MMP FOR GENERALIZED PAIRS ON KÄHLER 3-FOLDS

OMPROKASH DAS, CHRISTOPHER HACON, AND JOSÉ IGNACIO YÁÑEZ

Abstract. In this article we define generalized pairs $(X, B+\boldsymbol{\beta})$ where X is an analytic variety and $\boldsymbol{\beta}$ is a b-(1,1) current. We then prove that almost all standard results of the MMP hold in this generality for compact Kähler varieties of $\dim X \leq 3$. More specifically, we prove the cone theorem, existence of flips, existence of log terminal models, log canonical models and Mori fiber spaces, geography of log canonical and log terminal models, etc.

Contents

Omprokash Das was partially supported by the Start–Up Research Grant(SRG), Grant No. # SRG/2020/000348 of the Science and Engineering Research Board (SERB), Govt. Of India.

Christopher Hacon and José Ignacio Yáñez were partially supported by NSF research grants no: DMS-1952522, DMS-1801851 and by a grant from the Simons Foundation; Award Number: 256202.

1. Introduction

In this article we will develop the minimal model program for generalized Kähler surfaces and threefolds. Generalized pairs naturally arise in the context of Kawamata's canonical bundle formula and adjunction to lc centers, and have been playing an increasingly important role in the birational geometry of complex projective varieties (see [Kaw98], [FM00], [BZ16], [Bir21] and references therein). It is natural to hope that these results carry over to the context of Kähler manifolds, especially for surfaces and threefolds where the usual minimal model program is known to hold (see [HP16, HP15, CHP16], [DO23], [DH20] and references therein). We introduce generalized Kähler pairs (Definition 2.7), in a context which is more general than the usual definition of generalized pairs from projective geometry. Roughly speaking, a generalized pair $(X/S, B+\beta)$ consists of a proper morphism $X \to S$ of normal Kähler varieties, a pair (X, B), and a closed positive (1,1) current $\beta \in H^{1,1}_{\rm BC}(X)$ which is (bimeromorphically) nef over S (we refer the reader to Definition 2.7 for the technical nuances; we will denote the corresponding closed positive b-(1,1) current by $\boldsymbol{\beta}$, but for the purposes of this introduction, we will sometimes abuse notation and just refer to $\beta = \boldsymbol{\beta}_X$, the trace of $\boldsymbol{\beta}$ on X). Note that in the case of projective varieties one requires the more restrictive condition that β is a $\mathbb{R}$-divisor (birationally nef over S). Thus, if $H^2(X, \mathcal{O}_X) \neq 0$ (and hence $\mathrm{NS}(X)_{\mathbb{R}} \neq H^{1,1}_{\rm BC}(X)$), this allows us more flexibility even in the projective case. This is particularly important in the Kähler case as there may be very few $\mathbb{R}$-divisors whilst $H^{1,1}_{\rm BC}(X)$ may contain many interesting classes. For example, working in this generality allows us to:

(1) Prove the finiteness of certain 3-fold minimal models (see Theorem 3.26).
(2) Show that different 3-fold minimal models are connected by flops (see Theorems 3.26 and 3.29).
(3) Run the minimal model program with scaling of a Kähler form ω (see Theorems 3.21 and 3.23).

It is then possible to consider the various flavors of singularities of the minimal model program for generalized pairs (klt, lc, dlt etc.) and to show several natural properties (in all dimensions), such as the fact that generalized klt singularities are rational, and if X is Stein, then a generalized klt pair $(X, B+\boldsymbol{\beta})$ is equivalent to a usual klt pair $(X, B+\Delta)$ and in particular it admits a $\mathbb{Q}$-factorization (see Theorem 2.19). In Section 2.4 we give a treatment of the generalized surface MMP including the cone theorem, the existence of minimal models and Mori fiber spaces, and the existence of log canonical models when $K_X+B+\beta$ is big. In Section 3.1 we then develop the minimal model program for 3-fold generalized klt pairs. We show that 3-fold klt flips exist:

