \times_{\mathcal{Z}} \mathcal{Z}'$ and Z'' be an appropriate resolution of $Z \times_{\mathcal{Z}} \mathcal{Z}''$ (so that the inverse image of $\mathcal{Z}'' \setminus U_{\mathcal{Z}''}$ is a divisor with simple normal crossings) and $X'' = X \times_Z Z''$, $B'' = B \times_Z Z''$, then $f'' : X'' \to Z''$ is weakly semistable and (X'', B'') has good horizontal divisors where B'' is the inverse image of B plus the $X'' \to X$ exceptional divisors. □

Remark 1.18. *The morphism $f'' : X'' \to Z''$ is quasi-smooth in the following sense (see [Taka22] pages 1736-1737). For every $x'' \in X''$, there exists an open subset $x'' \in U'' \subset X''$ such that $(U'', \mathrm{Supp}(D'')|_{U''}) = (\tilde{U}'', D_{\tilde{U}''})/G$ where $(\tilde{U}'', D_{\tilde{U}''})$ is a log smooth toric variety and G is a finite abelian group and $f'' \circ \pi : (\tilde{U}'', D_{\tilde{U}''}) \to Z''$ is toric and flat and $\pi : \tilde{U}'' \to U''$ is the quotient map. In particular we can pick local coordinates $(x_1, \ldots, x_{n+m})$ on U'' and $(t_1, \ldots, t_m)$ on Z'' such that $(f'' \circ \pi)^*(t_i) = \prod x_j^{k_{i,j}})$ where the $k_{i,j}$ are non-negative integers such that $k_{i,j} \neq 0$ for at most one i for each index j.*

2. Adjunction of generalized pairs

In this section we will prove a version of Kawamata's canonical bundle formula and of Kawamata's sub-adjunction for generalized Kähler pairs. We will also prove plt and log canonical inversion of adjunction in this setting.

2.1. **Towards a canonical bundle formula.** Let $f: X \to Z$ and $Z \to S$ be proper morphisms of normal relatively compact Kähler varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Z$ and $(X/S, B+\boldsymbol{\beta})$ is a generalized pair. Suppose that $K_X + B + \boldsymbol{\beta}_X = f^*\gamma$ for some $\bar{\partial}, \partial$ closed current γ on Z. Let $\boldsymbol{\gamma} = \overline{\gamma}$ so that $\boldsymbol{\gamma}_{Z'} = \mu^*\gamma$ for any bimeromorphic map $\mu: Z' \to Z$. We define the boundary b-divisor $\mathbf{B}^Z$ as above and for any base change diagram (1), we define $B_{Z'}$ accordingly. In particular if $\mu: Z' \to Z$ is birational, then $B_{Z'} = \mathbf{B}^Z_{Z'}$. We also define the moduli part $\boldsymbol{\beta}^Z$ via

$$\boldsymbol{\beta}^Z = \boldsymbol{\gamma} - (\mathbf{K} + \mathbf{B}^Z), \qquad i.e. \qquad K_{Z'} + \mathbf{B}^Z_{Z'} + \boldsymbol{\beta}^Z_{Z'} = \boldsymbol{\gamma}_{Z'}.$$

One expects, similarly to the algebraic case, that $\mathbf{B}^Z, \boldsymbol{\beta}^Z$ descend to some model Z', $(Z', \mathbf{B}^Z_{Z'})$ is klt (lc) and that $\boldsymbol{\beta}^Z_{Z'}$ is nef. In other words we conjecture the following.

Conjecture 2.1. *Let $f: (X, B+\boldsymbol{\beta}) \to Z$ be a generalized klt (lc) pair as above, then $(Z, B_Z + \boldsymbol{\beta}^Z)$ is a generalized klt (lc) pair.*

We will prove this conjecture under additional conditions. As a first step, we prove the following result which is a generalization of the main result of [Gue20].

Theorem 2.2. *Let $f: X \to Z$ be a surjective projective map with connected fibers between normal compact Kähler varieties, $(X, B+\boldsymbol{\beta})$ a generalized pair which is log canonical over an open subset of Z, Z is smooth, γ a real (1,1)-class on Z, and L a $\mathbb{Q}$-line bundle such that $h^0(X_z, L^{[m]}|_{X_z}) \neq 0$ for $m \gg 0$ sufficiently divisible, $z \in Z$ general. If $K_X + B + \boldsymbol{\beta}_X = f^*\gamma + L$, then $K_{X/Z} + B + \boldsymbol{\beta}_X$ is pseudo-effective.*

Proof. Let $\nu: X' \to X$ be a resolution and write $K_{X'} + B' + \boldsymbol{\beta}_{X'} = \nu^*(K_X + B + \boldsymbol{\beta}_X) + E$ where $B', E \geq 0$ are effective divisors with no common components. We may assume that B' has simple normal crossings support. Since $E \geq 0$ is ν-exceptional, by the projection formula it suffices to show that $K_{X'/Z} + B' + \boldsymbol{\beta}_{X'}$ is pseudo-effective. Let $B' = B^h + B^v$ where the components of B^h dominate Z and the components of B^v do not dominate Z. Note that (X, B^h) is log canonical and it suffices to show that $K_{X'/Z} + B^h + \boldsymbol{\beta}_{X'}$ is pseudo-effective. For every $m > 0$ we will write

$$mF^m := \lfloor B^h \rfloor + \{mB^h\} \qquad \text{and} \qquad mB^m := m(B^h - F^m) = \lfloor mB^h \rfloor - \lfloor B^h \rfloor.$$

In particular (X', B^m) is klt and F^m has fixed support with coefficients in $[0, 1/m]$. Let H' be a relatively ample divisor and ω be a Kähler form on Z such that $f'^*\omega + H'$ is Kähler. For any $\epsilon > 0$ there is an $m > 0$ such that $\beta^{m,\epsilon} := \boldsymbol{\beta}_{X'} + \epsilon(f'^*\omega + H') + F^m$ is Kähler and $L^\epsilon := \nu^*L + E - B^v + \epsilon H'$ is a $\mathbb{Q}$-line bundle such that $h^0(X'_z, (L^\epsilon)^{[m]}|_{X'_z}) \neq 0$ for $m \gg 0$ sufficiently divisible, $z \in Z$ general. We may write

$$K_{X'} + B^m + \beta^{m,\epsilon} = f'^*(\gamma + \epsilon\omega) + L^\epsilon.$$

By Theorem 6.2, $K_{X'/Z} + B^m + \beta^{m,\epsilon}$ is pseudo-effective. Since being pseudo-effective is a closed condition, and

$$K_{X'/Z} + B^h + \boldsymbol{\beta}_{X'} = \lim \left(K_{X'/Z} + B^m + \beta^{m,\epsilon} \right),$$

it follows that $K_{X'/Z}+B^h+\boldsymbol{\beta}_{X'}$ is pseudo-effective. $\square$

Theorem 2.3. *Let $(X, B+\boldsymbol{\beta})$ be a generalized klt (lc) pair, $f : X \to Z$ a surjective projective morphism of normal compact Kähler varieties with connected fibers, such that $K_X+B+\boldsymbol{\beta}_X = f^*\gamma$ for some $\bar\partial, \partial$ closed current γ on Z. Then $(Z, B_Z+\boldsymbol{\beta}^Z)$ is a generalized klt (lc) pair, i.e.*

(1) $\boldsymbol{\beta}^Z$ is b-nef, that is $\boldsymbol{\beta}^Z$ descend to some model Z' so that $\boldsymbol{\beta}^Z_{Z'}$ is nef, and
(2) $(Z', \mathbf{B}^Z_{Z'})$ is sub-klt (sub-lc).

The proof of this result is somewhat technical but the strategy that we will now illustrate is fairly natural. We already know that the moduli part $\boldsymbol{\beta}^Z$ descends to some model, so assume for simplicity that it descends to Z. Similarly assume that $\beta = \boldsymbol{\beta}_X$ is nef and B has simple normal crossings. We must show that $\beta_Z := \boldsymbol{\beta}^Z_Z$ is nef. By [DHP22], it suffices to show that $\beta_Z|_W$ is pseudo-effective on (the normalization of) any subvariety $W \subset Z$. When $W = Z$, this follows from Theorem 2.2. If $W \neq Z$ the situation is more delicate. Assume further that f is weakly semistable and that B_Z is a reduced divisor supported on $Z \setminus U_Z$ (cf. Theorems 1.14, 1.17). If T is any component of B_Z, then there is a component S of $B^{=1}$ dominating T. By adjunction $(K_X+B+\beta)|_S = K_S+B_S+\beta_S = (f|_S)^*(\gamma|_T)$ and so $\beta_T := \beta(S/T, B_S+\beta_S)$ is nef by induction on the dimension. Since $\beta_Z|_T = \beta_T$, then $\beta_Z|_W$ is nef and hence pseudo-effective for any $W \subset T$. Finally, suppose that $W \not\subset \mathrm{Supp}(B_Z)$. Let $X_W = X \times_Z W$, then we expect that $(X_W, B_W+\beta_W)$ is a generalized pair with mild singularities and $K_{X_W}+B_W+\beta_W = (f|_W)^*(\gamma|_W)$ so that by induction on the dimension $\beta_Z|_W = \beta(X_W/W, B_W+\beta_W)$ is nef. Of course there are many technical issues that we will have to address in the proof.

Proof of Theorem 2.3. By Proposition 1.16, the boundary b-divisor $\mathbf{B}^Z$ for $(X, B+\boldsymbol{\beta}/Z)$ descends to a model Z', and hence so does the moduli part $\boldsymbol{\beta}^Z$. It is well known (and easy to see) that $(Z', \mathbf{B}^Z_{Z'})$ is sub-klt (sub-lc). Thus, it suffices to show that, after possibly replacing Z' by a higher model, $\boldsymbol{\beta}^Z_{Z'}$ is nef.

Let $f'' : X'' \to Z''$ and $f' : X' \to Z'$ be the bi-meromorphic models of $f : X \to Z$ defined in Theorem 1.17 and let $K_{X''}+B''+\boldsymbol{\beta}_{X''}$ and $K_{X'}+B'+\boldsymbol{\beta}_{X'}$ be the log pull backs of $K_X+B+\boldsymbol{\beta}_X$. We may assume that $\boldsymbol{\beta}$ descends to X'. It suffices to show that $\beta_{Z'}$ is nef where $\beta_{Z'} := \boldsymbol{\beta}^Z_{Z'}$ is the moduli part of $(X'/Z', B'+\boldsymbol{\beta}_{X'})$. Let $B_{Z''}$ and $\beta_{Z''} := \boldsymbol{\beta}^Z_{Z''}$ be the boundary and moduli parts of $(X''/Z'', B''+\boldsymbol{\beta}_{X''})$. Since $\mu_{Z'} : Z'' \to Z'$ is finite, then $\beta_{Z''} = \mu^*_{Z'}\beta_{Z'}$ Thus it suffices to show that $\beta_{Z''}$ is nef (see eg. [DHP22, Lemma 2.38]).

Let $B'' = B^+ - B^-$ where B^+, B^- are effective with no common components. We may assume that $B_{Z''}$ is the reduced divisor supported on $Z'' \setminus U_{Z''}$, and we let F be the sum of the components of $(f'')^*(B_{Z''})$ that are not contained in $\lfloor B^+ \rfloor$. Pick $0 < \delta \ll 1$, H'' a relatively ample divisor and let $\psi : X'' \dashrightarrow \bar{X}$ be a relative minimal model for $(X'', B^+ + \delta F + \boldsymbol{\beta}_{X''} + \epsilon H'')$ where $0 < \epsilon \ll \delta$. We note here that

$$K_{X''}+B^+ + \delta F + N_\epsilon \equiv K_{X''}+B^+ + \delta F + \boldsymbol{\beta}_{X''} + \epsilon H'' \equiv B^- + \delta F,$$

where $N_\epsilon := -(K_{X''}+B'') + \epsilon H'' \equiv_{Z''} \boldsymbol{\beta}_{X''} + \epsilon H''$ is a relatively ample $\mathbb{Q}$-divisor and hence the corresponding minimal model exists by [Fuj22].

We claim that $\psi_*(B^- + \delta F) = 0$. To see this, note that since for general $z \in Z$, $K_{X_z} + B_z + \boldsymbol{\beta}_X|_{X_z} \equiv 0$, then $K_{X_{z''}} + B^+_{z''} + \boldsymbol{\beta}_{X''}|_{X''_{z''}} \equiv B^-_{z''}$ where z'' denotes a preimage of z on Z''. As $B^-_{z''}$ is exceptional for $X''_{z''} \to X_z$ and $F_{z''} = 0$, then by standard arguments ψ will contract every component of $B^-_{z''}$. In particular $K_{\bar{X}_{z''}} + \bar{B}_{z''} + \boldsymbol{\beta}_{\bar{X}}|_{\bar{X}_{z''}} \equiv 0$ and so $K_{\bar{X}_{z''}} + \bar{B}_{z''} + \boldsymbol{\beta}_{\bar{X}}|_{\bar{X}_{z''}} + \epsilon \bar{H}|_{\bar{X}_{z''}}$ is nef for all $0 \le \epsilon \ll 1$. Since $\bar{B}^- + \bar{F}$ is vertical, up to adding a vertical divisor which is $\equiv_{Z''} 0$, we may assume that for any divisor P in the support of $\bar{f}_*(\bar{B}^- + \bar{F})$, every divisor Q dominating P has $\mathrm{mult}_Q(\bar{B}^- + \bar{F}) \ge 0$ and $\mathrm{mult}_Q(\bar{B}^- + \bar{F}) = 0$ for one such component. By [Lai11, Lemma 2.9], if $\mathrm{mult}_Q(\bar{B}^- + \bar{F}) > 0$ for one such Q, then there is a component Q' dominating P which is contained in $\mathbf{B}_-(\bar{B}^- + \delta\bar{F}/Z'')$. Since $\bar{B}^- + \delta\bar{F} + \epsilon H''$ is nef over Z'' (for $0 < \epsilon \ll 1$), it follows that no such divisor exists and hence $\bar{B}^- + \bar{F}$ is exceptional (over Z''). Since $X'' \to Z''$ is weakly semistable, $\bar{B}^- + \delta\bar{F} = 0$ and so $K_{\bar{X}} + \bar{B} + \boldsymbol{\beta}_{\bar{X}} \equiv \bar{f}^*\gamma''$. Note that $\bar{B} \ge \bar{f}^* B_{Z''}$.

By [DHP22, Theorem 2.36], it suffices to show that for any subvariety $W \subset Z''$ with normalization $W^\nu \to W$, then $\boldsymbol{\beta}_{Z''}|_{W^\nu}$ is pseudo-effective. Let $T \subset Z''$ be any component of $B_{Z''}$. There exists a component S of $(\bar{B})^{=1}$ such that $\mathrm{mult}_S(\bar{B}) = 1$ and S dominates T. We replace S by a minimal stratum of $(\bar{B})^{=1}$ dominating T. Then S is normal and by adjunction $(K_{\bar{X}} + \bar{B} + \boldsymbol{\beta}_{\bar{X}})|_S = K_S + B_S + \beta_S$ where $(S, B_S + \beta_S)$ is generalized dlt, $(K_{Z''} + B_{Z''})|_T = K_T + B_T$ where (T, B_T) is log canonical, $B_T = (B_{Z''} - T)|_T$. Then

$$K_S + B_S + \beta_S = g^*(K_{Z''} + B_{Z''} + \beta_{Z''})|_T = g^*(K_T + B_T + \beta_T)$$

where $\beta_S = \boldsymbol{\beta}_{\bar{X}}|_S$, $g = \bar{f}|_S$. By standard arguments (see for example the claim in the proof of Theorem 1.1 of [FG14]), we may assume that g has connected fibers. Since $(K_{Z''} + B_{Z''})|_T = K_T + B_T$, it follows that $\beta_{Z''}|_T = \beta_T$ is the mobile part of $(S, B_S + \beta_S/T)$. By induction on the dimension, β_T is nef.

Thus, we may now assume that W is not contained in $Z'' \setminus U_{Z''}$. Replacing $\bar{B}$ by $\bar{B} - \bar{f}^* B_{Z''}$ we will also assume that $B_{Z''} = 0$. It suffices to show that $\boldsymbol{\beta}_{Z''}|_{\tilde{W}}$ is pseudo-effective where $\mu_W : \tilde{W} \to W$ is a resolution or equivalently that $(K_{\tilde{X}''/\tilde{Z}} + B_{\tilde{X}''} + \boldsymbol{\beta}_{\tilde{X}''})|_{\tilde{X}''_{\tilde{W}}}$ is pseudo-effective (see Theorem 6.1.ii). We may assume that $\tilde{W} \subset \tilde{Z}$ where $\mu_{Z''} : \tilde{Z} \to Z''$ is a resolution. We write $\tilde{X} = \bar{X} \times_{Z''} \tilde{Z}$, $\tilde{X}'' = X'' \times_{Z''} \tilde{Z}$ and $\tilde{X}_{\tilde{W}} = \bar{X} \times_{Z''} \tilde{W}$, $\tilde{X}''_{\tilde{W}} = X'' \times_{Z''} \tilde{W}$. We let $\mu^*_{\bar{X}}(K_{\bar{X}/Z''} + \bar{B} + \boldsymbol{\beta}_{\bar{X}}) = K_{\tilde{X}/\tilde{Z}} + \tilde{B} + \boldsymbol{\beta}_{\tilde{X}}$ and $K_{\tilde{X}''/\tilde{Z}} + \tilde{B}'' + \boldsymbol{\beta}_{\tilde{X}''} = \tilde{\psi}^*(K_{\tilde{X}/\tilde{Z}} + \tilde{B} + \boldsymbol{\beta}_{\tilde{X}})$ where $\mu_{\bar{X}} : \tilde{X} \to \bar{X}$ and $\tilde{\psi} : \tilde{X}'' \dashrightarrow \tilde{X}$. By our construction $\boldsymbol{\beta}_{X''}$ is nef over Z'' and $\boldsymbol{\beta} = \overline{\boldsymbol{\beta}_{X''}}$. Note that $\tilde{X}''_{\tilde{W}}$ is not contained in the support of $\tilde{B}''$ and that $\tilde{X}''_{\tilde{W}}$ and $\tilde{X}_{\tilde{W}}$ are normal. This last claim follows since $\tilde{X}''_{\tilde{W}}$ and $\tilde{X}_{\tilde{W}}$ are regular in codimension 1 and all corresponding fibers are S_2. We now let

$$(K_{\tilde{X}''/\tilde{Z}} + B_{\tilde{X}''} + \boldsymbol{\beta}_{\tilde{X}''})|_{\tilde{X}''_{\tilde{W}}} = K_{\tilde{X}''_{\tilde{W}}/\tilde{W}} + B_{\tilde{X}''_{\tilde{W}}} + \boldsymbol{\beta}_{\tilde{X}''_{\tilde{W}}}$$

where $K_{\tilde{X}''_{\tilde{W}}/\tilde{W}} = K_{\tilde{X}''/\tilde{Z}}|_{\tilde{X}''_{\tilde{W}}}$ (see [Kollar22, Theorem 2.68]), $B_{\tilde{X}''_{\tilde{W}}} = B_{\tilde{X}''}|_{\tilde{X}''_{\tilde{W}}}$ and $\boldsymbol{\beta}_{\tilde{X}''_{\tilde{W}}} = \boldsymbol{\beta}_{\tilde{X}''}|_{\tilde{X}''_{\tilde{W}}}$ is nef. Similarly we let $(K_{\tilde{X}/\tilde{Z}} + B_{\tilde{X}} + \boldsymbol{\beta}_{\tilde{X}})|_{\tilde{X}_{\tilde{W}}} = K_{\tilde{X}_{\tilde{W}}/\tilde{W}} + B_{\tilde{X}_{\tilde{W}}} + \boldsymbol{\beta}_{\tilde{X}_{\tilde{W}}}$. Note that $K_{\tilde{X}''_{\tilde{W}}/\tilde{W}} + B_{\tilde{X}''_{\tilde{W}}} + \boldsymbol{\beta}_{\tilde{X}''_{\tilde{W}}} = \psi^*_{\tilde{W}}(K_{\tilde{X}_{\tilde{W}}/\tilde{W}} + B_{\tilde{X}_{\tilde{W}}} + \boldsymbol{\beta}_{\tilde{X}_{\tilde{W}}})$ and so we have a generalized pair

$$\psi_{\tilde W} : (\tilde X''_{\tilde W}, B_{\tilde X''_{\tilde W}} + \boldsymbol\beta_{\tilde X''_{\tilde W}}) \to (\tilde X_{\tilde W}, (\psi_{\tilde W})_*(B_{\tilde X''_{\tilde W}} + \boldsymbol\beta_{\tilde X''_{\tilde W}})).$$

Since $(\psi_{\tilde W})_*(B_{\tilde X''_{\tilde W}}) \geq B_{\tilde X_{\tilde W}}$, then $(\tilde X_{\tilde W}, (\psi_{\tilde W})_*(B_{\tilde X''_{\tilde W}} + \boldsymbol\beta_{\tilde X''_{\tilde W}}))$ is generalized log canonical over an open subset of $\tilde W$. By Theorem 2.2, $K_{\tilde X''_{\tilde W}/\tilde W} + B_{\tilde X''_{\tilde W}} + \boldsymbol\beta_{\tilde X''_{\tilde W}}$ is pseudo-effective. This concludes the proof. □

Remark 2.4. *We believe that if moreover $\boldsymbol\beta$ admits a smooth positive representative, then so does $\boldsymbol\beta^Z$ (cf. Theorem 5.2).*

2.2. **Adjunction.** In what follows, for ease of exposition, we will denote a generalized pair $(X/Z, B + \boldsymbol\beta)$ simply by $(X, B + \boldsymbol\beta)$.

