

# ON THE MINIMAL MODEL PROGRAM FOR KÄHLER 3-FOLDS

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ABSTRACT. In this article we prove the existence of pl-flipping and divisorial contractions and pl flips in dimension  $n$  for compact Kähler varieties, assuming results of the minimal model program in dimension  $n - 1$ . We also give a self contained proof of the cone theorem, the existence of flipping and divisorial contractions, of flips and minimal models in dimension 3.

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## 1. INTRODUCTION

In recent years there has been substantial progress towards the minimal model program for Kähler varieties. The 3-fold minimal model program is established in increasing generality in [CP97], [Pet98], [Pet01], [HP15], [HP16], [CHP16], [DO24], [DH20], and [DHY23]. Further progress was then made in

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[Fuj22b], [DHP24], and [LM22], where the minimal model program for projective morphisms between analytic varieties is established in arbitrary dimensions. Note that flipping contractions are, by definition, projective morphisms and hence the above references imply the existence of flips in all dimensions. Unluckily, we do not have a Kähler analog of the base point free and cone theorems and hence this is the main difficulty in proving the minimal model program in higher dimensions.

In dimension 3, the existence of flipping and divisorial contractions for terminal pairs was claimed in [HP16]. Flipping contractions and divisorial contractions to a point for klt 3-folds follow from [CHP16] and klt divisorial contractions to a curve follow from [DH20]. Unluckily, the proof of the existence of divisorial contractions to a curve claimed in [HP16] seems to be incomplete (and relies on [AT84] which also seems to contain an error). To make matters more complicated, the arguments of [CHP16] heavily rely on those of [HP16], and [DH20] also indirectly relies on [HP16] and [CHP16].

The purpose of this paper is to give a brief, unified, and mostly self contained treatment of the minimal model program for klt Kähler 3-folds. In particular, we give new proofs of the cone theorem, and the existence of flipping contractions and divisorial contractions for Kähler klt 3-folds. In the process, we also establish an inductive framework that we expect to be useful in proving the existence of flipping contractions and divisorial contractions for higher dimensional Kähler klt pairs. The strategy of this paper is primarily inspired by the approaches of [BCHM10], [DH20], and [DHP24].

We will now describe the main results contained in this paper. We begin by proving the existence of pl-divisorial contractions and pl-flipping contractions and the corresponding flips in dimension  $n$  assuming a base point free result in dimension  $n - 1$ .

**Theorem 1.1.** *Let  $(X, S + B)$  be a  $\mathbb{Q}$ -factorial compact Kähler plt pair of arbitrary dimension, and  $\pi : S \rightarrow T$  a contraction such that  $-(K_X + S + B)|_S$  and  $-S|_S$  are  $\pi$ -ample. Then there exists a projective bimeromorphic morphism  $p : X \rightarrow Z$  with connected fibers such that  $p|_S = \pi$  and  $p|_{X \setminus S}$  is an isomorphism. If  $p$  is of flipping type, then the flip  $p^+ : X^+ \rightarrow Z$  exists.*

In fact we prove a more general version of this results that applies to generalized plt pairs, see Theorems 5.8 and 5.12. Note that the existence of flips for plt pairs was shown in [DHP24] and [Fuj22b]. In order to apply the above result, we must guarantee the existence of the morphism  $\pi : S \rightarrow T$ . If there exists a Kähler form  $\omega$  such that  $\alpha = [K_X + S + B + \omega]$  is nef and  $R = \alpha^\perp \cap \overline{\text{NA}}(X)$  is an extremal ray, then we must show that  $R$  can be contracted in  $S$ . To this end, we consider the pair  $K_S + B_S + \omega_S = (K_S + S + B + \omega)|_S$  induced by adjunction, and we aim to prove that there is a morphism  $\pi : S \rightarrow T$  such

that  $K_S + B_S + \omega_S \equiv \pi^* \omega_T$  where  $\omega_T$  is a Kähler form on  $T$ . In dimension  $\dim S \leq 3$ ,  $\pi$  exists by [DHY23] and therefore Theorem 1.1 holds in dimension  $\dim X \leq 4$ . (We remark that the results of [DHY23] only rely on the 3-fold MMP established in this paper.) We also note that if such a morphism  $\pi$  exists, then  $-(K_S + B_S) \equiv_T \omega_S$  is ample over  $T$  and hence this morphism is projective and so, a posteriori, the existence of  $\pi$  is implied by the usual base point free theorem.

We now turn our attention to the full minimal model program for dlt Kähler 3-folds. We first show the existence of flipping and divisorial contractions (see Theorems 6.2 and 6.9).

**Theorem 1.2.** *Let  $(X, B)$  be a  $\mathbb{Q}$ -factorial compact Kähler 3-fold dlt pair such that  $K_X + B$  is pseudo-effective. Let  $\omega$  be a Kähler class such that  $\alpha = K_X + B + \omega$  is a nef and big class and  $\alpha^\perp \cap \overline{\text{NA}}(X) = R$  is an extremal ray. Then there exists a projective bimeromorphic morphism  $f : X \rightarrow Z$  with connected fibers such that  $\alpha = f^* \alpha_Z$ , where  $\alpha_Z$  is a Kähler class on  $Z$ . Moreover, if  $f$  is a divisorial contraction, then  $(Z, f_* B)$  has  $\mathbb{Q}$ -factorial klt singularities.*

Using only a special case of the above theorem (namely, assuming  $X$  is *strongly*  $\mathbb{Q}$ -factorial), we give a quick proof of the cone theorem when  $K_X + B$  is pseudo-effective.

**Theorem 1.3.** *Let  $(X, B)$  be a compact Kähler 3-fold dlt pair such that  $K_X + B$  is pseudo-effective. Then there are at most countably many rational curves  $\{\Gamma_i\}_{i \in I}$  in  $X$  such that  $-(K_X + B) \cdot \Gamma_i \leq 6$  for all  $i \in I$  and*

$$\overline{\text{NA}}(X) = \overline{\text{NA}}(X)_{(K_X + B) \geq 0} + \sum_{i \in I} \mathbb{R}^+[\Gamma_i].$$

Finally, using the cone and contraction theorems above we prove the existence of minimal models when  $K_X + B$  is pseudo-effective.

**Theorem 1.4.** *Let  $(X, B)$  be a  $\mathbb{Q}$ -factorial compact Kähler 3-fold dlt pair such that  $K_X + B$  is pseudo-effective. Then after finitely many  $(K_X + B)$ -flips and divisorial contractions*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \dots \dashrightarrow X_n$$

*we obtain that  $K_{X_n} + B_n$  is nef.*

The existence of Mori-fiber spaces also follows, but we do not pursue it here as it also relies on the results of [Bru06], [HP15], and [DH20].

*Remark 1.5.* We also note that Theorems 2.28, 2.30 and 2.31 in [DO24] rely on the contraction results of [HP16] and [CHP16], however these theorems are also implied by Theorem 1.2, and so the main results of [DO24] are unaffected.

This article is organized in the following manner. In Section 2 we recall some definitions and provide appropriate references, in Section 4 we prove some contraction theorems when the exceptional locus maps to finitely many points. Section 5 is the core of this article. In this section we prove Theorems 1.1 and 1.2. Finally, in Section 7 we prove the cone Theorem 1.3 and the existence of minimal models, Theorem 1.4.

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## 2. PRELIMINARIES

We will follow the usual conventions of the minimal model program. In particular, we refer the reader to [KM98, Chapter 2] for the definition of pairs and their singularities (klt, lc, plt etc.), [DHY23, Subsection 2.1] for the definitions of generalized pairs and related singularities (glc, gklt, gdlt, etc.), [Fuj22b, Chapters 2, 3, 11] for many key concepts of the minimal model program for analytic varieties, to [DHP24, Definition 2.8] for the definitions of nef, minimal and log terminal models, [HP16] for a discussion of Bott-Chern cohomology  $H_{BC}^{1,1}(X)$ , the Kähler and nef cones  $\mathcal{K} \subset \bar{\mathcal{K}}$ , the Mori cone and the cone of positive closed currents  $\overline{\text{NE}}(X) \subset \overline{\text{NA}}(X) \subset N_1(X)$ . Analytic pl-flipping contractions and pl-flips are defined in [Fuj22b, Chapter 15].

**Definition 2.1.** A compact analytic variety  $X$  is said to be in Fujiki's class  $\mathcal{C}$  if  $X$  is bimeromorphic to a compact Kähler manifold.

**Definition 2.2.** Let  $X$  be a normal compact analytic variety. A closed positive  $(1, 1)$  current  $T$  on  $X$  with local potentials is called a Kähler current if  $T \geq \omega$  for some smooth Hermitian form  $\omega$  on  $X$ . A  $(1, 1)$  class  $\alpha \in H_{BC}^{1,1}(X)$  is called big if it contains a Kähler current. A class  $\beta \in H_{BC}^{1,1}(X)$  is called a *modified Kähler class* if there is a Kähler current  $T$  such that the Lelong numbers satisfy  $\nu(T, P) = 0$  for all prime Weil divisors  $P \subset X$ .

*Remark 2.3.* Note that if  $X$  is a compact complex manifold in Fujiki's class  $\mathcal{C}$ , then clearly  $X$  carries a big  $(1, 1)$  class  $\alpha \in H_{BC}^{1,1}(X)$ . However, if  $X$  is singular, then it is not clear whether  $X$  carries any big  $(1, 1)$  class or not, the issue here is that the pushforward of a smooth Kähler form from a resolution of  $X$  is a positive current on  $X$  which may not have local potentials.

**Definition 2.4.** Let  $X$  be a normal compact analytic variety and  $\alpha \in H_{BC}^{1,1}(X)$  a pseudo-effective class. Then the *negative part*  $N(\alpha)$  of the Boucksom-Zariski decomposition of  $\alpha$  is an effective  $\mathbb{R}$ -divisor (see [DHY23, Definition A.6] and [Bou04, Definition 3.7]). We note that in [DHY23, Definition A.6] it is assumed

that  $X$  is Kähler, however, it was never used. If  $X$  is  $\mathbb{Q}$ -factorial,  $N(\alpha)$  is  $\mathbb{R}$ -Cartier, and so  $\alpha - [N(\alpha)]$  is a  $(1, 1)$  class in  $H_{\text{BC}}^{1,1}(X)$ . In this case we define  $P(\alpha) := \alpha - [N(\alpha)]$  and call it the *positive part* of  $\alpha$ . Lemma 2.6 below shows that  $P(\alpha)$  is a modified nef class (see [Bou04, Definition 2.2(ii)]) when  $X$  is a Kähler variety. We call  $\alpha = [N(\alpha)] + P(\alpha)$ , the *Boucksom-Zariski decomposition* of  $\alpha$ .

**Lemma 2.5.** *Let  $X$  be a normal  $\mathbb{Q}$ -factorial compact analytic variety in Fujiki's class  $\mathcal{C}$ , and  $\alpha \in H_{\text{BC}}^{1,1}(X)$  a big  $(1, 1)$  class. Then  $X$  contains a modified Kähler class.*

*Proof.* Let  $f : Y \rightarrow X$  be a resolution of singularities of  $X$  such that  $f$  is an isomorphism over  $X_{\text{sm}}$ . By Demailly's regularization theorem, there is a Kähler current  $T$  in  $f^*\alpha \in H_{\text{BC}}^{1,1}(Y)$  with analytic singularities such that  $T \geq \omega_Y$  for some smooth Hermitian form  $\omega_Y$  on  $Y$ . Let  $T = D + R$  be the Siu decomposition of  $T$ , where  $D$  is an effective  $\mathbb{R}$ -divisor and  $R$  is the residue current. Then  $R \geq \omega_Y$ . Let  $\omega$  be a smooth Hermitian form on  $X$ , then there is a  $C > 0$  such that  $\omega_Y \geq Cf^*\omega$ . Note that by [BG13, Proposition 4.6.3(i)],  $f_*T$  has local potentials on  $X$  and  $[f_*T] = \alpha$ . Now since  $X$  is  $\mathbb{Q}$ -factorial,  $f_*D$  is  $\mathbb{R}$ -Cartier, and hence  $f_*R$  has local potentials on  $X$ . However,  $f_*R \geq C\omega$ , and the generic Lelong numbers  $\nu(f_*R, P) = 0$  for all prime Weil divisors  $P$  on  $X$ . Therefore  $[f_*R] \in H_{\text{BC}}^{1,1}(X)$  is a modified Kähler class.  $\square$

**Lemma 2.6.** *Let  $X$  be a normal  $\mathbb{Q}$ -factorial compact analytic variety in Fujiki's class  $\mathcal{C}$ , such that  $H_{\text{BC}}^{1,1}(X)$  contains a big  $(1, 1)$  class. If  $\alpha \in H_{\text{BC}}^{1,1}(X)$  is a pseudo-effective class, then the positive part  $P(\alpha)$  of the Boucksom-Zariski decomposition of  $\alpha$  defined above is a modified nef class.*

*Proof.* Let  $f : Y \rightarrow X$  be a resolution of singularities of  $X$  such that  $Y$  is a Kähler manifold and  $f^*\alpha = N(f^*\alpha) + P(f^*\alpha)$  the Boucksom-Zariski decomposition of  $f^*\alpha$  as in [Bou04, Definition 3.7]. Then from [DHY23, Definition A.6] it follows that  $P(\alpha) = f_*(P(f^*\alpha))$ . Note that  $P(f^*\alpha)$  is modified nef by [Bou04, Proposition 3.8(i)]. Let  $\beta$  be a modified Kähler class on  $X$ , its existence is guaranteed by Lemma 2.5. It suffices to show that  $P(\alpha) + \beta$  is modified Kähler, as the modified nef cone is the closure of the modified Kähler cone (see [Bou04, §2.7]). Let  $T$  be a Kähler current in  $\beta$  with generic Lelong numbers  $\nu(T, P) = 0$  for all prime Weil divisors  $P$  on  $X$ . Let  $f^*T = D + R$  be the Siu decomposition, where  $D$  is the divisorial part and  $R$  is the residue current. Then  $D$  is a  $\mathbb{R}$ -Cartier divisor such that  $\text{Supp}(D) \subset \text{Ex}(f)$  and  $R$  is a Kähler current. In particular,  $[R] \in H_{\text{BC}}^{1,1}(Y)$  is a modified Kähler class. Then by [DH20, Lemma 2.35] applied to the class  $[R]$  and replacing  $Y$  by a higher resolution we may assume that there is a Kähler class  $\omega_Y$  and an effective  $f$ -exceptional  $\mathbb{R}$ -divisor  $E$  on  $Y$  such that  $f^*\beta = \omega_Y + E$ . Since the modified nef

cone is the closure of the modified Kähler cone, it follows that  $P(f^*\alpha) + \omega_Y$  is modified Kähler on  $Y$ . Let  $\Theta$  be a Kähler current in  $P(f^*\alpha) + \omega_Y$  such that the generic Lelong numbers  $\nu(\Theta, Q) = 0$  for all prime Weil divisors  $Q$  on  $Y$ . Note that  $P(f^*\alpha) = f^*P(\alpha) + F$  for some  $f$ -exceptional  $\mathbb{R}$ -divisor  $F$  (but not necessarily effective). Then the current  $\Theta + E + F$  is in the class  $f^*(P(\alpha) + \beta) + [F]$ . Let  $U := X \setminus f(\text{Ex}(f))$ ; then  $(f_*\Theta)|_U$  is a closed positive  $(1, 1)$  current with local potentials representing the class  $(P(\alpha) + \beta)|_U$ . Since  $\text{codim}_X f(\text{Ex}(f)) \geq 2$ , by the same argument as in the proof of [BG13, Proposition 4.6.3(i)] it follows that  $f_*\Theta|_U$  extends to a unique closed positive  $(1, 1)$  current  $f_*\Theta$  with local potentials representing the class  $P(\alpha) + \beta$ . From the definition of  $\Theta$  it follows that  $\nu(f_*\Theta, P) = 0$  for all prime Weil divisors  $P$  on  $X$ . Thus  $P(\alpha) + \beta$  is a Kähler class.  $\square$

### 3. SINGULARITIES OF CURRENTS ON COMPLEX SPACES

**Definition 3.1.** Let  $X$  be a normal analytic variety and  $T$  a closed positive  $(1, 1)$  current on  $X$  with local potentials. Then for each  $c > 0$  we define

$$E_c := \{x \in X : \nu(T, x) > c\},$$

where  $\nu(T, x)$  denote the Lelong number of  $T$  at  $x \in X$ .

This is an analytic subset of  $X$  by [Siu74], also see [HP24, Lemma 4.16]. We define

$$E_+(T) = \bigcup_{c \in \mathbb{R}^+} E_c.$$

If additionally  $X$  is compact and  $\alpha \in H_{\text{BC}}^{1,1}(X)$  is a big class, then we define the *non-Kähler* locus of  $\alpha$  as follows:

$$E_{nK}(\alpha) := \bigcap_{T \in \alpha} E_+(T),$$

where  $T$  runs over all Kähler currents contained in the class  $\alpha$ .

We will also need to consider the currents with *admissible singularities* and *weakly analytic singularities* as in [HP24].

**Definition 3.2.** [HP24, Definition 4.10] Let  $X$  be a normal analytic variety and  $\varphi : X \rightarrow [-\infty, \infty)$  a function on  $X$ . We say that  $\varphi$  has *admissible singularities* if

$$\varphi := \max\{\varphi_1, \varphi_2, \dots, \varphi_k\},$$

where each  $\varphi_i : X \rightarrow [-\infty, \infty)$  is a function with weakly analytic singularities in the sense [Dem92], i.e. locally

$$\varphi_i = \lambda_i \log \left( \sum_{j=1}^{n_j} |f_j|^2 \right) + g_i,$$

where  $\lambda_i \geq 0$ ,  $f_j$  are holomorphic functions and  $g_i$  are bounded functions.

In this case we say that the positive current  $T = \alpha + i\partial\bar{\partial}\varphi$ , where  $\alpha$  is a real closed smooth  $(1, 1)$  form, has *admissible singularities*.

**Definition 3.3.** [HP24, Definition 4.11] Let  $X$  be a normal analytic variety, and  $T = \alpha + i\partial\bar{\partial}\varphi$  a closed positive  $(1, 1)$  current on  $X$ , where  $\alpha$  is a smooth closed positive  $(1, 1)$  form on  $X$  with local potentials. We say that  $T$  has *weakly analytic singularities* if the following conditions are satisfied:

- (1) there is a proper bimeromorphic morphism  $\pi : \widehat{X} \rightarrow X$  from a normal analytic variety  $\widehat{X}$ , and
- (2) a closed positive  $(1, 1)$  current  $\widehat{T}$  on  $\widehat{X}$  such that

$$\widehat{T} = \pi^*\alpha + i\partial\bar{\partial}\psi \geq 0,$$

where  $\psi$  has admissible singularities and  $\pi_*\widehat{T} = T$ .

From [HP24, Remark 4.12] it follows that  $\widehat{T} = \pi^*T$ . Also, note that if  $T$  has weakly analytic singularities, then  $E_+(T)$  is a closed analytic subset of  $X$  by [HP24, Lemma 4.16].

Next we define the *restricted* non-Kähler locus of a current as in [HP24].

**Definition 3.4.** [HP24, Definition 4.18] Let  $X$  be a normal compact Kähler analytic variety and  $\alpha \in H_{BC}^{1,1}(X)$  a nef and big class. The restricted non-Kähler locus of  $\alpha$  is the following set

$$E_{nK}^{as}(\alpha) := \bigcap_{T \in \alpha} E_+(T),$$

where  $T$  is a Kähler current with weakly analytic singularities.

Next we recall two very important results from [HP24].

**Lemma 3.5.** [HP24, Corollary 4.20] Let  $X$  be a normal compact Kähler variety and  $\alpha \in H_{BC}^{1,1}(X)$  a big class. There is a Kähler current  $T$  on  $X$  with weakly analytic singularities such that  $E_{nK}^{as}(\alpha) = E_+(T)$ ; in particular,  $E_{nK}^{as}(\alpha)$  is a closed analytic subset of  $X$ .

**Theorem 3.6.** [HP24, Theorem 4.21] Let  $X$  be a normal compact Kähler variety and  $\alpha \in H_{BC}^{1,1}(X)$  a nef and big class. Then  $E_{nK}^{as}(\alpha) = \text{Null}(\alpha)$ , in particular,  $\text{Null}(\alpha)$  is an analytic set.

We will also need the following lemmas.

**Lemma 3.7.** Let  $X$  be normal compact Kähler variety,  $\alpha$  a real smooth closed  $(1, 1)$  form, and  $T = \alpha + i\partial\bar{\partial}\varphi \geq \omega$  a closed positive  $(1, 1)$  current with weakly analytic singularities, where  $\omega$  is a fixed Kähler form on  $X$ . Let  $Z := E_+(T)$ . Then there exists a resolution  $f : Y \rightarrow X$  such that

$$f^*T = F + \Theta \geq f^*\omega$$

where  $F$  is an effective  $\mathbb{R}$ -divisor with  $f(\text{Supp } F) = Z$ , and  $\Theta \geq f^*\omega$  is a closed positive  $(1, 1)$  current such that the classes  $[\Theta]$  and  $[\Theta - f^*\omega] \in H_{\text{BC}}^{1,1}(Y)$  are both nef.

*Proof.* Since  $T$  has weakly analytic singularities, we may assume that there is a resolution  $f : Y \rightarrow X$  such that the Siu decomposition is  $f^*T = F + R_0$ , where  $F$  is an effective  $\mathbb{R}$ -divisor and  $R_0$  is the residue current whose local potentials are bounded functions. Since  $f^*T \geq f^*\omega$ , then  $R_0 \geq f^*\omega$  and so the local potentials of  $R := R_0 - f^*\omega$  are also bounded functions. In particular, the Lelong numbers satisfy  $\nu(R, y) = 0$  for all  $y \in Y$ , and we can write  $R = \beta + i\partial\bar{\partial}\rho \geq 0$  (see [DP04, §3]), where  $\beta$  is a real smooth closed  $(1, 1)$  form and  $\rho : Y \rightarrow \mathbb{R}$  is a bounded quasi-psh function. Therefore we have  $f^*T = F + (\beta + f^*\omega) + i\partial\bar{\partial}\rho = F + \Theta \geq f^*\omega$ , where  $\Theta := \beta + f^*\omega + i\partial\bar{\partial}\rho = R + f^*\omega \geq f^*\omega$ . Since  $E_+(\Theta - f^*\omega) = E_+(R) = \emptyset$  by our construction, by [Pau98, Theorem 2, Page 418] the class  $[\Theta - f^*\omega] \in H_{\text{BC}}^{1,1}(Y)$  is nef, and so  $[\Theta] = [\Theta - f^*\omega] + [f^*\omega]$  is also nef.  $\square$

**Lemma 3.8.** *Let  $f : Y \rightarrow X$  be a proper bimeromorphic morphism from a compact complex manifold  $Y$  in Fujiki's class  $\mathcal{C}$  to a normal compact analytic variety  $X$  with rational singularities. Let  $\alpha$  be a  $(1, 1)$  big class on  $X$ . Then  $f^{-1}(E_{nK}(\alpha)) \cup \text{Ex}(f) \subset E_{nK}(f^*\alpha)$ .*

*Proof.* Supposing that  $y \notin E_{nK}(f^*\alpha)$ , we must show that  $y \notin f^{-1}(E_{nK}(\alpha)) \cup \text{Ex}(f)$ . Since  $Y$  is a compact complex manifold in Fujiki's class  $\mathcal{C}$ , by [Bou04, Theorem 3.27(ii)] (also see [CT15, Theorem 2.2]), there is a Kähler current  $T$  in the class  $f^*\alpha$  with analytic singularities such that  $E_{nK}(f^*\alpha) = E_+(T)$ . Thus  $y \notin E_{nK}(f^*\alpha)$  implies that  $T$  is a smooth form near  $y$ . Suppose that  $y \in \text{Ex}(f)$ . We can write  $T = f^*\theta + i\partial\bar{\partial}\varphi$ , where  $\theta$  is a real closed smooth  $(1, 1)$  form on  $X$  with local potentials representing the class  $\alpha$  and  $\varphi$  is a quasi-psh function on  $Y$ . Since  $T$  is a Kähler current, there is a smooth Hermitian form  $\omega_Y$  such that  $f^*\theta + i\partial\bar{\partial}\varphi \geq \omega_Y$ . Let  $E$  be an irreducible component of  $f^{-1}(f(y))$  containing  $y$ . Since  $T$  is smooth near  $y$ ,  $E$  is not contained in the pluripolar locus  $\{y' \in Y \mid \varphi(y') = -\infty\}$ , in particular,  $T|_E$  is well defined and we have

$$(3.1) \quad T|_E = (f^*\theta + i\partial\bar{\partial}\varphi)|_E = i\partial\bar{\partial}(\varphi|_E) \geq \omega_Y|_E.$$

Thus  $\varphi|_E$  is a strictly psh function. Since  $E$  is compact, from the maximal principle of psh functions it follows that  $\varphi|_E$  is a constant function, which is a contradiction to the above inequality. In particular,  $y \notin \text{Ex}(f)$ , and so  $f$  is an isomorphism near  $y \in Y$ . By [HP16, Lemma 3.4],  $S := f_*T$  is a closed positive  $(1, 1)$  current on  $X$  with local potentials such that  $[S] = \alpha$ ; clearly  $S$  is a Kähler current. Since  $f$  is an isomorphism near  $y \in Y$ , it follows that  $S$  is

a smooth form near  $x = f(y) \in X$ . In particular,  $x \notin E_{nK}(\alpha)$ , and the proof of the inclusion  $f^{-1}(E_{nK}(\alpha)) \cup \text{Ex}(f) \subset E_{nK}(f^*\alpha)$  is complete.  $\square$

*Remark 3.9.* With the same notations and hypothesis as in Lemma 3.8 if additionally we assume that  $X$  is Kähler and  $\alpha$  is nef (in addition to being big), then using [HP24, Corollary 4.20] it follows easily that  $E_{nK}^{\text{as}}(f^*\alpha) = f^{-1}(E_{nK}^{\text{as}}(\alpha)) \cup \text{Ex}(f)$ .