Theorem 1.1. *Let $(X, B+\boldsymbol{\beta})$ be a compact Kähler $\mathbb{Q}$-factorial 3-fold generalized klt pair, and $f : X \to Z$ a flipping contraction, then the flip $X^+ \to Z$ exists.*

Proving the termination of flips in this generality however turns out to be too difficult. Instead, following the approach of [BCHM10], we show that certain generalized minimal model programs with scaling terminate. For example, if $(X, B+\boldsymbol{\beta})$ is a compact Kähler $\mathbb{Q}$-factorial 3-fold generalized klt pair and $\beta = \boldsymbol{\beta}_X$ is Kähler, then $K_X + B + t\beta$ is Kähler for $t \gg 0$, and a $K_X + B + \beta$ mmp with scaling of $(t-1)\beta$ is also a $K_X + B$ mmp with scaling of $t\beta$ and so, in this case, termination follows from standard results on the termination of flips for the usual klt 3-fold pair (X, B). This allows us to prove the existence of minimal and canonical models.

Theorem 1.2. *Let $(X, B+\boldsymbol{\beta})$ be a generalized compact klt Kähler 3-fold pair.*

1. *If $K_X + B + \boldsymbol{\beta}_X$ is big, then $(X, B+\boldsymbol{\beta})$ has a log terminal model $f : X \dashrightarrow X^{\mathrm{m}}$ and a unique log canonical model $g : X^{\mathrm{m}} \to X^{\mathrm{c}}$.*
2. *If $K_X + B + \boldsymbol{\beta}_X$ is pseudo-effective and $\boldsymbol{\beta}_X$ is big, then $K_X + B + \boldsymbol{\beta}_X$ has a log terminal model $f : X \dashrightarrow X^{\mathrm{m}}$ and there is a contraction $g : X^{\mathrm{m}} \to Z$ such that $f_*(K_X + B + \boldsymbol{\beta}_X) \equiv g^*\omega_Z$ where ω_Z is a Kähler form on Z.*

For more general minimal model programs with scaling, termination of flips is achieved by studying Shokurov polytopes and the geography of minimal models. In particular we show the following (please see Theorems 3.19 and 3.26 for a more comprehensive statement).

Theorem 1.3. *Let X be a smooth compact Kähler 3-fold, B a simple normal crossings divisor, and Ω a compact convex polyhedral set of closed positive (1,1)-currents such that $[\beta]$ is nef and $[K_X + B + \beta]$ is big for all $\beta \in \Omega$. Then there exist a finite polyhedral decomposition $\Omega = \cup \Omega_i$ and finitely many bimeromorphic maps $\psi_{i,j} : X \dashrightarrow X_{i,j}$ such that if $\psi : X \dashrightarrow Y$ is a weak log canonical model for some $\beta \in \Omega$, then $\psi = \psi_{i,j}$ for some i, j.*

Building on this result, we are able to show that good minimal models are connected by flops (and in general minimal models are connected by flips, flops and anti-flips).

Theorem 1.4. *Let $(X_i, B_i + \boldsymbol{\beta}_{X_i})$ be compact $\mathbb{Q}$-factorial generalized klt Kähler 3-folds, where $K_{X_i} + B_i + \boldsymbol{\beta}_{X_i}$ is nef (resp. $(X_i, B_i + \boldsymbol{\beta}_{X_i})$ are good minimal models) for $i = 1, 2$ and $\phi : X_1 \dashrightarrow X_2$ a bimeromorphic map which is an isomorphism in codimension 1. Then ϕ can be decomposed as flips, flops and inverse flips, see Definition 3.28 (resp. ϕ can be decomposed as a sequence of flips).*

When $K_X + B + \boldsymbol{\beta}_X$ is not pseudo-effective, we show the existence of Mori fiber space, see Theorem 3.23.