Definition 2.5. *Suppose that $(X, B + \boldsymbol\beta)$ is a generalized pair and S is a component of B of coefficient 1 with normalization $\nu : S^\nu \to S$, then we define a generalized pair*

$$K_{S^\nu} + B_{S^\nu} + \boldsymbol\beta^{S^\nu} = (K_X + B + \boldsymbol\beta)|_{S^\nu}$$

as follows. Let $f : X' \to X$ be a log resolution of $(X, B + \boldsymbol\beta)$ so that X' is smooth, B' has simple normal crossings, $\boldsymbol\beta_{X'}$ is nef, $\boldsymbol\beta$ descends to X', and $K_{X'} + B' + \boldsymbol\beta_{X'} \equiv f^*(K_X + B + \boldsymbol\beta_X)$. Let $S' = f^{-1}_* S$, then S' is smooth and by the usual adjunction for sub-klt pairs, we can write $K_{S'} + B_{S'} = (K_{X'} + B')|_{S'}$ where $B_{S'} = (B' - S')|_{S'}$ has simple normal crossings. We may assume that β' is smooth and we let $\boldsymbol\beta^{S^\nu} = \overline{\beta_{S'}}$ where $\beta_{S'} := \beta'|_{S'}$ is the induced nef (1,1) form. Notice that $[K_{S'} + B_{S'} + \beta_{S'}] = g^*(\gamma|_{S^\nu})$ where $g : S' \to S^\nu$ is the induced morphism and $\gamma = [K_X + B + \beta]$. Let $K_{S^\nu} + B_{S^\nu} + \beta_{S^\nu} := g_*(K_{S'} + B_{S'} + \beta_{S'})$, then $g : (S', B_{S'} + \beta_{S'}) \to (S, B_{S^\nu} + \beta_{S^\nu})$ defines a generalized pair; equivalently $(S', B_{S'} + \boldsymbol\beta^{S'})$ is a generalized pair.

If $K_X + B$ is $\mathbb{R}$-Cartier, then we will write $K_{X'} + B^\sharp = f^*(K_X + B)$ and $\beta_{X'} = f^*\beta_X - E$ where $E \geq 0$ is exceptional. Note however that $E_{S'} := E|_{S'}$ may not be g-exceptional where $g : S' \to S^\nu$. If $K_{S^\nu} + B_{S^\nu} = (K_X + B)|_{S^\nu}$ is the usual adjunction, then

$$(K_X + B + \beta_X)|_{S^\nu} = K_{S^\nu} + B_{S^\nu} + g_* E_{S'} + g_* \beta_{S'}.$$

By Lemma 1.4, if $(X, B + \boldsymbol\beta)$ is generalized lc in codimension 2, then $K_X + B$ is $\mathbb{R}$-Cartier in codimension 2 and hence the above formula can always be used to compute $(K_X + B + \boldsymbol\beta)|_{S^\nu}$.

It is clear that if $(X, B + \boldsymbol\beta)$ is generalized log canonical (resp. generalized plt) then $(S^\nu, B_{S^\nu} + \beta_{S^\nu})$ is generalized log canonical (resp. generalized klt). We now will verify that the reverse implication also holds. The following statement is often referred to at *plt inversion of adjunction*.

Theorem 2.6. *Let $(X, B + \boldsymbol\beta)$ be a generalized pair and S a component of B of coefficient 1 with normalization $\nu : S^\nu \to S$. Then $(X, B + \boldsymbol\beta)$ is generalized plt on a neighborhood of S iff S is normal and $(S, B_S + \boldsymbol\beta^S)$ is generalized klt.*

Proof. Let $f : X' \to X$ be a log resolution of the generalized pair $(X, B + \boldsymbol\beta)$. We may assume that f is a projective morphism. We write $K_{X'} + B' + \boldsymbol\beta_{X'} = f^*(K_X + B + \boldsymbol\beta_X)$ and $S' = f^{-1}_* S$. We have a short exact sequence

$$0 \to \mathcal{O}_{X'}(-\lfloor B' \rfloor) \to \mathcal{O}_{X'}(-\lfloor B' \rfloor + S') \to \mathcal{O}_{S'}(-\lfloor B' \rfloor + S') \to 0.$$

Since, $-\lfloor B'\rfloor \equiv_X K_{X'}+\{B'\}+\boldsymbol{\beta}_{X'}$ where $(X',\{B'\})$ is klt and $\boldsymbol{\beta}_{X'}$ is nef (and big) over X, then $R^1f_*\mathcal{O}_{X'}(-\lfloor B'\rfloor)=0$ and so we have a surjection

$$\phi : f_*\mathcal{O}_{X'}(-\lfloor B'\rfloor+S')\to f_*\mathcal{O}_{S'}(-\lfloor B'\rfloor+S').$$

If $(X,B+\boldsymbol{\beta})$ is generalized plt (on a neighborhood of S), then $-\lfloor B'\rfloor+S'\geq 0$ is effective and exceptional (over a neighborhood of S) and hence $\lfloor B_{S'}\rfloor=(\lfloor B'\rfloor-S')|_{S'}\leq 0$ so that $(S^\nu,B_{S^\nu}+\boldsymbol{\beta}^S)$ is generalized klt. We also have that $f_*\mathcal{O}_{X'}(-\lfloor B'\rfloor+S')=\mathcal{O}_X$ and so ϕ factors through

$$f_*\mathcal{O}_{X'}(-\lfloor B'\rfloor+S')=\mathcal{O}_X\to\mathcal{O}_S\subset\nu_*\mathcal{O}_{S^\nu}=f_*\mathcal{O}_{S'}\subset f_*\mathcal{O}_{S'}(-\lfloor B'\rfloor+S')$$

and therefore $\mathcal{O}_S=\nu_*\mathcal{O}_{S^\nu}$ and S is normal.

If $(S,B_S+\beta_S)$ is generalized klt (and in particular S is normal), then $0\leq-\lfloor B_{S'}\rfloor=(-\lfloor B'\rfloor+S')|_{S'}$ and $f_*\mathcal{O}_{S'}(-\lfloor B'\rfloor+S')=\mathcal{O}_S$, and so we have a surjection $f_*\mathcal{O}_{X'}(-\lfloor B'\rfloor+S')\to\mathcal{O}_S$. Since $f_*\mathcal{O}_{X'}(-\lfloor B'\rfloor+S')\subset\mathcal{O}_X$, it follows that $f_*\mathcal{O}_{X'}(-\lfloor B'\rfloor+S')=\mathcal{O}_X$ on a neighborhood of S and so $-\lfloor B'\rfloor+S'\geq 0$ over a neighborhood of S, i.e. $(X,B+\boldsymbol{\beta})$ is generalized plt on a neighborhood of S.

□

The following result is known as *log canonical inversion of adjunction*. In the usual pair setting, it was first addressed in [Kawakita07] and refined in[Hacon14] and [Fil20]. The following proof is based on the ideas of [Hacon14].

Theorem 2.7. *Let $(X,B+\boldsymbol{\beta})$ be a generalized pair and S a component of B of coefficient 1 with normalization $\nu:S^\nu\to S$. Then $(X,B+\boldsymbol{\beta})$ is generalized lc on a neighborhood of S iff $(S^\nu,B_{S^\nu}+\boldsymbol{\beta}^{S^\nu})$ is generalized lc.*

Proof. Following the arguments above, it is easy to see that if $(X,B+\boldsymbol{\beta})$ is generalized lc on a neighborhood of S, then $(S^\nu,B_{S^\nu}+\boldsymbol{\beta}^{S^\nu})$ is generalized lc. Thus it suffices to show that if $(S^\nu,B_{S^\nu}+\boldsymbol{\beta}^{S^\nu})$ is generalized lc, then $(X,B+\boldsymbol{\beta})$ is generalized lc on a neighborhood of S. The question is local over X and so we may assume that X is a relatively compact Stein variety.

Let $\mu:Y\to X$ be a generalized dlt model given by Theorem 1.6 so that μ is projective, Y is $\mathbb{Q}$-factorial, all exceptional divisors have discrepancy $a\leq-1$, $(Y,B'_Y+\boldsymbol{\beta})$ is generalized dlt, and $B'_Y:=\mu^{-1}_*B+\mathrm{Ex}(\mu)\leq B_Y$ where $K_Y+B_Y+\boldsymbol{\beta}_Y=\mu^*(K_X+B+\boldsymbol{\beta}_X)$. Let $S_Y:=\mu^{-1}_*S$, $B'_Y=S_Y+\Gamma$ and $B_Y=S_Y+\Gamma+\Sigma$. Let H be a sufficiently ample divisor and run the $K_Y+S_Y+\Gamma+\boldsymbol{\beta}_Y$ mmp with scaling of H.

Claim 2.8. *There is a sequence of flips and contractions $\phi_i:Y_i\dashrightarrow Y_{i+1}$ and real numbers $s_0=1$, $s_i\geq s_{i+1}\geq 0$ such that $K_{Y_i}+S_i+\Gamma_i+\boldsymbol{\beta}_{Y_i}+sH_i$ is nef over X for $s_i\geq s\geq s_{i+1}$. If the minimal model program terminates, then we may assume that $s_{n+1}=0$, otherwise the we have $\lim s_i=0$.*

Proof. For any $s>0$, $\boldsymbol{\beta}_Y+sH\equiv_X-(K_Y+B_Y)+sH$ is an ample $\mathbb{R}$-divisor and hence

$$S_Y+\Gamma+\boldsymbol{\beta}_Y+sH\equiv_X\Delta_s$$

where (Y,Δ_s) is klt. By [DHP22] or [Fuj22], for any $0<\epsilon\ll 1$, we can run the $K_Y+\Delta_\epsilon$ mmp over X with scaling of $(1-s)H$. The claim now follows easily. □

We may assume that for $i \gg i_0$, all ϕ_i are flips.

Claim 2.9. *For any $t > 0$, there is an $\mathbb{R}$-divisor $\Theta_t \equiv_X \Gamma + \boldsymbol{\beta}_Y + tH$ such that $(Y, S_Y + \Theta_t)$ is plt.*

Proof. Since H is ample and $\boldsymbol{\beta}_Y$ is nef, then $\boldsymbol{\beta}_Y + tH \equiv_X -(K_Y + B_Y) + tH$ is an ample $\mathbb{R}$-divisor and hence

$$S_Y + \Gamma + \boldsymbol{\beta}_Y + tH \equiv_X S_Y + \Theta_t$$

where $(Y, S_Y + \Theta_t)$ is plt. $\square$

If $t \leq s_i$, then $(Y_i, S_i + \Theta_{t,i})$ is plt and in particular S_i is normal. Suppose that $\Sigma_i \cap S_i \neq \emptyset$, then

$$\mu_i^*(K_X + B + \boldsymbol{\beta}_X)|_{S_i} = (K_{Y_i} + S_i + \Gamma_i + \Sigma_i + \boldsymbol{\beta}_{Y_i})|_{S_i} = K_{S_i} + \mathrm{Diff}_{S_i}(\Gamma_i + \Sigma_i) + \boldsymbol{\beta}_{Y_i}|_{S_i}$$

where $K_{S_i} + \mathrm{Diff}_{S_i}(\Gamma_i + \Sigma_i)$ is not log canonical as $\mathrm{mult}_P(\Gamma_i + \Sigma_i) > 1$, and this implies that $\mathrm{mult}_Q \mathrm{Diff}_{S_i}(\Gamma_i + \Sigma_i) > 1$ where Q is any component of $S_i \cap P$. But then $K_{S_i} + \mathrm{Diff}_{S_i}(\Gamma_i + \Sigma_i) + \boldsymbol{\beta}_{Y_i}|_{S_i}$ is not generalized log canonical and so neither is $(S^\nu, B_{S^\nu} + \boldsymbol{\beta}^{S^\nu})$. Therefore $\Sigma_i \cap S_i = \emptyset$ for all $i \geq 0$.

Fix $m > 0$ such that $m\Sigma$ is an integral divisor and $s_i > \frac{1}{m} \geq s_{i+1}$. Then

$$H_i - m\Sigma_i - S_i \equiv_X K_{Y_i} + \Theta_{\frac{1}{m},i} + (m-1)(K_{Y_i} + S_i + \Gamma_i + \frac{1}{m}H_i + \boldsymbol{\beta}_{Y_i})$$

where $K_{Y_i} + \Theta_{\frac{1}{m},i}$ is klt and $K_{Y_i} + S_i + \Gamma_i + \frac{1}{m}H_i + \boldsymbol{\beta}_{Y_i}$ is nef and big over X so that $R^1(\mu_i)_*\mathcal{O}_{Y_i}(H_i - m\Sigma_i - S_i) = 0$ and hence we have a surjection

$$(\mu_i)_*\mathcal{O}_{Y_i}(H_i - m\Sigma_i) \to (\mu_i)_*\mathcal{O}_{S_i}(H_i - m\Sigma_i) = (\mu_i)_*\mathcal{O}_{S_i}(H_i).$$

Note that as $Y_{i_0} \dashrightarrow Y_i$ is a small bimeromorphic map, then the sheaves

$$(\mu_i)_*\mathcal{O}_{Y_i}(H_i - m\Sigma_i) = (\mu_{i_0})_*\mathcal{O}_{Y_{i_0}}(H_{i_0} - m\Sigma_{i_0}) \subset (\mu_{i_0})_*\mathcal{O}_{Y_{i_0}}(H_{i_0})$$

are contained in $\mathcal{I}_V \cdot (\mu_{i_0})_*\mathcal{O}_{Y_{i_0}}(H_{i_0})$ for $m \gg 0$ where $V = \mu_{i_0}(\Sigma_{i_0})$ and if $V \cap S \neq \emptyset$, then this contradicts the above surjection. Therefore $\mu_{i_0}(\Sigma_{i_0}) \cap S = \emptyset$ and so $(X, B + \boldsymbol{\beta})$ is generalized log canonical on a neighborhood of S.

$\square$

Proof of Theorem 0.1. Immediate from Theorems 2.6 and 2.7. $\square$

Proposition 2.10. *Let $(X, B + \boldsymbol{\beta})$ be a generalized log canonical pair and $(X, B_0 + \boldsymbol{\beta}_0)$ a generalized klt pair. If V a minimal log canonical center of $(X, B + \boldsymbol{\beta})$, then V is normal.*

Proof. The question is local on X and hence we may assume that X is relatively compact and Stein. By the usual tie breaking arguments, we may assume that there is a unique log canonical place for $(X, B + \boldsymbol{\beta})$ and that this place dominates V. Thus there is a log resolution of $(X, B + \boldsymbol{\beta})$, $f : X' \to X$ such that $K_{X'} + B' + \boldsymbol{\beta}_{X'} = f^*(K_X + B + \beta)$ where $B' = S' + \Delta'$ and S' is the unique LC place for $(X, B + \boldsymbol{\beta})$ so that $\lfloor \Delta' \rfloor \leq 0$ is an exceptional divisor. By the proof of Theorem 2.6, we have a surjection

$$\phi : f_*\mathcal{O}_{X'}(-\lfloor \Delta' \rfloor) \to f_*\mathcal{O}_{S'}(-\lfloor \Delta' \rfloor).$$

Since $-\lfloor\Delta'\rfloor$ is effective and exceptional, then $f_*\mathcal{O}_{X'}(-\lfloor\Delta'\rfloor) = \mathcal{O}_X$. Since S' is smooth, $S' \to V$ factors through the normalization $\nu : V^\nu \to V$ and so ϕ factors via $f_*\mathcal{O}_{X'}(-\lfloor\Delta'\rfloor) = \mathcal{O}_X \to \mathcal{O}_V$ and the natural inclusions

$$\mathcal{O}_V \subset \nu_*\mathcal{O}_{V^\nu} \subset f_*\mathcal{O}_{S'} \subset f_*\mathcal{O}_{S'}(-\lfloor\Delta'\rfloor).$$

It follows that $\mathcal{O}_V \cong \nu_*\mathcal{O}_{V^\nu}$ and hence V is normal. $\square$

Theorem 2.11. *Let $(X, B+\boldsymbol{\beta})$ be a generalized log canonical pair then* $\mathrm{nklt}(X, B+\boldsymbol{\beta})$ *is seminormal.*

Proof. The corresponding result for log canonical pairs is contained in [Amb98]. We will follow the approach of [Kollar07]. Let $\nu : X' \to X$ be a log resolution of $(X, B+\boldsymbol{\beta})$ and write $K_{X'} + B' + \boldsymbol{\beta}_{X'} = \nu^*(K_X + B + \boldsymbol{\beta}_X)$. If $B' = S - A + \{B'\}$ where S, A are effective Weil divisors without common components, then S is seminormal as it is a divisor with simple normal crossings and hence $f|_S : S \to W$ factors through the seminormilazation $h : W^{\rm sn} \to W$ via $g : S \to W^{\rm sn}$. We have a short exact sequence

$$0 \to \mathcal{O}_{X'}(A - S) \to \mathcal{O}_{X'}(A) \to \mathcal{O}_S(A|_S) \to 0.$$

Since $A - S \equiv_X K_{X'} + \{B'\} + \boldsymbol{\beta}_{X'}$ and $\boldsymbol{\beta}_{X'}$ is nef and big over X, it follows that $R^1\nu_*\mathcal{O}_{X'}(A - S) = 0$ and hence $\nu_*\mathcal{O}_{X'}(A) \to \nu_*\mathcal{O}_S(A|_S)$. Since A is ν-exceptional, $\nu_*\mathcal{O}_{X'}(A) = \mathcal{O}_X$ and hence $\nu_*\mathcal{O}_S(A|_S) = \mathcal{O}_W$. But $\nu_*\mathcal{O}_S(A|_S) \supset h_*g_*\mathcal{O}_S = h_*\mathcal{O}_{W^{\rm sn}}$ and so $h_*\mathcal{O}_{W^{\rm sn}} = \mathcal{O}_W$ i.e. $h : W^{\rm sn} \to W$ is an isomorphism. $\square$

We will now prove that Theorem 0.2 follows from Theorem 0.3.

Proof of Theorem 0.2. We may assume that W has codimension ≥ 2. By Proposition 2.10, W is normal and by a standard tie breaking argument, we may assume that there is a unique log canonical place for an auxiliary pair $(X, B^\sharp + \boldsymbol{\beta}^\sharp)$. Let $f : X' \to X$ be the corresponding dlt model (Theorem 1.6), then f has a unique exceptional divisor S' and $f^*(K_X + B + \boldsymbol{\beta}_X) = K_{X'} + S' + B' + \boldsymbol{\beta}_{X'}$ where $(X', S' + B' + \boldsymbol{\beta}')$ is plt. Note that by the proof of Proposition 2.10, $S' \to W$ has connected fibers. By Theorem 2.6, $(S', B_{S'} + \boldsymbol{\beta}_{S'})$ is generalized klt where $K_{S'} + B_{S'} + \boldsymbol{\beta}_{S'} = (K_{X'} + S' + B' + \boldsymbol{\beta}_{X'})|_{S'}$. Let $\nu : S' \to W$, then $K_{S'} + B_{S'} + \boldsymbol{\beta}_{S'} = \nu^*((K_X + B + \boldsymbol{\beta}_X)|_W)$ and so by Theorem 0.3, $(K_X + B + \boldsymbol{\beta}_X)|_W \equiv K_W + B_W + \boldsymbol{\beta}_W$ where $(W, B_W + \boldsymbol{\beta}_W)$ is generalized klt. $\square$

Remark 2.12. *The above arguments show that if $(X, B + \boldsymbol{\beta})$ and $(X, B' + \boldsymbol{\beta}')$ are generalized pairs, $V \subset X$ is a subvariety and $U \subset X$ an open subset such that $(U, (B+\boldsymbol{\beta})|_U)$ is generalized lc, $(U, (B'+\boldsymbol{\beta}')|_U)$ is generalized klt and $V \cap U$ is a minimal log canonical center of $(U, (B + \boldsymbol{\beta})|_U)$, then $(K_X + B + \boldsymbol{\beta})|_{V^\nu} = K_{V^\nu} + B_{V^\nu} + \boldsymbol{\beta}^{V^\nu}$ where $V^\nu \to V$ is the normalization and $(V^\nu, B_{V^\nu} + \boldsymbol{\beta}^{V^\nu})$ is a generalized pair.*

3. Cone Theorem for generalized klt pairs

The results in this section are inspired by [CH20]. They suggest that one of the main obstructions to the higher dimensional minimal model program for Kähler varieties is Conjecture 0.4.