**Lemma 3.10.** *Let  $f : Y \rightarrow X$  be a proper bimeromorphic morphism between two normal compact analytic varieties and  $\alpha$  a nef and big  $(1, 1)$ -class on  $X$ . Then  $\text{Null}(f^*\alpha) = f^{-1}(\text{Null}(\alpha)) \cup \text{Ex}(f)$ .*

*Proof.* First we will show that  $\text{Null}(f^*\alpha) \subset f^{-1}(\text{Null}(\alpha)) \cup \text{Ex}(f)$ . To that end assume that  $y \notin f^{-1}(\text{Null}(\alpha)) \cup \text{Ex}(f)$ . Then  $f$  is an isomorphism near  $y \in Y$ . Assume by contradiction that  $y \in \text{Null}(f^*\alpha)$ , then there is a subvariety  $V \subset Y$  containing  $y$  such that  $(f^*\alpha)^{\dim V} \cdot V = 0$ . Then by the projection formula we have  $\alpha^{\dim V} \cdot f_* V = 0$ . However, since  $f$  is an isomorphism near  $y$ ,  $f|_V : V \rightarrow f(V)$  is bimeromorphic, and hence  $f_* V \neq 0$ . Therefore  $f(y) \in f(V) \subset \text{Null}(\alpha)$ , and so  $y \in f^{-1}(\text{Null}(\alpha))$ , which is a contradiction.

Now we will show the reverse inclusion. Let  $y \in \text{Ex}(f)$  and  $E$  an irreducible component of  $f^{-1}(f(y))$  containing  $y$ . Then clearly,  $(f^*\alpha)^{\dim E} \cdot E = 0$ , and thus  $y \in \text{Null}(f^*\alpha)$ . Now choose  $y \in f^{-1}(\text{Null}(\alpha)) \setminus \text{Ex}(f)$ ; then  $f$  is an isomorphism near  $y$ . Since  $f(y) \in \text{Null}(\alpha) \setminus f(\text{Ex}(f))$ , there is a subvariety  $f(y) \in W \subset X$  not contained in  $f(\text{Ex}(f))$  such that  $\alpha^{\dim W} \cdot W = 0$ . Let  $V \subset Y$  be the strict transform of  $W$  under  $f$ . Then by the projection formula we have  $(f^*\alpha)^{\dim V} \cdot V = 0$ ; since  $y \in V$ , it follows that  $y \in \text{Null}(f^*\alpha)$ . This completes the proof.  $\square$

### 3.1. Generalized pairs.

**Definition 3.11.** Let  $\pi : X \rightarrow S$  be a proper morphism of normal Kähler varieties such that  $S$  is relatively compact and  $\nu : X' \rightarrow X$  a resolution of singularities,  $B'$  an  $\mathbb{R}$ -divisor on  $X'$  with simple normal crossings support, and  $\beta'$  a real closed smooth  $(1, 1)$ -form on  $X'$ , such that

- (1)  $B := \nu_* B' \geq 0$ ,
- (2)  $[\beta'] \in H_{\text{BC}}^{1,1}(X')$  is nef over  $S$ , and
- (3)  $[K_{X'} + B' + \beta'] = \nu^* \gamma$ , where  $\gamma \in H_{\text{BC}}^{1,1}(X)$ .

Then we let  $\beta = \nu_* \beta'$  and we say that  $\nu : (X', B' + \beta') \rightarrow (X, B + \beta)$  is a generalized pair. We will often abuse notation and say that  $(X, B + \beta)$  (or  $(X, B + \beta/S)$ ) is a generalized pair (over  $S$ ) and  $\nu : (X', B' + \beta') \rightarrow (X, B + \beta)$  is a log resolution. We will often assume that  $X$  is compact and  $S$  is a point (or that  $X = S$ ) and we omit  $\pi : X \rightarrow S$ .

Note that given  $\beta'$ , we can define the corresponding nef b-(1,1) current  $\beta := \overline{\beta'}$  as follows. For any bimeromorphic morphism  $p : X'' \rightarrow X'$  we define  $\beta_{X''} = p^*\beta'$  and for any bimeromorphic morphism  $q : X'' \rightarrow X'''$  we let  $\beta_{X'''} = q_*\beta_{X''}$  (in general this is a closed (1,1)-current, not necessarily a (1,1)-form). Using the projection formula, one easily checks that  $q_*\beta_{X''}$  is well defined (i.e.  $\beta_{X'''}$  does not depend on the choice of the common resolution  $X''$  of  $X'$  and  $X''$ ) and that for any bimeromorphic morphism  $r : X_1 \rightarrow X_2$  of bimeromorphic models of  $X$ , we have  $r_*\beta_{X_1} = \beta_{X_2}$ . We say that  $\overline{\beta'}$  descends to  $X'$ . Note that for any bimeromorphic morphism  $p : X'' \rightarrow X'$ , we also have  $\overline{\beta'} = \overline{\beta''}$  where  $\beta'' = \beta_{X''}$ , and so  $\beta$  also descends to  $X''$ .

Similarly, if  $\nu : Y \rightarrow X'$  is a proper bimeromorphic morphism, then write  $K_Y + B_Y = \nu^*(K_{X'} + B')$ . For any proper bimeromorphic morphism  $\mu : Y \rightarrow Y'$  we let  $B_{Y'} = \mu_*B_Y$ . In this way we have defined a b-divisor  $\mathbf{B}$  (whose trace  $\mathbf{B}_Y$  on  $Y$  is  $B_Y$ ). Since the b-divisor  $\mathbf{K} + \mathbf{B} = \overline{K_{X'} + B_{X'}}$  and the b-(1,1)-form  $\beta = \overline{\beta_{X'}}$  descend to  $X'$ , we say that the generalized pair  $(X, B + \beta)$  descends to  $X'$ .

We will often denote the generalized pair  $\nu : (X', B' + \beta') \rightarrow (X, B + \beta)$  by  $(X, B + \beta)$  where  $\beta = \overline{\beta'}$ . Note that then  $\beta' = \beta_{X'}$  and  $B' = \nu^*(K_X + B + \beta) - (K_{X'} + \beta')$ .

We define the *generalized discrepancies*  $a(P; X, B + \beta) = -\text{mult}_P(\mathbf{B}_Y)$ , where  $P$  is a prime divisor on a bimeromorphic model  $Y$  of  $X$ . We say that  $(X, B + \beta)$  is *generalized klt* or generalized Kawamata log terminal (resp. *generalized lc* or generalized log canonical) if for any log resolution  $\nu : X' \rightarrow X$ , we have  $\lfloor \mathbf{B}_{X'} \rfloor \leq 0$ , i.e.  $a(P; X, B + \beta) > -1$  for all prime divisors  $P$  over  $X$  (resp.  $a(P; X, B + \beta) \geq -1$  for all prime divisors  $P$  over  $X$ ). This can be checked on a single given log resolution. We say that  $(X, B + \beta)$  is *generalized dlt* (divisorially log terminal) if there is an open subset  $U \subset X$  such that  $(U, (B + \beta)|_U)$  is a log resolution (of itself) and  $a(P; X, B + \beta) \geq -1$  for every prime divisor  $P$  over  $X$  such that the generic point of  $\text{center}_X(P)$  is contained in  $U$  and  $a(P; X, B + \beta) > -1$  for every prime divisor  $P$  over  $X$  with  $\text{center}_X(P) \subset X \setminus U$ .

We remark that if  $\alpha'$  is a real closed smooth (1,1)-form in the class  $[\beta']$  and  $\alpha = \overline{\alpha'}$ , then we have  $a(P; X, B + \beta) = a(P; X, B + \alpha)$ , i.e. the generalized discrepancies do not depend on the choice of the representative of  $[\beta']$  (recall that  $\beta$  descends to  $\beta'$ ).

*Remark 3.12.* Following standard convention, in the rest of the article we will say that a generalized pair  $(X, B + \beta)$  is sub-gklt, sub-glc, etc. if  $B$  is not necessarily effective, and gklt, glc, etc. if  $B$  is effective.

**3.2. Adjunction for Generalized Pairs.** In this subsection we will define adjunction for generalized pairs.

- Recall that, if  $(X, S+B)$  is a log canonical pair such that  $\lfloor S+B \rfloor = S$  and  $S^\nu \rightarrow S$  is the normalization, then by adjunction (see [Kol13, Chapter 4]) there is an effective  $\mathbb{Q}$ -divisor  $\text{Diff}_{S^\nu}(B) \geq 0$  on  $S^\nu$ , called the *different*, such that

$$(K_X + S + B)|_{S^\nu} = K_{S^\nu} + \text{Diff}_{S^\nu}(B).$$

Now let  $(X, B + \beta)$  be a generalized pair such that the coefficients of  $B$  are  $\leq 1$  and  $S$  a component of  $\lfloor B \rfloor$ . Let  $f : X' \rightarrow X$  be a log resolution of the  $(X, B + \beta)$ ,  $S'$  is the strict transform of  $S$  and  $K_{X'} + B' + \beta_{X'} = f^*(K_X + B + \beta_X)$ . Let  $S^n \rightarrow S$  be the normalization morphism; then  $f|_{S'} : S' \rightarrow S$  factors through  $S^\nu$ , and we denote the induced morphism by  $g : S' \rightarrow S^n$ . We define

$$K_{S^n} + B_{S^n} + \beta_{S^n} := (K_X + B + \beta_X)|_{S^n} = g_*(K_{S'} + (B' - S')|_{S'} + \beta_{X'}|_{S'}).$$

It is easy to see that this definition does not depend on the log resolution  $f$ , and  $(S^n, B_{S^n} + \beta_{S^n})$  is a generalized pair. If  $(X, B + \beta)$  is a glc pair (in particular,  $B$  is a boundary divisor), then from Proposition 4.5 and 4.7 of [Kol13] it follows that  $B_{S^n}$  is a boundary divisor. Readers may compare this with the more familiar case of adjunction for generalized pairs on algebraic varieties, see [BZ16, Remark 4.8].

*Remark 3.13.* If  $(X, B + \beta)$  is a glc pair and  $K_X + B$  is  $\mathbb{R}$ -Cartier, then we can write  $K_{X'} + B' - E = f^*(K_X + B)$ , where  $0 \leq E = f^*\beta_X - \beta_{X'}$  is an effective  $f$ -exceptional  $\mathbb{R}$ -divisor (by the negativity lemma). It then follows easily from the definition above that

$$B_{S^n} = \text{Diff}_{S^n}(B) + g_*(E|_{S'}) \geq \text{Diff}_{S^n}(B).$$

*Remark 3.14.* Note that  $B_{S^n}$  is determined by a surface computation, and therefore if  $(X, B + \beta)$  is generalized log canonical in codimension 2, then for the purpose of computing  $B_{S^n}$  we may assume that  $X$  is a surface. A simple computation shows that  $(X, B)$  has numerically log canonical singularities, so that  $X$  is  $\mathbb{Q}$ -Gorenstein,  $K_X + B$  is  $\mathbb{R}$ -Cartier and  $(X, B)$  is log canonical.

We will also need the following lemma.

**Lemma 3.15.** *Let  $(X, B + \beta_X)$  be a generalized dlt pair. Then for every point  $x \in X$ , there is a relatively compact Stein open neighborhood  $x \in U \subset X$  and a  $\mathbb{R}$ -divisor  $\Delta \geq 0$  on  $U$  such that  $(U, \Delta)$  is a klt pair.*

*Proof.* By [DHY23, Remark 2.18] each point  $x \in X$  has a relatively compact Stein open neighborhood  $U \subset X$  satisfying Property **P** (see [DHY23, Definition 2.17]). Thus replacing  $X$  by  $U$  we may assume that  $X$  is a relative

compact Stein space satisfying Property **P**. By standard arguments (cf. the proof of [KM98, Proposition 2.43]), we may assume that  $(X, B + \beta_X)$  is a generalized klt pair. Then by [DHY23, Theorem 2.19(4)] it follows that there is a  $\Delta \geq 0$  such that  $(U, \Delta)$  is klt.  $\square$

**3.3. Varieties in Fujiki's class  $\mathcal{C}$ .** Recall that an analytic variety  $X$  is said to be in Fujiki's class  $\mathcal{C}$  if  $X$  is bimeromorphic to a Kähler manifold. In what follows, we will study some basic properties of these varieties which will be useful in running the MMP.

**Lemma 3.16.** *Let  $X$  be a normal compact analytic variety of dimension 3 in Fujiki's class  $\mathcal{C}$  and  $f : Y \rightarrow X$  a proper bimeromorphic morphism from a normal variety  $Y$ . Then a class  $\alpha \in H_{BC}^{1,1}(X)$  is nef if and only if  $f^*\alpha$  is nef on  $Y$ .*

*Proof.* This is [HP16, Lemma 3.13].  $\square$

**Lemma 3.17.** *Let  $(X, B)$  be a  $\mathbb{Q}$ -factorial dlt pair, where  $X$  is a compact analytic variety of dimension  $\leq 3$  in Fujiki's class  $\mathcal{C}$ . If  $K_X + B$  is pseudo-effective, then  $K_X + B$  is nef if and only if  $(K_X + B) \cdot C \geq 0$  for every compact curve  $C \subset X$ .*

*Proof.* We will prove the dimension 3 case and leave the surface case as an easy exercise for the reader. The only if part follows easily by passing to a resolution of singularities  $f : X' \rightarrow X$  such that  $X'$  is a Kähler manifold. Conversely, assume that  $(K_X + B) \cdot C \geq 0$  for all curves  $C \subset X$  and  $K_X + B$  is not nef. Then by [DHP24, Theorem 2.35 and Remark 2.36], there is a surface  $S \subset X$  such that  $(K_X + B)|_S$  is not pseudo-effective. Let  $K_X + B \equiv \sum \lambda_i S_i + \beta$  be the Boucksom-Zariski decomposition of  $K_X + B$  as in Definition 2.4. Then by a similar argument as in the proof of [HP16, Lemma 4.1] it follows that  $S = S_i$  for some  $\lambda_i > 0$ . By adjunction on the normalization  $S^\nu \rightarrow S$  and the above decomposition of  $K_X + B$  it follows that there is an effective  $\mathbb{Q}$ -divisor  $\Delta \geq 0$  such that  $K_{S^\nu} + \Delta$  is  $\mathbb{Q}$ -Cartier but not pseudo-effective. Let  $\tilde{S} \rightarrow S^\nu$  be the minimal resolution of  $S^\nu$ . Then  $K_{\tilde{S}}$  is also not pseudo-effective, and from surface classification (see [BHPVdV04, Table 10, page 244]) it follows that  $\kappa(\tilde{S}) = -\infty$ . Since  $\tilde{S}$  is in Fujiki's class  $\mathcal{C}$ , by [Fuj21, Theorem 1.3],  $\tilde{S}$  is projective, and thus  $S$  is Moishezon. Then we arrive at a contradiction by a similar argument as in the proof of [HP16, Corollary 4.2].  $\square$

**Lemma 3.18.** *Let  $(S, B + \beta)$  be a gdlt pair, where  $S$  is a compact analytic surface in Fujiki's class  $\mathcal{C}$ . Then  $S$  is a Kähler surface with  $\mathbb{Q}$ -factorial rational singularities.*

*Proof.* Following a similar argument as in [KM98, Proposition 2.43] we may assume that  $(S, B + \beta)$  is locally gklt. Thus by [DHY23, Lemma 2.20],  $S$

has  $\mathbb{Q}$ -factorial rational singularities. Now let  $\nu : S' \rightarrow S$  be a resolution of singularities of  $S$  such that  $S'$  is a Kähler manifold. Let  $\omega'$  be a Kähler class on  $S'$ . By the Hodge index theorem, the intersection matrix of the exceptional curves of  $\nu$  is negative definite. Thus there is a unique  $\nu$ -exceptional  $\mathbb{R}$ -divisor  $E$  such that  $\omega' + E \equiv_S 0$ . Since  $S$  has rational singularities and it belongs to Fujiki's class  $\mathcal{C}$ , by [HP16, Lemma 3.3] there is a class  $\omega \in H_{\text{BC}}^{1,1}(S)$  such that  $\omega' + E = \nu^*\omega$ . From the negativity lemma it follows that  $E \geq 0$ . Thus  $\omega = \nu_*(\omega' + E) = \nu_*\omega'$  is a big class on  $S$ . If  $C \subset S$  is any (compact) curve and  $C' := \nu_*^{-1}C$ , then  $\omega \cdot C = (\omega' + E) \cdot C' > 0$ . Then by [DHP24, Theorem 2.36 and Remark 2.37],  $\omega$  is nef, and by [DHP24, Theorem 2.30],  $\omega$  is a Kähler class.  $\square$

**Theorem 3.19** (Termination of Flips). *Let  $(X, B)$  be a  $\mathbb{Q}$ -factorial dlt pair, where  $X$  is a compact analytic 3-fold in Fujiki's class  $\mathcal{C}$ . Then every sequence of  $(K_X + B)$ -flips terminates.*

*Proof.* First we note that Fujino's proof of Special Termination (see [Fuj07]) holds here as the log MMP for surfaces in Fujiki's class  $\mathcal{C}$  is known again due to Fujino, see [Fuj21]. So we may assume that  $(X, B)$  is a klt pair. Next we claim that the proof presented in [DO24, Theorem 3.3] (which is based on the proof of [Kaw92]) holds in our case without any change. This is because even though  $X$  is assumed to be Kähler in [DO24], that property was never used in the proof, and all the necessary MMPs used in that proof are relative MMPs for projective morphisms which are known to exist due to [Nak87] and more recently by [DHP24] and [Fuj22b].  $\square$

#### 4. BLOWING DOWN ANALYTIC SPACES TO POINTS

Recall the following result, generalizing [Gra62], that is claimed in [HP16, Proposition 7.4].

*Claim 4.1.* [HP16, Proposition 7.4] Let  $X$  be a normal compact complex space and  $S$  a  $\mathbb{Q}$ -Cartier prime Weil divisor on  $X$  with Cartier index  $m$ . Suppose that  $S$  admits a morphism with connected fibres  $f : S \rightarrow T$  such that  $\mathcal{O}_S(-mS)$  is  $f$ -ample. Then there exists a bimeromorphic morphism  $\varphi : X \rightarrow Y$  to a normal compact complex space  $Y$  such that  $\varphi|_S = f$  and  $\varphi|_{X \setminus S}$  is an isomorphism onto  $Y \setminus T$ .

The above claim contradicts [Fuj75, Proposition 3], and the proof seems to confuse the divisor  $S$  with its thickenings (a similar issue occurs in [CHP16]). To be more precise the strategy of [HP16, Proposition 7.4] is based on [AT84] which also appears to contain an error. One should instead use the corresponding result in [Fuj75, Theorem 2] which states:

**Theorem 4.2.** [Fuj75, Theorem 2] Let  $X$  be a reduced complex space,  $A$  an effective Cartier divisor on  $X$  and  $f : A \rightarrow A'$  a proper surjective morphism of  $A$  onto another complex space  $A'$ . Assume that the following conditions hold:

- (1)  $\mathcal{O}_A(-A)$  is  $f$ -ample, and
- (2)  $R^1 f_* \mathcal{O}_A(-kA) = 0$  for all  $k > 0$ .

Then there exists a blowing down  $F : X \rightarrow X'$  such that  $F_* \mathcal{S}_{X,A,f} = \mathcal{O}_{X'}$ , where  $\mathcal{S}_{X,A,f}$  is the coherent sheaf defined by the following short exact sequence

$$0 \rightarrow \mathcal{S}_{X,A,f} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_A/f^{-1}\mathcal{O}_{A'} \rightarrow 0.$$

The proof of [HP16, Proposition 7.4] begins by checking that  $R^1 f_* \mathcal{O}_S(-km'S) = 0$  for  $k > 0$ , and given  $\mathcal{I} = \mathcal{O}_X(-m'S)$  it is claimed that

$$R^1 f_*(\mathcal{I}^k / \mathcal{I}^{k+1}) \cong R^1 f_* \mathcal{O}_S(-km'S)$$

however the right hand side should presumably be replaced by  $R^1 f_* \mathcal{O}_{kS}(-km'S)$  (note that the prime divisor  $S$  is not assumed to be Cartier in [HP16], while [Fuj75, Theorem 2] requires Cartier divisors). When  $\dim A' = 0$ , one can however deduce the following result of Grauert [Gra62].

**Lemma 4.3.** Let  $X$  be a normal analytic variety and  $S \subset X$  an effective  $\mathbb{Z}$ -divisor such that  $\text{Supp}(S)$  is compact and has  $k$  connected components. Assume that  $S$  is  $\mathbb{Q}$ -Cartier of Cartier index  $m > 0$  such that  $\mathcal{O}_S(-mS|_S)$  is ample. Then there exists a proper bimeromorphic morphism  $F : X \rightarrow Y$  to a normal analytic variety  $Y$  such that  $F(S) = \{p_1, \dots, p_k\} \subset Y$  are points and  $F|_{X \setminus S} : X \setminus S \rightarrow Y \setminus \{p_1, \dots, p_k\}$  is an isomorphism.

*Proof.* Since  $\mathcal{O}_S(-mS|_S)$  is ample, from [Laz04, Proposition 1.2.16(i)] it follows that  $\mathcal{O}_{mS}(-mS)$  is ample. Thus, by Serre's vanishing theorem, there is a positive integer  $k_0 > 0$  such that  $H^i(mS, \mathcal{O}_{mS}(-kmS)) = 0$  for all  $i > 0$  and  $k \geq k_0$ . Then from the short exact sequences (for  $j \in \mathbb{Z}^{\geq 0}$ )

$$0 \rightarrow \mathcal{O}_{mS}(-(k+j)mS) \rightarrow \mathcal{O}_{(j+1)mS}(-kmS) \rightarrow \mathcal{O}_{jmS}(-kmS) \rightarrow 0$$

it follows that  $H^i((j+1)mS, \mathcal{O}_{(j+1)mS}(-kmS)) = 0$  for all  $i > 0$ ,  $j > 0$  and  $k \geq k_0$ . In particular,  $H^i(k_0 mS, \mathcal{O}_{k_0 mS}(-jk_0 mS)) = 0$  for all  $i, j > 0$ . Replacing  $m$  by  $k_0 m$ , we may assume that  $H^i(mS, \mathcal{O}_{mS}(-jmS)) = 0$  for all  $i, j > 0$ .

Suppose  $A := mS$ . Then  $A$  is Cartier and  $\mathcal{O}_A(-A|_A)$  is ample since  $\mathcal{O}_S(-mS|_S)$  ample (see [Laz04, Proposition 1.2.16(i)]). Note that  $A$  is a projective scheme (of finite type) over  $\mathbb{C}$  with  $k$  connected components. Let  $f : A \rightarrow \text{Spec } \mathbb{C}$  be the structure morphism. Then we have  $R^1 f_* \mathcal{O}_A(-kA) = H^1(mS, \mathcal{O}_{mS}(-kmS)) = 0$  for all  $k > 0$ , and thus by Theorem 4.2 there is a blowing down  $F : X \rightarrow X'$  such that  $F(A) = \{p_1, \dots, p_k\} \subset X'$  and  $F$  satisfies the other required properties. Since  $\text{Supp}(S) = \text{Supp}(A)$ , we are done. □

As a corollary we will show below that if  $\alpha$  is a nef  $(1, 1)$  class and  $S := \text{Null}(\alpha)$  is an irreducible subvariety of codimension 1 such that  $\alpha|_{S^\nu} \equiv 0$ , where  $S^\nu \rightarrow S$  is the normalization, then we can contract  $S$ .