Theorem 1.5. *Let $(X, B + \boldsymbol{\beta})$ be a generalized klt Kähler 3-fold such that $K_X + B + \boldsymbol{\beta}_X$ is not pseudo-effective. Then we can run a $K_X + B + \boldsymbol{\beta}_X$-MMP $X \dashrightarrow X'$ ending with a Mori fiber space $X' \to Z$.*

We also establish the following cone theorem.

Theorem 1.6. *Let $(X, B + \boldsymbol{\beta}_X)$ be a $\mathbb{Q}$-factorial generalized klt pair, where X is a compact Kähler 3-fold. Then there are at most countably many rational curves $\{\Gamma_i\}_{i \in I}$ such that*

$$\overline{\mathrm{NA}}(X) = \overline{\mathrm{NA}}(X)_{K_X+B+\boldsymbol{\beta}_X \geq 0} + \sum_{i \in I} \mathbb{R}^+[\Gamma_i],$$

and $-(K_X + B + \boldsymbol{\beta}_X) \cdot \Gamma_i \leq 6$. Moreover, if $\boldsymbol{\beta}_X$ is big, then I is finite.

We believe that the added flexibility afforded by working with nef classes in $H^{1,1}_{\mathrm{BC}}$ will be useful in a variety of contexts. For example, we use this when showing that bimeromorphic Calabi-Yau threefolds are connected by flops (Theorem 3.29), and we expect that it will be important in the proof of the minimal model program for klt pseudo-effective Kähler 4-folds [DH23]. Note that the case of effective klt Kähler 4-folds was addressed in [DHP22].

This article is organized in the following manner: In Section 2 we define generalized pairs, generalized models and establish the generalized MMP for Kähler surfaces. We also the prove Theorem 1.1 in this section. Section 3 is the heart of our article, Theorem 1.2 is proved in Subsection 3.2, Theorem 1.5 is proved in Subsection 3.3, Theorem 1.6 is proved in Subsection 3.4, and Theorem 1.4 is proved in Subsection 3.6.

Acknowledgement We would like to thank Mihai Păun for answering our questions.

2. Preliminaries

An *analytic variety* or simply a *variety* X is a reduced and irreducible complex space. A holomorphic map $f : X \to Y$ between two complex spaces is called a *morphism*. A *small bimeromorphic map* or a *small map* is a bimeromorphic map $\phi : X \dashrightarrow X'$ between two normal analytic varieties such that ϕ is an isomorphism in codimension 1, i.e. there are closed analytic subsets $Z \subset X$ and $Z' \subset X'$ such that $\operatorname{codim}_X Z \geq 2$ and $\operatorname{codim}_{X'} Z' \geq 2$ and $\phi|_{X \setminus Z} : X \setminus Z \to X' \setminus Z'$ is an isomorphism. A $(1,1)$ class $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ is called *general* (resp. *very general*) if α is not contained in any finite union (resp. countable union) of analytic subvarieties of $H^{1,1}_{\mathrm{BC}}(X)$.

Definition 2.1. Let X be a normal analytic variety. The canonical sheaf ω_X is defined as $\omega_X := (\wedge^{\dim X}\Omega^1_X)^{**}$. Note that unlike the case of algebraic varieties, ω_X here does not necessarily correspond to a Weil divisor K_X such that $\omega_X \cong \mathcal{O}_X(K_X)$. However, by abuse of notation we will say that K_X is a canonical divisor when we actually mean the canonical sheaf ω_X. This doesn't create any problem in general as running the minimal model program involves intersecting subvarieties with ω_X.
A $\mathbb{Q}$-divisor (resp. an $\mathbb{R}$-divisor) on a normal analytic variety (non necessarily compact) is a *finite sum* of prime Weil divisor with $\mathbb{Q}$-coefficients (resp. $\mathbb{R}$-coefficients). A compact normal analytic variety X is called $\mathbb{Q}$-*factorial* if for every prime Weil divisor $D \subset X$ there is a $m \in \mathbb{Z}^+$ such that mD is Cartier and there is a $k \in \mathbb{Z}^+$ such that $(\omega_X^{\otimes k})^{**}$ is a line bundle on X.
For a normal analytic variety X and a $\mathbb{R}$-divisor B we say that $K_X + B$ is $\mathbb{R}$-Cartier, if locally around any point $x \in X$ we can choose a divisor K_X such that $\mathcal{O}_X(K_X) \cong \omega_X$ and $K_X + B$ is $\mathbb{R}$-Cartier. In this case, we define the singularities of the pair (X, B) as in [KM98]. Note that throughout this article, by a pair (X, B), we will always mean that X is normal, $B \geq 0$ is an effective $\mathbb{R}$-divisor. If B is not effective, then we will refer to the corresponding singularities of (X, B) as sub-klt, sub-dlt, etc.