Proposition 3.1. *Assume Conjecture 0.4 in dimension $\leq n-1$. Let X be a compact $\mathbb{Q}$-factorial Kähler n-fold such that $(X, B + \boldsymbol{\beta})$ is generalized klt, $K_X + B + \boldsymbol{\beta}_X$ is* *pseudo-effective, and ω be a Kähler form such that $\alpha := [K_X + B + \boldsymbol{\beta}_X + \omega]$ is nef and big but not Kähler. Then there is an α-trivial rational curve C such that $0 < -(K_X + B + \boldsymbol{\beta}_X) \cdot C = \omega \cdot C \leq 2 \dim X$.*

Proof. By Proposition 4.21 (see also [CT15, Theorem 1.1] and [Bou04, Theorem 3.17] in the smooth case) the restricted non-Kähler locus $E_{nK}^{as}(\alpha)$ coincides with the null-locus $\mathrm{Null}(\alpha)$, and there exists a Kähler current η with weak analytic singularities in the class α such that the Lelong set coincides with $E_{nK}^{as}(\alpha)$. Since α is not Kähler, then $\mathrm{Null}(\alpha)$ has a positive dimensional component. Let Z be a maximal dimensional irreducible component of $\mathrm{Null}(\alpha)$ and c be the log canonical threshold of $(X, B+\boldsymbol{\beta}_X)$ with respect to η on a neighborhood of general points of Z. This means that if we pick a log resolution $\nu : X' \to X$ such that $K_{X'}+B_{X'}+\boldsymbol{\beta}_{X'} = \nu^*(K_X+B+\boldsymbol{\beta}_X)$ where $\boldsymbol{\beta}_{X'}$ is nef and $\nu^*\eta = \eta' + F$ where F is an effective $\mathbb{R}$-divisor, $\eta' \geq 0$ and $F + B_{X'}$ has simple normal crossings, then Z is an irreducible component of $\nu((B_{X'} + cF)^{=1})$ and Z is not contained in $\nu((B_{X'} + cF)^{>1})$. If $\boldsymbol{\eta} = \overline{\eta'}$, then Z is a generalized log canonical center of the generalized pair $(X, B + c\nu_*F + \boldsymbol{\beta} + c\boldsymbol{\eta})$. By Remark 2.12,

$$(K_X + B + c\nu_*F + \boldsymbol{\beta} + c\boldsymbol{\eta})|_{Z^\nu} = K_{Z^\nu} + B_{Z^\nu} + \gamma_{Z^\nu}$$

where $(Z^\nu, B_{Z^\nu} + \gamma_{Z^\nu})$ is a generalized pair.

By assumption we have $k := \nu_{\mathrm{num}}(\alpha|_{Z^\nu}) < \dim Z$ so that $(\alpha|_{Z^\nu})^k \not\equiv 0$ and $(\alpha|_{Z^\nu})^{k+1} \equiv 0$. But then

$$(K_{Z^\nu} + B_{Z^\nu} + \gamma_{Z^\nu}) \cdot \alpha_{Z^\nu}^k \cdot \omega_{Z^\nu}^{\dim Z-k-1} = -\alpha_{Z^\nu}^k \cdot \omega_{Z^\nu}^{\dim Z-k} < 0$$

where $\alpha_{Z^\nu} = \alpha|_{Z^\nu}$ and $\omega_{Z^\nu} = \omega|_{Z^\nu}$. Since $B_{Z^\nu} \geq 0$ and γ_{Z^ν} is pseudo-effective, then K_{Z^ν} is not pseudo-effective and hence neither is $K_{Z'}$ for any resolution $Z' \to Z^\nu$.

Consider now the MRC fibration $Z' \to Y$, which is non-trivial as $K_{Z'}$ is not pseudo-effective. Passing to a higher model we may assume that it is a morphism with general fiber F. Note that F is rationally connected and hence $h^2(\mathcal{O}_F) = 0$ and so F is algebraic. Arguing as above, for any $\epsilon > 0$ we have

$$(K_{Z'} + B_{Z'} + \gamma_{Z'} + (1-\epsilon)\omega_{Z'} + t\alpha_{Z'}) \cdot \alpha_{Z'}^k \cdot \omega_{Z'}^{\dim Z-k-1} =$$

$$(K_{Z^\nu} + B_{Z^\nu} + \gamma_{Z^\nu} + (1-\epsilon)\omega_{Z^\nu}) \cdot \alpha_{Z^\nu}^k \cdot \omega_{Z^\nu}^{\dim Z-k-1} = -\epsilon\alpha_{Z^\nu}^k \cdot \omega_{Z^\nu}^{\dim Z-k} < 0,$$

where $\omega_{Z'} = \omega|_{Z'}$ and $\alpha_{Z'} = \alpha|_{Z'}$. Since $\gamma_{Z'}$ is pseudo-effective and $B_{Z'}^{\leq 0}$ is $Z' \to Z^\nu$ exceptional, it follows that $(K_{Z'} + (1-\epsilon)\omega_{Z'} + t\alpha_{Z'}) \cdot \alpha_{Z'}^k \cdot \omega_{Z'}^{\dim Z-k-1} < 0$ and hence $K_{Z'} + (1-\epsilon)\omega_{Z'} + t\alpha_{Z'}$ is not pseudo-effective for any $t > 0$. Since Y is not uniruled, K_Y is pseudo-effective and hence by [CH20, Theorem 5.2], $K_F + (1-\epsilon)\omega_F + t\alpha_F$ is also not pseudo-effective for any $t > 0$ and in particular α_F is not big. By the cone theorem, there are finitely many $K_F + (1-\epsilon)\omega_F$ negative extremal rays, and there is a non-empty finite collection of $K_F + (1-\epsilon)\omega_F$ negative extremal rays that are α trivial. Let $\eta : F \to \bar{F}$ be the induced non-trivial morphism contracting this face. Then $\alpha_F = \eta^*\alpha_{\bar{F}}$ where $\alpha_{\bar{F}}$ is ample on $\bar{F}$. If η is birational, then α_F is big which is a contradiction. Thus, η is of fiber type and hence F is covered by α_F-trivial rational curves C. Note that by bend and break, we may assume that $0 < -K_F \cdot C \leq 2 \dim F$ and hence $(1-\epsilon)\omega_F \cdot C < 2 \dim F$. But then Z' is covered by α_F-trivial rational curves and finally Z is covered by α-trivial rational curves C such that $0 < \omega \cdot C < \frac{2}{1-\epsilon} \dim X$. Since these curves belong to finitely many numerical classes, we may assume that $0 < \omega \cdot C \le 2 \dim X$. Finally, we observe that

$$0 < -(K_X + B + \boldsymbol{\beta}_X) \cdot C = \omega \cdot C \le 2 \dim X.$$

□

Corollary 3.2. *Let $(X, B + \boldsymbol{\beta})$ be a compact Kähler 4-fold generalized klt pair such that $K_X + B$ is pseudo-effective. Then there are at most countably many rational curves $\{\Gamma_i\}_{i \in I}$ such that $-(K_X + B + \boldsymbol{\beta}_X) \cdot \Gamma_i \le 8$ for all $i \in I$ and*

$$\overline{\mathrm{NA}}(X) = \overline{\mathrm{NA}}(X)_{(K_X + B + \boldsymbol{\beta}_X) \ge 0} + \sum_{i \in I} \mathbb{R}^+[\Gamma_i].$$

Proof. This follows by standard arguments from Proposition 3.1 (see eg the proof of [DH23, Theorem 1.3]). □

4. Null loci

In this section we generalize the main result of [CT15] to the singular setting. To start with, we recall the notion of Lelong number of a closed positive current $T \ge 0$ on a normal complex space, since it plays a crucial role in the formulation of Proposition 4.21 below.

4.1. **Closed positive currents on normal complex spaces.** Let $\Omega \subset \mathbb{C}^n$ be the unit ball, and let ϕ be a psh function on Ω. Then

$$T := \sqrt{-1}\partial\overline{\partial}\phi$$

defines a closed positive current of $(1,1)$-type on Ω. For example, if we take $\phi = \log|f|^2$, with f holomorphic, then up to a multiple, T is equal to the current of integration along the analytic set $f = 0$ (taking the multiplicities into account). Thus, closed positive $(1,1)$ currents can be seen as natural generalizations of effective divisors.

It turns out that the function

$$r \to \sup_{|z|=r} \phi(z)$$

is convex increasing of $\log r$, i.e. if $x := \log r$, then the function $x \to \sup_{|z|=e^x} \phi(z)$ is convex increasing. It therefore follows that the limit

$$\nu(T, 0) := \liminf_{z \to 0} \frac{\phi(z)}{\log|z|}$$

exists, and it is called the Lelong number of T at $z = 0$. In the example above, this is precisely the multiplicity of the divisor $(f = 0)$ at 0. Moreover, one can see that the equality

$$\nu(T, 0) = \sup\{\nu \ge 0 | \phi(z) \le \nu \log|z| + \mathcal{O}(1)\} \tag{3}$$

holds true.

We collect next a few facts about currents which will be needed later on.

Theorem 4.1. [DeBook] *Let X be a complex manifold, and let $T \geq 0$ be a closed positive current of $(1,1)$-type on X. We consider an analytic hypersurface $A \subset X$, and let χ_A be the characteristic function of A. Then the following assertions hold true.*

(1) *The function $a \to \nu(T,a)$ defined on the analytic set A is constant in the complement of an at most countable union of analytic subsets of A, and it defines the generic Lelong number of T along A.*
(2) *The currents $\chi_A T$ and $\chi_{X \setminus A} T$ obtained by multiplying T with the characteristic function of A and its complement, respectively are closed (and of course, positive). Moreover, we have*

$$\chi_A T = \nu(T,A)[A],$$

where $\nu(T,A)$ is the generic Lelong number of T along A.

Next, consider a surjective map $f : X \to Y$ between two compact complex manifolds. Given a closed $(1,1)$ current T on the base Y, the pull-back $f^\star T$ is a well-defined, closed current on X (this is not necessarily true for currents of other bi-degrees). The following result clarifies the connection between the Lelong numbers of T and those of its inverse image $f^\star T$.

Theorem 4.2. [Fav99] *Under the assumptions above, there exists a positive constant $C > 0$ such that we have*

$$C\nu(f^\star T, x) \leq \nu(T,y) \leq \nu(f^\star T, x),$$

for all $x \in X$ and $y = f(x)$.

Note that the right-hand side inequality in Theorem 4.2 follows immediately form the definition (3). Also, this sort of comparison inequalities is far from true in case of currents obtained by direct images, i.e. if one wishes to compare the Lelong numbers of a current Θ on X with those of its direct image $f_\star\Theta$. For example, let $f : X \to Y$ be the blow-up of a point y of Y. We assume that Y is Kähler; then given any Kähler metric ω on X, the direct image $f_\star\omega$ has a positive Lelong number at y.
Still in this context (i.e. f is the blow-up at a point and $T \geq 0$ is a closed positive current on Y), we have the equality

$$f^\star T = \nu(T,y)[E] + R \tag{4}$$

where E is the exceptional divisor of the blow-up f, and $R \geq 0$ is a closed positive current whose generic Lelong number along E is equal to zero. In particular, we have

$$\chi_E f^\star T = \nu(T,y)[E]. \tag{5}$$

Let $f : X \to Y$ be a surjective map between compact complex manifolds, and let ρ be a real, $(1,1)$ cohomology class on the target manifold Y. We have the following well-known remark.

Lemma 4.3. *Given any* $(1,1)$ *closed positive current*

$$T \in f^\star \rho,$$

there exists a $(1,1)$ *closed positive current* R *on* Y *such that*

$$T = f^\star R. \tag{6}$$

Proof. The matter is indeed clear: according to our hypothesis there exists an L^1 function ϕ on X and a smooth representative $a \in \rho$ such that the following equality

$$T = f^\star(a) + \sqrt{-1}\partial\overline{\partial}\phi$$

holds. Now the restriction $T|_{X_y}$ of the current T to the general fibers X_y of f is a well-defined, closed positive current. On the other hand, we have

$$T|_{X_y} = \sqrt{-1}\partial\overline{\partial}\phi|_{X_y}$$

which shows that $\phi|_{X_y}$ must be constant on X_y. Therefore our assertion follows. □

The notion of Lelong number of a closed positive current on a normal space will be needed in order to formulate the main result of this section. We recall it next.

Definition 4.4. [Dem82]. *Let* X *be a normal complex space, and let* $T \geq 0$ *be a closed positive* $(1,1)$*-current on* X*. We consider a positive function* $\varphi \in \mathcal{C}^2(X, \mathbb{R}_+)$*, such that* $\log\varphi$ *is psh and such that* $\mathrm{Supp}(T) \cap (\varphi < R)$ *is relatively compact in* X *for all* $0 < R \ll 1$ *sufficiently small. The limit*

$$\nu(T,\varphi) := \lim_{r\to 0} \frac{1}{(2\pi r)^{2n-2}} \int_{\varphi<r} T \wedge (\sqrt{-1}\partial\overline{\partial}\varphi)^{n-1}$$

is called the Lelong number of T *with respect to* φ*.*

Let $y \in X$ be an arbitrary point. If we consider an embedding

$$(X, y) \hookrightarrow (\mathbb{C}^N, 0) \tag{7}$$

then the coordinate functions $(z_i)_{i=1,\dots,N}$ on $\mathbb{C}^N$ restricted to X induce a generating system $(g_i)_{i=1,\dots,N}$ of the maximal ideal of the ring $\mathcal{O}_{X,y}$. The function

$$\varphi_y := \sum_i |g_i| \tag{8}$$

is defined on some small open subset U containing x, and then the Lelong number of T at y is defined as follows

$$\nu(T, y) := \nu(T|_U, \varphi_y), \tag{9}$$

where $T|_U$ is the restriction of T to U, so the RHS is defined as in Definition 4.4.

Remark 4.5. *It is not immediate that the limit in Definition 4.4 exists, but this is a consequence of the Jensen formula established in* [Dem82]*, Théorème 3. Moreover, note that the Lelong number* $\nu(T, y)$ *is independent of the embedding* (7)*, as consequence of Théorème 4 in* loc. cit.

Let n be the dimension of X. By composing the embedding map (7) with a generic linear projection on $\mathbb{C}^n$, we obtain a proper, finite map

$$p : (X, y) \to (\mathbb{C}^n, 0) \tag{10}$$

such that $p^{-1}(0) = y$. The function

$$\widetilde{\varphi}_y := \sum |p_j| \tag{11}$$

(where p_j are the components of p) verifies the inequalities

$$\varphi_y^M \leq \widetilde{\varphi}_y \leq C\varphi_y$$

locally near x, for some positive constants C and M. We can assume that it holds on the open subset U. By the comparison theorem for Lelong numbers (cf. Théorème 4, page 46 in [Dem82]), we have

$$\frac{1}{M}\nu(T, \widetilde{\varphi}_y) \leq \nu(T, y) \leq \nu(T, \widetilde{\varphi}_y). \tag{12}$$

Remark 4.6. *If $y \in X$ is a regular point, the equality*

$$\nu(T, y) = \liminf_{z \to y} \frac{\varphi_T}{\log |z - y|}$$

holds, and it provides an alternative definition for the Lelong number of T at y, as we have already mentioned. Simple examples ([BEGZ10], Appendix A) show that as soon as $y \in X_{\text{sing}}$, the relation above is no longer verified in general. However, we always have the inequality $\nu(T, y) \geq \liminf_{z \to y} \frac{\varphi_T}{\log |z - y|}$.

In connection with these topics, the following result was obtained very recently in [P24].

Lemma 4.7. [P24] *Let X be a normal complex space, and let $T \geq 0$ be a closed positive current on X. The following equivalence*

$$\nu(T, y) > 0 \iff \liminf_{z \to y} \frac{\varphi_T}{\log |z - y|} > 0 \tag{13}$$

holds true for any point $y \in X$. In other words, the Lelong number of T at y and the slope of its potential (i.e. the RHS of (13)) are simultaneously positive or zero.

Remark 4.8. *Of course, one expects an inequality of type*

$$\nu(T, y) \leq C \liminf_{z \to y} \frac{\varphi_T}{\log |z - y|} \tag{14}$$

to be true for some constant $C > 0$, uniform on compact subsets of X.

To finish this subsection we consider the following the set-up.

- X is a normal, compact Kähler space and $\pi : \widehat{X} \to X$ is generically finite, such that $\widehat{X}$ is also normal (and Kähler).

- $T = \alpha + \sqrt{-1}\partial\overline{\partial}\varphi$ is a closed current on X, where α is smooth and locally $\sqrt{-1}\partial\overline{\partial}$-exact, and such that

$$T \geq \gamma$$

for some smooth locally $\sqrt{-1}\partial\overline{\partial}$-exact $(1,1)$-form γ.
- We assume moreover that $\pi^\star T = [E] + \theta + \sqrt{-1}\partial\overline{\partial}\rho$, where E is effective, ρ is a bounded real function on $\widehat{X}$, and the form θ is locally given by the Hessian of a function bounded from above.

Then we claim that the following inequality

$$\theta + \sqrt{-1}\partial\overline{\partial}\rho \geq \pi^\star\gamma \tag{15}$$

holds true (this will be useful in the next sections). This is seen as follows: we only have to verify (15) locally near a point $x_0 \in \mathrm{Supp}(E)$ in the support of the divisor E. Let U be an open subset of X such that $\pi(x_0) \in U$ and such that γ restricted to U is given by the Hessian of the smooth function f_γ. Assume that there exists an open subset V containing x_0 such that we have

$$\theta|_V = \sqrt{-1}\partial\overline{\partial} f_\theta$$

where f_θ is bounded from above. It then follows that the function

$$f_\theta - f_\gamma \circ \pi + \rho|_{V_0} \tag{16}$$

is psh, where $V_0 := \big(V \cap \pi^{-1}(U)\big) \setminus \mathrm{Supp}\, E$. On the other hand, the function in (16) is bounded from above on V_0. Since $\widehat{X}$ is normal, it extends as psh function locally near this point, by result due to Hartogs in the smooth case, see [Dem85], Théorème 1.7 for the version we need here. This is the analogue of the fact that bounded holomorphic functions on normal spaces extend.

In conclusion, the $(1,1)$-current obtained by taking the $\sqrt{-1}\partial\overline{\partial}$ of the function (16) is positive – but this is simply

$$\theta + \sqrt{-1}\partial\overline{\partial}\rho - \pi^\star\gamma|_V,$$

and our claim is proved.

4.2. **Main results.** Prior to stating our results we set a few notations and conventions.

Definition 4.9. *Given a real* $(1,1)$*-class* α *we denote by* $\mathrm{Null}(\alpha)$ *the* null locus *of* α*, which is given by the union of analytic subsets* $V \subset X$ *such that* $\int_V \alpha^{\dim V} = 0$*.*

We next introduce and establish basic properties of a class of closed positive currents on normal varieties which will play the role of *Kähler currents* in [CT15].

4.2.1. *Currents with admissible singularities.* We introduce the following class of singularities.

Definition 4.10. *Let X be a normal, compact Kähler space, and let $\varphi : X \to [-\infty, \infty[$ be a function on X. We say that φ has admissible singularities if*

$$\varphi = \max(\varphi_1, \dots, \varphi_k)$$

where each φ_j has analytic singularities in the sense of [Dem92] *i.e. it can be locally expressed as*

$$\gamma \log\left(\sum |f_k|^2\right)$$

modulo a bounded function. In the expression above, $\gamma \geq 0$ is a real number and the functions (f_k) are holomorphic.

A more flexible version of this notion reads as follows.

Definition 4.11. *Let X be a normal, compact Kähler space, and let*

$$T = \alpha + \sqrt{-1}\partial\overline{\partial}\varphi$$

be a closed positive current of type $(1, 1)$ on X, where we denote by α a smooth, real $(1, 1)$-form on X, which is locally $\sqrt{-1}\partial\overline{\partial}$–exact. We say that T has weak analytic singularities if there exists:

- *a biholomorphic map $\pi : \widehat{X} \to X$ such that $\widehat{X}$ is normal, and*
- *a closed positive current*

$$\widehat{T} = \pi^\star\alpha + \sqrt{-1}\partial\overline{\partial}\psi \geq 0$$

on $\widehat{X}$ such that ψ has admissible singularities and such that we have $\pi_\star\widehat{T} = T$.