**Corollary 4.4.** *Let  $X$  be a normal  $\mathbb{Q}$ -factorial compact Kähler variety and  $\alpha \in H_{\text{BC}}^{1,1}(X)$  a nef and big class such that the following hold:*

- (1)  $\alpha^\perp$  defines an extremal ray  $R$  in  $\overline{\text{NA}}(X)$ ,
- (2) the curves in  $R$  cover a prime Weil divisor  $S \subset X$ , and
- (3)  $\alpha|_{S^\nu} \equiv 0$ , where  $S^\nu \rightarrow S$  is the normalization morphism.

Then there exists a proper bimeromorphic contraction  $f : X \rightarrow Y$  which contracts  $S$  to a point and is the identity morphism on the complement of  $S$ .

*Proof.* Since  $\alpha$  is big, we may write  $\alpha = \gamma + \omega$ , where  $\omega$  is a Kähler class and  $\gamma$  is a big class. Let  $\gamma = D + \beta$  be the Boucksom-Zariski decomposition of  $\gamma$  as defined above, where  $D$  is an  $\mathbb{R}$ -divisor and  $\beta$  is a modified nef class (see Lemma 2.6). Then  $\beta|_{S^\nu}$  is pseudo-effective. If  $S$  is not contained in the support of  $D$ , then  $\alpha|_{S^\nu} \equiv (D + \beta)|_{S^\nu} + \omega|_{S^\nu}$  is big, which is impossible as  $\alpha|_{S^\nu} \equiv 0$ . Thus  $S$  is contained in the support of  $D$ . We have  $0 = \alpha \cdot R = (D + \beta + \omega) \cdot R$  and so  $T \cdot R < 0$ , where  $T = \beta$  or  $T$  is a component of  $D$ . Since  $\beta$  is modified nef,  $\beta|_{S^\nu}$  is pseudo-effective and so  $T$  is a component of  $D$ . Note that  $\overline{\text{NA}}(X) = \overline{\text{NA}}(X)_{T \geq 0} + \overline{\text{NA}}(X)_{T < 0}$ , so from a standard argument (e.g. see the proof of [DHY23, Claim 3.25]) it follows that  $\alpha - \delta T$  is positive on  $\overline{\text{NA}}(X) \setminus \{0\}$  for some  $\delta > 0$ , and hence Kähler. But then  $(\alpha - \delta T)|_{S^\nu} \equiv -\delta T|_{S^\nu}$  is Kähler. Thus  $T = S$ , as otherwise  $-T|_{S^\nu}$  is an anti-effective class on  $S^\nu$  and thus cannot be Kähler. In particular,  $-S|_S$  is Kähler, and hence  $\mathcal{O}_S(-mS|_S)$  is an ample line bundle, where  $m$  is the Cartier index of  $S$  on  $X$ . Then by Lemma 4.3 there is a bimeromorphic contraction  $f : X \rightarrow Y$  such that  $f(S) = \{p_0\}$  is a point and  $f|_{X \setminus S} : X \setminus S \rightarrow Y \setminus \{p_0\}$  is an isomorphism.  $\square$

A second consequence of Lemma 4.3 is the existence of flipping contractions for terminal 3-folds (or more generally log canonical 3-folds with isolated rational singularities).

**Proposition 4.5.** *Let  $X$  be a normal compact Kähler variety of arbitrary dimension with isolated rational singularities and  $\alpha$  a nef and big  $(1, 1)$  class on  $X$ . If  $\text{Null}(\alpha)$  is a pure 1-dimensional analytic subset of  $X$ , then there is a proper bimeromorphic morphism  $f : X \rightarrow Y$  such that  $\text{Ex}(f) = \text{Null}(\alpha)$  and  $f(\text{Null}(\alpha))$  is a finite set of points.*

*Proof.* Let  $\nu : X' \rightarrow X$  be a log resolution of  $X$  and  $\text{Null}(\alpha)$ , which is an isomorphism on the complement of  $X'_{\text{sing}} \cup \text{Null}(\alpha)$ . Then  $\nu^*\alpha$  is also nef and big and hence, by [CT15, Theorem 1.1] and Lemma 3.10 we have

$$(4.1) \quad E_{nK}(\nu^*\alpha) = \text{Null}(\nu^*\alpha) = \text{Ex}(\nu) \cup \nu^{-1}(\text{Null}(\alpha)).$$

Since  $E_{nK}(\alpha) \subset \nu(E_{nK}(\nu^*\alpha))$  by Lemma 3.8, from eqn. (4.1) above it follows that  $E_{nK}(\alpha)$  doesn't contain any divisor of  $X$ . By Demailly's regularization theorem (see [Bou04, Theorem 3.17(ii)]), there exists a Kähler current  $T$  in  $\nu^*\alpha$  with analytic singularities such that  $E_+(T) = E_{nK}(\nu^*\alpha)$ . Let  $\mu : X'' \rightarrow X'$  be a resolution of the singularities of  $T$  so that  $\mu^*T = \phi + F$ , where  $F \geq 0$  is an effective  $\mathbb{R}$ -divisor and  $\phi$  is a smooth form (see [Bou04, Definition 2.5.1]); note that  $F$  is  $\nu \circ \mu$ -exceptional. Let  $\omega'$  be a Kähler form on  $X'$  such that  $T \geq \varepsilon\omega'$  for some  $\varepsilon > 0$ . Then  $\phi \geq \varepsilon\mu^*\omega'$ , and thus by [Bou02, Lemma 2.9] we see that there is an effective  $\mu$ -exceptional  $\mathbb{R}$ -divisor  $E \geq 0$  such that  $\phi - \delta E$  is cohomologous to a Kähler form for  $0 < \delta \ll 1$ . Then we can write  $(\nu \circ \mu)^*\alpha \equiv \omega'' + G$ , where  $\omega''$  is a Kähler class and  $G \geq 0$  is effective  $\mathbb{Q}$ -divisor such that  $\text{Supp}(G) = \text{Ex}(\nu \circ \mu)$ . Let  $C^n \rightarrow C$  be the normalization of  $C$ , a connected component of  $\mathcal{N} := \text{Null}(\alpha)$ . Then  $\alpha|_{C^n} \equiv 0$  (since  $C$  is a curve) and so if  $D := (\nu \circ \mu)^{-1}(C)$ , then  $D$  is a reduced effective divisor contained in the support of  $G$  and  $-G|_D \equiv \omega''|_D$  is ample. Now from our construction above it follows that  $(\nu \circ \mu)(\text{Ex}(\nu \circ \mu)) = X_{\text{sing}} \cup \text{Null}(\alpha)$ . Thus the components of  $G$  which do not map into  $\text{Null}(\alpha)$  are contracted to (the singular) points of  $X$ , and hence  $-G|_{\text{Supp}(G)}$  is ample. So by Lemma 4.3 there is a contraction  $X'' \rightarrow Y$  which contracts  $\text{Supp}(G)$  to a finite set of points and is an isomorphism on the complement of  $\text{Supp}(G)$ . Then by the rigidity lemma (see [BS95, Lemma 4.1.13]), we obtain the required morphism  $f : X \rightarrow Y$ .  $\square$

**Corollary 4.6.** *Let  $(X, B)$  be a compact Kähler 3-fold terminal pair. Let  $\omega$  be a Kähler class on  $X$  and  $\alpha = K_X + B + \omega$  a nef but not Kähler such that  $\text{Null}(\alpha)$  is a pure 1-dimensional analytic subset of  $X$  and  $\overline{\text{NA}}(X) \cap \alpha^\perp = R$  is an extremal ray. Then the flipping contraction  $f : X \rightarrow Z$  and the corresponding flip  $f^+ : X^+ \rightarrow Z$  both exist.*

*Proof.* As  $(X, B)$  is a terminal 3-fold,  $X$  has only isolated rational singularities. Thus Proposition 4.5 applies and hence the flipping contraction  $f : X \rightarrow Z$  exists. By [DHP24, Theorem 1.3] or [Fuj22b, Theorem 1.8], we can construct the log canonical model of  $(X, B)$  locally over  $Z$  and then glue these together to obtain the flip  $f^+ : X^+ \rightarrow Z$ .  $\square$

In order to complete the 3-fold MMP for klt pairs, it is necessary to construct divisorial contractions where  $n(\alpha) = 1$ , i.e. divisorial contractions to a curve, and flipping contractions in the non-terminal case. Divisorial contractions to a curve were constructed in [DH20] and flipping contractions are constructed in [CHP16]. Note however that the proof of [CHP16] is somewhat technical as it involves the use of ample sheaves (which are not necessarily vector bundles). In what follows we will give a unified approach to the construction of flipping contractions and divisorial contractions to a curve. This approach relies on first proving that the pl-extremal contractions exist. Our approach seems to

be well suited for higher dimensional arguments and settles several important higher dimensional cases.

## 5. THE MAIN CONTRACTION THEOREMS

The goal of this section is to prove the contraction Theorems 1.1 and 1.2.

**5.1. Existence of pl-contractions and flips for generalized pairs.** In this subsection we will prove Theorem 1.1. First we need some preparatory results. The following theorem is a version of the relative Kawamata-Viehweg vanishing theorem which holds for generalized pairs.

**Lemma 5.1.** *Let  $(X, B + \beta)$  be a gklt pair,  $g : X \rightarrow T$  a projective morphism of analytic varieties, and  $D$  a  $\mathbb{Q}$ -Cartier  $\mathbb{Z}$ -divisor such that  $D - (K_X + B + \beta_X)$  is nef and big over  $T$ . Then  $R^i g_* \mathcal{O}_X(D) = 0$  for all  $i > 0$ .*

*Proof.* Since the question is local on the base, we may assume that  $T$  is a relatively compact Stein space. Let  $\nu : (X', B' + \beta_{X'}) \rightarrow (X, B + \beta_X)$  be a log resolution of the gklt pair  $(X, B + \beta)$  such that  $K_{X'} + B' + \beta_{X'} = \nu^*(K_X + B + \beta_X)$ . We have  $\lceil \nu^* D \rceil = \nu^* D + \{-\nu^* D\}$ , and we let  $E := \lfloor B' + \{-\nu^* D\} \rfloor$ , where  $\{\cdot\}$  denotes the fractional part.

We make the following claim.

*Claim 5.2.*  $\lceil \nu^* D \rceil - E \geq \lfloor \nu^* D \rfloor$ .

*Proof of Claim 5.2.* Let  $P$  be any prime Weil divisor on  $X'$ . First assume that  $P$  is not contained in the support of  $\{-\nu^* D\}$ . Then  $\text{mult}_P(-E) = \text{mult}_P(-\lfloor B' \rfloor) \geq 0$  and  $\text{mult}_P(\lceil \nu^* D \rceil) = \text{mult}_P(\nu^* D) = \text{mult}_P(\lfloor \nu^* D \rfloor)$ , and the claim follows. Suppose now that  $P$  is contained in the support of  $\{-\nu^* D\}$ . Then  $\text{mult}_P(\{\nu^* D\}) = (1 - \text{mult}_P(\{-\nu^* D\})) > 0$ , so that  $\text{mult}_P(\lceil \nu^* D \rceil) = \text{mult}_P(\lfloor \nu^* D \rfloor) + 1$ . Since  $\text{mult}_P(-E) = \text{mult}_P(-\lfloor B' + \{-\nu^* D\} \rfloor) \geq -1$ , the required inequality follows.  $\square$

Now recall that  $(X, B + \beta)$  is a gklt pair and  $D$  is a  $\mathbb{Z}$ -divisor. Therefore  $E = \lfloor B' + \{-\nu^* D\} \rfloor$  is  $\nu$ -exceptional,  $\nu_*(\lceil \nu^* D \rceil - E) = D$  and  $\nu_* \lfloor \nu^* D \rfloor = D$ . Since  $\nu$  is bimeromorphic, the natural morphism  $\nu_* \mathcal{O}_{X'}(\lceil \nu^* D \rceil - E) \rightarrow \mathcal{O}_X(\nu_*(\lceil \nu^* D \rceil - E)) = \mathcal{O}_X(D)$  is injective, and  $\nu_* \mathcal{O}_{X'}(\lfloor \nu^* D \rfloor) = \mathcal{O}_X(D)$  (see [Das21, Lemma 3.2]). By Claim 5.2 it follows that

$$\mathcal{O}_X(D) \supset \nu_* \mathcal{O}_{X'}(\lceil \nu^* D \rceil - E) \supset \nu_* \mathcal{O}_{X'}(\lfloor \nu^* D \rfloor) = \mathcal{O}_X(D).$$

Therefore we have

$$(5.1) \quad \nu_* \mathcal{O}_{X'}(\lceil \nu^* D \rceil - E) = \mathcal{O}_X(D).$$

Now observe that

$$\lceil \nu^* D \rceil - E = K_{X'} + B^* + \beta_{X'} + \nu^*(D - (K_X + B + \beta_X)),$$

where  $B^* := B' + \{-\nu^*D\} - E = \{B' + \{-\nu^*D\}\}$ . But then  $(X', B^*)$  is klt and the  $\mathbb{R}$ -divisor

$$\lceil \nu^*D \rceil - E - (K_{X'} + B^*) \equiv \beta_{X'} + \nu^*(D - (K_X + B + \beta_X))$$

is nef and big over  $T$ , and so, by the relative Kawamata-Viehweg vanishing theorem [Nak87, Theorem 3.7] (see also [Fuj22b, Theorem 5.2]) for all  $i > 0$  we have

$$R^i \nu_* \mathcal{O}_{X'}(\lceil \nu^*D \rceil - E) = 0, \quad R^i(g \circ \nu)_* \mathcal{O}_{X'}(\lceil \nu^*D \rceil - E) = 0.$$

By a standard spectral sequence argument and eqn. (5.1) it follows that

$$R^i g_* \mathcal{O}_X(D) = R^i g_*(\nu_* \mathcal{O}_{X'}(\lceil \nu^*D \rceil - E)) = 0 \quad \text{for all } i > 0.$$

□

**Lemma 5.3.** *Let  $(X, S + B + \beta)$  be a gpdt pair such that  $\lceil S + B \rceil = S$  is irreducible. Then  $S$  is normal and  $(S, B_S + \beta_S)$  is gklt, where  $K_S + B + \beta_S = (K_X + S + B + \beta)|_S$  is defined by adjunction as in Subsection 3.2.*

*Proof.* Since the question is local on  $X$ , we may assume that  $X$  is a relatively compact Stein space. Let  $f : X' \rightarrow X$  be a log resolution of the generalized pair  $(X, B + \beta)$  and

$$K_{X'} + S' + B' + \beta_{X'} = f^*(K_X + S + B + \beta_X) + E'$$

where  $S' = f_*^{-1}S$ ,  $B' \geq 0$ ,  $E' \geq 0$ ,  $f_*B' = B$ ,  $f_*E' = 0$  and  $S' + B'$  and  $E'$  do not share any common component.

Then  $-S' + \lceil E' \rceil \equiv_f K_{X'} + B' + \{-E'\} + \beta_{X'}$  and  $(X', B' + \{-E'\})$  is a klt pair. Since  $\lceil \beta_{X'} \rceil \in H_{BC}^{1,1}(X')$  is a nef over  $X$ , from Lemma 5.1 it follows that  $R^i f_* \mathcal{O}_{X'}(-S' + \lceil E' \rceil) = 0$  for all  $i > 0$ . Now consider the following exact sequence

$$0 \longrightarrow \mathcal{O}_{X'}(\lceil E' \rceil - S') \longrightarrow \mathcal{O}_{X'}(\lceil E' \rceil) \longrightarrow \mathcal{O}_{S'}(\lceil E' \rceil|_{S'}) \longrightarrow 0.$$

Since  $R^1 f_* \mathcal{O}_{X'}(-S' + \lceil E' \rceil) = 0$ , from the long exact sequence of cohomology we get the surjection

$$f_* \mathcal{O}_{X'}(\lceil E' \rceil) \twoheadrightarrow f_* \mathcal{O}_{S'}(\lceil E' \rceil|_{S'}).$$

Since  $\lceil E' \rceil|_{S'}$  is an effective divisor, this map factors as

$$\mathcal{O}_X = f_* \mathcal{O}_{X'}(\lceil E' \rceil) \rightarrow \mathcal{O}_S \hookrightarrow \nu_* \mathcal{O}_{S^\nu} \hookrightarrow f_* \mathcal{O}_{S'}(\lceil E' \rceil|_{S'}),$$

where  $\nu : S^\nu \rightarrow S$  is the normalization morphism. Thus we have  $\mathcal{O}_S \cong \nu_* \mathcal{O}_{S^\nu}$ , and hence  $S$  is normal.

For the second part by contradiction assume that there is a prime Weil divisor  $F$  over  $S$  such that  $a(F, S, B_S + \beta_S) \leq -1$ . Passing to a sufficiently high resolution  $f : X' \rightarrow X$  we may assume that we are in the setup of

Subsection 3.2 and that there is a  $f$ -exceptional divisor  $E$  on  $X'$  such that  $E \cap S' = F$  and  $f(E) = f|_{S'}(F)$ . It then follows that  $a(E, X, S + B + \beta_X) = a(F, S, B_S + \beta_S) \leq -1$ , a contradiction, since  $(X, S + B + \beta)$  is gplt.

□

**Lemma 5.4.** *Let  $(X, B + \beta)$  be a generalized dlt pair, and  $D$  a  $\mathbb{Q}$ -Cartier  $\mathbb{Z}$ -divisor. Then  $\mathcal{O}_X(D)$  is Cohen-Macaulay.*

*Proof.* Since the question is local on  $X$ , by Lemma 3.15 we may assume that  $X$  is a relatively compact Stein space and  $(X, B)$  is klt. Then the proof follows from [KM98, Corollary 5.25]. We explain the necessary changes needed for analytic varieties. Let  $m > 0$  be the Cartier index of  $D$ , then  $\mathcal{O}_X(-mD)$  is a line bundle on  $X$ . Since  $X$  is Stein,  $\mathcal{O}_X(-mD)$  is generated by its global sections. Using  $\mathcal{O}_X(-mD)$  in place of  $\mathcal{O}_X(mL - mD)$  in the proof of [KM98, Corollary 5.25], and using [DHP24, Theorem 2.21] for the necessary Bertini-type theorems, the arguments of [KM98, Corollary 5.25] work here.

□

Proposition 5.6 below is an analytic version of a key result from [HW19] (see also [BK23]), and an important step towards the proof of Theorem 1.1. First we fix some notations.

**Definition 5.5.** Let  $X$  be a normal analytic variety.

- (1) A non-empty open subset  $U \subset X$  called *big*, if  $X \setminus U$  is a countable union of locally closed (analytic) subsets of  $X$  of codimension at least 2.
- (2) A sheaf  $\mathcal{L}$  on  $X$  is called a  *$\mathbb{Q}$ -line bundle*, if  $\mathcal{L}$  is a reflexive sheaf of rank 1 and  $(\mathcal{L}^{\otimes m})^{**}$  is locally free for some  $m > 0$ . Note that if  $D$  is a  $\mathbb{Q}$ -Cartier Weil divisor, then  $\mathcal{L} = \mathcal{O}_X(D)$  is a  $\mathbb{Q}$ -line bundle. On the other hand, if  $X$  is Stein, then for any  $\mathbb{Q}$ -line bundle  $\mathcal{L}$  there exists a  $\mathbb{Q}$ -Cartier  $\mathbb{Z}$ -divisor  $D$  such that  $\mathcal{L} = \mathcal{O}_X(D)$ . Moreover, If  $\mathcal{L}$  is a  $\mathbb{Q}$ -line bundle and  $G$  is a  $\mathbb{Z}$ -divisor, then we define  $\mathcal{L}(G) := (\mathcal{L} \otimes \mathcal{O}_X(G))^{**}$  and  $\mathcal{L}(K_X + G) := (\mathcal{L} \otimes \omega_X \otimes \mathcal{O}_X(G))^{**}$ .

Recall from Subsection 3.2 and Lemma 5.3 that if  $(X, S + B + \beta)$  is a gplt pair, then  $S$  is normal and  $(S, B_S + \beta_S)$  is klt, where  $(K_X + S + B + \beta)|_S = K_S + B_S + \beta_S$ .

**Proposition 5.6.** *Suppose that  $(X, S + B + \beta)$  is a relatively compact gplt pair, and  $S$  a  $\mathbb{Q}$ -Cartier prime Weil divisor. Let  $\mathcal{L}$  be an rank one reflexive sheaf on  $X$  such that  $\mathcal{L}(K_X)$  is a  $\mathbb{Q}$ -line bundle on  $X$ . Then there exists an effective  $\mathbb{Q}$ -divisor  $0 \leq \Delta_S \leq B_S$ , a positive integer  $m > 0$ , and a reflexive*

rank 1 sheaf  $\mathcal{E}$  on  $S$  such that the following sequence

$$(5.2) \quad 0 \rightarrow \mathcal{L}(K_X) \rightarrow \mathcal{L}(K_X + S) \rightarrow \mathcal{E} \rightarrow 0$$

is exact, and there is a smooth big open subset  $S^0 \subset S$  such that  $\mathcal{E}|_{S^0}$  and  $(\mathcal{L}^{\otimes m})|_{S^0}$  are locally free,  $m\Delta_S$  is a  $\mathbb{Z}$ -divisor, and

$$(5.3) \quad (\mathcal{E}|_{S^0})^{\otimes m} \cong (\mathcal{L}^{\otimes m})|_{S^0} \otimes \mathcal{O}_{S^0}(m(K_S + \Delta_S)).$$

*Proof.* This proof follows along the lines of [HW19, Proposition 3.1]. We include the details as various arguments require more care than their algebraic counterparts.

Let  $X = \cup X_i$  be a (finite) cover by relatively compact Stein open subsets  $X_i$ . Then  $\mathcal{L}(K_X)|_{X_i} \cong \mathcal{O}_{X_i}(G_i)$  for some  $\mathbb{Q}$ -Cartier  $\mathbb{Z}$ -divisor  $G_i$  whose support does not contain  $S_i := S \cap X_i$ . Let  $f_i : X'_i \rightarrow X_i$  be a log resolution of  $(X_i, S_i)$ . We define  $\Delta_{S_i} := (f_i|_{S'_i})_*(\{-f_i^*G_i\}|_{S'_i})$ , where  $S'_i := (f_i)^{-1}S_i$ . We first claim that  $\Delta_{S_i}$  is independent of the log resolution  $f_i$ . To this end let  $X''_i \rightarrow X_i$  be another log resolution of  $(X_i, S_i)$  that factors through  $f_i$ , and  $g_i : X''_i \rightarrow X'_i$  the induced morphism. It suffices to show that

$$(5.4) \quad ((f_i \circ g_i)|_{S''_i})_*(\{-(f_i \circ g_i)^*G_i\}|_{S''_i}) = (f_i|_{S'_i})_*(\{-f_i^*G_i\}|_{S'_i})$$

where  $S''_i := (g_i)^{-1}S'_i$ . Let  $P \subset S_i$  be a prime divisor on  $S_i$ , and  $P'$  and  $P''$  its strict transforms on  $S'_i$  and  $S''_i$ , respectively. Note that, since  $G_i$  is a  $\mathbb{Z}$ -divisor, the support of  $\{-f_i^*G_i\}$  and  $\{-(f_i \circ g_i)^*G_i\}$  are contained in the exceptional loci of  $f_i$  and  $f_i \circ g_i$ , respectively. Since  $X'_i$  is smooth,  $\lfloor f_i^*G_i \rfloor$  is Cartier and so is  $g_i^*(\lfloor f_i^*G_i \rfloor)$ . Thus  $\{-(f_i \circ g_i)^*G_i\} = g_i^*\{-f_i^*G_i\}$ . But then

$$\{-(f_i \circ g_i)^*G_i\}|_{S''_i} = (g_i^*\{-f_i^*G_i\})|_{S''_i} = (g_i|_{S''_i})^*(\{-f_i^*G_i\}|_{S'_i}).$$

Pushing forward by  $g_i|_{S''_i}$  we have  $(g_i|_{S''_i})_*(\{-(f_i \circ g_i)^*G_i|_{S''_i}\}) = \{-f_i^*G_i\}|_{S'_i}$ . Pushing forward by  $f_i|_{S'_i}$  we obtain (5.4).