Definition 2.2. An analytic variety X is *Kähler* or a *Kähler space* if there exists a positive closed real $(1,1)$ form $\omega \in \mathcal{A}^{1,1}_{\mathbb{R}}(X)$ such that the following holds: for every point $x \in X$ there exists an open neighborhood $x \in U$ and a closed embedding $\iota_U : U \hookrightarrow V$ into an open set $V \subset \mathbb{C}^N$, and a strictly plurisubharmonic $\mathcal{C}^\infty$ function $f : V \to \mathbb{R}$ such that $\omega|_{U\cap X_{\rm sm}} = (i\partial\bar{\partial} f)|_{U\cap X_{\rm sm}}$. Here $X_{\rm sm}$ is the smooth locus of X.

(1) On a normal compact analytic variety X we replace the use of Néron-Severi group $\mathrm{NS}(X)_{\mathbb{R}}$ by $H^{1,1}_{\rm BC}(X)$, the Bott-Chern cohomology of real closed $(1,1)$ forms with local potentials or equivalently, the closed bi-degree $(1,1)$ currents with local potentials. See [HP16, Definition 3.1 and 3.6] for more details. More specifically, we define
$$N^1(X) := H^{1,1}_{\rm BC}(X).$$
(2) If X is in Fujiki's class $\mathcal{C}$ and has *rational singularities*, then from [HP16, Eqn. (3)] we know that $N^1(X) = H^{1,1}_{\rm BC}(X) \subset H^2(X, \mathbb{R})$. In particular, the intersection product can be defined in $N^1(X)$ via the cup product of $H^2(X, \mathbb{R})$.
(3) For the definitions of nef, pseudo-effective class, etc. see [DH20, Definition 2.2].
(4) We define $\overline{\mathrm{NA}}(X) \subset N_1(X)$ to be the closed cone generated by the classes of positive closed currents of bi-dimension $(1,1)$, see [HP16, Defintion 3.8]. The Mori cone $\overline{\mathrm{NE}}(X) \subset \overline{\mathrm{NA}}(X)$ is defined as the

closure of the cone of currents of integration T_C, where $C \subset X$ is an irreducible curve.

Definition 2.3. If X is a normal Kähler variety and $\omega \in H^{1,1}_{\mathrm{BC}}(X)$, then we say that ω is modified Kähler if there exists a bimeromorphic morphism $\nu : X' \to X$ and Kähler form ω' on X' such that $\nu_*\omega' = \omega$. By [Bou04, Proposition 2.3], if X is compact, then this is equivalent to requiring that ω contains a Kähler current T with Lelong number $\nu(T, D) = 0$ for all prime divisors D in X.