Remark 4.12. *As consequence of the fact that the current $\widehat{T}$ is assumed to belong to the class $\pi^\star\alpha$, we show that* the function ψ above (in the second point of Definition 4.11) is constant on every connected component of positive dimensional fibers of π. *This can be seen as follows: assume that the restriction of ψ to a fiber F of π is not identically $-\infty$. Then by Théorème 1.10 in* [Dem85] *combined with the fact that ψ has admissible singularities we infer that $\psi|_F$ is a psh function defined on a compact analytic space – hence, it must be constant by the maximum principle. So, there exists a function φ_1 on X such that $\varphi_1 \circ \pi = \psi$, and moreover, the difference $\varphi - \varphi_1$ is smooth (because it belongs to the kernel of the operator $\sqrt{-1}\partial\overline{\partial}$).*[1] *It follows that Definition 4.11 is equivalent to the existence of a birational map $\pi : \widehat{X} \to X$ together with a current $\widehat{T} = \pi^\star\alpha + \sqrt{-1}\partial\overline{\partial}\psi \geq 0$ on $\widehat{X}$ such that $\pi^\star T = \widehat{T}$. In other words, the "singular analogue" of Lemma 4.3 holds true.*

It turns out that this class of currents behaves very well under a few natural operations which will be needed in the proof of Proposition 4.21 below. In particular we have the following statement.

[1]Since ψ has admissible singularities, we expect that this should be the case for φ_1 (and φ) as well, but it is not clear how such a statement can be proved.

Lemma 4.13. *Let X and Y be normal compact Kähler spaces, and let $p : Y \to X$ be a holomorphic map. Let β be a smooth, real and closed $(1,1)$–form on X. We have the following assertions.*

(a) *Let T be a current with weak analytic singularities on X. Then the inverse image $p^\star T$ has weak analytic singularities.*

(b) *For $i = 1, 2$ let $T_i := \beta + \sqrt{-1}\partial\overline{\partial}\varphi_i$ be two currents with weak analytic singularities in the class induced by the smooth form β. If we define $\varphi := \max(\varphi_1, \varphi_2)$, then the current $T := \beta + \sqrt{-1}\partial\overline{\partial}\varphi$ has weak analytic singularities.*

(c) *Assume moreover that p is birational. If $\Theta \in p^\star(\beta)$ is a closed positive current with weak analytic singularities in the class $p^\star(\beta)$ on Y, then the direct image $T := p_\star\Theta$ has weak analytic singularities.*

Proof. Concerning the first point (a), consider the map $\pi_X : \widehat{X} \to X$ given in Definition 4.11, so that the inverse image

$$\widehat{T} := \pi_X^\star T$$

is a closed positive current on $\widehat{X}$, whose potential has admissible singularities. We can construct holomorphic maps

$$\pi_Y : \widehat{Y} \to Y, \qquad \widehat{p} : \widehat{Y} \to \widehat{X}$$

such that the equality $p \circ \pi_Y = \pi_X \circ \widehat{p}$ holds, the space $\widehat{Y}$ is normal and π_Y is birational.

Consider the inverse image $\Theta := \widehat{p}^\star \widehat{T}$. By Definition 4.10, it is clear that Θ has admissible singularities. On the other hand, by Remark 4.12, we may assume that

$$\Theta = \widehat{p}^\star\big(\pi_X^\star T\big) = \pi_Y^\star(p^\star T)$$

which shows that the current $p^\star T$ has weak analytic singularities.

Point (b) is a direct consequence of (a), so we will not give any further details.

For assertion (c) we argue as follows. Given that Θ has weak analytic singularities, there exist a map $\pi_Y : \widehat{Y} \to Y$ and a closed positive current

$$\widehat{\Theta} \in \pi_Y^\star(p^\star\beta) = (p \circ \pi_Y)^\star\beta$$

as in Definition 4.11, such that

$$\pi_{Y\star}\widehat{\Theta} = \Theta$$

and moreover we have $\widehat{\Theta} = (p \circ \pi_Y)^\star\beta + \sqrt{-1}\partial\overline{\partial}\psi$ for a function ψ with admissible singularities. The map $p \circ \pi_Y : \widehat{Y} \to X$ is birational, and we clearly have

$$(p \circ \pi_Y)_\star\widehat{\Theta} = p_\star\Theta.$$

It therefore follows that the direct image $p_\star\Theta$ has weak analytic singularities. □

Remark 4.14. *Assume that X is a manifold and that $T \in \alpha$ is a Kähler current with weak analytic singularities. By the regularisation results in [Dem92], the class α contains a current with analytic singularities, meaning that the equality*

$$\varphi|_{U_i} = \gamma_i \log\big(\sum_\alpha |f_{i\alpha}|^2\big) + \psi_i$$

holds, where the $f_{i\alpha}$ are holomorphic and ψ_i is bounded. We expect that this still holds in our context, i.e. in case X is a normal, compact Kähler space.

Remark 4.15. *It follows from Lemma 4.13 that in the definition of a current with weak analytic singularities (4.11) we can assume that the space $\widehat{X}$ is non-singular.*

We prove next another property of currents with weak analytic singularities. In the proof below, we use the generic notation "C" for a constant that can change from one line to another.

Lemma 4.16. *Let T be a current with weak analytic singularities on a normal compact Kähler space X. Then the set $E_+(T) := \bigcup_{c>0} E_c(T)$ is a closed, analytic subset of X.*

Proof. We are using the notations in the previous Remark 4.12. In particular we assume that $\widehat{X}$ is non-singular, and moreover the set $(\varphi = -\infty)$ coincides with the image of $(\psi = -\infty)$ via the map π, since we have

$$\psi = \varphi \circ \pi \tag{17}$$

modulo a bounded quantity. Given that ψ has admissible singularities, it follows that we have the equality

$$(\psi = -\infty) = E_+(\widehat{T}),$$

and therefore we have $E_+(T) \subset \pi\big(E_+(\widehat{T})\big)$, since $E_+(T) \subset (\varphi = -\infty)$ as a consequence of upper-semicontinuity of φ.
Actually, more is true, namely the equality

$$E_+(T) = \pi\big(E_+(\widehat{T})\big)$$

holds. To see this, it would be enough to show the existence of a constant $C > 0$ such that we have

$$C\nu(T, y) \geq \nu(\pi^\star T, x) = \nu(\widehat{T}, x) \tag{18}$$

where $y \in X$ is an arbitrary point and $x \in \pi^{-1}(y)$. If we admit this for the moment, the proof of our lemma is complete.
In order to establish inequality (18), we proceed as follows. Consider a local parametrization $\tau : (X, y) \to (\mathbb{C}^n, 0)$. It is a proper, finite map such that $\tau^{-1}(0) = y$ (as sets). In this context we have the following important estimate, cf. [Dem82], Théorème 6

$$C\nu(T, y) \geq \nu(\Theta, 0), \tag{19}$$

which we have already mentioned in (12), where the constant C here is very explicit (depending on a certain multiplicity associated to the map τ) and $\Theta := \tau_\star T$ is the direct image of T with respect to the proper map τ.
By the main result in [Fav99], we have

$$C\nu(\Theta, 0) \geq \nu\big((\tau \circ \pi)^\star \Theta, x\big), \tag{20}$$

which combined with (19) gives

$$C\nu(T, y) \geq \nu\big((\tau \circ \pi)^\star \Theta, x\big). \tag{21}$$

It would therefore be sufficient to show that the inequality

$$\tau^\star(\tau_\star T) \geq T, \tag{22}$$

because then it follows that $\nu\big((\tau\circ\pi)^\star\Theta, x\big) \geq \nu(\pi^\star T, x)$.

Let $T|_U = \sqrt{-1}\partial\overline{\partial}\varphi_T$ be the local expression of the current T. Then we have

$$\tau_\star T = \sqrt{-1}\partial\overline{\partial}\psi$$

where $\psi(z) := \sum_{w\in\tau^{-1}(z)} \varphi_T(w)$ is the trace of the local potential φ_T of T. It follows that the following formula

$$\tau^\star(\tau_\star T) = \sqrt{-1}\partial\overline{\partial}\psi\circ\tau \geq \sqrt{-1}\partial\overline{\partial}\varphi_T, \tag{23}$$

holds. Indeed, the difference

$$\psi\circ\tau(w) - \varphi_T(w) = \sum_{x\in F_w^\star} \varphi_T(x) \tag{24}$$

is a psh function on U, where $F_w^\star := \{x \in U : \tau(x) = \tau(w), x \neq w\}$. The argument for this last claim is as follows: on the unramified locus of τ things are clear, and on the other hand the RHS of (24) is uniformly bounded from above. Our proof is finished. □

Remark 4.17. *The inequality (22) does not holds in general. For example, if instead of being finite and proper the map τ is the blow-up of $\mathbb{C}^2$ at 0, then (22) certainly fails in case T is the current of integration on the exceptional divisor.*

Finally, we introduce the following notion.

Definition 4.18. *Let X be a normal compact Kähler space, and let α be a nef and big real $(1,1)$-class on X (in the Bott-Chern cohomology). The restricted non-Kähler locus of α is the following set*

$$E_{nK}^{as}(\alpha) := \bigcap_{T\in\alpha} E_+(T)$$

where T above is assumed to be a Kähler current with weak analytic singularities, and $E_+(T) \subset X$ is the (analytic) subset of X for which the Lelong numbers of T are strictly positive.

Remark 4.19. *In the case of a non-singular Kähler space X, one defines $E_{nK}(\alpha)$ as the intersection of $E_+(T)$ for all Kähler currents $T \in \alpha$. Thus, the difference between $E_{nK}^{as}(\alpha)$ and $E_{nK}(\alpha)$ is that in the definition of the former we restrict ourselves to currents with weak analytic singularities. If X is non-singular, then we have $E_{nK}^{as}(\alpha) = E_{nK}(\alpha)$, thanks to the regularisation results in [Dem92]. In the general case of a normal space, things are less clear, but we can at least say that*

$$E_{nK}(\alpha) \subset E_{nK}^{as}(\alpha) \subset E_{nK}(\alpha) \cup X_{\text{sing}}$$

holds true. We actually expect the first inclusion to be an equality.

The following statement will be important in what follows.

Corollary 4.20. *Let X be a normal compact complex Kähler space, and let α be a nef and big $(1,1)$-class. Then $E_{nK}^{as}(\alpha)$ is an analytic subset of X, and there is a Kähler current with weak analytic singularities $T \in \alpha$ such that $E_+(T) = E_{nK}^{as}(\alpha)$.*

Proof. By (b) of Lemma 4.13, given two Kähler currents with weak analytic singularities $T_i \in \alpha$, we can construct $T \in \alpha$ such that

$$E_+(T) \subset E_+(T_1) \cap E_+(T_2)$$

and moreover T is again a Kähler current with weak analytic singularities. We can therefore construct a sequence $T_k \in \alpha$ of such currents, for which the following assertions

$$E_+(T_{m+1}) \subset E_+(T_m), \qquad E_{nK}^{as}(\alpha) \subset \bigcap_k E_+(T_k)$$

are true for each $m \geq 1$. Since X is compact and $E_+(T_m)$ are analytic sets, the corollary follows because by notherian induction there exists an integer m_0 such that $E_+(T_{m+1}) = E_+(T_m)$ for all $m \geq m_0$. □

In this context, we have the following statement, which represents the main result of this section.

Theorem 4.21. *Let X be a compact normal Kähler variety, and let α be a smooth (1,1)-form, which is locally $\sqrt{-1}\partial\overline{\partial}$-exact and such that the corresponding class is nef and big. Then $E_{nK}^{as}(\alpha) = \mathrm{Null}(\alpha)$. In particular, the set $\mathrm{Null}(\alpha)$ is analytic.*

Remark 4.22. *As we have already mentioned in Remark 4.6, in singular setting the Lelong number of a positive current can be different from the slope of its potential at a given point. Therefore one might wonder why the former notion appears in Proposition 4.21 and not the latter. The explanation is given by Lemma 4.7: considering slopes instead of Lelong numbers would lead to the same set $E_{nK}^{as}(\alpha)$.*

Proof. The arguments which will follow combine [CT15] (where this statement is established in case X is a manifold), with additional inputs from [DHP22].

Step 1: the inclusion $\mathrm{Null}(\alpha) \subset E_{nK}^{as}(\alpha)$ *holds.* Indeed, let $V \subset X$ be an irreducible component of $\mathrm{Null}(\alpha)$ so that

$$\int_{V_{\mathrm{reg}}} \alpha^d = 0, \tag{25}$$

where d is the dimension of V. If $V \not\subset E_{nK}^{as}(\alpha)$, then by the definition of this subset, there exists a Kähler current $\Theta \in \alpha$, whose potentials have weak analytic singularities and whose Lelong number at a generic point of V is equal to zero. These two properties show that the restriction $\Theta|_V$ is a Kähler current, so in particular $\alpha|_V$ is nef and big. This contradicts the equality (25).

Step 2: the set $E_{nK}^{as}(\alpha)$ *does not have isolated points.* Let

$$T := \alpha + \sqrt{-1}\partial\overline{\partial}\phi$$

be a Kähler current in the class α (in the equality above we abusively denote by "α" a smooth representative of this class). Assume that the function ϕ has weak analytic singularities and moreover $x \in X$ is an isolated point in the set $\phi^{-1}(-\infty)$. Then we can remove the pole of ϕ at x as follows (see [Dem92] and the references therein).

Consider $(X, x) \subset U \subset (\mathbb{C}^N, 0)$ a local embedding of X, such that

$$\alpha|_{U\cap X} = \sqrt{-1}\partial\overline{\partial}\tau_x$$

for some smooth function τ_x. The sum $\tau_x + \phi$ is the restriction of a function ψ defined on U and whose Hessian is bigger than a positive multiple of the Euclidean metric on $\mathbb{C}^N$. We now define

$$\widetilde{\psi} := \max(\psi, C_1\|Z\|^2 - C_2)$$

where C_1 is strictly positive and $C_2 \gg 0$ is large enough, so that

$$\psi|_{\partial(U\cap X)} > (C_1\|Z\|^2 - C_2)|_{\partial(U\cap X)}$$

holds. This is indeed possible, since the restriction of ψ to a small enough neighborhood of the boundary of $U \cap X$ in X is smooth. We have denoted by Z the coordinates in $\mathbb{C}^N$ and for each open set Ω we denote by $\partial(\Omega)$ its boundary.

We then define $\widetilde{\phi}$ the function on X given by $\widetilde{\psi}|_{U\cap X} - \tau_x$ on $U \cap X$ and ϕ on $X \setminus U$. This function has weak analytic singularities, and the current

$$\widetilde{T} := \alpha + \sqrt{-1}\partial\overline{\partial}\widetilde{\phi}$$

is greater than a small multiple of the Kähler metric on X. Moreover, $x \notin \widetilde{\phi}^{-1}(-\infty)$. Therefore, the set $E_{nK}^{as}(\alpha)$ cannot contain isolated points, all its irreducible components must have dimension at least one.

Step 3: the inclusion $E_{nK}^{as}(\alpha) \subset \mathrm{Null}(\alpha)$ holds. Let $V \subset E_{nK}^{as}(\alpha)$ be any irreducible component. We have to show that $\displaystyle\int_{V_{\mathrm{reg}}} \alpha^d = 0$, where $d \geq 1$ is the dimension of V. Assume that this equality does not hold. Then given that the restriction $\alpha|_V$ is nef, the only alternative is

$$\int_{V_{\mathrm{reg}}} \alpha^d > 0 \tag{26}$$

and we show next that (26) leads to a contradiction. This will be done if we are able to construct a Kähler current in the class α, whose potential has weak analytic singularities and such that it is bounded locally at some point of V.

A first important reduction is that we can assume that V is non-singular. Indeed, as shown in the proof of [DHP22, Theorem 2.29] there exists a modification $p : \widehat{X} \to X$ such that the following hold.

- The complex Kähler space $\widehat{X}$ is normal.
- The proper transform $\widehat{V}$ of V is a smooth submanifold of $\widehat{X}$.
- The map p is an isomorphism locally over an open subset of V.

If we are able to construct a Kähler current $\Theta \in p^\star\alpha$ with weak analytic singularities such that Θ is bounded locally at a very general point of $\widehat{V}$, then we are done by considering the direct image $p_\star\Theta$, cf. Lemma 4.13, (b). Thus replacing X with $\widehat{X}$, we can assume from this point on that V is non-singular.

Thanks to inequality (26), it follows that we can construct a Kähler current

$$\Theta_V := \alpha|_V + \sqrt{-1}\partial\overline{\partial} f$$

such that $f : V \to [-\infty, 0]$ has *analytic singularities*, cf. Remark 4.14.

On the other hand, the class α is nef and big on X, so there exists a Kähler current

$$\Theta := \alpha + \sqrt{-1}\partial\overline{\partial} F$$

where F has weak analytic singularities along $V \cup Z$, where $Z \subset X$ is an analytic subset of X which does not contain V.

The idea is to remove the singularities of Θ at the generic point of V (as we did in Step 2) by using Θ_V, so that the resulting current will provide the sought-after contradiction. However, in the actual context additional complications arise due the singularities of F.

A particular case. As "warm-up" for the rest of the proof, we provide here an argument in case the function f is smooth. By the proof of [Dem90, Theorem 4] there exists a smooth extension $\widetilde{f}$ defined in an open subset U of V such that

$$\alpha|_U + \sqrt{-1}\partial\overline{\partial}\widetilde{f}$$

is a Kähler current on U. Let now ψ_Z be a quasi-psh function on X, with analytic poles along Z. If $\delta_0 > 0$ is sufficiently small, then

$$\alpha|_U + \sqrt{-1}\partial\overline{\partial}(\widetilde{f} + \delta_0\psi_Z|_U) \tag{27}$$

is a Kähler current on U, and it has poles along the analytic set $Z \cap U$.

Next we will use the hypothesis "α nef" in order to diminish the order of poles of F. This is necessary, as otherwise we will be unable to glue Θ with the form in (27). Indeed, for any positive $\varepsilon > 0$ there exists a smooth function G_ε on X such that

$$T_\varepsilon := \alpha + \sqrt{-1}\partial\overline{\partial} G_\varepsilon \geq -\varepsilon\omega_X$$

where ω_X is a reference Kähler metric on X. Let $\delta_1 > 0$ be a strictly positive real number such that $\Theta \geq \delta_1\omega_X$. The convex combination

$$\Theta_\varepsilon := (1 - \frac{\varepsilon}{\delta_1})T_\varepsilon + \frac{\varepsilon}{\delta_1}\Theta$$

is a Kähler current in the class α. It is easy to check that it is greater or equal than

$$-\varepsilon(1 - \frac{\varepsilon}{\delta_1})\omega_X + \frac{\varepsilon}{\delta_1}\delta_1\omega_X = \frac{\varepsilon^2}{\delta_1}\omega_X.$$

On the other hand, the singularities of Θ_ε are of order $\mathcal{O}(\varepsilon)$, in other words we can make them as small as we want/need. Let $F_\varepsilon := (1 - \varepsilon/\delta_1)G_\varepsilon + \varepsilon/\delta_1 F$ be the potential of Θ_ε. It follows that given a point $z \in Z \setminus V$, there exists an open subset U_z containing z such that the following inequality

$$F_\varepsilon > \delta_0\psi_Z \tag{28}$$

holds for all $0 < \varepsilon \ll 1$ small enough. This can be seen as consequence of Hilbert's Nullstellensatz, as follows.

We recall that $E_+(\Theta)$ is equal to $Z \cup V$. Moreover there exists a birational map $\pi : \widehat{X} \to X$ such that $\pi^\star\Theta$ has admissible singularities, meaning in particular that locally near $w \in \pi^{-1}(z)$ we have

$$F \circ \pi|_{(\widehat{X},w)} = \log(\sum |f_i|^{2r_i}) + b$$

where b is bounded, (f_i) is a set of holomorphic functions and $r_i \geq 0$ are positive real numbers.

In the proof of Lemma 4.16 we showed that $E_+(\Theta) = \pi\big(E_+(\pi^\star\Theta)\big)$, which in particular implies that we have $\pi^{-1}\big(E_+(\Theta)\big) \supset E_+(\pi^\star\Theta)$. In fact, the equality

$$\pi^{-1}\big(E_+(\Theta)\big) = E_+(\pi^\star\Theta) \tag{29}$$

holds: let $x \in \pi^{-1}\big(E_+(\Theta)\big)$, so that $\nu\big(\Theta, \pi(x)\big) > 0$. We claim that $\nu(\pi^\star\Theta, x) > 0$ is strictly positive as well. If this is not the case, the local potential of $\pi^\star\Theta$ is *locally bounded from below* near x. By the analogue of (17) in our setting here, it follows that the potential of Θ is equally locally bounded from below near $\pi(x)$. But this contradicts the fact that the Lelong number of Θ at $\pi(x)$ is positive.