Next, observe that if  $G'_i$  is another  $\mathbb{Z}$ -divisor on  $X_i$  such that  $G_i \sim G'_i$ , then  $f_i^*G_i \sim f_i^*G'_i$ , and thus  $\{-f_i^*G_i\} = \{-f_i^*G'_i\}$ . In particular,  $\Delta_{S_i}$  depends only on  $\mathcal{L}(K_X)|_{X_i}$ . Next we will show that the  $\Delta_{S_i}$  can be glued together to a unique divisor  $\Delta_S$  on  $X$ . To this end, let  $X_{ij} := X_i \cap X_j$  and  $S_{ij} := S_i \cap S_j$ . Then  $G_i|_{X_{ij}} \sim G_j|_{X_{ij}}$ , and thus  $\Delta_{S_i}|_{S_{ij}} = \Delta_{S_j}|_{S_{ij}}$ , this follows from the fact that  $\Delta_{S_i}$  does not depend on the choice of log resolution, as we proved above. Thus  $\{\Delta_{S_i}\}_i$  glue together to give a unique  $\mathbb{Q}$ -divisor  $\Delta_S$  on  $S$ . If  $X = \cup X'_i$  is another (finite) cover of  $X$  by relatively compact Stein open subsets of  $X$  and if  $\Delta'_S$  is another  $\mathbb{Q}$ -divisor on  $S$  defined as above using the cover  $\{X'_i\}$ , then from our arguments above it follows that, for any  $i$ ,  $\Delta'_S|_{S \cap X'_i \cap X_j} = \Delta_S|_{S \cap X'_i \cap X_j}$  for all  $j$ . In particular,  $\Delta'_S = \Delta_S$ , i.e.  $\Delta_S$  is independent of the choice of the cover of  $X$ .

Next we will prove rest of the claims under the additional assumption that  $X$  is  $\mathbb{Q}$ -factorial and then deduce the general claim at the end. Assume therefore, for the time being that  $X$  is  $\mathbb{Q}$ -factorial.

Observe that the exactness of the sequence (5.2) can be checked locally on  $X$ . Moreover, notice that in order to produce a smooth big open subset  $S^0 \subset S$  and an integer  $m > 0$  satisfying the isomorphism in (5.3), it is enough to work on a finite open cover of  $X$ , and since  $X$  is relatively compact, it can be covered by finitely many relatively compact Stein open subsets. Thus from now on we assume that  $X$  itself is a relatively compact Stein space. We note here that  $X$  is no longer  $\mathbb{Q}$ -factorial (as  $\mathbb{Q}$ -factoriality is not an analytic local property), however,  $K_X, S$  and  $B$  are all  $\mathbb{Q}$ -Cartier and  $\mathcal{L}$  is represented by an effective  $\mathbb{Q}$ -Cartier  $\mathbb{Z}$ -divisor, say  $D$ , i.e.  $\mathcal{L} \cong \mathcal{O}_X(D)$ ; we may further assume here that  $D$  does not contain  $S$  in its support. In particular, we have  $\mathcal{L}(K_X) = \mathcal{O}_X(K_X + D)$  and  $\mathcal{L}(K_X + S) \cong \mathcal{O}_X(K_X + S + D)$ , where  $K_X + D$  and  $K_X + S + D$  are both  $\mathbb{Q}$ -Cartier Weil divisors. Note that since  $(X, S + B + \beta)$  is gplt, by Lemma 5.3,  $S$  is normal and  $(X, S)$  is plt. Let  $\Theta$  be an effective  $\mathbb{Q}$ -divisor on  $S$  such that  $(K_X + S)|_S = K_S + \Theta$ . Then  $\Theta \leq B_S$ , and thus by cutting down  $X$  by general hyperplanes it follows from [HW19, Lemma 3.3] that  $\Delta_S \leq \Theta \leq B_S$ .

Now consider the following exact sequence

$$(5.5) \quad 0 \rightarrow \mathcal{O}_X(K_X + D) \rightarrow \mathcal{O}_X(K_X + S + D) \rightarrow \mathcal{E} \rightarrow 0.$$

By Lemma 5.4,  $\mathcal{O}_X(K_X + D)$  and  $\mathcal{O}_X(K_X + S + D)$  are Cohen-Macaulay, and hence so is  $\mathcal{E}$  by [Kol13, Corollary 2.62]. Thus by [Har80, Proposition 1.3],  $\mathcal{E}$  is a reflexive sheaf on  $S$  of rank 1.

Let  $\mu : X' \rightarrow X$  be a log resolution of  $(X, S + D)$  and  $S' := \mu_*^{-1}S$ . Let  $B_{X'}$  be a  $\mathbb{Q}$ -divisor on  $X'$  defined by the equation

$$K_{X'} + S' + B_{X'} = \mu^*(K_X + S).$$

Note that since  $\mu : X' \rightarrow X$  is projective,  $K_{X'}$  is represented by a Weil divisor. Moreover, since  $(X, S)$  is plt,  $\lfloor B_{X'} \rfloor \leq 0$ , and it follows easily that  $\lceil \mu^*D \rceil \geq \lfloor B_{X'} + \mu^*D \rfloor$ . Thus

$$K_{X'} + S' + \lceil \mu^*D \rceil \geq \lfloor K_{X'} + S' + B_{X'} + \mu^*D \rfloor = \lceil \mu^*(K_X + S + D) \rceil, \quad \text{and}$$

$$K_{X'} + \lceil \mu^*D \rceil \geq \lfloor K_{X'} + B_{X'} + \mu^*D \rfloor \geq \lceil \mu^*(K_X + D) \rceil.$$

It follows (see for example [Das21, Lemma 3.2]) that

$$\mu_* \mathcal{O}_{X'}(K_{X'} + S' + \lceil \mu^*D \rceil) = \mathcal{O}_X(K_X + S + D), \quad \text{and}$$

$$\mu_* \mathcal{O}_{X'}(K_{X'} + \lceil \mu^*D \rceil) = \mathcal{O}_X(K_X + D).$$

Let  $G' := K_{S'} + \lceil \mu^*D \rceil|_{S'}$ . Then we have the following exact sequence

$$0 \rightarrow \mathcal{O}_{X'}(K_{X'} + \lceil \mu^*D \rceil) \rightarrow \mathcal{O}_{X'}(K_{X'} + S' + \lceil \mu^*D \rceil) \rightarrow \mathcal{O}_{S'}(G') \rightarrow 0.$$

Since  $K_{X'} + \lceil \mu^* D \rceil \equiv_X K_{X'} + \{-\mu^* D\}$ , where  $(X', \{-\mu^* D\})$  is klt (as  $\mu^* D$  has SNC support), and  $\mu$  is bimeromorphic, by the relative Kawamata–Viehweg vanishing theorem (see Lemma 5.1), we have  $R^1\mu_*\mathcal{O}_{X'}(K_{X'} + \lceil \mu^* D \rceil) = 0$ . Thus applying  $\mu_*$ , we obtain an exact sequence

$$(5.6) \quad 0 \rightarrow \mathcal{O}_X(K_X + D) \rightarrow \mathcal{O}_X(K_X + S + D) \rightarrow (\mu|_{S'})_*\mathcal{O}_{S'}(G') \rightarrow 0.$$

Now let  $G := (\mu|_{S'})_*G' = (\mu|_{S'})_*(K_{S'} + \{-\mu^* D\}|_{S'} + \mu^* D|_{S'}) = K_S + \Delta_S + D|_S$ . We will now proceed to show the isomorphism (5.3). To that end first observe that, since  $S$  is normal, it is smooth in codimension 1, and since  $\mathcal{E}$  is reflexive, it is locally free in codimension 1. Hence, there is a big open subset  $S^0 \subset S$  such that  $S^0$  is smooth,  $\mu|_{S'} : S' \rightarrow S$  is an isomorphism over  $S^0$ , and  $\mathcal{E}|_{S^0}$  is locally free. Recall that  $\mu : X' \rightarrow X$  is given by a sequence of blow ups along smooth centers. Let  $X^0 \subset X$  be the open subset of  $X$  obtained by removing the images of the exceptional divisors of  $\mu$  whose centers have codimension  $\geq 3$  in  $X$ . Thus, over  $X^0$ ,  $\mu$  is given by a sequence of blow ups of smooth centers of codimension 2. Furthermore, removing the closed analytic subset  $S \setminus S^0$  of  $X$  (which has comdim  $\geq 3$  in  $X$ ) from  $X^0$  we may assume that  $S^0 = S \cap X^0$ . By the classification of plt surface singularities it follows that we may also assume that there is an integer  $m > 0$  such that  $mK_X$ ,  $mS$  and  $mD$  are Cartier on  $X^0$ . Now from (5.5) and (5.6) it follows that  $\mathcal{E} \cong (\mu|_{S'})_*\mathcal{O}_{S'}(G')$ . Moreover, since  $\mu|_{S'}$  is an isomorphism over  $S^0$  by our construction, the natural morphism  $(\mu|_{S'})_*\mathcal{O}_{S'}(G') \rightarrow \mathcal{O}_S(G)$  is an isomorphism over  $S^0$ . Therefore we have  $\mathcal{E}|_{S^0} \cong \mathcal{O}_S(G)|_{S^0}$ ; in particular,  $\mathcal{E} \cong \mathcal{O}_S(G)$ , as both sheaves are reflexive and  $S^0$  is a big open subset of  $S$ . Restricting the sequence (5.6) on  $X^0$  we see that  $mG = m(K_S + \Delta_S + D|_S)$  is a Cartier divisor on  $S^0$ ; in particular,  $m\Delta_S$  is a  $\mathbb{Z}$ -divisor on  $S$ , as  $mD|_S$  is Cartier. Thus

$$(\mathcal{E}|_{S^0})^{\otimes m} \cong (\mathcal{O}_S(G)|_{S^0})^{\otimes m} \cong (\mathcal{L}^{\otimes m})|_{S^0} \otimes \mathcal{O}_S(m(K_S + \Delta_S))|_{S^0}.$$

Finally, we address the general (non- $\mathbb{Q}$ -factorial) case. As argued in the  $\mathbb{Q}$ -factorial case, we may assume that  $X$  is a relatively compact Stein space. Shrinking  $X$  further if necessary, we may assume by [DHY23, Theorem 2.19] that there is exists a small bimeromorphic morphism  $\mu : X' \rightarrow X$  such that  $X'$  is  $\mathbb{Q}$ -factorial. We let  $K_{X'} + S' + B' + \beta_{X'} = \mu^*(K_X + S + B + \beta_X)$  and  $\mathcal{L}'(K_{X'}) := (\mu^*\mathcal{L}(K_X))^{\text{**}}$  the reflexive hull of  $\mu^*\mathcal{L}(K_X)$ . Then  $\mathcal{L}'(K_{X'})$  is a  $\mathbb{Q}$ -line bundle and  $\mathcal{L}'(K_{X'}) \equiv_X 0$ . By what we have seen above, there is a short exact sequence

$$0 \rightarrow \mathcal{L}'(K_{X'}) \rightarrow \mathcal{L}'(K_{X'} + S') \rightarrow \mathcal{E}' \rightarrow 0$$

and a smooth big open subset  $S'^0 \subset S'$  such that  $\mathcal{E}'|_{S'^0}$  and  $(\mathcal{L}'^{\otimes m})|_{S'^0}$  are locally free,  $m\Delta'_S$  is a  $\mathbb{Z}$ -divisor,  $0 \leq \Delta'_S \leq \Theta' \leq B'_S$ , where  $(K_{X'} + S')|_{S'} = K_{S'} + \Theta'$ ,  $\Theta' \leq B'_S$ , and

$$(\mathcal{E}'|_{S'^0})^{\otimes m} \cong (\mathcal{L}'^{\otimes m})|_{S'^0} \otimes \mathcal{O}_{S'^0}(m(K_{S'} + \Delta_{S'})).$$

Since  $\mu$  is small,  $\mu_*\mathcal{L}'(K_{X'}) = \mathcal{L}(K_X)$  and  $\mu_*\mathcal{L}'(K_{X'} + S') = \mathcal{L}(K_X + S)$ . Note that, since  $X$  is Stein, by a similar argument as in the proof of [DHY23, Theorem 2.19] it follows that there is a  $\mathbb{Q}$ -divisor  $\Gamma'$  on  $X'$  such that  $\beta_{X'} \equiv_{X'} \Gamma'$ , and  $(X', S' + B' + \Gamma')$  and  $(X, S + B + \Gamma)$  are both plt, where  $\Gamma := \mu_*\Gamma'$ . Moreover, we have  $K_{X'} + S' + B' + \Gamma' \equiv \mu^*(K_X + S + B + \Gamma)$ . Finally, by a similar argument as in [KM98, Proposition 2.43], we see that there is an effective  $\mathbb{Q}$ -divisor  $G$  on  $X$  such that  $(X, G)$  is klt. Let  $K_{X'} + G' = \mu^*(K_X + G)$ . Then  $\mathcal{L}'(K_{X'}) \equiv_{X'} 0 \equiv_{X'} \mu^*(K_X + G) \equiv_{X'} K_{X'} + G'$ , where  $(X', G')$  is klt. Thus by the relative Kawamata-Viehweg vanishing theorem  $R^1\mu_*\mathcal{L}'(K_{X'}) = 0$ , and so there is a short exact sequence

$$0 \rightarrow \mathcal{L}(K_X) \rightarrow \mathcal{L}(K_X + S) \rightarrow \mu_*\mathcal{E}' \rightarrow 0.$$

Arguing as above we see that  $\mathcal{L}(K_X)$  and  $\mathcal{L}(K_X + S)$  are both Cohen-Macaulay, and hence so is  $\mathcal{E} := \mu_*\mathcal{E}' = \mathcal{L}(K_X + S)|_S$ . It follows, as explained above that  $\mathcal{E}$  is reflexive of rank 1. Since  $\mu(\text{Ex}(\mu))$  has codimension  $\geq 3$ ,  $S^0 := S \setminus (\mu(\text{Ex}(\mu) \cup (S' \setminus S'^0)))$  is a smooth big open subset of  $S$  such that  $\mu$  is an isomorphism on a neighborhood of  $S^0$ . Thus  $\mathcal{E}|_{S^0}$  and  $(\mathcal{L}^{\otimes m})|_{S^0}$  are locally free,  $m\Delta_S := (\mu|_{S'})_*(m\Delta'_S)$  is a  $\mathbb{Z}$ -divisor, where  $0 \leq \Delta_S = (\mu|_S)_*\Delta'_S \leq (\mu|_S)_*B'_S = B_S$ ,  $K_S + B_S + \beta_S = (K_X + S + B + \beta)|_S$ , and

$$(\mathcal{E}|_{S^0})^{\otimes m} \cong (\mathcal{L}^{\otimes m})|_{S^0} \otimes \mathcal{O}_{S^0}(m(K_S + \Delta_S)).$$

□

The following lemma (which is well known to experts) will be useful in our proof of the general contraction Theorem 5.8.

**Lemma 5.7.** *Let  $(X, B + \beta)$  be a relatively compact gklt pair and  $D_1, D_2, \dots, D_n$  a collection of  $\mathbb{Q}$ -divisors. Then there is a projective small bimeromorphic morphism  $f : X' \rightarrow X$  such that  $f^*D_i := f_*^{-1}D_i$  are  $\mathbb{Q}$ -Cartier divisors on  $X'$  for all  $1 \leq i \leq n$ .*

*Proof.* Note that by induction it is enough to prove the result for  $n = 1$ . Now replacing  $X$  by a small relatively compact Stein neighborhood we may assume by [DHY23, Theorem 2.16] that there is a projective small  $\mathbb{Q}$ -factorization  $f : X' \rightarrow X$  and an effective  $\mathbb{R}$ -divisor  $\Delta'$  such that  $\beta_{X'} \equiv_X \Delta'$ ,  $(X', B' + \Delta')$  and  $(X, B + \Delta)$  are both klt, and  $K_{X'} + B' + \Delta' = f^*(K_X + B + \Delta)$ , where  $\Delta := f_*\Delta'$ . Let  $D := nD_1$  so that  $D$  is a  $\mathbb{Z}$ -divisor, and  $D' := f_*^{-1}D$ ; then  $D'$  is  $\mathbb{Q}$ -Cartier. Choose  $0 < \epsilon \ll 1$  so that  $(X, B + \Delta + \epsilon D)$  is klt. Shrinking  $X$  further we may assume that  $K_X + B + \Delta \sim_{\mathbb{R}} 0$ . Then  $K_{X'} + B' + \Delta' + \epsilon D' \sim_{\mathbb{R}} \epsilon D'$ . Next, we run a  $D'$ -MMP over  $X$  as in [DHP24, Theorem 1.4] and then replacing  $X'$  by the output of this MMP we may assume that  $D'$  is nef over  $X$ . By the relative base-point free theorem (see [Fuj22b, Theorem 8.1]) we have that  $D'$  is semiample over  $X$ . In particular,  $\oplus_{m \geq 0} f_*\mathcal{O}_{X'}(mD')$  is a finitely generated  $\mathcal{O}_X$ -algebra. But since  $f$  is small,  $f_*\mathcal{O}_{X'}(mD') = \mathcal{O}_X(mD)$ . Thus  $\oplus_{m \geq 0} \mathcal{O}_X(mD)$

is a finitely generated  $\mathcal{O}_X$ -algebra. Now, since  $X$  is a relatively compact space, there is an analytic space  $W$  such that  $X \subset W$  is an open subset and  $\overline{X} \subset W$ . Let  $\overline{X} \subset U \subset W$  be a small neighborhood of  $\overline{X}$  such that  $U = \cup U_i$  is a finite union of relatively compact Stein open subsets and  $\oplus_{m \geq 0} \mathcal{O}_{U_i}(mD|_{U_i})$  are finitely generated  $\mathcal{O}_{U_i}$ -algebras. Then  $Y := \text{Projan } \oplus_{m \geq 0} \mathcal{O}_U(mD|_U)$  is a normal analytic variety such that the projection  $\pi : Y \rightarrow U$  is a projective small bimeromorphic morphism and  $\pi_*^{-1}D$  is  $\mathbb{Q}$ -Cartier (see [KM98, Lemma 6.2]). Let  $X'' := \pi^{-1}(X)$  and  $g := \pi|_{X''}$ ; then  $g : X'' \rightarrow X$  is a projective small bimeromorphic morphism of normal analytic varieties such that  $g_*^{-1}D$  is  $\mathbb{Q}$ -Cartier. This completes our proof.  $\square$

The following result is a generalization of Theorem 1.1 that works for generalized pairs.

**Theorem 5.8.** *Let  $(X, S + B + \beta)$  be a generalized plt pair, where  $X$  is a compact analytic variety such that  $S$  is  $\mathbb{Q}$ -Cartier. Suppose that*

- (1)  $\lfloor S + B \rfloor = S$  is irreducible,
- (2) there is a proper morphism  $\pi : S \rightarrow T$  such that  $\pi_* \mathcal{O}_S = \mathcal{O}_T$ ,
- (3)  $-(K_X + S + B + \beta_X)|_S$  is relatively Kähler (over  $T$ ), and
- (4)  $-S|_S$  is  $\pi$ -ample.

Then there exists a bimeromorphic morphism  $p : X \rightarrow Z$  such that  $p|_S = \pi$  and  $p|_{X \setminus S}$  is an isomorphism.

*Proof.* By adjunction (as in Lemma 5.3) we know that  $S$  is normal,  $(S, B_S + \beta_S)$  is gklt and  $-(K_S + B_S + \beta_S)$  is Kähler over  $T$ , where  $(K_X + S + B + \beta)|_S = K_S + B_S + \beta_S$ .

Choosing  $\mathcal{L} = \mathcal{O}_X(-K_X - (k+1)S)$  for  $k \geq 1$  in eqn. (5.2) we get the following short exact sequences

$$(5.7) \quad 0 \rightarrow \mathcal{O}_X(-(k+1)S) \rightarrow \mathcal{O}_X(-kS) \rightarrow \mathcal{E}_k \rightarrow 0.$$

Since  $S$  is  $\mathbb{Q}$ -Cartier,  $\mathcal{O}_X(S)$  is a  $\mathbb{Q}$ -line bundle, and thus by Proposition 5.6,  $\mathcal{E}_k$  is a reflexive rank 1 sheaf on  $S$ . Let  $S_k$  be the thickening of  $S$  defined by  $\mathcal{O}_{S_k} \cong \mathcal{O}_X/\mathcal{O}_X(-kS)$ . Then we have the following short exact sequence of sheaves of rings on  $S$

$$0 \rightarrow \mathcal{E}_k \rightarrow \mathcal{O}_{S_{k+1}} \rightarrow \mathcal{O}_{S_k} \rightarrow 0.$$

We can then consider the corresponding long exact sequence of sheaves of rings on  $T$  obtained by pushforward (note that identifying the topological spaces of the thickenings  $|S_k| = |S|$ , we may regard  $\pi : |S_k| \rightarrow |T|$  as a morphism of topological spaces). We claim that:

*Claim 5.9.*  $R^1 \pi_* \mathcal{E}_k = 0$ , and so  $\pi_* \mathcal{O}_{S_{k+1}} \rightarrow \pi_* \mathcal{O}_{S_k}$  is surjective for all  $k \geq 1$ .

Grant this for the time being. Since  $\pi_* \mathcal{O}_{S_{k+1}} \rightarrow \pi_* \mathcal{O}_{S_k}$  is surjective for  $k \geq 1$ , we have inclusions  $T_k \hookrightarrow T_{k+1}$ , where  $T_k := \text{Specan}(\pi_* \mathcal{O}_{S_k})$ . Since  $\pi_* \mathcal{O}_{S_1} = \pi_* \mathcal{O}_S = \mathcal{O}_T$ , the  $T_k$  define thickenings of  $T$  with underlying topological space  $|T|$ . The natural isomorphisms  $H^0(\mathcal{O}_{T_k}) \cong H^0(\pi_* \mathcal{O}_{S_k}) \cong H^0(\mathcal{O}_{S_k})$  induce morphisms of complex spaces  $\pi_k : S_k \rightarrow T_k$  (e.g. see [Har77, Exercise II.2.4]). The induced composite morphisms  $S_k \rightarrow T_k \hookrightarrow T_{k+1}$  factors as  $S_k \rightarrow S_{k+1} \rightarrow T_{k+1}$ . We would like to emphasize here that  $\pi_k : S_k \rightarrow T_k$  are morphisms of non-reduced complex spaces for all  $k \geq 1$ , not just maps of topological spaces. Fix  $\ell \geq 1$  such that  $A = \ell S = S_\ell$  is Cartier and let  $\pi_A : A \rightarrow T_\ell$  be the induced morphism (here  $\pi_A = \pi_\ell$ ).

Consider the following commutative diagram

$$(5.8) \quad \begin{array}{ccc} S = A_{\text{red}} & \xrightarrow{\varphi} & A \\ \pi \downarrow & & \downarrow \pi_A \\ T = (T_\ell)_{\text{red}} & \xrightarrow{\psi} & T_\ell. \end{array}$$

We claim that  $\mathcal{O}_A(-A)$  is  $\pi_A$ -ample. Indeed, since  $\varphi^* \mathcal{O}_A(-A) = \mathcal{O}_S(-\ell S|_S)$  is  $\pi$ -ample, and  $\psi$  is a proper finite morphism,  $\varphi^* \mathcal{O}_A(-A)$  is  $(\pi_A \circ \varphi)$ -ample. Since  $\varphi : A_{\text{red}} \rightarrow A$  is the reduction of nilpotent elements of  $\mathcal{O}_A$ , from an argument similar to the proof of [Laz04, Proposition 1.2.16(i)] it follows that  $\mathcal{O}_A(-A)$  is  $\pi_A$ -ample.

Arguing as in the proof of Lemma 4.3, we will show the following.

*Claim 5.10.* Replacing  $A$  by a multiple, we may assume that  $R^i \pi_{A,*} \mathcal{O}_A(-kA) = 0$  for all  $i, k \geq 1$ .

By Theorem 4.2, it then follows that there exists a proper bimeromorphic morphism  $p : X \rightarrow Z$  such that  $p|_{X \setminus A}$  is an isomorphism and  $p|_A = \pi_A$ . Finally,  $p|_S = (p|_A)|_S = \pi_A|_S = \pi$ .

*Proof of Claim 5.9.* To check the claim, note that the vanishing  $R^1 \pi_* \mathcal{E}_k = 0$  can be checked locally on the base  $T$ , and hence we may assume that  $T$  is a relatively compact Stein space. Since  $\pi$  is a projective morphism, it follows that  $\mathcal{E}_k = \mathcal{O}_S(G_k)$  for some Weil divisor  $G_k$  on  $S$ . By Proposition 5.6,  $G_k \sim_{\mathbb{Q}} -kS|_S - \Delta_k$ , where  $\Delta_k \geq 0$  is an effective  $\mathbb{Q}$ -divisor on  $S$  such that  $0 \leq \Delta_k \leq B_S$ . (Note that with the notation of Proposition 5.6,  $\Delta_k = B_S - \Delta_S$ .)