Consider next the ideal $\mathcal{I} = (f_i)$ generated by the functions appearing in the expression of $F \circ \pi$ above. We remark that the set of zeros of $\mathcal{I}$ is contained in $(\pi^{-1}(Z), w)$ (this is only true because we are "far" from V). The function ψ_Z is obtained by gluing functions of type $\log(\sum |h_j|^2)$, where (h_j) are the local equations of Z and so there exists an integer $N > 0$ such that $(h_j \circ \pi)^N \in \mathcal{I}$. In particular we have $\sum |h_j \circ \pi|^{2N} \leq C(\sum |f_i|^2)$ for some positive constant $C > 0$. By arranging the constants, and taking into account the fact that $\pi^{-1}(z)$ is a compact set, this implies (28).

We fix a value, say ε_0, of ε small enough, such that (28) holds true for every point of $\partial U \cap Z$: this is possible, since $\partial U \cap V = \emptyset$. Then we claim that there exists a positive constant $C > 0$ such that the inequality

$$\widetilde{f} + \delta_0\psi_Z < F_0 + C \tag{30}$$

holds true pointwise in a small open subset containing the boundary ∂U of U, where $F_0 := F_{\varepsilon_0}$.

In order to verify this claim, we note that ∂U is a compact set, and let $x \in \partial U$ be one of its elements. If $x \in \partial U \cap Z$, we clearly have $\delta_0\psi_Z < F_0$ locally near x, as consequence of the inequality (28). Moreover, the function $\widetilde{f}$ is non-singular, so the existence of the constant "C" such that (30) holds in a small open subset of x is guaranteed. If $x \notin \partial U \cap Z$, then things are clear. By compactness, we have established our claim.

The next claim is that the function

$$\widehat{F} := \max(\widetilde{f} + \delta_0\psi_Z, F_0 + C)$$

is defined on the whole space X, is bounded at the generic point of V, and moreover

$$\alpha + \sqrt{-1}\partial\overline{\partial}\widehat{F}$$

is a Kähler current. Indeed, near the boundary ∂U of the set U where $\widetilde{f}$ is defined the inequality (30) shows that $\widehat{F} = F_0 + C$, and thus we define $\widehat{F} = F_0 + C$ on the complement of U.

End of the proof. In general, the function f has no reason to be smooth, but nevertheless the line of arguments above remains valid, thanks to a very important remark in [CT15]. The point is that instead of obtaining a smooth extension $\widetilde{f}$ of f defined on an open subset $U \subset X$ as above, in the actual context we only get a function with log poles $\widetilde{f}$ and a "pinched" neighborhood U of $V \setminus W$, where W is an analytic subset of V, such that the analog expression (27) is a Kähler current. In our context, this can be seen as follows.

Consider finitely many open subsets (A_i) of X, such that each A_i is an analytic subset the unit ball of some Euclidean space, and such that $V \subset \cup A_i$. We can assume that the equality

$$f|_{A_i \cap V} = \delta_i \log(\sum_\alpha |g_{i\alpha}|^2) + \tau_i$$

holds, where $\delta_i > 0$, the functions $g_{i\alpha}$ defined on $A_i \cap V$ are holomorphic and τ_i are bounded (given that f has analytic singularities). In particular, the equations $g_{i\alpha} = 0$ define a global analytic set W contained in V.

Let $\mathcal{I}_W \subset \mathcal{O}_V$ be the ideal sheaf of W, and let $\mathcal{J}_W \subset \mathcal{O}_X$ be its pull-back via the projection map $\mathcal{O}_X \to \mathcal{O}_V$. Then there exists a birational map

$$\pi : \widehat{X} \to X$$

obtained by blowing-up smooth centres contained in V, such that the following hold.

(a) *The space $\widehat{X}$ is normal and the proper transform $\widehat{V}$ of V is non-singular. Moreover, the map π is biholomorphic near the general point of $\widehat{V}$.*

(b) *The inverse image of $\mathcal{J}_W$ via the restriction of π to $\widehat{V}$ is equal to $\mathcal{O}_{\widehat{V}}(-D)$, where D is an effective divisor on $\widehat{V}$ whose support is snc.*

For the construction of the map π we use the same argument as in [DHP22]. We start by constructing a principalization of the ideal $\mathcal{I}_W \subset \mathcal{O}_V$: this is achieved by a finite sequence of blow-ups

$$p_k : V_{k+1} \to V_k$$

of smooth centres $\Sigma_k \subset V_k$, for $k = 0, \dots, N-1$ with $V_0 := V$. Next we interpret $\Sigma_0 \subset V$ as analytic subspace of X, and we blow-up X along Σ_0; let

$$\pi_1 : X_1 \to X$$

be the corresponding map. We get a closed immersion $V_1 \to X_1$ (whose image is simply the proper transform of V), and we repeat this operation with Σ_1. Notice that the space X_N is not necessarily normal, but the map $X_N \to X$ is biholomorphic at the generic point of V_N. Since X is normal, it follows that the complement of a proper analytic subset of V_N is contained in the set of normal points of X_N. Therefore, the normalization $\nu : \widehat{X} \to X_N$ is biholomorphic near the proper transform $\widehat{V}$ of V_N (cf. [GR], Corollary, page 164). It follows that the map

$$\pi : \widehat{X} \to X$$

obtained by composing the normalization of X_N with the previous sequence of blow-ups has all the properties we need.

Consider next the pull-back current $(\pi|_{\widehat{V}})^\star\Theta_V$. Its singularities are concentrated along the snc divisor D in $\widehat{V}$, whose support is $\Lambda_1+\cdots+\Lambda_N$. For each $i=1,\ldots,N$ we denote by β_i an arbitrary, smooth representative of the first Chern class of Λ_i. By blowing up $\widehat{X}$ along Λ_i, we can add the following item to the properties of π above:

(c) *There exists a set $(\rho_i)_{i=1,\ldots,N}$ of smooth $(1,1)$-forms on $\widehat{X}$ such that the equality*

$$\rho_i|_{\widehat{V}} \equiv \beta_i$$

holds for each $i=1,\ldots,N$, meaning that the restriction of ρ_i to $\widehat{V}$ belongs to the class β_i.

Indeed, locally analytically we blow up the non-singular set Λ_i in $\mathbb{C}^N$. The exceptional set of $\widetilde{\mathbb{C}^N}\to\mathbb{C}^N$ is a smooth divisor $F_i\to\Lambda_i$. Then $E_i:=\widetilde{X}\cap F_i$ where $\widetilde{X}\to\widehat{X}$ is the strict transform and even if E_i is neither reduced nor irreducible, it is nevertheless a Cartier divisor. Then $\mathcal{O}(E_i)$ is locally free, and so we can endow it with a smooth metric denoted by h_i (this notion is defined precisely as in the usual case of a line bundle on a manifold). The curvature form corresponding to h_i will be our ρ_i, for each index i. Moreover, notice that these additional transformations do not affect $\widehat{V}$, since Λ_i has codimension one in $\widehat{V}$.

Given the properties (a)–(c) above, we obtain the decomposition

$$(\pi|_{\widehat{V}})^\star\Theta_V = \alpha_1 + \sum a_i[\Lambda_i] + \sqrt{-1}\partial\overline{\partial}\widehat{f}$$

where the notations/conventions are as follows:

- The same symbol e.g. α is used to denote a cohomology class and some fixed representative contained in it (in case we do not intend to emphasize a particular representative of the said class).
- Λ_i are the hypersurfaces of $\widehat{V}$ introduced before.
- The smooth $(1,1)$-form α_1 is defined as

$$\alpha_1 = (\pi|_{\widehat{V}})^\star(\alpha|_V) - \sum a_i\rho_i|_{\widehat{V}},$$

  where ρ_i are the forms in (c).
- The function $\widehat{f}$ is bounded.

We note that since Θ_V is a Kähler current, the inequality

$$(\pi|_{\widehat{V}})^\star\Theta_V \geq \delta(\pi|_{\widehat{V}})^\star(\omega_X|_V)$$

holds, for some $\delta>0$. It follows that we have

$$\widehat{\Theta}_V := \alpha_1 + \sqrt{-1}\partial\overline{\partial}\widehat{f} \geq \delta(\pi|_{\widehat{V}})^\star(\omega_X|_V)$$

as well, since $\widehat{\Theta}_V$ is smooth, and the inequality above holds in the complement of an analytic set of $\widehat{V}$.

Then we claim that for each $\varepsilon > 0$ there exists an open subset $U_\varepsilon \subset \widehat{X}$ containing $\hat{V}$, together with a smooth function $\widehat{f}_\varepsilon : U_\varepsilon \to \mathbb{R}$ such that

$$\pi^\star(\alpha) - \sum a_i\rho_i + \sqrt{-1}\partial\overline{\partial}\widehat{f}_\varepsilon \geq \delta\pi^\star(\omega_X) - \varepsilon\omega_{\widehat{X}} \tag{31}$$

pointwise on U_ε, where $\omega_{\widehat{X}}$ is a fixed Kähler metric on $\widehat{X}$. Indeed, this is done in two steps: we first apply the regularisation result in [Dem92] on $\widehat{V}$ in order to convert $\widehat{f}$ to a smooth function. The price to pay is a loss of positivity, which can be assumed to be of size $\frac{\varepsilon}{2}\omega_{\widehat{X}}|_{\widehat{V}}$. Then by the argument in [DeBook] already used in the particular case above, we obtain U_ε and $\widehat{f}_\varepsilon$.
The inequality (31) above is in particular true if we construct the metric $\omega_{\widehat{X}}$ as follows:

$$\omega_{\widehat{X}} := \pi^\star\omega_X - \sum \varepsilon_i\rho_i$$

where $\varepsilon_i > 0$ are well-chosen real numbers. Then we take $\varepsilon := \frac{\delta}{2}$ and if we denote by U and $\widetilde{f}$ the corresponding set and function, respectively, then all in all we have

$$\pi^\star(\alpha) + \sqrt{-1}\partial\overline{\partial}\big(\widetilde{f} + \sum(a_i + \frac{\delta}{2}\varepsilon_i)\log|s_{E_i}|^2_{h_i}\big) \geq \frac{\delta}{2}\pi^\star(\omega_X) \tag{32}$$

in the sense of currents on U.
On the other hand, we also have at our disposal the inverse image current $\pi^\star\Theta \geq \delta_1\pi^\star(\omega_X)$ which has log poles along $\widehat{V} \cup Z$, where $Z \subset \widehat{X}$ is an analytic set. The procedure we have used in the previous particular case applies here: indeed, we have only used the fact that f is smooth in order to construct the open subset U. Then adding the function $\delta_0\psi_Z$ to it has the effect of diminishing a bit more the lower bound in (32), but we can afford this since $\delta > 0$.

The current obtained after the gluing procedure on $\widehat{X}$ has log poles and it is non-singular at the generic point of $\widehat{V}$. But as we have already mentioned, the birational map π is a biholomorphism at the general point of V, so the direct image of the said current will be smooth at the generic point of V. Moreover, it has weak analytic singularities by definition –actually this is the main reason why we have introduced this class of singularities. □

5. On subadjunction and the canonical bundle formula

Let X be a compact Kähler manifold, and consider a real $(1,1)$-class α on X. We assume that α contains a closed positive current $R \geq 0$ with admissible singularities. This means that we can write

$$R = \alpha + \sqrt{-1}\partial\overline{\partial}\phi \tag{33}$$

where (abusing notation) α is a smooth representative of the class α and the function ϕ verifies the conditions in Definition 4.10.
We denote by $\pi : \widehat{X} \to X$ a log-resolution of the integral closure of the ideal generated locally by the functions (f_i) in Definition 4.10. The pull back of the current R admits

the decomposition

$$\pi^\star R = [D] + R_0 \tag{34}$$

where we denote by $[D]$ the current of integration along the effective divisor D and $R_0 \geq 0$ is a closed, smooth and semi-positive $(1,1)$-form on $\widehat{X}$. By analogy with the case in which R is induced by an effective divisor we call such a map π a log-resolution of R.

We assume next that there exists a log-resolution of R, such that the following additional requirements are satisfied.

(1) The support of the π-exceptional divisor E has simple normal crossings, and π is obtained as composition of blow-ups of smooth centers.

(2) We have $\pi^\star(K_X + \alpha) \simeq K_{\widehat{X}} + \beta + S + \Xi_1 - \Xi_2$, where:
- The notation above means that the relative canonical class $K_{\widehat{X}/X}$ of π plus the divisor $S + \Xi_1 - \Xi_2$ coincides with $\pi^\star(\alpha) - \beta$.
- β is a nef $(1,1)$-class on $\widehat{X}$.
- S is a smooth hypersurface on $\widehat{X}$ whose image is denoted by $T := \pi(S)$.
- The restriction $\pi|_S$ admits a decomposition

  $$\pi|_S = \pi_W \circ f$$

  where $\pi_W : W \to T$ is a desingularisation of T and $f : S \to W$ is holomorphic.
- The Ξ_i's are effective $\mathbb{R}$-divisors on $\widehat{X}$ such that their supports do not have common components, $S + \Xi_1 + \Xi_2$ is snc, $(\widehat{X}, S + \Xi_1)$ is plt and any hypersurface Y contained in the support of Ξ_2 is π-exceptional.

We let

$$\gamma := \pi_W^\star(K_X + \alpha)|_T - K_W, \tag{35}$$

and we write

$$f^\star(\gamma) + \Xi_2|_S \simeq K_{S/W} + (\beta + \Xi_1)|_S. \tag{36}$$

Note that the pair $(\widehat{X}, \Xi := \{\Xi_1 - \Xi_2\})$ is klt and that T is the unique center of log canonical singularities for the generalized pair $(\hat{X}, S + \Xi_1 - \Xi_2 + \beta)$. In particular T is normal.

3. The class β contains a smooth, positive representative.
4. The coefficients of the divisors Ξ_i are rational.

A first result we establish here is the following.

Theorem 5.1. *Assume that the requirements 1-4 above are satisfied. Then the class $K_{S/W} + (\beta + \Xi_1)|_S$ contains a closed positive current $\Theta \geq 0$ such that:*

(1) *For each general fiber $S_w = f^{-1}(w)$, the restriction $\Theta|_{S_w}$ is induced by the space of sections of the line bundle associated to $m\Xi_2|_{S_w}$, for m large and sufficiently divisible.*

(2) *Consider the divisor* $\Xi_2' \leq \Xi_2|_S$ *obtained by discarding the components of* $\Xi_2|_S$ *whose image is contained in the singular subset of* T. *Then,* $\Theta \geq [\Xi_2']$ – *that is to say, the current* Θ *is singular along the divisor* Ξ_2'.

The following results can be seen as "transcendental" versions of the canonical bundle formula. They can be used to refine Theorem 5.1, but they are of independent interest as they apply to a variety of other contexts.

Let $f : S \to W$ be a surjective map of compact Kähler manifolds. Let $P := \sum P_i$ and $Q = \sum Q_j$ be two reduced, snc divisors on S and W, respectively such that moreover $f^{-1}Q \subset P$. We decompose the divisor P

$$P = P^h + P^v$$

into f-horizontal and f-vertical parts, and we assume moreover that the restriction of f to the support of P^h is relatively snc on the complement of the support of Q, and moreover $f(\operatorname{Supp} P^v) = Q$.

Let $B = \sum d_i P_i$ be a $\mathbb{Q}$-divisor on S, and let β be a $(1,1)$-class such that the following requirements are satisfied.

(a) The pair (S, B) is sub-klt.

(b) The morphism $\mathcal{O}_W \to f_\star \mathcal{O}_S(\lceil -B \rceil)$ is surjective at general points of W.

(c) We have $K_S + B + \beta \simeq f^\star \gamma$ and moreover β contains a smooth positive representative.

($\star$) For any point $z_0 \in S$ and $w_0 = f(z_0) \in W$ there exist local coordinates $(x_1, \dots, x_{n+m})$ on S centred at z_0 and $(t_1, \dots, t_m)$ on W centred at w_0 such that $t_i \circ f(x) = \prod x_j^{k_{ij}}$ where the k_{ij} are non-negative integers such that $k_{ij} \neq 0$ for at most one i for each index j.

Then the following result holds true – the case $\beta = 0$ corresponds to the original result of Y. Kawamata in [Kawamata98].

Theorem 5.2. *Assume that conditions* (a), (b), (c) *as well as* ($\star$) *hold. Then the class* $\{\gamma\}$ *can be decomposed as* $K_W + B_W + \beta_W$ *where* B_W *is the discriminant divisor and* β_W *is a cohomology class containing a closed positive current with zero Lelong numbers. In particular,* β_W *is a nef class.*

Remark 5.3. *Note that the hypothesis* (a), (b), (c) *are very natural, identical to the set-up in* [Kawamata98]. *We expect Theorem 5.2 to hold without the additional hypothesis* ($\star$), *but there are serious technical difficulties to overcome.*

One could ask the same type of questions in a more flexible and natural context, in which β is only assumed to be nef (so that we start with a nef class on S and the "output" is a nef class on W). It turns out that the situation is a bit more complicated –the reason being that a perturbation of β could destroy the first hypothesis in (c)–, and in order to treat the nef case we consider the following assumptions.

(d) There exists a Kähler metric ω on S such that

$$\omega = f^\star g + \theta,$$

where g is a Kähler metric on W and θ is a *rational* $(1,1)$-form on S. Therefore, for any coordinate subset $\Omega \subset W$ biholomorphic to a ball we have a $\mathbb{Q}$-line bundle A_Ω on $V := f^{-1}(\Omega)$ whose curvature equals $\omega|_V$.

(b') For any fixed, sufficiently big and divisible integer $m_0 > 0$, there is an integer k_0 such that the natural inclusion $f_\star\mathcal{O}_V(m_0 A_\Omega + k_0\lceil -B\rceil) \subset f_\star\mathcal{O}_V(m_0 A_\Omega + k\lceil -B\rceil)$ is an isomorphism for all $k \geq k_0$ (i.e. the local sections vanish along $(k-k_0)\lceil -B\rceil$), where A_Ω is defined above.

(c') We have $K_S + B + \beta \simeq f^\star\gamma$ and β is a nef class.

Theorem 5.4. *Assume that conditions* (a), (b'), (c'), (d)*, as well as* $(\star)$ *hold. Then the class* $\{\gamma\}$ *can be decomposed as* $K_W + B_W + \beta_W$ *where* B_W *is the discriminant divisor and* β_W *is a nef class.*

Remark 5.5. *The proof of Theorem 5.1 will show that the hypothesis (b') is quite natural, in the sense that if the map $f : S \to W$ is induced by a birational transformation π, then these hypothesis hold true.*

The content of the following sections is organized as follows. There are two techniques of constructing closed positive currents in twisted relative classes of a map between compact Kähler manifolds. One can either use fiberwise holomorphic sections (normalized in a canonical manner), or fiberwise Kähler-Einstein metrics, cf. [Gue20] and the references therein. Here we will use the former, since the latter is not sufficiently general to be implemented in our context.

Indeed, given a holomorphic surjective map $f : S \to W$ between two Kähler manifolds and a Hermitian line bundle $(L, h_L) \to X$, the spaces

$$H^0\big(S_w, (K_{S_w} + L|_{S_w}) \otimes \mathcal{I}(h_L|_{S_w})\big)$$

of L^2 sections (for $w \in W$ general) can be "pieced together" in order to construct a metric on $K_{S/W} + L$, which is semi-positively curved e.g. in case the curvature current $\sqrt{-1}\Theta(L, h_L) \geq 0$ is positive, cf. [BP08]. The same is true in the pluricanonical case, i.e. we can construct a positively curved metric on $mK_{S/W} + L$, by replacing the L^2 normalization with an $L^{\frac{2}{m}}$ condition. As a result, the rational class $K_{S/W} + \dfrac{1}{m}L$ contains a closed positive current, whose restriction to the general fiber of f is induced by the subspace of sections of

$$H^0\big(S_w, mK_{S_w} + L|_{S_w}\big)$$

which satisfy an $L^{\frac{2}{m}}$ integrability condition.

In section 6.1 we show that the same holds true if we replace $\frac{1}{m}L$ with a $(1,1)$-class α, provided that the we can still define the space above, i.e. the restriction $\alpha|_{S_w}$ of our class to the fibers of f is rational. This will settle the first part of Theorem 5.1. The singularities of the current constructed are analyzed by using techniques borrowed from extension of pluricanonical forms.