If every component of  $B_S$  is  $\mathbb{Q}$ -Cartier, then  $\Delta_k$  is  $\mathbb{Q}$ -Cartier and so is  $G_k$ . Moreover, in this case we can also write

$$G_k \sim_{\mathbb{Q}} K_S + B_S - \Delta_k + \beta_S - (K_S + B_S + \beta_S) - kS|_S$$

so that  $(S, (B_S - \Delta_k) + \beta_S)$  is gklt (by Lemma 5.3) and  $-(K_S + B_S + \beta_S) - kS|_S$  is relatively Kähler over  $T$ , and hence the claim now follows from Lemma 5.1.

Otherwise, since  $(S, B_S + \beta_S)$  is a relatively compact generalized klt pair, by Lemma 5.7 there is a projective small bimeromorphic morphism  $q : \tilde{S} \rightarrow S$  such that every component of  $q_*^{-1}B_S$  is  $\mathbb{Q}$ -Cartier. Let  $\tilde{G}_k := q_*^{-1}G_k$ ,  $\tilde{B}_S := q_*^{-1}B_S$ ,  $\tilde{\Delta}_k := q_*^{-1}\Delta_k$  so that  $K_{\tilde{S}} + \tilde{B}_S - \tilde{\Delta}_k + \beta_{\tilde{S}} = q^*(K_S + B_S - \Delta_k + \beta_S)$  and  $\tilde{\mathcal{E}}_k := \mathcal{O}_{\tilde{S}}(\tilde{G}_k)$ . Then

$$\tilde{G}_k \sim_{\mathbb{Q}} K_{\tilde{S}} + \tilde{B}_S - \tilde{\Delta}_k + \beta_{\tilde{S}} - q^*(K_S + B_S + \beta_S + kS|_S),$$

where  $(\tilde{S}, (\tilde{B}_S - \tilde{\Delta}_k) + \beta_{\tilde{S}})$  is gklt and  $-q^*(K_S + B_S + \beta_S + kS|_S)$  is relatively nef and big over  $T$  (hence also over  $S$ ). By Lemma 5.1,  $R^i q_* \mathcal{O}_{\tilde{S}}(\tilde{G}_k) = 0$  and  $R^i(\pi \circ q)_* \mathcal{O}_{\tilde{S}}(\tilde{G}_k) = 0$  for all  $i > 0$ . Since  $q$  is small, then  $q_* \mathcal{O}_{\tilde{S}}(\tilde{G}_k) = \mathcal{O}_S(G_k)$ , and so by an easy spectral sequence argument we have

$$R^i \pi_* \mathcal{O}_S(G_k) = R^i \pi_*(q_* \mathcal{O}_{\tilde{S}}(\tilde{G}_k)) = 0 \quad \text{for all } i > 0.$$

□

*Proof of Claim 5.10.* Since the statement is local over  $T_\ell$ , replacing  $T_\ell$  by a Stein open subset  $U$  and  $X$  by a neighborhood of  $\pi_A^{-1}(U)$ , we may assume that  $T_\ell$  is Stein and the statement is equivalent to  $H^i(A, \mathcal{O}_A(-kA)) = 0$  for all  $k > 0$ . Since  $\mathcal{O}_A(-A)$  is ample, by Serre vanishing, there is an integer  $k_0 > 0$  such that  $H^i(A, \mathcal{O}_A(-kA)) = 0$  for all  $k \geq k_0$  and  $i > 0$ . For  $j \geq 1$ , consider the short exact sequence

$$0 \rightarrow \mathcal{O}_A(-(k+j)A) \rightarrow \mathcal{O}_{(j+1)A}(-kA) \rightarrow \mathcal{O}_{jA}(-kA) \rightarrow 0.$$

Proceeding by induction we get that  $H^i(\mathcal{O}_{(j+1)A}(-kA)) = 0$  for all  $k \geq k_0$ ,  $j \geq 1$ , and  $i > 0$ . Replacing  $A$  by  $k_0 A$ , the claim now follows. □

□

*Remark 5.11.* Assume that we are in the settings of Theorem 5.8, and additionally assume that  $X$  is a Kähler space. Then the morphism  $p$  is projective (since  $-S$  is relatively ample over  $Z$ ), and  $Z$  is in Fujiki's class  $\mathcal{C}$ . Working locally over  $Z$ , we may pick a Kähler form  $\omega$  such that  $\omega \equiv_Z -(K_X + S + B + \beta_X)$ . Let  $\nu : X' \rightarrow X$  be a log resolution of  $(X, S + B + \beta)$  and write  $K_{X'} + S' + B' + \beta_{X'} = \nu^*(K_X + S + B + \beta_X)$ , where  $[\beta_{X'}] \in H_{BC}^{1,1}(X')$  is nef. Then  $\beta_{X'} + \nu^*\omega$  is numerically equivalent (over  $Z$ ) to a nef and big  $\mathbb{R}$ -divisor  $G' \geq 0$  such that  $K_X + S + B + G = \nu_*(K_{X'} + S' + B' + G')$  is plt. By the Base-point free theorem (see [Fuj22b, Theorem 8.1])  $K_Z + S_Z + B_Z + G_Z = \pi_*(K_X + S + B + G)$  is plt and in particular has rational singularities. Then by [HP16, Lemma 3.4] and [DHP24, Lemma 8.7], the image of  $p^* : H_{BC}^{1,1}(Z) \rightarrow H_{BC}^{1,1}(X)$  is given by

$$\text{Im}(p^*) = \{\alpha \in H_{BC}^{1,1}(X) : \alpha \cdot C = 0, \text{ for all } C \subset X \text{ curves s.t. } p(C) = \text{pt}\}.$$

The next result shows that flips for generalized pairs exist in all dimensions.

**Theorem 5.12.** *Let  $(X, B + \beta)$  be a  $\mathbb{Q}$ -factorial gdlt pair, where  $X$  is a compact analytic variety belonging to Fujiki's class  $\mathcal{C}$ , and  $f : X \rightarrow Z$  a flipping contraction. Then the flip of  $f$  exists.*

Recall that a flipping contraction  $f : X \rightarrow Z$  is a small bimeromorphic morphism such that  $\rho(X/Z) = 1$  and  $-(K_X + B + \beta_X)$  is Kähler over  $Z$ . The corresponding flip (if it exists) is a small bimeromorphic morphism  $f^+ : X^+ \rightarrow Z$  such that  $\rho(X^+/Z) = 1$  and  $K_{X^+} + B^+ + \beta_{X^+}$  is Kähler over  $Z$ , where  $B^+$  is the strict transform of  $B$ .

*Proof.* Replacing  $B$  by  $(1 - \epsilon)B$  for some  $0 < \epsilon < 1$ , we may assume that  $(X, B + \beta)$  is gklt. Let  $\nu : X' \rightarrow X$  be a log resolution of  $(X, B + \beta)$  such that  $f \circ \nu$  is a projective morphism and  $K_{X'} + B' + \beta_{X'} = \nu^*(K_X + B + \beta_X)$ . Let  $\beta_{X'} + E = \nu^*\beta_X$ , then  $E \geq 0$  is an effective  $\mathbb{R}$ -divisor by the negativity lemma. Since  $\rho(X/Z) = 1$ , it follows that if  $K_X + B \not\equiv_Z 0$ , then  $X \rightarrow Z$  is projective and  $\beta_X \equiv_Z \lambda(K_X + B)$  for some  $\lambda \in \mathbb{R}$ . For any point  $z \in Z$  we fix a neighborhood  $z \in W \subset Z$  such that  $W$  is relatively compact, Stein and satisfies Property (P) (see [DHY23, Definition 2.17 and Remark 2.18]). Let  $X_W = f^{-1}(W)$ , then we may assume that  $K_{X_W}$  is a  $\mathbb{Q}$ -Cartier divisor and we let  $D := \lambda(K_{X_W} + B_W)$  where  $B_W = B|_{X_W}$ . We also let  $X'_W = \nu^{-1}(X_W)$ ,  $B'_W = B'|_{X'_W}$ ,  $E_W = E|_{X'_W}$  and  $D' := -\nu_W^*D - E_W$ . Then  $D' \equiv_W \beta_{X'_W}$  is nef and big over  $W$ . Since  $D' \equiv_W \beta_{X'_W}$  is nef and big over  $W$ , we may assume that  $D' \equiv_W \Delta'_W \geq 0$  so that  $(X'_W, B'_W + \Delta'_W)$  is sub-klt, and hence  $(X_W, B_W + \Delta_W)$  is klt, where  $B_W + \Delta_W := \nu_{W,*}(B'_W + \Delta'_W)$ . But then the relative log canonical model  $(X_W^+, B_W^+ + \Delta_W^+)$  of  $(X_W, B_W + \Delta_W)$  exists by [DHP24, Theorem 1.3] and [Fuj22b, Theorem 1.8], and it is the relative log canonical model of  $(X_W, B_W + \beta_W)$ . Since these relative log canonical models are unique by [DHY23, Lemma 2.12(3)], they glue together, and we obtain a relative log canonical model  $(X^+, B^+ + \beta)$  of  $(X, B + \beta)$  over  $Z$ .

Suppose now that  $K_X + B \equiv_Z 0$ . Then  $K_Z + B_Z := f_*(K_X + B)$  is klt and  $Z$  has rational singularities. Fix  $z \in Z$ , we will construct the flip locally over a neighborhood of  $z$ . We may assume that a resolution  $\nu : X' \rightarrow X$  is projective over  $Z$  and  $X'_z$  is a divisor with simple normal crossing support. Let  $f' = f \circ \nu$ . Since  $Z$  has rational singularities, from the short exact sequence ([HP16, §3, eqn. (2)])

$$0 \rightarrow \mathbb{R} \rightarrow \mathcal{O}_{X'} \rightarrow \mathcal{H}_{X'} \rightarrow 0$$

it follows that  $R^1 f'_* \mathcal{H}_{X'} \rightarrow R^2 f'_* \mathbb{R}$  is an isomorphism. Note that  $R^2 f'_* \mathbb{R} \cong R^2 f'_* \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}$  by the universal coefficient theorem, as  $f'$  is proper. Now consider the class  $[\beta_{X'}] \in H_{\text{BC}}^{1,1}(X') \cong H^1(\mathcal{H}_{X'})$ . By [KM92, Lemma 12.1.1], there is a neighborhood  $z \in W \subset Z$  and a  $\mathbb{R}$ -divisor  $D'_W$  on  $X'_W$  such that the images of  $\beta'_W := \beta_{X'}|_{X'_W}$  and  $D'_W$  in  $R^2 f'_* \mathbb{R}$  coincide. But then  $\beta'_W \equiv_W D'_W$ . Shrinking  $W$ , we may assume that it is relatively compact, Stein and satisfies Property

(P). We now argue as above and construct the flip  $X \dashrightarrow X^+$  locally over  $Z$  and then glue these together.  $\square$

*Remark 5.13.* Even though it is not clear that the flipping and flipped contractions are projective over  $Z$ , the proof shows that they are locally projective over  $Z$ .

We will also need the following technical alternate version of Theorem 5.12.

**Theorem 5.14.** *Let  $f : X \rightarrow Z$  be a small proper bimeromorphic morphism of normal relatively compact analytic varieties in Fujiki's class  $\mathcal{C}$ . Let  $(X, B + \beta)$  be a gdlt pair,  $\lfloor B \rfloor$  is  $\mathbb{Q}$ -Cartier, and  $K_X + B + \beta_X \equiv_Z -G$ , where  $G$  is an  $\mathbb{R}$ -Cartier divisor. Then there exists  $f^+ : X^+ \rightarrow Z$  a small bimeromorphic morphism such that  $K_{X^+} + B^+ + \beta_{X^+}$  is Kähler over  $Z$ . We say that  $f'$  is the flip of  $f$ .*

*Proof.* Replacing  $B$  by  $B - \epsilon \lfloor B \rfloor$  for some  $0 < \epsilon < 1$ , we may assume that  $(X, B + \beta)$  is gklt. By standard arguments, it is easy to see that it suffices to construct the flip  $f'$  locally over  $Z$  and hence we may assume that  $Z$  is relatively compact and Stein.

Let  $\nu : X' \rightarrow X$  be a log resolution of  $(X, B + \beta)$  such that  $f \circ \nu$  is a projective morphism and  $K_{X'} + B' + \beta_{X'} = \nu^*(K_X + B + \beta_X)$ . Since  $\beta_{X'} \equiv_Z -(K_{X'} + B' + \nu^*G)$  is nef and big (over  $Z$ ), then  $-(K_{X'} + B' + \nu^*G) \equiv_Z \Delta'$  where  $(X', B' + \Delta')$  is sub-klt. If  $\Delta = \nu_* \Delta'$ , then  $(X, B + \Delta)$  is klt and it suffices to set  $X^+ := \text{Proj}_Z R(X, K_X + B + \Delta)$  by [DHP24, Theorem 1.3].  $\square$

*Proof of Theorem 1.1.* This follows immediately from Theorems 5.8 and 5.12.  $\square$

## 6. EXISTENCE OF FLIPS AND DIVISORIAL CONTRACTIONS IN DIMENSION 3

Suppose that  $X$  is a normal compact analytic variety in Fujiki's class  $\mathcal{C}$ , and  $\alpha \in H_{BC}^{1,1}(X)$  is nef and big but not Kähler, then by [DHP24, Theorem 2.30],  $\text{Null}(\alpha)$  is non-empty. Recall that  $\text{Null}(\alpha)$  is the union of (positive dimensional) analytic subvarieties  $Z \subset X$  such that  $\alpha^{\dim Z} \cdot Z = 0$ . If  $X$  is Kähler, then from Lemma 3.5 and Theorem 3.6 it follows that  $\text{Null}(\alpha)$  is a closed analytic subset of  $X$ . Suppose that  $\alpha^\perp \cap \overline{\text{NA}}(X) = R$  is an extremal ray, then we say that  $R$  is divisorial if  $\dim \text{Null}(\alpha) = \dim X - 1$  (i.e.  $\text{Null}(\alpha)$  contains a divisor) and  $R$  is of flipping type if  $\dim \text{Null}(\alpha) < \dim X - 1$  (i.e.  $\text{Null}(\alpha)$  contains no divisors). If  $\alpha = [K_X + B + \beta_X + \omega]$ , where  $(X, B + \beta)$  is a glc pair and  $\omega$  is Kähler form, then it is expected that  $\text{Null}(\alpha)$  is given a by the union of all curves  $C$  such that  $[C] \in R$ , i.e.  $\alpha \cdot C = 0$ .

The result below shows that flipping contractions and flips exist in dimension 3 for *strongly  $\mathbb{Q}$ -factorial* pairs.

**Definition 6.1.** Let  $X$  be a normal compact analytic variety. Then  $X$  is called *strongly  $\mathbb{Q}$ -factorial* if for every reflexive sheaf  $\mathcal{L}$  of rank 1, there is a positive integer  $m \in \mathbb{Z}^+$  such that  $\mathcal{L}^{[m]} := (\mathcal{L}^{\otimes m})^{**}$  is a line bundle.

Note that complex manifolds are strongly  $\mathbb{Q}$ -factorial, and this property is preserved under the steps of MMP, for more details see Lemmas 2.3–2.5 of [DH20].

**Theorem 6.2.** Let  $(X, B)$  be a strongly  $\mathbb{Q}$ -factorial compact Kähler 3-fold klt pair such that

- (1)  $K_X + B$  is pseudo-effective,
- (2)  $\alpha = [K_X + B + \beta]$  is nef and big for some Kähler form  $\beta$ , and
- (3)  $\alpha^\perp \cap \overline{\text{NA}}(X) = R$  is an extremal ray of flipping type.

Then the flipping contraction  $f : X \rightarrow Z$  and the flip  $X \dashrightarrow X^+$  exist, and there is a Kähler class  $\alpha_Z$  on  $Z$  such that  $\alpha = f^*\alpha_Z$ .

*Proof.* Let  $\nu : X' \rightarrow X$  be a log resolution of  $(X, B)$ . We may assume that  $\alpha' := \nu^*\alpha = \omega' + G'$ , where  $G' \geq 0$  is  $\nu$ -exceptional and  $\omega'$  is a Kähler class (see the arguments in the proof of Proposition 4.5). Let  $\omega' := \overline{\omega}'$  (see Subsection 3.11), and  $\Delta' := \nu_*^{-1}B + \text{Ex}(\nu)$ , then  $(X', \Delta')$  is dlt and we may write  $K_{X'} + \Delta' = \nu^*(K_X + B) + E$ , where  $E \geq 0$  and  $\text{Supp}(E) = \text{Ex}(\nu)$ . Note that  $K_{X'} + \Delta' + t\omega'$  is Kähler for  $t \gg 0$ . Thus the following hypothesis are satisfied:

- (1)  $\alpha'$  is nef and  $\omega'$  is a modified Kähler class,
- (2)  $\alpha' = \omega' + G'$ , where  $G' \geq 0$  is supported on  $\lfloor \Delta' \rfloor$ ,
- (3)  $K_{X'} + \Delta' + t\omega' + a\alpha'$  is nef for some  $t > 0$  and  $a > 0$ ,
- (4)  $(X', \Delta' + t\omega')$  is gdlt, and
- (5)  $X'$  is strongly  $\mathbb{Q}$ -factorial.

Note that even though  $\omega'$  is a Kähler class here, in the rest of proof we will only use the modified Kähler property of  $\omega'$ , which is preserved by steps of the minimal model program.

If  $S'$  is any component of  $\lfloor \Delta' \rfloor$ , then by adjunction we can write

$$K_{S'} + \Delta_{S'} + t\omega'_{S'} = (K_{X'} + \Delta' + t\omega'_{X'})|_{S'},$$

where  $\Delta_{S'} := \text{Diff}_{S'}(\Delta')$ . Then  $(S', \Delta_{S'})$  is a dlt surface and  $\omega'_{S'} = \omega'_{X'}|_{S'}$  is a big class, since  $\omega'_{X'}$  is a modified Kähler class.

Let  $\alpha_{S'} := \alpha'|_{S'}$  so that  $\alpha_{S'}$  is nef and define

$$\tau_{S'} := \inf\{s \geq 0 \mid K_{S'} + \Delta_{S'} + s\omega_{S'} + a\alpha_{S'} \text{ is nef for some } a > 0\}.$$

*Claim 6.3.* If  $\tau_{S'} > 0$ , then  $K_{S'} + \Delta_{S'} + \tau_{S'}\omega_{S'} + a'\alpha_{S'}$  is nef for  $a' \gg 0$  and there is a  $(K_{S'} + \Delta_{S'})$ -negative extremal ray  $R_{S'}$  of  $\overline{\text{NA}}(S')$  such that

$$(K_{S'} + \Delta_{S'} + \tau_{S'}\omega_{S'}) \cdot R_{S'} = \alpha_{S'} \cdot R_{S'} = 0.$$

*Proof.* Since  $\omega_{S'}$  is big and  $\tau_{S'} > 0$ , then there are finitely many  $(K_{S'} + \Delta_{S'} + \frac{\tau_{S'}}{2}\omega_{S'})$ -negative extremal rays in  $\overline{\text{NA}}(S')$  (see [DHY23, Corollary 2.32]). Let  $\Sigma_1, \dots, \Sigma_k$  be curves that span those rays and let

$$\Lambda := \{i : \alpha_{S'} \cdot \Sigma_i = 0 \text{ for some } 1 \leq i \leq k\}.$$

We define

$$\sigma_{S'} := \min \left\{ s \geq \frac{\tau_{S'}}{2} \mid (K_{S'} + \Delta_{S'} + s\omega_{S'}) \cdot \Sigma_i \geq 0 \text{ for all } i \in \Lambda \right\}.$$

If  $\Lambda = \emptyset$ , i.e.  $\alpha_{S'} \cdot \Sigma_i > 0$  for  $i = 1, \dots, k$ , then we define  $\sigma_{S'} := \frac{\tau_{S'}}{2}$ . Clearly  $\sigma_{S'} \leq \tau_{S'}$  and if  $\sigma_{S'} = \tau_{S'}$ , then there is a curve  $\Sigma_i$  for some  $i \in \Lambda$  such that  $(K_{S'} + \Delta_{S'} + \tau_{S'}\omega_{S'}) \cdot \Sigma_i = \alpha_{S'} \cdot \Sigma_i = 0$ . Since  $\Sigma_i$  is  $(K_{S'} + \Delta_{S'} + \frac{\tau_{S'}}{2}\omega_{S'})$ -negative, it follows that  $(K_{S'} + \Delta_{S'}) \cdot \Sigma_i < 0$  and the claim holds. Therefore, it suffices to show that  $\sigma_{S'} < \tau_{S'}$  is impossible. To the contrary assume that  $\sigma_{S'} < \tau_{S'}$ . Then for  $a' \gg 0$  we have

$$(K_{S'} + \Delta_{S'} + \sigma_{S'}\omega_{S'} + a'\alpha_{S'}) \cdot \Sigma_i \geq 0 \quad \text{for } i = 1, \dots, k.$$

For  $0 < \epsilon \ll 1$  and  $a' \gg 0$ , we claim that  $K_{S'} + \Delta_{S'} + (\tau_{S'} - \epsilon)\omega_{S'} + a'\alpha_{S'}$  is non-negative on

$$\overline{\text{NA}}(S')_{K_{S'} + \Delta_{S'} + \frac{\tau_{S'}}{2}\omega_{S'} \geq 0}.$$

Indeed,  $\alpha_{S'}$  is nef and  $K_{S'} + \Delta_{S'} + (\tau_{S'} - \epsilon)\omega_{S'} + a'\alpha_{S'}$  is a convex linear combination of  $K_{S'} + \Delta_{S'} + \frac{\tau_{S'}}{2}\omega_{S'}$  and the nef classes  $K_{S'} + \Delta_{S'} + t\omega_{S'} + a\alpha_{S'}$  and  $\alpha_{S'}$ . More specifically, if  $\eta_{s,a} := K_{S'} + \Delta_{S'} + s\omega_{S'} + a\alpha_{S'}$  and  $\lambda := \frac{t - \tau_{S'} + \epsilon}{t - \tau_{S'}/2}$ , then we can write

$$\eta_{\tau_{S'} - \epsilon, a'} = \lambda\eta_{\tau_{S'}/2, 0} + (1 - \lambda)\eta_{t, a} + (a' - (1 - \lambda)a)\alpha_{S'}.$$

Note that  $\lambda > 0$ ,  $1 - \lambda = \frac{\tau_{S'}/2 - \epsilon}{t - \tau_{S'}/2} > 0$ , since  $0 < \epsilon \ll 1$ , and  $a' - (1 - \lambda)a \geq 0$  for  $a' \gg 0$ . Since  $K_{S'} + \Delta_{S'} + (\tau_{S'} - \epsilon)\omega_{S'} + a'\alpha_{S'}$  is non-negative on the  $\Sigma_i$  for all  $1 \leq i \leq k$ , it is non-negative on

$$\overline{\text{NA}}(S') = \overline{\text{NA}}(S')_{K_{S'} + \Delta_{S'} + \frac{\tau_{S'}}{2}\omega_{S'} \geq 0} + \sum_{i=1}^k \mathbb{R}^+[\Sigma_i].$$

Thus  $K_{S'} + \Delta_{S'} + (\tau_{S'} - \epsilon)\omega_{S'} + a'\alpha_{S'}$  is nef, which is a contradiction to the definition of  $\tau_{S'}$ , and the claim follows. Note that  $K_{S'} + \Delta_{S'} + \tau_{S'}\omega_{S'} + a'\alpha_{S'}$  is nef for all  $a' \gg 0$ , as  $\alpha_{S'}$  is nef.  $\square$

Let  $\tau := \max\{\tau_{S'}\}$  as  $S'$  runs through all the components of  $\lfloor \Delta' \rfloor$ . Then  $K_{S'} + \Delta_{S'} + \tau\omega_{S'} + a'\alpha_{S'}$  is nef for each component  $S'$  of  $\lfloor \Delta' \rfloor$  and  $a' \gg 0$ , as  $K_{S'} + \Delta_{S'} + t\omega_{S'} + a'\alpha_{S'}$  is nef and  $\tau_{S'} \leq \tau \leq t$  for every component  $S'$  of  $\lfloor \Delta' \rfloor$ .

*Claim 6.4.* If  $\tau > 0$ , then  $K_{X'} + \Delta' + \tau\omega' + a\alpha'$  is nef for some  $a > 0$ .