Concerning Theorem 5.2, recall that any nef class is pseudo-effective, but in general the two cones are different. Nevertheless, if a $(1,1)$ class contains a closed positive current whose Lelong numbers at each point of the ambient space are equal to zero, then the class in question is psef, cf. [Dem92]. The nefness of the moduli part (in our notations, the class β_W) in the canonical bundle formula was established by S. Takayama in [Taka22] along these lines. Here we will adopt the same strategy –i.e., we will conclude by showing that β_W contains a closed positive current with zero Lelong numbers–, and by the same token, simplify a little the arguments in *loc. cit.*

6. Positivity of the relative adjoint transcendental classes

We begin this section by recalling the following results.

Theorem 6.1. *Let $f : S \to W$ be a surjective map between two compact complex manifolds. Let α be a real class of type $(1,1)$ on W. Then*

(1) α is nef if and only if $f^\star\alpha$ is nef.
(2) α is nef if and only if $\alpha|_Z$ is pseudo-effective for all irreducible proper subvarieties $Z \subset W$ and $f^\star\alpha$ is pseudo-effective.

Proof. (1) follows immediately from [DHP22, Lemma 2.38].

We will now show that (2) also follows from the proofs of [DHP22, Theorem 2.36, Lemma 2.38]. By [DHP22, Theorem 2.36], we know that if $Z^\nu \to Z$ is the normalization of an irreducible proper subvariety of W, then $\alpha|_{Z^\nu}$ is nef. By [DHP22, Corollary 2.39], it suffices to show that if ω is Kähler on W and $\dim W = d$, then $\int_W \alpha^k \wedge \omega^{d-k} \geq 0$ for $0 < k \leq d$. Suppose that α is not nef, then we let $t > 0$ be the nef threshold so that $\alpha + t\omega$ is nef but not Kähler. Clearly $(\alpha + t\omega)|_{Z^\nu}$ is Kähler and so by [DHP22, Corollary 2.39] $\int_W (\alpha + t\omega)^k \wedge \omega^{d-k} = 0$ for some $0 < k \leq d$. Let F be a general fiber of $S \to W$, η a Kähler class on S and $\lambda = \int_F \eta^{n-d}$ where $n = \dim S$. Then

$$\lambda\cdot\int_W (\alpha+t\omega)^k\wedge\omega^{d-k} = \int_S f^*((\alpha+t\omega)^k\wedge\omega^{d-k})\wedge\eta^{n-d} \geq \int_S f^*(t^k\omega^d)\wedge\eta^{n-d} = \lambda t^k\cdot\int_W \omega^d$$

which is impossible as the LHS equals 0 and the RHS is strictly positive. Thus $t = 0$ and α is nef.

Another way of establishing the point (2) is by using [DHP22, Corollary 2.32] : this shows that α contains a closed positive current, i.e. it is pseudo-effective. The conclusion follows by using [DHP22, Theorem 2.36]. □

Therefore, in order to show that the class β_W in Theorem 5.2 is nef, it is sufficient to show that this is true for its pull back via f. This will follow from the main results in the next two subsections. In the first one we collect a few results about the construction of closed positive currents.

6.1. **Closed positive currents in twisted relative canonical classes.** To start with, we introduce the following set of notations, *which will only be used in this subsection.*

(1) $f : X \to Y$ is a surjective map with connected fibers between compact Kähler manifolds.

(2) $D = \sum a_i D_i$ is an effective, snc divisor with rational coefficients $0 < a_1 < 1$ and L is a $\mathbb{Q}$-line bundle on X. Thus there exists a positive integer m_0 such that the reflexive hull $L^{[m_0]} := (L^{\otimes m_0})^{\vee\vee}$ is a genuine line bundle, and a metric on L will simply be given by a collection of functions $\left(\frac{\varphi_i}{m_0}\right)_{i\in I}$, where $(\varphi_i)_{i\in I}$ are the weights of a metric on $L^{[m_0]}$. By abuse of notation we will often denote $L^{[m_0]}$ by $m_0 L$.
(3) α and γ are real $(1,1)$–classes on X and Y, respectively. Moreover, α contains a smooth positive representative denoted by θ.
(4) The class $f^\star\gamma - \alpha$ coincides with the first Chern class of $K_{X/Y} + D - L$.

In this context we prove the next statement.

Theorem 6.2. *Assume that conditions* (1), (2), (3) *and* (4) *hold and that for some sufficiently big and divisible integer $m > 0$ we have $H^0(X_y, mL|_{X_y}) \neq 0$ for general $y \in Y$. Then the class*

$$K_{X/Y} + D + \alpha$$

contains a closed positive current $\Theta \geq 0$ whose restriction to the general fiber of f is (well-defined and) induced by the sections of mL restricted to the fibers of f.

Proof. To begin with, we remark that if $D = 0$ and if some multiple of α belongs to $H^2(X, \mathbb{Z})$, then the matter is clear. Indeed, in this case we can choose a $\mathbb{Q}$-line bundle F on X whose Chern class is α and such that

$$(K_{X/Y} + D + F)|_{X_y} \simeq L|_{X_y}$$

for all general $y \in Y$. The results proved in [PT18] show that the current Θ constructed fiber-wise by the m^{th} root of the sections of $mL|_{X_y}$ is positive.

Even though α may not be a rational class, hypothesis (4) implies that this is the case locally over Y. This will allow us to conclude via an approach similar to the one in [PT18], [CH20] (with slight modifications). The details are as follows.

We denote by γ any closed, real $(1,1)$-form contained in the class γ given by hypothesis (3) -and apologize for the abuse of notation. As consequence of the hypothesis (4) above, the $(1,1)$-form μ defined by the equality

$$\mu := \theta - f^\star(\gamma) \tag{37}$$

is closed, real, and its corresponding class is *rational*.

Let h be a metric on $L - (K_{X/Y} + D)$ whose corresponding curvature form equals μ (here we are using the convention in (2) above). We consider a finite open cover $(U_i)_{i\in I}$ of Y such that

$$\gamma|_{U_i} = dd^c\tau_i \tag{38}$$

for some smooth real function τ_i defined on U_i.

For each index i we endow the restriction

$$(L - (K_{X/Y} + D))|_{f^{-1}(U_i)}$$

with the metric $h_i := e^{-\tau_i\circ f} h$. The equality

$$\theta|_{f^{-1}(U_i)} = dd^c(f^\star\tau_i) + \mu|_{f^{-1}(U_i)} \tag{39}$$

shows that the curvature corresponding to the metric h_i is equal to $\theta|_{f^{-1}(U_i)}$.

All in all, we can define a Hermitian $\mathbb{Q}$-line bundle (F_i, h_i) on the inverse image $f^{-1}(U_i)$ such that the following hold.

(a) For each $i \in I$ the following holds

$$(K_{X/Y} + D)|_{f^{-1}(U_i)} + F_i \simeq L|_{f^{-1}(U_i)} \tag{40}$$

especially the sections of $mL|_{X_y}$ correspond to sections of $m(K_{X/Y}+D+F_i|_{X_y})$ for each general point $y \in U_i$, where $X_y := f^{-1}(y)$.

(b) We describe here more precisely the metric h_i. Let $(V_j)_{j\in J}$ be an open covering of X, such that the restriction of the bundles $K_X, f^\star K_Y, mD, mL$ to each V_j is trivial. Recall that we have fixed a metric h on $L - K_{X/Y} - D$, and denote by ρ_j its weight on the set V_j. Then the weight of the metric h_i on the set $V_j \cap f^{-1}(U_i)$ is

$$\varphi_{ij} := \rho_j|_{V_j\cap f^{-1}(U_i)} + \tau_i \circ f|_{V_j\cap f^{-1}(U_i)}.$$

We stress the fact that the only "non-global" part of the metric h_i corresponds to the pull-back of τ_i.

(c) It follows that we have

$$\sqrt{-1}\Theta(F_i, h_i) = \theta|_{f^{-1}(U_i)}. \tag{41}$$

and remark that even if (F_i, h_i) is only locally defined (with respect to the base Y), the corresponding curvature is a global form on X.

Relation (40) allows us to define a metric $h_{X/Y,i}$ on $(K_{X/Y} + D)|_{f^{-1}(U_i)} + F_i$ whose corresponding curvature is positive. This was done in [PT18], and we recall next the construction. Let $x_0 \in f^{-1}(U_i)$ be an arbitrary point. We fix coordinates (t_k) and (z_l) on U_i and near x_0, respectively. Assume that f is smooth over $Y \setminus \Sigma$. For each $y \in U_i \setminus \Sigma_i$ and $\xi \in V_{m,y}$ let

$$\|\xi\|^{\frac{2}{m}}_{y,i} := \int_{X_y} |\xi|^{\frac{2}{m}} e^{-\varphi_D - \varphi_i} \tag{42}$$

be the $L^{2/m}$-seminorm on the space of sections

$$V_{m,y} := H^0\big(X_y, mL_y\big) = H^0\big(X_y, m(K_{X_y} + D_y + F_{i,y})\big)$$

where the notations are explained below. The subscript $(\dots)_y$ denotes restriction to the fiber X_y and m is assumed to be sufficiently divisible so that all divisors in question are Cartier. In (42) the symbol $e^{-\varphi_i}$ means that we are using the metric h_i (cf. (b) above) on the bundle F_i. The section ξ in (42) is interpreted as a twisted pluricanonical form, so that the quantity under the integral is a (n, n)–form.

Then the weight of the metric $h_{X/Y,i}$ at the point x_0 is equal to

$$e^{\varphi_{X/Y,i}(x_0)} := \sup_{\|\xi\|_{y_0,i}=1} |\xi_0(x_0)|^{\frac{2}{m}} \tag{43}$$

where $y_0 := f(x_0)$ and ξ_0 is given by the equality

$$\xi \wedge f^\star(dt^{\otimes m}) = \xi_0 dz^{\otimes m}$$

written locally near x_0.

We have given such a detailed description of the metric $h_{X/Y,i}$ because thanks to it, it is easy to deduce its dependence on the index i. Indeed, we assume that we choose the same coordinates t and z on $U_i \cap U_k$ and near $x_0 \in V_j$, respectively, where it is understood that $f(x_0) \in U_i \cap U_k$. Notice that the space of holomorphic sections $V_{m,y}$ involved in the definition of the relative metric is independent of i, but this may be not the case for the semi-norm (42). By (b) of (40), we can write

$$\int_{X_y} |\xi|^{\frac{2}{m}} e^{-\varphi_D - \varphi_i} = e^{-\tau_i(y)} \int_{X_y} |\xi|^{\frac{2}{m}} e^{-\varphi_D - \rho_j}, \tag{44}$$

where the notation is indicating the weight $\varphi_D + \rho_j$ we are using on the set $V_j \cap X_y$. This is a consequence of the definition of the metric h_i in (b). Moreover, we remark that the second factor of the product on the RHS of (44) is independent on the index "i".

Thus, by (42) we infer that the equality

$$\|\xi\|_{y,i}^{\frac{2}{m}} = e^{\tau_k(y) - \tau_i(y)} \|\xi\|_{y,k}^{\frac{2}{m}} \tag{45}$$

holds for any point $y \in U_i \cap U_k$ and $\xi \in V_{m,y}$.

Moreover, we can assume that the difference

$$\tau_k - \tau_i = \Re(\tau_{ik}) \tag{46}$$

is the real part of some holomorphic function τ_{ik} defined on the intersection $U_i \cap U_k$, since their respective Hessian forms coincide by (39).

It follows that

$$\sup_{\|\xi\|_{y_0,i}=1} |\xi_0(x_0)| = \sup_{\|\xi\|_{y_0,k}=e^{-\Re(\tau_{ik}(y_0))}} |\xi_0(x_0)| = e^{-m\Re(\tau_{ik}(y_0))} \sup_{\|\xi\|_{y_0,k}=1} |\xi_0(x_0)|\,. \tag{47}$$

Finally, we get

$$\varphi_{X/Y,i}(x_0) = \varphi_{X/Y,k}(x_0) - \frac{1}{2}\Re(\tau_{ik}(y_0)) \tag{48}$$

and since the point x_0 was arbitrary and $y_0 = f(x_0)$ it follows that we have

$$\varphi_{X/Y,i} = \varphi_{X/Y,k} - \frac{1}{2}\Re(\tau_{ik} \circ f) \tag{49}$$

locally near a fixed point on $f^{-1}(U_i \cap U_k)$. In particular we obtain the equality

$$dd^c \varphi_{X/Y,i} = dd^c \varphi_{X/Y,k}, \tag{50}$$

on the overlapping U_i's.

In conclusion, (50) shows that the curvature currents we construct locally on the base agree on the intersection of the corresponding sets, and the construction of Θ is finished, since the positivity of this current was already established in [PT18, Theorem 4.2.7]. $\square$

6.2. **Singularities of the metric.** In order to prove Theorem 5.1 we can apply Theorem 6.2 for the following data: $X := S, Y := W, D = \Xi_1$ and $L = \Xi_2$, together with $\alpha := \beta$. The output is a current

$$\Theta \in c_1(K_{S/W}) + (\beta + \Xi_1)|_S$$

with the properties stated in the point (1) of Theorem 5.1. The assertion (2) will be established along the following lines.

Proof of Theorem 5.1, (2). To begin with, we recall an important class of manifolds on which L^2 methods can be applied.

Definition 6.3. *A manifold/complex space X is called weakly pseudo-convex if it admits a smooth, plurisubharmonic exhaustion function ψ, so that the closure of the sets $(\psi < C) \Subset X$ are compactly contained in X, for any constant C.*

Obviously, compact holomorphic manifolds have this property, but this is equally the case for any complex space which admits a proper map into a Stein manifold. In particular, consider the map $f : S \to W$ given in Theorem 5.1; for any Stein open subset $U \subset W$ the inverse image $f^{-1}(U) \subset S$ is an example of weakly pseudo-convex manifold which will be important in what follows.

Consider next the blow-up map $\pi : \widehat{X} \to X$ introduced at the beginning of Section 5, and denote by $E = \sum E_j$ the corresponding exceptional divisor. We define the following form

$$\omega_{\widehat{X}} := \pi^\star \omega_X + \sum a_i \theta_i \tag{51}$$

where ω_X is a Kähler metric on the base X, the coefficients a_i are positive rational numbers and the forms θ_i belong to the Chern class of $\mathcal{O}_{\widehat{X}}(-E_i)$. By an appropriate choice of the coefficients a_i, we can assume that $\omega_{\widehat{X}} > 0$ – so we have a Kähler metric on $\widehat{X}$ for which the only "transcendental" part is pulled-back from the base X.

Let $w_0 \in W$ be an arbitrary point, and let $\Omega \subset X$ be a Stein co-ordinate subset which contains the image $\iota_W(w_0)$, cf. diagram (52) below.

$$\begin{array}{ccc} S & \xrightarrow{\iota_S} & \widehat{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi} \\ W & \xrightarrow{\iota_W} & X \end{array} \tag{52}$$

Consider the following sets

$$\widehat{X}_\Omega := \pi^{-1}(\Omega), \qquad U := \iota_W^{-1}(\Omega), \qquad S_U := f^{-1}(U) \tag{53}$$

contained in $\hat{X}$, W, and S, respectively. We have the following statement.

Lemma 6.4. *There exist Hermitian line bundles $(A_\Omega, h) \to \widehat{X}_\Omega$ and $(A_U, h) \to U$ on $\widehat{X}_\Omega$ and U, respectively such that the corresponding curvature forms are multiple of Kähler forms, i.e.*

$$\sqrt{-1}\Theta(A_\Omega, h) = N\omega_{\widehat{X}}|_{\widehat{X}_\Omega}, \qquad \sqrt{-1}\Theta(A_U, h) = N\omega_W|_U$$

where N is positive and sfficiently divisible.

Proof. This follows by standard arguments. Since the restriction $\omega_X|_\Omega$ is dd^c-exact, it can be interpreted as trivial bundle over Ω endowed with a non-trivial metric whose corresponding curvature form is $\omega_X|_\Omega$. The rest follows as consequence of (51) – in particular we only need the positive integer N in order to clear the denominators of the coefficients a_i. A similar argument applies for ω_W. $\square$

Remark 6.5. *Note that we may assume $\iota_W : W \to X$ is given by a finite sequence of blow ups whose centers are contained in the singular locus $T_{\rm sing}$ of T. The metric ω_W can be obtained by the same formula as in (51), so that the corresponding exceptional divisors $E_i \subset W$ map into the singular locus of the centre X.*

After these preparations, we proceed with the second part of Theorem 5.1. Let $\Omega \subset X$ be an open subset as above. By the same procedure as in the proof of Theorem 6.2, we can construct a $\mathbb{Q}$-line bundle $(F_\Omega, h_F) \to \widehat{X}_\Omega$ such that the following relations hold

$$K_{\widehat{X}} + S + \Xi_1 + F_\Omega \simeq \Xi_2, \qquad \sqrt{-1}\Theta(F_\Omega, h_F) = \beta \tag{54}$$

on $\widehat{X}_\Omega \subset \widehat{X}$.

On the other hand, let ρ be any non-singular $(1,1)$-form on W, such that $\rho \in c_1(K_W)$. We can assume that the constant N in Lemma 6.4 is large enough, so that the following inequality

$$\rho + N\omega_W \geq 0 \tag{55}$$

holds point-wise on W. We then consider the closed positive current

$$\Theta + f^\star(\rho + N\omega_W) \geq 0 \tag{56}$$

which belongs to the class $K_S + (\Xi_1 + \beta)|_S + N f^\star(\omega_W)$.
Next, given the expression of the metric ω_W combined with Remark 6.5, there exist integers k_i and divisors E_i such that the current

$$\widehat{\Theta} := \Theta + f^\star(\rho + N\omega_W) + \sum k_i [E_i|_S] \tag{57}$$

has the following properties

- It belongs to the cohomology class $(K_X + S + \Xi_1 + \beta + N\pi^\star(\omega_X))|_S$
- The divisors E_i appearing in (57) project into $T_{\rm sing}$.

When restricted to the set

$$S_U = S \cap \pi^{-1}\Omega$$

(cf. (53) for the notations), the class β corresponds to the line bundle $F_\Omega|_S$, so by abuse of notation we can write

$$\widehat{\Theta}|_{S_U} \simeq K_X + S + \Xi_1 + F_\Omega|_{S_U} \tag{58}$$

by which we mean that the bundle on the RHS admits a singular metric $h_\theta = e^{-\varphi_\theta}$ defined on S_U whose curvature form is precisely the restriction $\widehat{\Theta}|_{S_U}$.
For each $k \geq 1$ sufficiently divisible we consider the line bundle

$$L_k := (k(K_X + S + \Xi_1 + F_\Omega) + A_\Omega)|_{S_U}$$

and the corresponding Hilbert space of homolorphic sections

$$\mathcal{H}_k := \{s \in H^0(S_U, L_k) / \int_{S_U} |s|^2 e^{-k\varphi_\theta - \varphi_A} dV < \infty\} \tag{59}$$

in the multiplier ideal induced by the current $k\widehat{\Theta}|_{S_U}$. Then we recall the following result, basically proved in [Dem09, Section 13] and references therein.

Theorem 6.6. *Let $\widehat{\Theta}_k \geq 0$ be the closed positive current on S_U given by a family of orthonormal sections of $\mathcal{H}_k$. Then we have*

$$\nu(\widehat{\Theta}, x) = \lim_k \frac{1}{k} \nu(\widehat{\Theta}_k, x)$$

where $x \in S_U$ is an arbitrary point and where we denote by $\nu(\widehat{\Theta}, x)$ the Lelong number of $\widehat{\Theta}$ at x.

Remark 6.7. *We note that in* loc. cit. *the result above is established in the setting of bounded pseudo-convex subsets in $\mathbb{C}^n$, but the proof applies in the context of Theorem 6.6, so we will not reproduce it here. As a matter of fact, it is at this point that the pseudo-convexity of the set S_U (cf. Definition 6.3) is very important.*

In other words, in order to evaluate the singularities of $\widehat{\Theta}$ it would suffice to have a uniform lower bound for the vanishing orders of the sections $s \in \mathcal{H}_k$ as $k \to \infty$. To this end, we recall that as a consequence of the results e.g. in [BP10] the following local version of the *invariance of plurigenera* holds true.

Theorem 6.8. *In the above set-up, any holomorphic section s of the bundle L_k extends to $\widehat{X}_\Omega$ as section $\widetilde{s}$ of $k(K_X + S + \Xi_1|_{\widehat{X}_\Omega} + F_\Omega) + A_\Omega$.*

We offer next a few explanations about 6.8 in the very particular case in which we have to extend a section s of the bundle $k(K_{X_\Omega} + S + L)|_{S_U}$, where (L, h_L) is a semi-positively curved line bundle, such that h_L is non-singular and it is defined over $X_{\Omega'}$ for some $\Omega \Subset \Omega'$. As we have seen above, we have an ample line bundle A_Ω over X_Ω, and thus, in order to construct the extension of s we need the following.