*Proof.* Suppose that  $K_{X'} + \Delta' + \tau\omega' + a'\alpha'$  is not nef for a given  $a' \gg 0$ , then by [DHP24, Theorem 2.36 and Remark 2.37] there is a subvariety  $Z' \subset X'$  such that  $(K_{X'} + \Delta' + \tau\omega' + a'\alpha')|_{Z'}$  is not pseudo-effective. Since

$$K_{X'} + \Delta' + \tau\omega' + a'\alpha' \equiv K_{X'} + \Delta' + t\omega' + (a' - t + \tau)\alpha' + (t - \tau)G'$$

and  $K_{X'} + \Delta' + t\omega' + (a' - t + \tau)\alpha'$  is nef for  $a' \gg 0$ , then  $Z'$  is contained in the support of  $G'$  and hence in a component  $S'$  of  $\lfloor \Delta' \rfloor$  and this contradicts what we have shown above (see Claim 6.3).  $\square$

If  $\tau > 0$ , then from Claim 6.3 it follows that there is a component  $S'$  of  $\lfloor \Delta' \rfloor$  and a  $(K_{S'} + \Delta_{S'})$ -negative extremal ray  $R_i = \mathbb{R}^+ \cdot [\Sigma_i]$  such that  $(K_{X'} + \Delta' + \tau\omega') \cdot R_i = \alpha' \cdot R_i = 0$ . Let  $\pi : S' \rightarrow T$  denote the corresponding contraction. Since  $\omega' \cdot R_i > 0$  and  $\alpha' \cdot R_i = (\omega' + G') \cdot R_i = 0$ , it follows that  $G' \cdot \Sigma_i < 0$ . Thus  $\Sigma_i$  is contained in a component of  $G'$ , say  $S''$  such that  $S'' \cdot \Sigma_i < 0$ . Since  $\text{Supp}(G') = \lfloor \Delta' \rfloor$ ,  $\Sigma_i$  is contained in a  $(K_{S''} + \Delta_{S''})$ -negative extremal face of  $\overline{\text{NA}}(S'')$ , where  $K_{S''} + \Delta_{S''} := (K_{X'} + \Delta')|_{S''}$ . This face contains a negative extremal ray  $R''_i = \mathbb{R}^+ \cdot [\Sigma''_i]$  such that  $S'' \cdot R''_i < 0$ . Thus replacing  $S'$  by  $S''$  and  $R_i$  by  $R''_i$  we may assume that  $S' \cdot R_i < 0$ , and hence  $-S'|_{S'}$  is  $\pi$ -ample. Now write  $\Delta' = S' + \Delta''$  so that  $S' \not\subset \text{Supp}(\Delta'')$ . Since  $(X', \Delta' + \omega')$  is gdlt, for  $0 < \epsilon \ll 1$  we have  $(X', S' + (1 - \epsilon)\Delta'' + \omega')$  is gplt and  $(K_{X'} + S' + (1 - \epsilon)\Delta'' + \omega') \cdot R_i < 0$ . Thus by Theorem 5.8, there exists a bimeromorphic morphism  $p : X' \rightarrow Z'$  such that  $p|_{S'} = \pi$  and  $p|_{X' \setminus S'}$  is an isomorphism.

*Claim 6.5.* The morphism  $p$  is either a divisorial or flipping contraction, and in the latter case the corresponding flip exists.

*Proof.* Since  $-(K_{X'} + \Delta')$  is  $p$ -ample,  $Z'$  has rational singularities by [DH20, Lemma 2.44]. Let  $\gamma \in H_{\text{BC}}^{1,1}(X')$  such that  $\gamma \cdot R_i = 0$ , then  $\gamma \cdot C = 0$  for any  $p$ -exceptional curve. This is because all  $p$ -exceptional curves are contained in  $S'$  and  $\rho(S'/T) = 1$ . By [HP16, Lemma 3.3], there is a class  $\gamma_{Z'} \in H_{\text{BC}}^{1,1}(Z')$  such that  $p^*\gamma_{Z'} = \gamma$ . It follows that  $\rho(X'/Z') = 1$ . This shows that  $p$  is either a divisorial or flipping contraction. By Theorem 5.12, if it is a flipping contraction, then the corresponding flip exists.  $\square$

We may then replace  $X'$  by the image of this flip or divisorial contraction. Observe that, at this stage  $X'$  is in Fujiki's class  $\mathcal{C}$  and not necessarily Kähler, however, by Lemma 3.18 the components of  $\lfloor \Delta' \rfloor$  are still Kähler surfaces with  $\mathbb{Q}$ -factorial rational singularities. So we can repeat the above arguments.

*Claim 6.6.* Fix  $0 < \epsilon \ll 1$ . After finitely many steps, we obtain a bimeromorphic map  $\phi : X' \dashrightarrow X^+$  such that for any  $0 < t \leq \epsilon$

$$K_{X^+} + \Delta^+ + t\omega^+ + a\alpha^+ = \phi_*(K_{X'} + \Delta' + t\omega' + a\alpha')$$

is nef for some  $a > 0$  (depending on  $t$ ).

*Proof.* Properties (1-5) stated at the beginning of our proof continue to be satisfied after each flip or divisorial contraction. So we may repeat the procedure obtaining an  $\alpha'$ -trivial,  $(K_{X'} + \Delta')$ -minimal model program. Since termination of flips holds by Theorem 3.19, after finitely many steps we may assume that  $\tau = 0$ . Thus we obtain an  $\alpha'$ -trivial  $(K_{X'} + \Delta')$ -MMP  $X' \dashrightarrow X^+$  such that  $K_{X^+} + \Delta^+ + \epsilon\omega^+ + a\alpha^+$  is nef for some  $\epsilon > 0$ ,  $a > 0$  and

$$\inf\{s \geq 0 \mid K_{X^+} + \Delta^+ + s\omega^+ + a\alpha^+ \text{ is nef for some } a > 0\} = 0.$$

The claim now follows easily by taking convex linear combinations.  $\square$

Let  $\psi : X \dashrightarrow X^+$  and  $\phi : X' \dashrightarrow X^+$  be the induced bimeromorphic map,  $U := X \setminus \text{Null}(\alpha)$  and  $X'_U := \nu^{-1}(U)$ . Next we will show that every step of this MMP is vertical over  $U$ .

*Claim 6.7.* The map  $\phi_U =: \phi|_{X'_U}$  is a proper bimeromorphic map defined over  $U$ , i.e. there exist proper bimeromorphic morphisms  $X'_U \rightarrow U$  and  $X_U^+ \rightarrow U$  commuting with  $\phi_U$ .

*Proof.* Suppose that  $X' =: X^0 \dashrightarrow X^1 \dashrightarrow \dots \dashrightarrow X^n =: X^+$  is the MMP obtained above and  $\phi^i : X' \dashrightarrow X^i$  are the induced bimeromorphic maps. Proceeding by induction, it suffices to show that if  $\phi_U^i := \phi^i|_{X'_U} : X'_U \dashrightarrow X_U^i$  is a proper bimeromorphic map defined over  $U$ , then so is  $\phi_U^{i+1}$ . We will denote by  $\nu_U^i : X_U^i \rightarrow U$  the corresponding proper morphism. Since  $X^i \dashrightarrow X^{i+1}$  is an  $\alpha^i$ -trivial flip or divisorial contraction, if  $f^i : X^i \rightarrow Z^i$  is the corresponding contraction, and  $C$  is a contracted curve, then  $\alpha^i \cdot C = 0$ . Suppose that  $C_U := C \cap X_U^i \neq \emptyset$ . Let  $p : W \rightarrow X^i$ ,  $q : W \rightarrow X^i$  be the normalization of the graph of  $\phi^i$ . If  $C_U$  is not  $\nu_U^i$  vertical, then let  $C' \subset W$  be a curve dominating  $C$  so that  $q_*C' = dC$  for some  $d > 0$ . Then  $\nu_*p_*C' \neq 0$  and so  $\nu_*p_*C' \cdot \alpha \neq 0$  which is a contradiction since then

$$0 = C \cdot \alpha^i = \frac{1}{d}C' \cdot q^*\alpha^i = p_*(\frac{1}{d}C') \cdot \alpha' = \nu_*p_*(\frac{1}{d}C') \cdot \alpha \neq 0.$$

Thus every such  $C_U$  is  $\nu_U^i$  vertical. It follows that  $f_U^i := f^i|_{X_U^i}$  is a (proper bimeromorphic) morphism over  $U$  and hence  $X_U^{i+1} \rightarrow U$  is a proper bimeromorphic map over  $U$ .  $\square$

*Claim 6.8.* The divisors contracted by  $\phi : X' \dashrightarrow X^+$  coincide with the set of  $\nu : X' \rightarrow X$  exceptional divisors and hence  $\psi : X \dashrightarrow X^+$  is a small bimeromorphic map of strongly  $\mathbb{Q}$ -factorial varieties.

*Proof.* We will first show that  $\text{Supp}N(K_{X'} + \Delta' + a\alpha') = \text{Ex}(\nu)$  for any  $a \gg 0$ . Recall that for any pseudo-effective  $(1, 1)$  class  $\gamma$ ,  $N(\gamma)$  is defined as the negative part of the Boucksom-Zariski decomposition of  $\gamma$ , see Definition 2.4.

Now since  $\alpha$  is big,  $K_X + B + a\alpha$  is big for  $a \gg 0$ . Moreover, since  $K_{X'} + \Delta' + a\alpha' = \nu^*(K_X + B + a\alpha) + E$ , where  $\text{Supp}(E) = \text{Ex}(\nu)$ , from [DHY23, Lemma A.5] it follows that  $\text{Supp}N(K_{X'} + \Delta' + a\alpha') \supset \text{Ex}(\nu)$ . Since  $\alpha' = G' + \omega'$  where  $\omega'$  is Kähler and  $\text{Supp}(G') = \text{Ex}(\nu)$ ,  $\omega^* := \omega' + \frac{1}{a}(K_{X'} + \Delta')$  is also Kähler for  $a \gg 0$  and so  $K_{X'} + \Delta' + a\alpha' \equiv a\omega^* + aG'$ . Thus  $\text{Supp}N(K_{X'} + \Delta' + a\alpha') \subset \text{Supp}(N(G')) \subset \text{Ex}(\nu)$  and hence  $\text{Supp}N(K_{X'} + \Delta' + a\alpha') = \text{Ex}(\nu)$  for any  $a \gg 0$ . Since  $\omega'$  is a modified Kähler class,  $\text{Supp}N(K_{X'} + \Delta' + a\alpha' + \epsilon\omega') = \text{Ex}(\nu)$  for all  $0 < \epsilon \ll 1$  (e.g. see the proof of [DHY23, Claim 3.18]). Since  $\phi : X' \dashrightarrow X^+$  is a  $(K_{X'} + \Delta' + \epsilon\omega' + a\alpha')$ -MMP, then  $K_{X^+} + \Delta^+ + \epsilon\omega^+ + a\alpha^+$  is nef, and by [DHY23, Theorem A.11],  $\phi$  contracts  $\text{Ex}(\nu)$ .  $\square$

By [DH20, Lemma 2.32],  $\psi_* : H_{\text{BC}}^{1,1}(X) \rightarrow H_{\text{BC}}^{1,1}(X^+)$  is an isomorphism. Let  $R^\perp \subset N^1(X)$  be the codimension 1 subspace of classes  $\gamma \in N^1(X)$  such that  $\gamma \cdot R = 0$ . Let  $W$  be the normalization of the graph of  $\psi : X \dashrightarrow X^+$  and  $p : W \rightarrow X$  and  $q : W \rightarrow X^+$  the induced morphisms. Since  $\psi$  is not a morphism, then there is a curve  $\tilde{C} \subset W$  such that  $p_*\tilde{C} = 0$  and  $C^+ = q_*\tilde{C} \neq 0$ . It follows that  $\alpha^+ \cdot C^+ = 0$ . Let  $R^+ \subset N_1(X^+)$  be the ray spanned by  $C^+$ . If  $\gamma \in R^\perp$ , then  $\nu^*\gamma \cdot \Sigma \equiv 0$  for any curve  $\Sigma$  such that  $\alpha' \cdot \Sigma = 0$  where  $\alpha' = \nu^*\alpha$ . Since the minimal model program  $X' \dashrightarrow X^+$  is  $\alpha'$ -trivial, i.e. it only contracts  $\alpha'$ -trivial curves, it follows that it is also  $\gamma' := \nu^*\gamma$ -trivial and hence that  $\gamma^+ \cdot C^+ = 0$  where  $\gamma^+ = \phi_*\gamma' \in H_{\text{BC}}^{1,1}(X^+)$ . Therefore,  $\psi_*(R^\perp) = (R^+)^\perp$ , and hence every  $p$ -exceptional curve  $\tilde{C}'$  satisfies  $q_*\tilde{C}' \in R^+$ . Since  $\psi_*$  is an isomorphism and  $(K_X + B) \cdot R \neq 0$ , then  $(K_{X^+} + B^+) \cdot R^+ \neq 0$ . Suppose that  $(K_{X^+} + B^+) \cdot R^+ < 0$ , then  $(K_{X^+} + B^+ + t\omega^+) \cdot R^+ < 0$  for  $0 < t \ll 1$ . Since  $\alpha^+ \cdot R^+ = 0$ ,  $(K_{X^+} + B^+ + t\omega^+ + a\alpha^+) \cdot R^+ < 0$  for any  $a > 0$ , contradicting Claim 6.6. Therefore  $(K_{X^+} + B^+) \cdot R^+ > 0$ .

Let  $\eta := \alpha + \delta(K_X + B)$  for some  $0 < \delta \ll 1$  and  $\eta^+ := \psi_*\eta \in H_{\text{BC}}^{1,1}(X^+)$ . Consider  $E := p^*\eta - q^*\eta^+$ , then we claim that  $-E$  is  $p$ -ample. Indeed, if  $\tilde{C} \subset W$  is a  $p$ -vertical curve, then  $q_*\tilde{C} \neq 0$ , and as we have seen above that  $[q_*\tilde{C}] \in R^+$ , so  $\eta^+ \cdot q_*\tilde{C} > 0$  and hence  $-E \cdot \tilde{C} > 0$ . It follows by the negativity lemma that  $E \geq 0$  and the support of  $E$  contains  $\text{Ex}(p)$ . But then  $\text{Supp}(E) = \text{Ex}(p)$ . Clearly  $\text{Supp}(E) \subset \text{Ex}(q)$ , as  $\psi$  is an isomorphism in codimension 1. We claim that  $\text{Ex}(q) = \text{Supp}(E) = \text{Ex}(p)$ . Indeed, let  $\tilde{C} \subset W$  be a  $q$ -vertical curve. Then  $p_*\tilde{C} \neq 0$  and in fact  $p_*\tilde{C} \subset \text{Null}(\alpha)$  as  $\psi : X \dashrightarrow X^+$  is an isomorphism on  $X \setminus \text{Null}(\alpha)$ . Thus  $E \cdot \tilde{C} = \eta \cdot p_*\tilde{C} = (\alpha + \delta(K_X + B)) \cdot p_*\tilde{C} < 0$ , and hence  $\text{Ex}(q) \subset \text{Supp}(E)$ .

We will now show that  $-E|_{\text{Supp}(E)}$  is ample. Let  $E_i$  be a component of  $\text{Ex}(p) = \text{Ex}(q)$ . We have  $C_i := p(E_i)$  and  $C_i^+ := q(E_i)$  are curves and  $\theta := (p|_{E_i}, q|_{E_i}) : E_i \rightarrow C_i \times C_i^+$  is finite. Let  $\tilde{C}_i$  (resp.  $\tilde{C}_i^+$ ) be a curve on  $E_i$  dominating  $C_i \times \{x\}$  (resp.  $\{x\} \times C_i^+$ ) for general  $x \in C_i^+$  (resp.  $x \in C_i$ ), then  $q_*\tilde{C}_i = 0$  (resp.  $p_*\tilde{C}_i^+ = 0$ ). Thus  $[C_i^+] \in R^+$  and  $C_i \cdot \alpha = 0$  so that

$[C_i] \in R$ . It follows that  $-\eta|_{C_i^+}$  (resp.  $\eta|_{C_i}$ ) is Kähler. Since  $\theta$  is finite, then  $-E|_{E_i} \equiv (q^*\eta^+ - p^*\eta)|_{E_i} = \theta^*(-\eta|_{C_i}, \eta^+|_{C_i^+})$  is ample and so  $-E|_{\text{Supp}(E)}$  is ample. Perturbing the coefficients of  $E$  slightly we may assume that  $E$  is a  $\mathbb{Q}$ -Cartier divisor. But then we can contract  $E$  to a point by Lemma 4.3.

Let  $g : W \rightarrow Z$  be the induced bimeromorphic morphism, then we claim that we have induced morphisms  $f : X \rightarrow Z$  and  $f^+ : X^+ \rightarrow Z$ . To this end, if  $C \subset W$  is a  $q$  (respectively  $p$ ) exceptional curve, then  $C$  is contained in  $\text{Supp}(E) = \text{Ex}(p) = \text{Ex}(q)$ . Since  $g$  contracts  $E$  to a point, then  $g_*C = 0$  and so by the rigidity lemma  $X \rightarrow Z$  and  $X^+ \rightarrow Z$  are morphisms.

Note that by construction  $f : X \rightarrow Z$  contracts a non-empty set of  $\alpha$ -trivial curves (i.e. curves in  $R$ ). We claim that in fact  $f$  contracts every curve in  $R$ . Let  $C \subset X$  be a curve in  $R$  so that  $\alpha \cdot C = 0$ . If  $C$  is not contracted by  $f$ , then  $p$  is an isomorphism over the general points of  $C$  and we let  $\tilde{C} := p_*^{-1}C$  and  $C^+ := q_*\tilde{C}$ . Then  $\alpha^+ \cdot C^+ = 0$ . Since  $K_{X^+} + B^+ + t\omega^+ + a\alpha^+$  is nef for any  $0 < t \ll 1$  and some  $a \gg 0$ , then

$$(K_{X^+} + B^+ + t\omega^+) \cdot C^+ = (K_{X^+} + B^+ + t\omega^+ + a\alpha^+) \cdot C^+ \geq 0$$

and taking the limit we have  $(K_{X^+} + B^+) \cdot C^+ \geq 0$ . It follows that

$$0 \leq E \cdot \tilde{C} = (p^*\eta - q^*\eta^+) \cdot \tilde{C} = (K_X + B) \cdot C - (K_{X^+} + B^+) \cdot C^+ < 0.$$

This is impossible and so  $C$  is contracted by  $f$ . It follows that  $f$  is  $(K_X + B)$ -negative. By [DH20, Lemma 2.44],  $Z$  has rational singularities and by [HP16, Lemma 3.3]  $f^*H_{BC}^{1,1}(Z) = R^\perp \subset H_{BC}^{1,1}(X)$ . Thus  $\rho(X/Z) = 1$ . Finally, since  $f$  contracts the set of  $\alpha$ -trivial curves, we have  $\alpha = f^*\alpha_Z$ .

We will now check that  $\alpha_Z$  is a Kähler class. To this end, let  $V \subset Z$  be a positive dimensional subvariety of  $Z$ . Note that  $\dim f(\text{Ex}(f)) = 0$ . Let  $V'$  be the strict transform of  $V$ . Then  $(\alpha_Z)^{\dim V} \cdot V = \alpha^{\dim V} \cdot V' > 0$  and  $V'$  is not contained in  $\text{Null}(\alpha)$ . Thus from [DHP24, Theorem 2.29] it follows that  $\alpha_Z$  is a Kähler class. Since  $\psi_*$  is an isomorphism, then  $\rho(X^+/Z) = 1$ . We have already seen that  $f^+$  contracts a  $(K_{X^+} + B^+)$ -positive curve and so  $K_{X^+} + B^+$  is  $f^+$ -ample. Finally, it is easy to see that  $(f^+)^*\alpha_Z + \delta(K_{X^+} + \Delta^+)$  is Kähler for  $0 < \delta \ll 1$ , and hence  $X^+$  is a Kähler 3-fold.

□

Next we prove the existence of divisorial contractions in dimension 3 for strongly  $\mathbb{Q}$ -factorial pairs.

**Theorem 6.9.** *Let  $(X, B)$  be a strongly  $\mathbb{Q}$ -factorial compact Kähler 3-fold klt pair such that*

- (1)  $K_X + B$  is pseudo-effective,
- (2)  $\alpha = [K_X + B + \beta]$  is nef and big for some Kähler form  $\beta$ , and

(3)  $\alpha^\perp \cap \overline{\text{NA}}(X) = R$  is an extremal ray of divisorial type.

Then the divisorial contraction  $f : X \rightarrow Z$  exists and there is a Kähler class  $\alpha_Z$  on  $Z$  such that  $\alpha = f^*\alpha_Z$ .

*Proof.* Recall that by definition  $\alpha^\perp \cap \overline{\text{NA}}(X) = R$  is an extremal ray of divisorial type if and only if  $\dim \text{Null}(\alpha) = 2$ . We begin by observing the following.

*Claim 6.10.* There is an irreducible divisor  $S$  such that  $\text{Null}(\alpha) = S$ ,  $S$  is a Moishezon surface covered by family of  $\alpha$ -trivial curves and  $S \cdot R < 0$ .

*Proof.* Let  $\omega$  be a Kähler class on  $X$ . Since  $\alpha$  is big,  $\alpha - \epsilon\omega$  is big for  $0 < \epsilon \ll 1$ , and we have the Boucksom-Zariski decomposition  $\alpha - \epsilon\omega = \sum s_j S_j + P$  where  $P$  is modified nef and big and in particular the restriction of  $P$  to any surface is pseudo-effective. Let  $S \subset \text{Null}(\alpha)$  be a divisor (a surface) and  $S' \rightarrow S$  the minimal resolution. Then  $\alpha|_{S'}$  is nef but not big (as  $(\alpha|_{S'})^2 = 0$ ), therefore  $S$  coincides with some component of  $\sum s_j S_j$ , which for simplicity we denote by  $S_1$ . We claim that  $S$  is Moishezon. To this end, consider  $b = \text{mult}_S(B)$  and

$$\left(1 + \frac{1-b}{s_1}\right) \alpha|_{S'} = \left(K_X + B + \beta + \left(\frac{1-b}{s_1}\right) \left(\sum s_j S_j + P + \epsilon\omega\right)\right) \Big|_{S'}.$$

Since  $\text{mult}_S\left(B + \frac{1-b}{s_1} \sum s_j S_j\right) = 1$ , we have  $\left(K_X + B + \frac{1-b}{s_1} \sum s_j S_j\right) \Big|_{S'} = K_{S'} + B_{S'}$  where  $B_{S'} \geq 0$ . Then

$$(6.1) \quad \left(1 + \frac{1-b}{s_1}\right) \alpha|_{S'} = K_{S'} + B_{S'} + \left(\beta + \frac{1-b}{s_1}(P + \epsilon\omega)\right) \Big|_{S'}.$$

Since  $B_{S'} \geq 0$ ,  $P|_{S'}$  is pseudo-effective,  $(\beta + \frac{1-b}{s_1}(P + \epsilon\omega))|_{S'}$  is big and  $\alpha|_{S'}$  is not big, it follows that  $K_{S'}$  is not pseudo-effective, and hence  $S'$  is projective by classification and  $H^2(S', \mathcal{O}_{S'}) = 0$  (e.g. see [DH20, Lemma 2.43]). In particular,  $S$  is Moishezon.

We claim that  $S$  is covered by an analytic family of  $\alpha$ -trivial curves  $\{C_t\}$ . Fix  $A_{S'}$  an ample divisor on  $S'$ . Since  $H^2(S', \mathcal{O}_{S'}) = 0$ ,  $\alpha|_{S'}$  is represented by an  $\mathbb{R}$ -divisor. Since  $\alpha|_{S'}$  is nef, it is in the cone  $\overline{NM}_1(S')$  of movable curves. By [Ara10, Theorem 1.3] and [Das20, Theorem 1.9], for any  $\epsilon > 0$ , we have a decomposition

$$\alpha|_{S'} \equiv C_\epsilon + \sum_{i=1}^k \lambda_i M_i,$$

where  $C_\epsilon \in (\overline{NE}_1(S')_{(K_{S'}) \geq 0} + \overline{NM}_1(S')_{(K_{S'} + \epsilon A_{S'}) \geq 0})$ ,  $\lambda_i > 0$ ,  $(K_{S'} + \epsilon A_{S'}) \cdot C_\epsilon \geq 0$  and the  $M_i$  are movable curves. If  $\alpha|_{S'} \cdot M_i = 0$ , then the pushforward of  $M_i$  onto  $S$  defines a movable curve on  $S$  which is  $\alpha$ -trivial and the claim follows. Therefore, we may assume that  $\alpha|_{S'} \cdot M_i > 0$  for all  $i$ . Now,  $0 = (\alpha|_{S'})^2 = \alpha|_{S'} \cdot (C_\epsilon + \sum_{i=1}^k \lambda_i M_i)$  and since  $\alpha|_{S'}$  is nef then  $k = 0$ , i.e.  $\alpha|_{S'} \equiv C_\epsilon$ .

But then  $\alpha|_{S'} \cdot (K_{S'} + \epsilon A_{S'}) \geq 0$ . Taking the limit as  $\epsilon \rightarrow 0$ , it follows that  $\alpha|_{S'} \cdot K_{S'} \geq 0$ . Since  $\alpha|_{S'}$  is nef, then  $\alpha|_{S'} \cdot \gamma \geq 0$  for any pseudo-effective class  $\gamma$ . Thus  $\alpha|_{S'} \cdot B_{S'} \geq 0$ ,  $\alpha|_{S'} \cdot P|_{S'} \geq 0$ , and  $\alpha|_{S'} \cdot \omega|_{S'} \geq 0$ . It follows from (6.1) and  $(\alpha'|_{S'})^2 = 0$  that all these terms are equal to 0. But then  $\alpha|_{S'} \cdot \omega|_{S'} = 0$  where  $S' \rightarrow S$  is the normalization. Since  $\omega|_{S'}$  is Kähler, then  $\alpha|_{S'} \equiv 0$ . This contradicts the inequality above  $\alpha|_{S'} \cdot M_i > 0$ , and so the claim holds.

Let  $\{C_t\}_{t \in T}$  be an analytic family of  $\alpha$ -trivial curves on  $S$  covering  $S$  as proved in the claim above. Then  $[C_t] \in R$  for all  $t \in T$  and we have

$$0 > -\epsilon \omega \cdot C_t = (\alpha - \epsilon \omega) \cdot C_t = \left( \sum s_j S_j + P \right) \cdot C_t \geq s_1 S \cdot C_t$$

as  $S_i \cdot C_t \geq 0$  for  $i \neq 1$  and  $S = S_1$ . Thus  $S \cdot C_t < 0$ , and hence  $S \cdot R < 0$  as  $R = \mathbb{R}^+ \cdot [C_t]$ . In particular, if  $C$  is any curve with  $[C] \in R$ , then  $S \cdot C < 0$  and hence  $S = \text{Null}(\alpha)$ .  $\square$

Let  $b := \text{mult}_S(B)$  and  $\Delta := B + (1 - b)S$ .

*Claim 6.11.* There exists a dlt model  $\mu : X' \rightarrow X$  of  $(X, \Delta)$ . In particular

- (1)  $(X', \Delta')$  is  $\mathbb{Q}$ -factorial and dlt, where  $\Delta' := \mu_*^{-1} \Delta + \text{Ex}(\mu)$ ,
- (2)  $K_{X'} + \Delta'$  is nef over  $X$  so that  $\mu^*(K_X + \Delta) - (K_{X'} + \Delta') \geq 0$ , and
- (3)  $K_{X'} + \Delta' = \mu^*(K_X + B) + (1 - b)S' + \sum a_j E_j$ , where  $S' = \mu_*^{-1} S$  and  $a_j > 0$  for all  $j$ .

*Proof.* Let  $\mu : X' \rightarrow X$  be a projective log resolution and  $\Delta' := \mu_*^{-1} \Delta + \text{Ex}(\mu)$ , then  $(X', \Delta')$  is dlt. We may run the  $(K_{X'} + \Delta')$ -MMP over  $X$  (the existence of this MMP is well known since  $\dim X = 3$  and  $\mu$  is projective, but see also [DHP24, Theorem 1.4] for the case  $\dim X > 3$ ). By [DO24, Theorem 3.3], we may assume that the corresponding MMP terminates and replacing  $(X', \Delta')$  by the output of this MMP, we may assume that  $K_{X'} + \Delta'$  is nef over  $X$ . By the negativity lemma,  $\mu^*(K_X + \Delta) - (K_{X'} + \Delta') \geq 0$  and so (1) and (2) hold. (3) follows since  $(X, B)$  is klt.  $\square$

We will now run the  $\alpha'$ -trivial  $(K_{X'} + \Delta')$ -minimal model program, where  $\alpha' = \mu^* \alpha$ . Note that if  $U := X \setminus S = X \setminus \text{Null}(\alpha)$ , then  $(U, \Delta|_U)$  is klt and so  $X'_U \rightarrow U$  is a small bimeromorphic  $(K_{X'} + \Delta')$ -trivial morphism. Since the  $\mu$ -exceptional locus is divisorial, then  $X'_U \rightarrow U$  is an isomorphism. If  $C$  is a curve not contained in  $W' := X' \setminus X'_U$ , then  $\alpha' \cdot C > 0$ . It follows easily that every  $\alpha'$ -trivial step of the  $(K_{X'} + \Delta')$ -MMP will involve only curves contained in the complement of  $X'_U$ , and hence the restriction of this MMP to  $X'_U$  is an isomorphism.

Let  $n(\alpha)$  be the nef dimension of the restriction of  $\alpha$  to the normalization  $S' \rightarrow S$ , see [BCE<sup>+</sup>02, Definition 2.7]. Since  $S$  is a Moishezon surface covered by a family of  $\alpha$ -trivial curves (as shown above), it follows that  $0 \leq n(\alpha) \leq 1$ .

If  $n(\alpha) = 0$ , then  $\alpha|_{S^\nu} \equiv 0$  and if  $n(\alpha) = 1$ , then the nef reduction map  $S^\nu \rightarrow T$  is a morphism to a smooth (projective) curve (see [BCE<sup>+</sup>02, 2.4.4]). Observe that if  $n(\alpha) = 0$ , then the nef reduction map  $S^\nu \rightarrow T$  is a morphism to a point and in this case the existence of the divisorial contraction  $f : X \rightarrow Z$  follows from Corollary 4.4. We however give an alternate argument below which proves both of the cases:  $n(\alpha) = 0$  and  $n(\alpha) = 1$ , simultaneously.

If  $\gamma \in H_{BC}^{1,1}(X)$ , then we will say that  $\gamma \equiv_\alpha 0$  if and only if  $\gamma \cdot C = 0$  for every  $\alpha$ -trivial curve  $C \subset X$ , i.e.  $\gamma \cdot C = 0$  whenever  $\alpha \cdot C = 0$  for a curve  $C \subset X$ . We have that  $(K_X + B - sS) \cdot R = 0$  for some  $s > 0$  (as both  $K_X + B$  and  $S$  are  $R$ -negative), and hence  $K_X + B - sS \equiv_\alpha 0$ . It follows that  $\mu^*(K_X + B - sS) \equiv_{\alpha'} 0$  and hence that

$$K_{X'} + \Delta' \equiv_{\alpha'} \Theta' := (1-b)S' + \sum a_j E_j + s\mu^*S \geq 0,$$

where  $\text{Supp}(\Theta') = \lfloor \Delta' \rfloor$ .

*Claim 6.12.* We can run the  $\alpha'$ -trivial  $(K_{X'} + \Delta')$ -MMP

$$X' = X^0 \dashrightarrow X^1 \dashrightarrow \dots \dashrightarrow X^m$$

so that if  $\phi^i : X' \dashrightarrow X^i$ , then

- (1) each flip and divisorial contraction is  $\alpha'$ -trivial, hence  $\alpha^i = \phi_*^i \alpha'$  is nef,
- (2)  $\phi^i|_{X'_U}$  is an isomorphism over  $U$  and we denote by  $X_U^i$  its image,
- (3)  $(X^i, \Delta^i := \phi_*^i \Delta')$  is dlt,  $X^i$  is strongly  $\mathbb{Q}$ -factorial, and  $\lfloor \Delta^i \rfloor \subset W^i := X^i \setminus X_U^i$ ,
- (4)  $K_{X^i} + \Delta^i - \Theta^i \equiv_{\alpha^i} 0$  and  $\text{Supp}(\Theta^i) = \lfloor \Delta^i \rfloor$ , and
- (5) if  $S^m$  is a component of  $\lfloor \Delta^m \rfloor$  then every  $K_{S^m} + \Delta_{S^m} := (K_{X^m} + \Delta^m)|_{S^m}$  negative extremal ray  $R^m$  is  $\alpha_{S^m} := \alpha_{X^m}|_{S^m}$ -positive, i.e.  $\alpha_{S^m} \cdot R^m > 0$ .

*Proof.* We will proceed by induction on  $i$ . Assume that we have already constructed  $X' \dashrightarrow X^1 \dashrightarrow \dots \dashrightarrow X^i$  with the required properties. Suppose that there is a component  $S^i$  of  $\lfloor \Delta^i \rfloor$  and a  $K_{S^i} + \Delta_{S^i} := (K_{X^i} + \Delta^i)|_{S^i}$ -negative extremal ray  $R^i = \mathbb{R}^+[\Sigma^i]$  which is  $\alpha_{S^i} := \alpha_{X^i}|_{S^i}$  non-positive and hence trivial, i.e.  $\alpha_{S^i} \cdot R^i = 0$ . Since  $K_{X^i} + \Delta^i \equiv_{\alpha^i} \Theta^i$  and  $\text{Supp}(\Theta^i) = \lfloor \Delta^i \rfloor$ , then  $\Theta^i \cdot \Sigma^i < 0$  and hence, there is a component  $\bar{S}^i$  of  $\lfloor \Delta^i \rfloor$  such that  $\bar{S}^i \cdot \Sigma^i < 0$  and so  $\Sigma^i \subset \bar{S}^i$ . Clearly  $\Sigma^i$  is also  $K_{\bar{S}^i} + \Delta_{\bar{S}^i} := (K_{X^i} + \Delta^i)|_{\bar{S}^i}$ -negative and hence is contained in an  $\alpha_{\bar{S}^i}$ -trivial  $(K_{\bar{S}^i} + \Delta_{\bar{S}^i})$ -negative extremal face. Let  $\bar{\Sigma}^i$  be a curve spanning an extremal ray in this face, we may assume that  $\bar{S}^i \cdot \bar{\Sigma}^i < 0$ . Let  $\pi : \bar{S}^i \rightarrow V$  be the corresponding extremal contraction, then  $-\bar{S}^i$  and  $-(K_{\bar{S}^i} + \Delta_{\bar{S}^i})$  are relatively ample over  $V$  and so by Theorem 5.8 there is a contraction  $p : X^i \rightarrow Z^i$  such that  $p|_{\bar{S}^i} = \pi$  and  $p|_{X^i \setminus \bar{S}^i}$  an isomorphism. If  $p$  is a flipping contraction, then the flip exists by Theorem 5.12. We let  $X^i \dashrightarrow X^{i+1}$  be the induced bimeromorphism (be it a flip or divisorial

contraction). By construction  $p$  is  $\alpha^i$ -trivial and  $p|_{X_U^i}$  is the identity. In particular, since  $Z^i$  has rational singularities (this follows from [DH20, Lemma 2.44]),  $\alpha^i \equiv p^* \alpha_{Z^i}$  where  $\alpha_{Z^i}$  is nef. (1-3) <sub>$i+1$</sub>  are easily seen to hold. Since  $K_{X^i} + \Delta^i - \Theta^i \equiv_{\alpha^i} 0$  and  $p$  is  $\alpha^i$ -trivial, it also follows that

$$K_{Z^i} + \Delta_{Z^i} - \Theta_{Z^i} := p_*(K_{X^i} + \Delta^i - \Theta^i) \equiv_{\alpha_{Z^i}} 0$$

and hence, pulling back to  $X^{i+1}$  we have  $K_{X^{i+1}} + \Delta^{i+1} - \Theta^{i+1} \equiv_{\alpha^{i+1}} 0$  and (4) <sub>$i+1$</sub>  holds. By termination of flips Theorem 3.19, after finitely many steps, we obtain  $X \dashrightarrow X^m$  such that (5) holds.  $\square$

Now recall the nef reduction morphism  $S^\nu \rightarrow T$ . Note that if  $P$  is a component of  $\mu^*S$ , then there is an induced morphism  $\sigma_P : P \rightarrow T$  such that for any curve  $C$  on  $P$  we have  $\alpha' \cdot C = 0$  if and only if  $C$  is vertical over  $T$  i.e.  $\sigma_{P,*}C = 0$ . Note that if  $n(\alpha|_{S^\nu}) = 0$ , then  $T$  is a point and every curve is vertical over  $T$ . Since  $\phi^i : X' \dashrightarrow X^i$  is  $\alpha'$ -trivial, it follows easily that if  $\phi_*^i P = P^i \neq 0$ , then the induced bimeromorphic map  $P \dashrightarrow P^i$  is defined over  $T$  and in particular  $P^i \dashrightarrow T$  is a morphism.

*Claim 6.13.* If  $n(\alpha|_{S^\nu}) = 0$ , then the map  $\phi^m : X' \dashrightarrow X^m$  contracts  $\mu^*S$  and if  $n(\alpha|_{S^\nu}) = 1$ , then the map  $\phi^m : X' \dashrightarrow X^m$  contracts  $S'$  and every component  $E_j$  of  $\mu^*S$  such that  $n(\alpha'|_{E_j}) = 1$ . Thus if  $n(\alpha|_{S^\nu}) = 0$ , then  $\phi : X \dashrightarrow X^m$  is a bimeromorphic map that contracts  $S$  (and no other divisors) and if  $n(\alpha|_{S^\nu}) = 1$ , then  $\phi : X \dashrightarrow X^m$  contracts only  $S$  and may extract some divisors of  $X'$  such that  $n(\alpha'|_{E_j}) = 0$ .

*Proof.* Recall that  $F' \geq 0$  is a  $\mu$ -exceptional  $\mathbb{Q}$ -divisor such that  $-F'$  is  $\mu$ -ample. Let  $\Xi' := \mu^*S + \epsilon F'$  for  $0 < \epsilon \ll 1$ . We claim that for  $t \gg 0$  we have that  $(-\Xi' + t\alpha')|_{S'}$  and  $(-\Xi' + t\alpha')|_{E_j}$  are Kähler for every component  $E_j$  of  $\text{Ex}(\mu)$ . To see that  $(-\Xi' + t\alpha')|_{S'}$  is Kähler, note that  $-S|_{S^\nu} \equiv_{\alpha|_{S^\nu}} \frac{-1}{s}(K_X + B)|_{S^\nu} \equiv_{\alpha|_{S^\nu}} \frac{1}{s}\beta|_{S^\nu}$  is ample over  $T$ . Now if  $\dim T = 0$ , then  $-S|_{S^\nu}$  ample and  $\alpha|_{S^\nu} \equiv 0$ , thus  $(-S + t\alpha)|_{S^\nu}$  is Kähler for any  $t > 0$ . If  $\dim T = 1$ , then  $\alpha|_{S^\nu} = \varphi^*\gamma$  for some Kähler class  $\gamma$  on  $T$  (see [BCE<sup>+</sup>02, Proposition 2.11]), where  $\varphi : S^\nu \rightarrow T$  is the nef reduction map. Thus  $(-S + t\alpha)|_{S^\nu}$  is Kähler for  $t \gg 0$ . Since  $-F'|_{S'}$  is  $\mu|_{S'}$ -ample and hence ample over  $S^\nu$ , we have  $(-\mu^*S + t\alpha' - \epsilon F')|_{S'}$  is Kähler for  $0 < \epsilon \ll 1$ . To see that  $(-\Xi' + t\alpha')|_{E_j}$  is Kähler, we consider 2 cases. If  $\mu(E_j)$  is a point, then  $(-\mu^*S + t\alpha')|_{E_j} \equiv 0$  and  $(-\epsilon F')|_{E_j}$  is Kähler and the claim follows. Therefore, we may assume that the normalization  $V_j \rightarrow \mu(E_j)$  is a smooth curve. If  $\alpha|_{V_j} \equiv 0$ , then  $\mu(E_j)$  is a curve in  $R$  and hence  $-S|_{V_j}$  is ample and so  $(-\mu^*S - \epsilon F')|_{E_j}$  is ample for  $0 < \epsilon \ll 1$ , and  $\alpha'|_{E_j} \equiv 0$ . The claim follows. Finally, if  $\alpha|_{V_j} \neq 0$ , then  $\alpha|_{V_j}$  is Kähler so that  $(-S + t\alpha)|_{V_j}$  is Kähler for  $t \gg 0$  and hence  $(-\mu^*S + t\alpha' - \epsilon F')|_{E_j}$  is Kähler for  $0 < \epsilon \ll 1$  and the claim follows.

We claim that  $\phi^m$  is  $\Xi'$  non-positive, i.e. if  $p : W \rightarrow X'$  and  $q : W \rightarrow X^m$  is the normalization of the graph of  $\phi^m$ , then we claim that  $p^*\Xi' - q^*(\phi_*^m\Xi') \geq 0$ . Recall that  $p$  and  $q$  are isomorphisms over  $X'_U = \mu^{-1}U = \mu^{-1}(X \setminus S)$ , since so is  $\phi^m$ . We now check that  $-p^*\Xi'$  is nef over  $X^m$ . To this end consider a curve  $C$  on  $W$  such that  $q_*C = 0$ , then  $p_*C \neq 0$  is contained in  $\text{Supp}(\mu^*S)$  and  $\alpha' \cdot p_*C = \alpha^m \cdot q_*C = 0$ . But then  $-p^*\Xi' \cdot C = -\Xi' \cdot p_*C > 0$ , and in particular,  $-p^*\Xi'$  is nef over  $X^m$ . Let  $\Xi^m := q_*p^*\Xi' = \phi_*^m\Xi'$ , then  $q^*\Xi^m - p^*\Xi'$  is  $q$ -exceptional and  $q$ -nef so that by the negativity lemma,  $p^*\Xi' - q^*\Xi^m \geq 0$ ; this proves our claim.

Suppose now that  $n(\alpha|_{S^\nu}) = 0$  and  $\phi_*^m(\mu^*S) \neq 0$  (resp.  $n(\alpha|_{S^\nu}) = 1$  and there is a component  $E$  of  $S' + \sum E_j$  such that  $n(\alpha'|_E) = 1$  and  $E^m := \phi_*^m E \neq 0$ ). Let  $\lambda$  be the smallest constant such that  $\Theta^m - \lambda\Xi^m \leq 0$  (resp.  $\text{mult}_{E^m}(\Theta^m - \lambda\Xi^m) \leq 0$ , where  $E^m = \phi_*^m E$ ). Thus in both cases, there is component  $E$  of  $S' + \sum E_j$  such that  $\text{mult}_{E^m}(\Theta^m - \lambda\Xi^m) = 0$ , and there is a family of  $\alpha^m$ -trivial curves  $\{C_t^m\}$  covering  $E^m$ . We claim that  $(K_{X^m} + \Delta^m) \cdot C_t^m \geq 0$ . If this is not the case, then

$$[C_t^m] \in (\alpha^m)^\perp \cap \overline{\text{NA}}(E^m)_{K_{E^m} + \Delta_{E^m} < 0},$$

where  $K_{E^m} + \Delta_{E^m} = (K_{X^m} + \Delta^m)|_{E^m}$ . Since  $\alpha^m$  is nef, by the cone theorem on  $E^m$ , there is a  $(K_{E^m} + \Delta_{E^m})$ -negative  $\alpha^m$ -trivial extremal ray, contradicting (5) above. If  $C_t$  is the strict transform of  $C_t^m$  on  $E = (\phi^m)_*^{-1}E^m$ , then

$$0 \geq (\Theta^m - \lambda\Xi^m) \cdot C_t^m = (K_{X^m} + \Delta^m - \lambda\Xi^m) \cdot C_t^m \geq -\lambda\Xi^m \cdot C_t^m \geq -\lambda\Xi' \cdot C_t > 0$$

where the first inequality follows by our definition of  $\lambda$ , the second as  $(K_{X^m} + \Delta^m - \Theta^m) \equiv_{\alpha^m} 0$ , the third was observed above, the fourth as  $p^*\Xi' - q^*\Xi^m \geq 0$  and  $\text{mult}_{E^m}(p^*\Xi' - q^*\Xi^m) = 0$  and the last as  $(-\Xi' + t\alpha')|_E$  is ample. This is the required contradiction and the claim is proven.  $\square$

*Claim 6.14.* The map  $\phi^m : X' \dashrightarrow X^m$  contracts every component of  $\mu^*S$  dominating  $T$ , and hence if  $n(\alpha) = 0$ , then  $\phi : X \dashrightarrow X^m$  is a bimeromorphic map that contracts  $S$  (and no other divisors) and if  $n(\alpha) = 1$ , then  $\phi : X \dashrightarrow X^m$  contracts only  $S$  and may extract some divisors of  $X'$  which are contained in  $\mu^{-1}(S)$  that do not dominate  $T$ .

*Proof.* Recall that  $F' \geq 0$  is a  $\mu$ -exceptional  $\mathbb{Q}$ -divisor such that  $-F'$  is  $\mu$ -ample. Since  $-S|_{S^\nu} \equiv_\alpha \frac{-1}{s}(K_X + B)|_{S^\nu}$  is ample over  $T$ , then  $\Xi' := \mu^*S + \epsilon F'$  is a  $\mathbb{Q}$ -divisor such that  $-\Xi'|_{S'}$  and  $-\Xi'|_{E_j}$  are ample over  $T$  for every component  $E_j$  of  $\text{Ex}(\mu)$  and all  $0 < \epsilon \ll 1$ .

We claim that  $\phi^m$  is  $\Xi'$  non-positive, i.e. if  $p : W \rightarrow X'$  and  $q : W \rightarrow X^m$  is the normalization of the graph of  $\phi^m$ , then we claim that  $p^*\Xi' - q^*(\phi_*^m\Xi') \geq 0$ . Recall that  $p$  and  $q$  are isomorphisms over  $X'_U = \mu^{-1}U = \mu^{-1}(X \setminus S)$ , since

so is  $\phi^m$ . We now check that  $-p^*\Xi'$  is nef over  $X^m$ . To this end consider a curve  $C$  on  $W$  such that  $q_*C = 0$ , then  $p_*C \neq 0$  is contained in  $\text{Supp}(\mu^*S)$  and  $\alpha' \cdot p_*C = \alpha^m \cdot q_*C = 0$ . But then  $p_*C$  is vertical over  $T$  and so  $-p^*\Xi' \cdot C = -\Xi' \cdot p_*C > 0$ ; in particular,  $-p^*\Xi'$  is nef over  $X^m$ . Let  $\Xi^m := q_*p^*\Xi' = \phi_*^m\Xi'$ , then  $q^*\Xi^m - p^*\Xi'$  is  $q$ -exceptional and  $q$ -nef so that by the negativity lemma,  $p^*\Xi' - q^*\Xi^m \geq 0$ ; this proves our claim.

Suppose now that  $\phi_*^m(\mu^*S)$  dominates  $T$  i.e. that  $\phi_*^m(\mu^*S) \rightarrow T$  is surjective and let  $\lambda$  be the smallest constant such that  $(\Theta^m - \lambda\Xi^m)_{\text{hor}} \leq 0$ , where  $(\dots)_{\text{hor}}$  denotes the horizontal components over  $T$  i.e. those components that dominate  $T$ . Then there is a component  $E^m$  of  $\phi_*^m(\mu^*S)$  dominating  $T$  such that  $\text{mult}_{E^m}(\Theta^m - \lambda\Xi^m) = 0$ , and there is a family  $C_t^m$  (contained in the fibers over  $T$ ) of  $\alpha^m$ -trivial curves covering  $E^m$ . We claim that  $(K_{X^m} + \Delta^m) \cdot C_t^m \geq 0$ . If this is not the case, then

$$[C_t^m] \in (\alpha^m)^\perp \cap \overline{\text{NA}}(E^m)_{K_{E^m} + \Delta_{E^m} < 0}$$

where  $K_{E^m} + \Delta_{E^m} = (K_{X^m} + \Delta^m)|_{E^m}$ . Since  $\alpha^m$  is nef, by the cone theorem on  $E^m$ , there is a  $(K_{E^m} + \Delta_{E^m})$ -negative  $\alpha^m$ -trivial extremal ray, contradicting (5) above. If  $C_t$  is the strict transform on  $E = (\phi^m)_*^{-1}E^m$ , then

$$0 \geq (\Theta^m - \lambda\Xi^m) \cdot C_t^m = (K_{X^m} + \Delta^m - \lambda\Xi^m) \cdot C_t^m \geq -\lambda\Xi^m \cdot C_t^m \geq -\lambda\Xi' \cdot C_t > 0$$

where the first inequality follows by our definition of  $\lambda$ , the second as  $(K_{X^m} + \Delta^m - \Theta^m) \equiv_{\alpha^m} 0$ , the third was observed above, the fourth as  $p^*\Xi' - q^*\Xi^m \geq 0$  and  $\text{mult}_{E^m}(p^*\Xi' - q^*\Xi^m) = 0$  and the last as  $-\Xi'|_E$  is ample over  $T$ . This is the required contradiction and the claim is proven.  $\square$

We will now show that in fact  $\phi^m$  contracts  $\mu^*S$ . To this end, it suffices to show that  $\Theta^m = 0$  or equivalently that  $q^*\Theta^m = 0$ . Since  $q^*\Theta^m$  is effective and exceptional over  $X$ , by the negativity lemma it suffices to show that  $q^*\Theta^m$  is nef over  $X$ . Let  $C \subset W$  be a curve such that  $\mu_*p_*C = 0$ . If  $q_*C = 0$ , then  $C \cdot q^*\Theta^m = 0$ . If instead  $q_*C \neq 0$ , then  $\alpha^m \cdot q_*C = \alpha \cdot \mu_*p_*C = 0$ . Clearly  $q^*\Theta^m \cdot C \geq 0$  if  $q_*C$  is not contained in any component  $P$  of the support  $\Theta^m$  (and hence of the support of  $[\Delta^m]$ ). So assume that  $q_*C$  is contained in a component  $P$  of  $\Theta^m$ . Since  $\Theta^m \equiv_{\alpha^m} K_{X^m} + \Delta^m$  and  $(K_{X^m} + \Delta^m)|_P$  is non-negative on  $\alpha^m$ -trivial curves (by (5) above), then  $q^*\Theta^m \cdot C = (K_{X^m} + \Delta^m) \cdot q_*C \geq 0$ . Therefore  $q^*\Theta^m$  is nef over  $X$  and hence  $\Theta^m = 0$  as observed above.