- *A local version of the Ohsawa-Takegoshi extension theorem.* The statement we need is available, cf. [DemOT].
- *A finite family of holomorphic sections for the bundles*

  $$(k + r)(K_{X_\Omega} + S + L) + C(k) A_\Omega|_{X_\Omega}$$

  for $r = 0, \ldots, k$ such that for each r, their common set of zeroes is empty. This is easy to see, despite of the fact that S_U and X_Ω are not compact: the point is that all the bundles/metrics extend over $X_{\Omega'}$ and we construct our sections by a quick compactness argument.

These two points granted, one follows the usual algorithm, see e.g.[BP10] and the references therein.

However, in our case there is an additional level of difficulty, induced by the presence of the $\mathbb{Q}$-divisor Ξ_1. This can also be treated by the known techniques (i.e. work by

Hacon-McKernan, Ein-Popa ...), given the fact that $S+\Xi_1+\Xi_2$ is snc and the relation (54) (and therefore the condition (7) in *loc. cit.* is automatically satisfied).

Now, by relation(54), the extension $\widetilde{s}$ can be seen as section of the bundle $k\Xi_2|_{\widehat{X}_\Omega}+A_\Omega$. Given that the support of Ξ_2 is π-contractible and that the set $\widehat{X}_\Omega$ is the inverse image of Ω by the map π, it follows that the vanishing order of $\widetilde{s}$ along Ξ_2 is at least $k-k_0$ (we "loose" a fixed amount m_0 because of the ample bundle A_Ω).

In conclusion, it follows that we have

$$\widehat{\Theta}_k \geq (k-k_0)[\Xi_2]|_{S_U} \tag{60}$$

and the proof is finished by using Theorem 6.6. □

7. Proof of Theorem 5.2

The main steps of the proof of our version of the canonical bundle formula – Theorem 5.2 – are as follows. Let $\Xi := \{B\}$ be the fractional part of B and we write $B = \Xi + \lfloor B\rfloor = \Xi - \lceil -B\rceil$ as difference of two effective $\mathbb{Q}$-divisors. We assume that the discriminant divisor B_W is equal to zero (we can do this without altering any of our hypothesis). We then have the numerical identity

$$K_{S/W}+\Xi+\beta \simeq f^\star\beta_W + \lceil -B\rceil.$$

Next, we apply the methods already used in the proof of Theorem 5.1 in order to construct a closed positive current $\Theta\geq 0$ in the class corresponding to the LHS of the relation above. The said current is proved to be singular along $\lceil -B\rceil$: this follows as consequence of the hypothesis (b) (which replaces the fact that the map $f: S\to W$ might not be induced by a log-resolution π).

The heart of the matter is to show the (highly non-trivial) fact that the Lelong numbers of the difference

$$\Theta - \lceil -B\rceil$$

are equal to zero. To this end we adapt the method used in [Taka22] in our context.

We start with a general discussion –and a simple result– concerning fiber integrals.

7.0.1. *Fiber integrals.* Let $p: X\to Y$ be a proper, surjective holomorphic map, where X is a $(n+m)$-dimensional Kähler manifold and Y is the unit disk in $\mathbb{C}^m$. We denote by $Y_0\subset Y$ the set of regular values of p. Let $t=(t_1,\dots,t_m)$ be coordinates on $\mathbb{C}^m$ induced by a fixed base. Consider a $\mathbb{Q}$-line bundle (L,h_L) on the total space X, endowed with a metric h_L eventually singular, but whose curvature is semi-positive. Let $s\in H^0\big(X,k(K_X+L)\big)$ be a pluricanonical form with values in kL, where k is a positive, sufficiently divisible integer so that kL is a line bundle. For each $y\in Y_0$ let $s_y\in H^0\big(X_y,k(K_{X_y}+L_y)\big)$ be the induced form on X_y, in the sense that

$$s|_{X_y} = s_y\wedge p^\star(dt)^{\otimes k}. \tag{61}$$

In this setting we show that the following holds true.

Lemma 7.1. *We assume moreover that there exists a section σ of a line bundle Λ such that the quotient $\dfrac{s}{\sigma}$ is a holomorphic section of $k(K_X+L)-\Lambda$. There exists a positive constant $C_0>0$ independent of s such that the inequality*

$$\int_{X_y}|s_y|^{\frac{2}{k}}e^{-\varphi_L}\geq C_0\sup_{X_y}\left|\frac{s}{\sigma}\right|^{\frac{2}{k}} \tag{62}$$

holds for any $y\in Y_0$ such that $|y|<\frac{1}{2}$. The norm on the RHS is with respect to a fixed, smooth metric on $k(K_X+L)-\Lambda$, and an upper bound for the constant C_0 can be obtained from the proof that follows.

Proof. Let $z_0\in X_y$ be a point such that

$$\sup_{X_y}\left|\frac{s}{\sigma}\right|^{\frac{2}{k}}=\left|\frac{s}{\sigma}\right|^{\frac{2}{k}}(z_0)$$

and let $y=f(z_0)$ be its image. We take the local coordinates $z=(z_1,\dots,z_{n+m})$ and $t=(t_1,\dots,t_m)$ centred at z_0 and y respectively. The t-coordinates are defined on some open set $\Omega\subset Y$, and the z-coordinates are defined on $V\subset f^{-1}(\Omega)$ biholomorphic to the unit ball in $\mathbb{C}^{n+m}$. Let ω be an arbitrary Kähler metric on X.
Corresponding to this data we define the function $\psi:V\to\mathbb{R}\cup\{-\infty\}$ as follows

$$\omega^n\wedge f^\star(\sqrt{-1}dt\wedge d\overline{t})=e^{\psi}\sqrt{-1}dz\wedge d\overline{z} \tag{63}$$

where we use the notations

$$\sqrt{-1}dt\wedge d\overline{t}:=\prod_{i=1}^{m}\sqrt{-1}dt_i\wedge d\overline{t}_i,\qquad \sqrt{-1}dz\wedge d\overline{z}:=\prod_{i=1}^{m+n}\sqrt{-1}dz_i\wedge d\overline{z}_i.$$

Therefore, the restriction of the form $e^{-\psi}\omega^n$ to the fiber $X_y\cap V$ is equal to the measure sometimes denoted with $\left|\dfrac{dz}{f^\star dt}\right|^2$ on X_y.
We assume that the bundles L and Λ are trivial when restricted to V, and let $u\in\mathcal{O}(V)$ be the local holomorphic function corresponding to the section $s|_V$. Then we clearly have the inequality

$$\int_{X_y\cap V}|u|^{\frac{2}{k}}e^{-\varphi_L-\psi}\omega^n\leq\int_{X_y}|s_y|^{\frac{2}{k}}e^{-\varphi_L}. \tag{64}$$

On the other hand, by the $L^{\frac{2}{k}}$-version of the Ohsawa-Takegoshi theorem established in [PT18], Proposition 1.2 there exists a function $U\in\mathcal{O}(V)$ such that

$$U|_{V\cap X_y}=u|_{V\cap X_y},\qquad \int_V|U|^{\frac{2}{k}}e^{-\varphi_L}d\lambda\leq C_{\mathrm{univ}}\int_{X_y\cap V}|u|^{\frac{2}{k}}e^{-\varphi_L-\psi}\omega^n, \tag{65}$$

where C_{univ} is a numerical constant.
Let now $\sigma_V\in\mathcal{O}(V)$ be the local holomorphic function induced by the section σ and let $N\gg 0$ be a large enough integer so that the integral

$$\int_V\frac{d\lambda}{|\sigma_V|^{\frac{2}{N}}}<\infty \tag{66}$$

is convergent. The first part of (65) combined with the fact that z_0 is the maximum point and the mean-value inequality gives

$$\sup_{X_y}\left|\frac{s}{\sigma}\right|^{\frac{2}{k}} \leq C\left|\frac{U}{\sigma_V}(z_0)\right|^{\frac{1}{kN}} \leq C\int_V\left|\frac{U}{\sigma_V}\right|^{\frac{1}{kN}} d\lambda \tag{67}$$

where the first constant is due to the fixed metric on $k(K_X+L)-\Lambda$ and it follows -thanks to Hölder inequality- that

$$\sup_{X_y}\left|\frac{s}{\sigma}\right|^{\frac{2}{k}} \leq C_0\int_V |U|^{\frac{2}{k}}e^{-\varphi_L}d\lambda, \tag{68}$$

where C_0 depends on (67), and an upper bound for φ_L. This inequality, combined with the estimate in (65) completes the proof of Lemma 7.1. □

7.0.2. *Pseudo-effectivity.* We remark that we can assume $B_W=0$, by simply replacing B with $B-f^\star(B_W)$ and noticing that under the transversality hypothesis in our statement, the new pair (S,B) is sub-klt and moreover the hypothesis (b) still holds.

Under the assumption that $B_W=0$, it follows from the hypothesis (c) of Theorem 5.2 that we have

$$K_{S/W}+B+\beta \simeq f^\star\beta_W. \tag{69}$$

Since the pair (S,B) is sub-klt, we can write $B=\{B\}+\lfloor B\rfloor := \Xi-\lceil -B\rceil$ and therefore we obtain

$$K_{S/W}+\Xi+\beta \simeq f^\star\beta_W+\lceil -B\rceil, \tag{70}$$

where (S,Ξ) is klt and $\lceil -B\rceil$ is effective, with integer coefficients. We apply Theorem 6.2 to the following data: $X:=S, Y:=W, \alpha:=\beta, \gamma:=\beta_W, L:=\mathcal{O}_S(\lceil -B\rceil)$ and finally $D:=\Xi$. It follows that there is a closed positive current $\Theta\geq 0$ in the class (70), induced by the sections of $\mathcal{O}_S(\lceil -B\rceil)|_{S_w}$ for $w\in W$ general.

We then formulate our next assertion:

Claim 7.2. *The inequality*

$$\Theta\geq\lceil -B\rceil$$

holds in the sense of currents on S, where the RHS is interpreted as current of integration on the divisor $\lceil -B\rceil$.

Proof of the Claim. We start with a little comment: if a hypersurface $Y\subset S$ belongs to the support of the divisor $\lceil -B\rceil$ is such that $f(Y)=W$ (i.e. Y is horizontal with respect to the map f), then the hypothesis (c) together with the construction of Θ show immediately that $\Theta\geq\mu[Y]$, where μ is the multiplicity of $\lceil -B\rceil$ along Y. However, things are less clear for the vertical part of the support of $\lceil -B\rceil$, since we only have the explicit expression of Θ over general points of W. It is at this point that the techniques from the subsection 7.0.1 come into play. The argument which follows has its origins in [BP08], as well as in [CH20]. The reason why we review it here is to show that it can be easily adapted to the pluricanonical case, needed a bit later.

Let $w \in W$ be any regular value of the map f, and let $w \in \Omega \subset W$ be a coordinate set of W, biholomorphic with the unit ball in $\mathbb{C}^m$. Recall from the proof of Theorem 6.2 that there exists a Hermitian line bundle (F, h_F) defined over $f^{-1}(\Omega)$, whose associated curvature form is equal to β, and such that restricting to $f^{-1}(\Omega)$ we have

$$K_{S/W} + \Xi + F \simeq \lceil -B \rceil. \tag{71}$$

Next, let u be any holomorphic section of $K_{S_w} + (\Xi + F)|_{S_w}$, such that

$$\int_{S_w} |u|^2 e^{-\varphi_\Xi - \varphi_F} = 1. \tag{72}$$

As recalled in 7.0.1, there exists a section U of $K_S + \Xi + F$ such that

$$U|_{S_w} = u \wedge f^\star(dt), \qquad \int_{f^{-1}(\Omega)} |U|^2 e^{-\varphi_\Xi - \varphi_F} \leq C_0. \tag{73}$$

Since the canonical bundle of W is trivial when restricted to Ω, we can interpret U as a section of $K_{S/W} + \Xi + F$, which is the same as $\lceil -B \rceil|_{f^{-1}(\Omega)}$ thanks to (71). In particular, by (b) the quotient

$$\tau := \frac{U}{s_{\lceil -B \rceil}}$$

becomes a *holomorphic* function on $f^{-1}(\Omega)$, where $s_{\lceil -B \rceil}$ is the canonical section of $\mathcal{O}(\lceil -B \rceil)$.
By Lemma 7.1 we infer the following inequality

$$\sup_{S_w} |\tau| \leq C, \tag{74}$$

–because of the normalisation (72)– where the constant C in (74) is independent of u.
As consequence of (72) combined with the definition of the relative metric we obtain

$$\varphi_{S/W} \leq C + \log |s_{\lceil -B \rceil}|^2, \tag{75}$$

from which our claim follows. □

7.0.3. *Lelong numbers.* Next we show that the Lelong numbers of the closed positive current

$$T := \Theta - \lceil -B \rceil \tag{76}$$

are equal to zero. To this end we will use an important result due to S. Takayama. Actually we will "extract" from the proof in [Taka22] the result below (which will be useful for the proof of Theorem 5.4 as well). To begin with, we recall the construction of a natural metric on $K_{S/W}$, cf. [MP12].
Let $z_0 \in S$ be an arbitrary point, and $t_0 = f(z_0)$ be its image. We take coordinates $z = (z_1, \ldots, z_{n+m})$ and $t = (t_1, \ldots, t_m)$ centred at z_0 and t_0 respectively. The t-coordinates are defined on some open set $\Omega \subset W$, and the z-coordinates are defined on $V \subset f^{-1}(\Omega)$. Let ω be an arbitrary Kähler metric on S. Corresponding to this data we define the function ψ as follows

$$\omega^n \wedge f^\star(\sqrt{-1} dt \wedge d\bar{t}) = e^\psi \sqrt{-1} dz \wedge d\bar{z} \tag{77}$$

where we use the notations

$$\sqrt{-1}dt \wedge d\bar{t} := \prod_{i=1}^{m} \sqrt{-1}dt_i \wedge d\bar{t}_i, \qquad \sqrt{-1}dz \wedge d\bar{z} := \prod_{i=1}^{m+n} \sqrt{-1}dz_i \wedge d\bar{z}_i.$$

We now consider a covering of S and W with coordinates sets as above. Given the equality (77), the resulting functions $e^{-\psi}$ define a metric h on the relative canonical bundle $K_{S/W}$, which is general is singular. Let $h_0 = e^{-\psi_0}$ be an arbitrary, smooth metric on $K_{S/W}$. The difference of the weights corresponding to the two metrics

$$\psi_f := \psi - \psi_0$$

is a global function on S.

For each regular value $t \in W$ of f we define the function

$$F(w) := \int_{S_w} e^{-\psi_f - \phi_B} \omega^n \tag{78}$$

where $\phi_B := \log |s_B|^2$ is the log of the norm of the canonical section of the divisor B (which we recall, is not necessarily effective).

We remark that so far the hypothesis $(\star)$ was not used in our arguments. It comes into play through the following important result, established in [Taka22]. Although it is not stated in this form explicitly, it is a direct consequence of the proof of Theorem 3.1 in *loc. cit.*

Theorem 7.3. [Taka22] *Assume that the hypothesis of Theorem 5.2 are satisfied, as well as the following.*

- *The divisor $B_W = 0$ is zero.*
- *given any point $z_0 \in S$ and $w_0 = f(z_0) \in W$ there exist local coordinates $(x_1, \dots, x_{n+m})$ on S centred at z_0 and $(t_1, \dots, t_m)$ on W centred at w_0 such that $t_i \circ f = \prod x_j^{k_{ij}}$ where the k_{ij} are non-negative integers such that $k_{ij} \neq 0$ for at most one i for each index j.*

Then for any point $w_0 \in W$ the following inequality holds

$$F(w) \leq C \prod_j \log \frac{1}{|w_j|}$$

where $C > 0$ is a positive constant and w are coordinates centred at any w_0.

Remark 7.4. *For the comfort of the readers, we will provide a complete proof of Theorem 7.3 in the Appendix of this article.*

Remark 7.5. *We note that the inequality in Theorem 7.3 holds for any w such that $\mu_W(w)$ belongs to the set of regular values of f, and the constant C is uniform.*

Consider $z_0 \in S$ and $w_0 := f(z_0)$, together with the corresponding local coordinates chosen as in Theorem 7.3. By the definition of the relative metric we have

$$e^{\varphi_{S/W}(z)} \geq \frac{|f_{\lceil -B \rceil}|^2}{\int_{S_w} |s_{\lceil -B \rceil, w}|^2 e^{-\varphi_\Xi - \varphi_F}} \tag{79}$$

where we denote by $s_{\lceil -B \rceil, w}$ the restriction of the section $s_{\lceil -B \rceil}$ to the fiber S_w, so that all in all the expression $|s_{\lceil -B \rceil, w}|^2 e^{-\varphi_\Xi - \varphi_F}$ is a volume form on S_w.
Moreover, given the definition of the divisors Ξ and $\lceil -B \rceil$, we have

$$|s_{\lceil -B \rceil, w}|^2 e^{-\varphi_\Xi - \varphi_F} \leq C e^{-\psi_f - \phi_B} \omega^n|_{S_w} \tag{80}$$

for some constant $C > 0$ (remark that the proximity of w to the singular loci of f is luckily irrelevant for the uniformity of C).

Then we have the following inequality for the potential φ_T of the current T introduced in (76)

$$e^{\varphi_T(z)} \geq \frac{C}{F(w)} \tag{81}$$

where $w = f(z)$. The second bullet in Theorem 7.3 together with the upper bound for the function F provided by this result show that

$$\nu(T, z_0) = 0, \tag{82}$$

and therefore Theorem 5.2 is completely proved, modulo the regularisation theorem in [Dem92] (a class containing a closed positive current whose Lelong numbers are equal to zero is nef)

Remark 7.6. *In general, a nef cohomology class does not necessarily contain a closed positive current with zero Lelong numbers. Therefore, the property we are establishing in the proof of Theorem 5.2 is stronger than neffness. Moreover, we construct the current T is a very explicit manner, so in principle it should be possible to further analyze its singularities.*

Remark 7.7. *We expect that a more general form of Theorem 5.2 holds true, namely one should obtain a version of this result in the absence of the hypothesis $\star$. This promises to be a difficult problem (given the arguments invoked to prove it in [Taka22]).*

8. Proof of Theorem 5.4

The main steps of the proof that follows are the same as in the previous subsection. To begin with, recall that by hypothesis (d) we have a Kähler metric ω on S such that

$$\omega = f^\star g + \theta, \tag{83}$$

where g is a Kähler metric on W and θ is a *rational* $(1,1)$-form on S.
Consider a positive integer m_0 divisible enough such that $m_0\theta \in H^2(S, \mathbb{Z})$. For each $k \geq 1$ the class

$$\beta_k := \beta + \frac{m_0}{k}\omega \tag{84}$$

contains a positive representative, since β is nef. Therefore, we can write

$$K_{S/W} + B + \beta_k = f^\star(\gamma_k) + A_k \tag{85}$$

where

$$\gamma_k := \gamma + \frac{m_0}{k} g, \qquad \frac{m_0}{k}\theta \in c_1(A_k)$$

i.e. A_k is a $\mathbb{Q}$-bundle such that kA_k becomes a holomorphic line bundle which admits a metric h_k whose curvature is precisely $m_0\theta$.

We therefore find ourselves in the framework of Theorem 6.2: we obtain a closed positive current Θ_k belonging to the class $K_{S/W} + \Xi + \beta_k$, constructed by using the global sections of

$$k(A_k + \lceil -B \rceil)|_{S_w}$$

for $w \in W$ generic.

Moreover, by hypothesis (b') together with the arguments in sub-section 7.0.1 and the Claim 7.2, we infer that we have

$$\Theta_k \geq (1 - \delta_k)\lceil -B \rceil \tag{86}$$

where $\delta_k \to 0$ as $k \to \infty$.

Finally, we analyse next the the singularities of the closed positive current

$$T_k := \Theta_k - (1 - \delta_k)\lceil -B \rceil, \qquad T_k \in K_{S/W} + B + \beta_k + \delta_k \lceil -B \rceil. \tag{87}$$

To this end, we first observe that for each co-ordinate ball $\Omega \subset W$ the restriction

$$kA_k|_V \tag{88}$$

admits a metric whose curvature is equal to $m_0\omega|_V$, where $V := f^{-1}(\Omega)$. We can assume that m_0 is large enough, so that the bundle in (88) is generated by its global sections.