We will now show that the induced bimeromorphic map  $\phi : X \dashrightarrow X^m$  is a morphism. Let  $p : W \rightarrow X$  and  $q : W \rightarrow X^m$  be a resolution of the graph of  $\phi$ . Pick  $\omega^m$  a Kähler class on  $X^m$  and  $E_1, \dots, E_r$  the  $p$ -exceptional divisors generating  $H_{\text{BC}}^{1,1}(W)/p^*H_{\text{BC}}^{1,1}(X)$  as in [DH20, Lemma 2.32]. Then we may write  $q^*\omega^m + \sum e_i E_i \equiv_X 0$  for some  $e_i \in \mathbb{R}$ , and hence there is a  $\omega \in H_{\text{BC}}^{1,1}(X)$  such that  $p^*\omega \equiv q^*\omega^m + \sum e_i E_i$ . Pick  $r \in \mathbb{R}$  such that  $(\omega + rS) \cdot R = 0$ , then

$p^*(\omega + rS) \equiv q^*\omega^m + \sum e_i E_i + rp^*S$ . Let  $C$  be a  $q$ -exceptional curve. If  $p_*C = 0$ , then  $(q^*\omega^m + \sum e_i E_i + rp^*S) \cdot C = (\omega + rS) \cdot p_*C = 0$ . If  $p_*C \neq 0$ , then  $p^*\alpha \cdot C = q^*\alpha^m \cdot C = 0$  and so  $p_*C \in R$  and hence  $(\omega + rS) \cdot p_*C = 0$  so that  $(q^*\omega^m + \sum e_i E_i + rp^*S) \cdot C = 0$ . Recall that  $\Theta^m = 0$ , in particular,  $\phi$  contracts  $S$  and doesn't extract any divisor. Thus we have shown that  $\sum e_i E_i + rp^*S$  is  $q$ -exceptional and  $q$ -numerically trivial. Then by the negativity lemma  $\sum e_i E_i + rp^*S = 0$  and so  $p^*(\omega + rS) \equiv q^*\omega^m$ . But then, if  $C$  is  $p$ -exceptional and not  $q$ -exceptional, we have

$$0 = C \cdot p^*(\omega + rS) = C \cdot q^*\omega^m = q_*C \cdot \omega^m > 0$$

which is impossible. Since no such curves exist, then  $\phi : X \dashrightarrow X^m$  is a morphism which we will now denote by  $f : X \rightarrow Z$ .

As we have seen above,  $f|_U$  is an isomorphism and  $f$  contracts a non-empty set of  $\alpha$ -trivial curves. Since  $\overline{\text{NA}}(X) \cap \alpha^\perp = R$ , then  $f$  contracts the set of all curves in  $R$  (and no other curves). Since  $K_X + B + \beta \equiv_Z 0$ , then  $-(K_X + B)$  is  $f$ -ample. Then from [DH20, Lemma 2.44] it follows that  $Z$  has rational singularities. By [HP16, Lemma 3.3],  $\rho(X/Z) = 1$  and  $\alpha \equiv f^*\alpha_Z$  for some  $\alpha_Z \in H_{\text{BC}}^{1,1}(Z)$ . By [DHP24, Theorem 2.29],  $\alpha_Z$  is Kähler if and only if for any positive dimensional subvariety  $W \subset Z$ ,  $\alpha_Z^{\dim W} \cdot W > 0$  holds. If  $W \not\subset f(S)$ , then the strict transform  $W' \subset X$  is not contained in  $S$  and so  $\alpha_Z^{\dim W} \cdot W = \alpha^{\dim W'} \cdot W' > 0$ . Therefore, we may assume that  $W = f(S)$  is a (projective) curve. Since  $f$  is a projective morphism,  $S$  is projective, and thus there is a (projective curve)  $W' \subset S$  such that  $f(W') = W$ . Then  $\alpha \cdot W' > 0$ , since by our construction above a curve in  $X$  is contracted by  $f$  if and only if it is  $\alpha$ -trivial. Thus by the projection formula  $\alpha_Z \cdot W > 0$ ; this completes our proof.  $\square$

We conclude this section by proving the existence of a flipping contraction for non  $\mathbb{Q}$ -factorial 3-folds.

**Lemma 6.15.** *Let  $(X, B)$  be a compact Kähler 3-fold klt pair, and  $\alpha$  a nef and big class such that  $\text{Null}(\alpha) = C$  is a finite union of curves. Then there exists a bimeromorphic morphism of normal Kähler varieties  $\nu : X' \rightarrow X$  and an effective  $\nu$ -exceptional  $\mathbb{Q}$ -Cartier divisor  $\Psi \geq 0$  such that  $\omega' = \nu^*\alpha - \Psi$  is Kähler and  $\Psi = \nu^{-1}(C)$ . In particular,  $-\Psi|_{\text{Supp}(\Psi)}$  is ample.*

*Proof.* Fix a Kähler form  $\omega$  on  $X$ , and let  $\mu : \tilde{X} \rightarrow X$  be a log resolution of  $(X, B)$  and  $C$ . By Lemma 3.5 and Theorem 3.6 there is a positive current  $T \in \alpha$  with weak analytic singularities such that  $E_+(T) = C$ , and  $T \geq \epsilon\omega$  for some  $\epsilon > 0$ . By Lemma 3.7 we may assume that we can write  $\mu^*\alpha \equiv [\tilde{\Theta}] + \Phi$ , where  $\Phi \geq 0$  is an effective  $\mathbb{R}$ -divisor such that  $\mu(\text{Supp}\Phi) = C$ , and  $\tilde{\Theta}$  is a closed positive  $(1, 1)$  current such that  $\tilde{\Theta} \geq \epsilon\mu^*\omega$  and the class  $[\tilde{\Theta} - \epsilon\mu^*\omega] \in H_{\text{BC}}^{1,1}(\tilde{X})$

is nef. In particular,  $-\Phi$  is nef over  $X$ , and thus from the negativity lemma it follows that  $\text{Supp}(\Phi) = \mu^{-1}(C)$ .

Let  $\Delta := \mu_*^{-1}B + \text{Ex}(\mu)$ , then

$$K_{\tilde{X}} + \Delta \equiv_X G := \sum_{i=1}^k g_i G_i \geq 0,$$

where  $\text{Supp}(G) = \text{Ex}(\mu)$ , as  $(X, B)$  is klt. Let  $E = \sum_{i=1}^k e_i G_i$ , where  $e_i > 0$ ,  $E$  is  $\mu$ -exceptional, and  $-E$  is  $\mu$ -ample. Perturbing the coefficients of  $E$ , we may assume that  $e_1, \dots, e_k$  are linearly independent over  $\mathbb{Q}(g_1, \dots, g_k)$ .

We now run the  $[\tilde{\Theta}]$ -trivial  $(K_{\tilde{X}} + \Delta)$ -Minimal Model Program over  $X$  with scaling of  $-E$  as in [Kol21, Theorem 2]. This means that at each step we only contract  $(K_{\tilde{X}} + \Delta)$ -negative extremal rays over  $X$  that are  $[\tilde{\Theta}]$ -trivial. All steps of this MMP exists by [Nak87], [DHP24] and [Fuj22a, Theorem 1.2]. Note that since each step of this mmp is  $E$  negative, the contracted locus is contained in the support of  $E$  and hence in the support of  $[\Delta]$ . Since the mmp for surfaces holds, termination follows easily from the usual arguments for special termination [Fuj07].

Let  $\phi : \tilde{X} \dashrightarrow X'$  be the output of this MMP and  $\nu : X' \rightarrow X$  the induced map. Following [Kol21, Theorem 2], we may assume that there is a  $\nu$ -exceptional  $\mathbb{Q}$ -Cartier divisor  $F \geq 0$  such that  $-F + \lambda[\Theta']$  is Kähler over  $X$  for some  $\lambda > 0$ , where  $\Theta' := \phi_*\tilde{\Theta}$ . Note that every step of this MMP is also  $[\tilde{\Theta} - \epsilon\mu^*\omega]$ -trivial, and hence the class  $[\Theta' - \epsilon\nu^*\omega]$  is nef.

Let  $U = X \setminus C$ ,  $\tilde{U} = \mu^{-1}(U)$ , and  $U' = \nu^{-1}(U)$ , then  $\Phi|_{\tilde{U}} = 0$  and so  $[\tilde{\Theta}]|_{\tilde{U}} \equiv_{\tilde{U}} 0$ , and hence over  $U$  this is the usual relative MMP. Since  $G'|_{U'} \geq 0$ , where  $G' = \phi_*G$ , and  $G'|_{U'}$  is nef, by the negativity lemma it follows that  $G'|_{U'} = 0$ . Thus, there are no  $\nu|_{U'}$ -exceptional divisors. As  $-F|_{U'}$  is relatively ample, we have  $U' = U$ .

It then follows that, for  $0 < \delta \ll 1$  the class of

$$(1 + \lambda\delta)\Theta' - \delta F = \epsilon\nu^*\omega - \delta(F - \lambda\Theta') + (\Theta' - \epsilon\nu^*\omega)$$

is Kähler (as  $[\Theta' - \epsilon\nu^*\omega]$  is a nef class and  $[\epsilon\nu^*\omega - \delta(F - \lambda\Theta')]$  is a Kähler class for  $0 < \delta \ll 1$ ). We let  $\omega' := [\Theta' - \frac{\delta}{1+\lambda\delta}F]$  and  $\Psi = \Phi' + \frac{\delta}{1+\lambda\delta}F$ . Then

$$\nu^*\alpha = \phi_*(\mu^*\alpha) = \phi_*([\tilde{\Theta}] + \Phi) = [\Theta'] + \Phi' = \omega' + \Psi,$$

and the claim follows. Note that perturbing  $\Psi$  and  $\omega'$ , we may assume that  $\Psi$  is a  $\mathbb{Q}$ -divisor.  $\square$

**Theorem 6.16.** *Let  $(X, B)$  be a compact Kähler 3-fold klt pair and  $K_X + B$  pseudo-effective. Suppose that there is a Kähler class  $\omega$  such that  $\alpha = K_X + B + \omega$  is a nef and big class but not Kähler, and  $\text{Null}(\alpha) = C$  is a finite union of curves. Then there exists a bimeromorphic morphism  $f : X \rightarrow Z$  of normal*

compact Kähler varieties and a Kähler class  $\alpha_Z$  on  $Z$  such that  $\alpha = f^*\alpha_Z$ ; in particular,  $\text{Ex}(f) = \text{Null}(\alpha)$ .

*Proof.* By Lemma 6.15, there is a bimeromorphic morphism of normal compact Kähler varieties  $\nu : X' \rightarrow X$  and an effective  $\nu$ -exceptional  $\mathbb{Q}$ -divisor  $\Psi$  such that  $\nu(\Psi) = C$  and  $-\Psi|_{\text{Supp}(\Psi)}$  is ample. By Lemma 4.3, there is a bimeromorphic morphism  $g : X' \rightarrow Z$  to a normal compact analytic variety  $Z$  such that  $g(\text{Supp}(\Psi))$  is a finite set of points and  $X \setminus \text{Supp}(\Psi) \rightarrow Z \setminus g(\text{Supp}(\Psi))$  is an isomorphism. By the rigidity lemma (see [BS95, Lemma 4.1.13]), we obtain a bimeromorphic morphism  $f : X \rightarrow Z$  such that  $f \circ \nu = g$ . In particular,  $\text{Ex}(f) = C = \text{Null}(\alpha)$ . Thus,  $-(K_X + B)$  is  $f$ -ample, hence  $Z$  has rational singularities by [DH20, Lemma 2.44]. Then by [HP16, Lemma 3.3], there is an  $(1, 1)$  class  $\omega_Z \in H_{\text{BC}}^{1,1}(Z)$  such that  $\alpha = f^*\omega_Z$ . Finally, by the projection formula and [DHP24, Theorem 2.29] it follows that  $\omega_Z$  is a Kähler class.  $\square$

## 7. CONE THEOREM AND MINIMAL MODELS

In this section we will establish the cone theorem and existence of minimal models for klt pairs  $(X, B)$  of dimension 3 such that  $K_X + B$  is pseudo-effective.

Let  $K \subset \mathbb{R}^n$  be a compact convex set. An *extremal point* of  $K$  is a point  $x$  such that if  $x = tx_1 + (1-t)x_2$  for  $x_i \in K$ , then  $x = x_1$  or  $x = x_2$ ; equivalently  $K \setminus \{x\}$  is convex. An *exposed point* of  $K$  is a point  $x \in K$  such that  $\{x\} = K \cap \{f = 0\}$  where  $\{f = 0\}$  is a supporting hyperplane for  $K$ . Every exposed point is extremal but not vice versa. We let  $\text{ext}(K)$  be the set of extremal points,  $\text{exp}(K)$  the set of exposed points and we denote by  $\text{conv}(X)$  the convex hull of a set  $X \subset \mathbb{R}^n$  and  $\text{cl}(X)$  the closure of  $X$ . For a detailed discussion on these topics see [HW20, Chapter 1]. We need the following result:

**Theorem 7.1.** *Let  $K \subset \mathbb{R}^n$  be a non-empty compact convex set. Then*

$$\text{cl}(\text{conv}(\text{exp}(K))) = \text{conv}(\text{ext}(K)) = K.$$

*In particular, the sets  $\text{exp}(K)$  and  $\text{ext}(K)$  are both non-empty.*

*Proof.* The equality  $\text{conv}(\text{ext}(K)) = K$  is well known and due to Minkowski (see [HW20, Theorem 1.21]), and the equality  $\text{cl}(\text{conv}(\text{exp}(K))) = K$  follows from [HW20, Theorem 1.23].  $\square$

*Proof of Theorem 1.3.* By standard arguments (see the proof of [DHP24, Corollary 5.3]), passing to a strongly  $\mathbb{Q}$ -factorial model (which exists by [DH20, Lemma 2.28]), we may assume that  $X$  is strongly  $\mathbb{Q}$ -factorial. Let  $K$  be a compact convex slice of  $\overline{\text{NA}}(X)$  of Euclidean dimension  $n - 1$ , where  $n = \dim_{\mathbb{R}} H_{\text{BC}}^{1,1}(X)$ , i.e. the cone generated by  $K$  is equal to  $\overline{\text{NA}}(X)$ , and  $R$  an

extremal ray corresponding to an exposed point  $\{x\} \in K$ . Then by definition, there is a  $(1, 1)$  class  $\alpha \in H_{\text{BC}}^{1,1}(X)$  such that  $K$  is contained in the half-space  $\alpha_{\geq 0} := \{\gamma \in N_1(X) : \alpha \cdot \gamma \geq 0\}$  and  $\{x\} = K \cap \alpha^\perp$ . In particular,  $R = \overline{\text{NA}}(X) \cap \alpha^\perp$ . Since  $\alpha$  is non-negative on  $\overline{\text{NA}}(X)$ , it is nef. If  $(K_X + B) \cdot R < 0$ , then for  $0 < a \ll 1$ , it follows easily that  $\beta := \alpha - a(K_X + B)$  is positive on  $\overline{\text{NA}}(X) \setminus \{0\}$ , and hence Kähler (e.g. see the proof of [DHY23, Claim 3.25]). Thus  $\alpha = a(K_X + B + \frac{1}{a}\beta)$ . Replacing  $\alpha$  by  $\frac{1}{a}\alpha$  and letting  $\omega = \frac{1}{a}\beta$  we may assume that  $\alpha \equiv K_X + B + \omega$ , where  $\omega$  is a Kähler class. By Theorems 6.2 and 6.9, there is a corresponding flipping or divisorial contraction and hence  $R = \mathbb{R}^+[\Gamma]$  for some curve  $\Gamma$  on  $X$ . By [DO24, Theorem 1.23], we may assume that  $\Gamma$  is a rational curve and  $0 > (K_X + B) \cdot \Gamma \geq -6$ . Since  $X$  is a Kähler variety, by a Douady space argument there are at most countably many such extremal rays  $\{\mathbb{R}^+ \cdot [\Gamma_i]\}_{i \in I}$ . Let  $V := \overline{\text{NA}}(X)_{K_X + B \geq 0} + \sum_{i \in I} \mathbb{R}^+[\Gamma_i]$ . Following [HP16, Lemma 6.1] it suffices to show that  $\overline{\text{NA}}(X) = \overline{V}$ . To the contrary assume that  $\overline{\text{NA}}(X) \setminus \overline{V} \neq \emptyset$ . Since  $\overline{V}$  is a convex set (as so is  $V$ ) and  $\overline{\text{NA}}(X)$  is generated by  $K$ , from Theorem 7.1 it follows that there is an exposed point  $x \in K$  not contained in  $\overline{V}$ . Let  $R$  be the  $(K_X + B)$ -negative extremal ray corresponding to the exposed point  $x \in K$ . Then from what we have seen above, there is a rational curve  $\Gamma$  such that  $0 < -(K_X + B) \cdot \Gamma \leq 6$  and  $R = \mathbb{R}^+ \cdot [\Gamma]$ , but all such extremal rays are contained in  $\overline{V}$  by definition of  $V$ , this is a contradiction.  $\square$

*Proof of Theorem 1.2.* Choose  $0 < \epsilon \ll 1$  so that  $\omega + \epsilon B$  is Kähler. Thus replacing  $B$  by  $(1 - \epsilon)B$  and  $\omega$  by  $\omega + \epsilon B$  we may assume that  $(X, B)$  is a klt pair. By [DH20, Lemma 2.27] there is a small projective bimeromorphic morphism  $g : X' \rightarrow X$  such that  $X'$  is strongly  $\mathbb{Q}$ -factorial and  $K_{X'} + B' = g^*(K_X + B)$ . Let  $\alpha' := g^*\alpha$  and run a  $\alpha'$ -trivial  $(K_{X'} + B')$ -MMP using the cone Theorem 1.3 and the contraction Theorems 6.2 and 6.9. Note that each step of this MMP preserves strong  $\mathbb{Q}$ -factoriality by [DH20, Lemma 2.5]. This MMP terminates by Theorem 3.19. Let  $\phi : X' \dashrightarrow X^n$  be the end result of this MMP,  $B^n := \phi_* B'$  and  $\alpha^n := \phi_* \alpha'$ . We claim that  $\text{Null}(\alpha^n)$  does not contain any surface. Assume to the contrary that there is a surface  $S^n \subset \text{Null}(\alpha^n)$ . Let  $S \subset \text{Null}(\alpha)$  be the strict transform of  $S^n$  under the induced map  $\psi := \phi \circ g^{-1} : X' \dashrightarrow X^n$ . We note that  $S$  exists, since  $\psi$  does not extract any divisor. Then from the proof of Claim 6.10 it follows that  $\text{Null}(\alpha) = S$ , and  $S$  is covered by an analytic family of  $\alpha$ -trivial curves  $\{C_t\}_{t \in T}$  such that  $S \cdot C_t < 0$  for all  $t \in T$ . Now recall that  $\alpha = K_X + B + \omega$ , where  $\omega$  is a Kähler class; let  $\omega^n := \psi_* \omega$  and  $C_t^n := \psi_* C_t$ . Then  $\{C_t^n\}_{t \in T}$  is a family of  $\alpha^n$ -trivial curves covering  $S^n$ . Now since  $\alpha^n = K_{X^n} + B^n + \omega^n$ , we have  $(K_{X^n} + B^n) \cdot C_t^n = (K_{X^n} + B^n)|_{S^n} \cdot C_t^n = -\omega^n|_{S^n} \cdot C_t^n < 0$ , since  $\omega^n|_{S^n}$  is

big as  $\omega^n$  is a modified Kähler class. This is a contradiction to the fact that  $(K_{X^n} + B^n) \cdot C \geq 0$  for all  $\alpha^n$ -trivial curves  $C$ .

Now by Theorem 3.6,  $\text{Null}(\alpha^n) = E_{nK}^{as}(\alpha^n)$  is an analytic subset of  $X^n$ , and hence  $\text{Null}(\alpha^n)$  is either an empty set or a finite union of curves. First assume that  $\text{Null}(\alpha^n)$  is a finite union of curves. In this case by Theorem 6.16, there is a small proper bimeromorphic morphism  $h : X^n \rightarrow Z$  to a normal analytic variety  $Z$  and a Kähler class  $\alpha_Z \in H_{BC}^{1,1}(Z)$  such that  $\alpha^n = h^*\alpha_Z$  and  $\text{Ex}(h) = \text{Null}(\alpha^n)$ . Let  $W$  be a resolution of the graph of  $\phi : X' \dashrightarrow X^n$  and  $p : W \rightarrow X'$  and  $q : W \rightarrow X^n$  are projection morphisms; then  $(g \circ p)^*\alpha = (h \circ q)^*\alpha_Z$ . Let  $C \subset W$  a curve contracted by  $g \circ p$ , then  $(h \circ q)^*\alpha_Z \cdot C = (g \circ p)^*\alpha \cdot C = 0$ . Since  $\alpha_Z$  is Kähler, this implies that  $C$  is also contracted by  $h \circ q$ . Then by the rigidity lemma [BS95, Lemma 4.1.13], there is a morphism  $f : X \rightarrow Z$  such that  $f \circ (g \circ p) = h \circ q$ . In particular,  $\alpha = f^*\alpha_Z$ , and thus a curve  $C \subset X$  is contracted by  $f$  if and only if  $\alpha \cdot C = 0$ , i.e.  $[C] \in R$ . The only thing that remains to be shown in this case is that  $f$  is projective. To see this, observe that  $-(K_X + B) \equiv_f \omega$ , and hence  $-(K_X + B)$  is a  $\mathbb{Q}$ -Cartier  $f$ -ample divisor on  $X$ , thus  $f$  is projective.

Now if  $\text{Null}(\alpha^n) = \emptyset$ , then by [DHP24, Theorem 2.29],  $\alpha^n$  is a Kähler class on  $X^n$ . We rename  $X^n$  by  $Z$ ; then by a similar argument as above it follows that there is a projective bimeromorphic morphism  $f : X \rightarrow Z$  such that  $\alpha = f^*\alpha^n$ . In particular, a curve  $C \subset X$  is contracted by  $f$  if and only if  $\alpha \cdot C = 0$ , i.e.  $[C] \in R$ .

Finally, if  $f$  is a divisorial contraction, then by a similar argument as in the proof of [KM98, Proposition 3.36] it follows that  $Z$  is  $\mathbb{Q}$ -factorial, and  $(Z, f_*B)$  is dlt (see [KM98, Lemma 3.38]). If  $f$  is a flipping contraction, then the flip  $f^+ : X^+ \rightarrow Z$  is defined as  $X^+ := \text{Projan } \bigoplus_{m \geq 0} f_*\mathcal{O}_X(md(K_X + B))$ , where  $d$  is the Cartier index of  $K_X + B$ ; the finite generation of this ring is due to Shokurov, see [CHP16, Theorem 4.3], also, for a more general result in all dimensions see [DHP24, Theorem 1.3].

□

*Proof of Theorem 1.4.* Suppose that  $K_X + B$  is not nef, then by Theorem 1.3 and its proof, there is a  $(K_X + B)$ -negative extremal ray  $R$  with nef supporting class  $\alpha$  such that  $R = \overline{\text{NA}}(X) \cap \alpha^\perp$ , and the corresponding flipping or divisorial contractions exist by Theorem 1.2. Let  $(X, B) \dashrightarrow (X_1, B_1)$  be the corresponding flip or divisorial contraction. We may replace  $(X, B)$  by  $(X_1, B_1)$  and repeat the above procedure. By termination of flips Theorem 3.19, after finitely many steps we obtain the required minimal model. □

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