As in the proof of Theorem 5.2 assume that the morphism f satisfies the additional hypothesis in the statement of Theorem 7.3. We consider $z_0 \in S$ such that $w_0 := f(z_0) \in \Omega$, and let x and t be coordinates having the second bullet property in Theorem 7.3. Let u be a holomorphic section of the bundle $kA_k|_V$, such that $z_0 \notin (u = 0)$. The product

$$\rho := u \otimes s_{\lceil -B \rceil}^{\otimes k}$$

can be interpreted as section of $k(K_{S/W} + \Xi + F_k|_V)$ (notations as in Section 6) and by the definition of the $L^{2/k}$-metric we get

$$e^{\varphi_{S/W}(z)} \geq \frac{|f_\rho(z)|^{\frac{2}{k}}}{\int_{S_w} |\rho_w|^{\frac{2}{k}} e^{-\varphi_\Xi - \varphi_F}} \tag{89}$$

for any point z near z_0 and $w := f(z)$. In (88) we denote by f_ρ the local holomorphic function corresponding to the section ρ.

Inequality (80) still applies, so we infer that

$$\nu(T_k, z_0) \leq \delta_k \nu(\lceil -B \rceil, z_0) \tag{90}$$

given the definition of ρ, provided that we assume beforehand that $B_W = 0$ so that we can use Theorem 7.3. Now the quantity δ_k is independent of the point z_0, and it follows that

$$\sup_{z\in Z} \nu(T_k, z) \leq \delta_k C, \tag{91}$$

where C is the maximum multiplicity of the divisor $\lceil -B \rceil$ at points of S.
We next use [Dem92] in order to obtain a smooth representative $\widetilde{T}_k \in K_{S/W} + B + \beta_k + \delta_k\lceil -B \rceil$ such that

$$\widetilde{T}_k \geq -C_S \delta_k \omega$$

where the constant C_S only depends on the geometry of (S, ω). Since k was arbitrary, it follows that the class

$$K_{S/W} + B + \beta = f^\star \gamma$$

is nef, hence the same is true for γ by Theorem 6.1.

9. Appendix

The main result of this subsection is a direct argument for Theorem 7.3. We first fix a few notations:

- $U \subset S$ is an open subset of S small enough so that we have the coordinates $x = (x_1, \ldots, x_{n+m})$ on U with the property that
$$f_1(x) = \prod_{i=1}^{l_1} x_i^{a_i}, \quad f_2(x) = \prod_{i=l_1+1}^{l_2} x_i^{a_i}, \quad \ldots, \quad f_m(x) = \prod_{i=l_{m-1}+1}^{l_m} x_i^{a_i}$$
where $a_i \geq 1$ and $0 = l_0 < l_1 < \cdots < l_m \leq n+m$ and $f_i := t_i \circ f$ for $i = 1, \ldots, m$.
- For every multi-index $I := (i_1, \ldots, i_m)$ such that $i_k \in J_k := \{l_k + 1, \ldots l_{k+1}\}$ we define a form of by-degree (n, n) through the formula
$$\sqrt{-1} dx_I \wedge d\overline{x}_I := \Big(\prod_{k=1}^{m} \prod_{i\in J_k, i\neq i_k} \sqrt{-1} dx_i \wedge d\overline{x}_i\Big) \wedge \prod_{i=l_m+1}^{n+m} \sqrt{-1} dx_i \wedge d\overline{x}_i$$
- $\omega := \sum_i \sqrt{-1} dx_i \wedge d\overline{x}_i$ is the local version of the reference Kähler metric. We set
$$\omega_l := \sum_I \sqrt{-1} dx_I \wedge d\overline{x}_I$$
and up to a constant, we see that we have

$$\omega_l = \Big(\sum_{i=1}^{l_1} \sqrt{-1} dx_i \wedge d\overline{x}_i\Big)^{l_1 - 1} \wedge \cdots \wedge \Big(\sum_{i=l_{m-1}+1}^{l_m} \sqrt{-1} dx_i \wedge d\overline{x}_i\Big)^{l_m - 1} \wedge \bigwedge_{i=l_m+1}^{n+m} \sqrt{-1} dx_i \wedge d\overline{x}_i$$

We now proceed to the evaluation of the function ψ defined in (77). The first remark is that

$$\frac{df_i}{f_i} = \sum_{j=l_{i-1}+1}^{l_i} a_i \frac{dx_i}{x_i} \tag{92}$$

and therefore a few simple calculations that we skip show that we have

$$e^{\psi} \simeq \big(\prod_{i=1}^{m} |f_i|^2\big) \prod_{k=1}^{m} \sum_{i=l_{k-1}+1}^{l_k} \frac{1}{|x_i|^2} \tag{93}$$

where the symbol " $\simeq$ " in (93) means that the quotient of the two functions is two-sided bounded away from zero.

The hypothesis $B_W = 0$ shows that the inequality

$$e^{-\psi-\phi_B} \leq C \prod_{k=1}^{m} \frac{1}{|X_k|^{2d_k-2}} \prod_{j\in J} \frac{1}{|x_j|^{2-2\beta_j}} =: \Lambda(x) \tag{94}$$

holds, where $d_k := l_k - l_{k-1}$ and $|X_k|^2 := \sum_{i=l_{k-1}+1}^{l_k} |x_i|^2$. Moreover, we have $J \subset \{l_m + 1, \dots, m+n\}$ and $\beta_j > 0$ for each j.

On the other hand, by the Poincaré-Lelong formula, the local version of the quantity we have to analyse equals

$$F_U(t) := \int_U \theta(x) e^{-\psi-\phi_B} \bigwedge_{i=1}^{m} dd^c \log|t_i - f_i(x)|^2 \wedge \omega^n \tag{95}$$

where θ is a truncation function defined as follows

$$\theta(x) := \theta(|X'|^2) \prod_{i=1}^{m} \theta(|X_i|^2)$$

and $|X'|^2 := \sum_{i=l_m+1}^{n+m} |x_i|^2$.

Now, given the expression of the functions f_i, we infer that the integral (95) is bounded by the following expression

$$\int_U \theta(x)\Lambda(x)\omega_l \wedge \bigwedge_{i=1}^{m} dd^c \log|t_i - f_i(x)|^2 \tag{96}$$

up to a constant independent of t, where we recall that the function Λ was defined in (94). Indeed, the equality

$$\bigwedge_{i=1}^{m} dd^c \log|t_i - f_i(x)|^2 \wedge \omega^n = \omega_l \wedge \bigwedge_{i=1}^{m} dd^c \log|t_i - f_i(x)|^2$$

holds modulo a constant, because f_i only depends on the variables $x_{l_{i-1}+1}, \dots, x_{l_i}$.

In order to evaluate (96) we use integration by parts: this expression is the same as

$$\int_U \log|t_1 - f_1(x)|^2 dd^c\big(\theta\Lambda\big) \wedge \omega_l \wedge \bigwedge_{i=2}^{m} dd^c \log|t_i - f_i(x)|^2 \tag{97}$$

and we will consider first the term containing $\theta(x)dd^c\Lambda$.

Since $dx^\alpha \wedge d\overline{x}^\beta \wedge \omega_l \wedge \bigwedge_{i=2}^m dd^c \log|t_i - f_i(x)|^2 = 0$ if $\max(\alpha, \beta) \geq l_1 + 1$, we see that the integral

$$dd^c\Lambda \wedge \omega_l \wedge \bigwedge_{i=2}^m dd^c \log|t_i - f_i(x)|^2$$

is equal to

$$\prod_{k=2}^m \frac{1}{|X_k|^{2d_k-2}} \prod_{j\in J} \frac{1}{|x_j|^{2-2\beta_j}} dd^c \frac{1}{|X_1|^{2d_1-2}} \wedge \omega_l \wedge \bigwedge_{i=2}^m dd^c \log|t_i - f_i(x)|^2. \tag{98}$$

We recall that we have the equality

$$dd^c \frac{1}{|X_1|^{2d_1-2}} \wedge dd^c|X_1|^2 = \delta_0,$$

the Dirac distribution at the origin in $\mathbb{C}^{l_1}$, so that we have

$$\int_U \log|t_1 - f_1(x)|^2 \theta dd^c\big(\Lambda\big) \wedge \omega_l \wedge \bigwedge_{i=2}^m dd^c \log|t_i - f_i(x)|^2 = \tag{99}$$

$$\log\frac{1}{|t_1|^2} \int_{U'} \theta_1 \Lambda_1 \bigwedge_{i=2}^m dd^c \log|t_i - f_i(x)|^2 \wedge \omega_{l'}$$

where $U' \subset \mathbb{C}^{n+m-l_1}$ is the unit ball, $x' = (x_{l_1+1}, \dots, x_{n+m})$ and

$$\theta_1(x') := \theta(|X'|^2) \prod_{i=2}^m \theta(|X_i|^2), \quad l' = (l_2, \dots, l_m), \qquad \Lambda_1 := \prod_{k=2}^m \frac{1}{|X_k|^{2d_k-2}} \prod_{j\in J} \frac{1}{|x_j|^{2-2\beta_j}}.$$

A quick argument by induction gives the expected estimate for the RHS of (99). The remaining terms involve the differential of $X_1 \to \theta(|X_1|^2)$, on the support of which $\dfrac{1}{|X_1|^{2d_1-2}}$ is smooth.

In conclusion, after the first integration by parts we get

$$\int_U \theta_1(X_1)\psi_1(X_1) \log|t_1 - f_1(x)|^2 \theta_1(x)\Lambda_1(X) \bigwedge_{i=1}^{l_1} dd^c|X_1|^2 \wedge \omega_{l'} \wedge \bigwedge_{j=2}^m dd^c \log|t_j - f_j(x)|^2 \tag{100}$$

modulo the terms (99).

We now repeat this procedure, integrating by parts using the factor $dd^c \log|t_2 - f_2(x)|^2$. Notice that, because of the form $\bigwedge_{i=1}^{l_1} dd^c|X_1|^2$ the derivatives of the function

$$\theta_1(X_1)\psi_1(X_1) \log|t_1 - f_1(x)|^2$$

don't come into play.

Thus, after a finite number of steps the last term we still have to deal with equals

$$\int_U \theta(x)\psi(x)\prod_{i=1}^{m}\log|t_i - f_i(x)|^2 \frac{d\lambda}{\prod_{j\in J}|x_j|^{2-2\beta_j}} \tag{101}$$

where ψ is a truncation function. Since $\beta_j > 0$, the integral (101) is uniformly bounded as soon as we fix a bound for $|t|$. Collecting all the terms, Theorem 7.3 is proved.

Remark 9.1. *We consider the function*

$$f : \mathbb{C}^3 \to \mathbb{C}^2, \qquad f(z) = (z_1 z_2, z_1 z_3).$$

Let θ be a truncation function which equals 1 near the origin of $\mathbb{C}^3$. A simple calculation shows that we have

$$\int_{\mathbb{C}^3} \theta e^{-\psi} dd^c \log|t_1 - f_1(x)|^2 \wedge dd^c \log|t_2 - f_2(x)|^2 \wedge \omega \simeq \frac{1}{|t_1|^2 + |t_2|^2}$$

therefore the hypothesis $(\star)$ *is crucial.*

Remark 9.2. *Let $f : S \to W$ be a morphism such that all the hypothesis of Theorem 5.2 except perhaps for* $(\star)$ *are satisfied. Then the techniques developed in our article show that the current T in* (76) *can only have positive Lelong numbers along an analytic subset of X which projects in codimension two. The reason is that one can construct a subset $W_0 \subset W$ whose codimension is at least two, and such that the morphism f satisfies* $(\star)$ *in the complement of W_0. It follows that if* $\dim W = 2$, *Theorem 5.2 holds true for morphisms which only satisfy the assumptions* (a), (b), (c), *thanks to the following general fact.*

Theorem 9.3. [Dem92] *Let X be a compact complex manifold, and let T be a* $(1,1)$–*closed positive current on X. If the level sets*

$$E_c(T) := \{x \in X : \nu(T, x) \geq c\}$$

have dimension zero for any $c > 0$, then the cohomology class of T is nef.

References

[AT19] D. Abramovich, M. Temkin, *Functorial factorization of birational maps for qe schemes in characteristic 0*, Algebra Number Theory 13(2): 379–424 (2019)

[Amb98] F. Ambro, *The locus of log canonical singularities.* 1998. arXiv:9806067.

[Amb99] F. Ambro. *The Adjunction Conjecture and its applications.* arXiv 9903060, 1999.

[ALT19] K. Adiprasito, G. Liu and M. Temkin, *Semistable reduction in characteristic 0*, Séminaire Lotharingien de Combinatoire 82B (2019).

[AK00] D. Abramovich and K. Karu, *Weak semistable reduction in characteristic 0*, Invent. Math. 139 (2000), no. 2, 241–273.

[BdSB23] André Belotto da Silva, Edward Bierstone, *Monomialization of a quasianalytic morphism*, to appear in the Annales scientifiques de l'École normale supérieure (2023). See arXiv version 1907.09502 [math.AG], 2019.

[BP08] B. Berndtsson, M. Păun, *Bergman kernels and the pseudoeffectivity of relative canonical bundles*, Duke Math. J. 145(2): 341–378.

[BP10] B. Berndtsson, M. Păun, *Quantitative extensions of pluricanonical forms and closed positive currents*, Nagoya Math. J. 205 (2012), 25–65.

[BCHM10] C. Birkar, P. Cascini, C. Hacon, Christopher, J. M^{c}Kernan, *Existence of minimal models for varieties of log general type.* J. Amer. Math. Soc. 23 (2010), no. 2, 405–468.

[Birkar12] C. Birkar, *Existence of log canonical flips and a special LMMP*, Pub. Math. IHES., 115 (2012), 325–368

[Birkar21] C. Birkar, *Generalised pairs in birational geometry.* EMS Surv. Math. Sci. 8 (2021), no. 1–2, 5–24.

[BZ16] C. Birkar, D.-Q. Zhang, *Effectivity of Iitaka fibrations and pluricanonical systems of polarized pairs*, Publications mathématiques de l'IHES, Volume 123, pages 283–331, (2016)

[Bou04] S. Boucksom, *Divisorial Zariski decompositions on compact complex manifolds,* Ann. Sci. Ecole Norm. Sup. (4) 37 (2004) No. 1, 45–76.

[BDPP13] S. Boucksom, J.-P. Demailly, M. Păun, and T. Peternell. *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension.* Journal of Algebraic Geometry, 22:201–248, 2013

[BEGZ10] S. Boucksom, P. Eyssidieux, V. Guedj, A. Zeriahi, *Monge-Ampère equations in big cohomology classes,* Acta Math. 205 (2010), no. 2, 199–262.

[CH20] J. Cao, A. Höring, *J. Differential Geom.*, 114, (2020), No. 1, 1–39.

[CT15] T. C. Collins and V. Tosatti, *Kähler currents and null loci*, Invent. Math. 202(3), 1167–1198 (2015).

[DeBook] J.-P. Demailly, *Complex differential and analytic geometry*, on author's web site.

[Dem82] J.-P. Demailly, *Sur les nombres de Lelong associés à l'image directe d'un courant positif fermé*, Ann. Inst. Fourier, 32 (1982), 37–66.

[Dem85] J.-P. Demailly, *Mesures de Monge-Ampère et caractérisation géométrique des variétés algébriques affines*, Mém. Soc. Math. France (N.S.) (19), 124 (1985), 1–126.

[Dem90] J.-P. Demailly, *Cohomology of q-convex spaces in top degrees*, Math. Z. 204(2), 283–295 (1990).

[Dem92] J.-P. Demailly, *Regularization of closed positive currents and Intersection Theory*, Jour. Alg. Geom, (1992), 361–409.

[DemOT] J.-P. Demailly, *On the Ohsawa-Takegoshi-Manivel extension theorem,* Proceedings of the Conference in honour of the 85th birthday of Pierre Lelong, Paris, September 1997, Progress in Mathematics, Birkaüser, Vol. 188 2000, 47-82.

[Dem09] J.-P. Demailly, *Analytic methods in algebraic geometry*, D17 on the author's web page.

[DH20] O. Das and C. Hacon, *The log minimal model program for Kähler 3-folds* , arXiv e-prints, https://arxiv.org/abs/2009.05924.pdf,

[DH23] O. Das and C. Hacon, *On the Minimal Model Program for Kähler 3-folds*, arXiv e-prints, https://arxiv.org/abs/2306.11708.pdf,

[DHP22] O. Das, C. Hacon and M. Păun, *On the 4-dimensional minimal model program for Kähler varieties*, arXiv e-prints, https://arxiv.org/pdf/2009.05924.pdf,

[DHP10] J.-P. Demailly, C. Hacon, M. Păun, *Extension theorems, Non-vanishing and the existence of good minimal models*, 2010, arXiv:1012.0493v2

[DHY23] O. Das, C. Hacon and J. Yanez, *Generalized Kähler 3-fold mmp*

[Fil20] S. Filipazzi, *On a generalized canonical bundle formula and generalized adjunction*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5).

[Fav99] *Note on pull-back and Lelong number of currents,* Bulletin de la Société Mathématique de France (1999).

[Fuj22] O. Fujino, *Minimal model program for projective morphisms between complex analytic spaces*, (2022).

[FG14] O. Fujino, Y. Gongyo, *On the moduli b-divisors of lc-trivial fibrations.* Ann. Inst. Fourier (Grenoble) 64 (2014), no.4, 1721–1735.

[GR] H. Grauert, R. Remmert, *Coherent analytic sheaves,* Springer-Verlag, 1984.

[Gue20] H. Guenancia, *Families of conic Kahler-Einstein metrics,* Math. Ann. 376 (2020), no. 1-2, 1–37.

[Hacon14] C. D. Hacon, *On the Log Canonical Inversion of Adjunction.* In: Proceedings of the Edinburgh Mathematical Society 57.1 (2014), pp. 139–143

[HX13] C. D. Hacon, C. Xu, *Existence of log canonical closures.* Invent. Math. 192 (2013), no. 1, 161–195.

[Hir75] H. Hironaka, *Flattening Theorem in Complex-Analytic Geometry*, American Journal of Mathematics, 1975, Vol. 97, No. 2, pp. 503–547

[HP16] A. Horing and T. Peternell, *Minimal models for Kahler threefolds*, Invent. Math. 203(1), 217–264 (2016).

[Karu99] K. Karu, *Semistable reduction in characteristic zero.* PhD thesis. Boston University, 1999.

[Kawakita07] M. Kawakita, *Inversion of adjunction on log canonicity.* Invent. Math. 167 (2007), no. 1, 129–133.

[Kawamata98] Y. Kawamata, *Subadjunction of log canonical divisors. II.* Amer. J. Math. 120 (1998), no. 5, 893–899.

[Kawamata99] Y. Kawamata, *Deformations of canonical singularities*, J. Amer. Math. Soc. 12 (1999), 85–92.

[Kollar86] J. Kollár, *Higher direct images of dualizing sheaves. I* Annals of Math., Volume 123 (1986), Issue 1, 11–42

[Kollar07] J. Kollár, *Kodaira's canonical bundle formula and adjunction.* Flips for 3-folds and 4-folds, 134–162, Oxford Lecture Ser. Math. Appl., 35, Oxford Univ. Press, Oxford, 2007

[Kollar22] J. Kollár, *Families of varieties of general type*, https://web.math.princeton.edu/ kollar/FromMyHomeI final.pdf

[KK10] J. Kollár and S. J. Kovács, *Log canonical singularities are Du Bois.* Journal of the American Mathematical Society 23.3 (2010), pp. 791–813

[Lai11] C. J. Lai, *Varieties fibered by good minimal models.* Math. Ann. 350, 533–547 (2011).

[Nak87] N. Nakayama, *The lower semicontinuity of the plurigenera of complex varieties*, in Algebraic geometry, Sendai, 1985, volume 10 of Adv. Stud. Pure Math.,36 pages 551–590, North-Holland, Amsterdam, 1987.

[P24] C.-M. Pan, *Demailly-Lelong numbers on complex spaces*, arXiv:2403.08620.

[MP12] M. Păun, *Relative adjoint transcendental classes and Albanese maps of compact Kähler manifolds with nef Ricci curvature*,, arXiv:1209.2195.

[PT18] M. Păun, S. Takayama, *Positivity of twisted relative pluricanonical bundles and their direct images* J. Alg. Geom. 27 (2018).

[W-N19] D. Witt-Nyström. *Duality between the pseudoeffective and the movable cone on a projective manifold.* J. Am. Math. Soc., 32(3): 675–689, 2019.

[PS14] M. Popa, C. Schnell, *On direct images of pluricanonical bundles.* Algebra Number Theory 8 (2014), no. 9, 2273–2295.

[Taka22] S. Takayama, *Singularities of the L^2-metric in the canonical bundle formula*, American Journal of Mathematics, Volume 144, Number 6, December 2022, pp. 1725–1743

Department of Mathematics, University of Utah, 155 S 1400 E, Salt Lake City, Utah 84112, USA
Email address: `hacon@math.utah.edu`

Institut für Mathematik, Universität Bayreuth, 95440 Bayreuth, Germany
Email address: `mihai.paun@uni-bayreuth.de`