Definition 2.4. Let $\pi : X \to S$ be a proper morphism of normal Kähler varieties such that S is relatively compact. Let β be a closed $(1,1)$ current with local potentials, i.e. a locally $\partial\bar\partial$-exact current on X. We say that the class $[\beta] \in H^{1,1}_{\mathrm{BC}}(X)$ is relatively Kähler (or Kähler over S) if $[\beta + \pi^*\omega_S] \in H^{1,1}_{\mathrm{BC}}(X)$ is a Kähler class for some Kähler form ω_S on S, and we say that the class $[\beta]$ is relatively nef if $[\beta + \omega]$ is relatively Kähler for every relatively Kähler class $[\omega]$ on X. Similarly, we say that β is relatively modified Kähler if $\beta + \pi^*\omega_S$ is modified Kähler for some Kähler form ω_S on S.

It is well known that if a class $[\beta] \in H^{1,1}_{\mathrm{BC}}(X)$ is relatively Kähler (resp. relatively nef), then its restriction to each fiber is Kähler (resp. nef). By abuse of notation we will say that a closed bi-degree $(1,1)$ current T with local potentials is relatively Kähler or relatively nef over S if so is its class $[T] \in H^{1,1}_{\mathrm{BC}}(X)$.

2.1. **Generalized Pairs.** Let X be a normal analytic variety. A *closed b-(1,1) current* $\boldsymbol{\beta}$ is a collection of closed bi-degree (1,1) currents $\boldsymbol{\beta}_{X'}$ on all proper bimeromorphic models $X' \to X$ such that if $p : X_1 \to X_2$ is a bimeromorphic morphism of proper models of X, then $p_*\boldsymbol{\beta}_{X_1} = \boldsymbol{\beta}_{X_2}$.

Suppose that β is a closed positive (1,1)-current on X with local (psh) potentials, then we may define a b-(1,1) current $\bar\beta$ as follows. For any bimeromorphic morphism $\nu : X' \to X$ we let $\bar\beta_{X'} := \nu^*\beta$. Explicitly, if $X = \cup U_i$ is an open cover and γ_i are psh functions on U_i such that $\beta = \partial\bar\partial\gamma_i$, then $\nu^*\beta$ is defined by letting $U_i' = \nu^{-1}U_i$, $\gamma_i' = \gamma_i \circ \nu|_{U_i'}$, and $\nu^*\beta = \partial\bar\partial\gamma_i'$ on U_i'. If $\mu : X' \to X''$ is another proper bimeromorphic morphism, then we let $\bar\beta_{X''} = \mu_*\beta_{X'}$. We note that

Claim 2.5. The closed b-(1,1) current $\bar\beta_{X''}$ is well defined.

Proof. Suppose that $\tilde\nu : \tilde X \to X$ and $\tilde\mu : \tilde X \to X''$ are also proper bimeromorphic morphisms of normal complex varieties. By a standard argument, passing to a common resolution, we may in fact assume that there is a bimeromorphic morphism $\rho : \tilde X \to X'$ such that $\tilde\nu = \nu \circ \rho$ and $\tilde\mu : \mu \circ \rho$. Then by the projection formula we have

$$\tilde\mu_*(\tilde\nu^*\beta) = \mu_*\rho_*(\rho^*\nu^*\beta) = \mu_*(\nu^*\beta).$$

□

If $\boldsymbol{\beta} = \bar{\beta}$ for some closed positive (1,1)-current β on X then we say that $\boldsymbol{\beta}$ *is a positive closed b-(1,1) current that descends to* X. Note that in this case for any bimeromorphic morphism $\nu : X' \to X$ we also have that $\boldsymbol{\beta} = \overline{\boldsymbol{\beta}_{X'}}$ i.e. $\boldsymbol{\beta}$ also descends to X'.

Remark 2.6. We make the following observations:

(i) Note that if $\gamma \in H^{1,1}_{\mathrm{BC}}(X')$ is nef, then it is pseudo-effective and so we may choose a positive closed (1,1) form β' on X' with psh local potentials such that $[\beta'] = \gamma$ and we may then set $\boldsymbol{\beta} := \bar{\beta}'$. Different choices of β' give rise to different (non-equivalent) generalized pairs.

(ii) Note that if $\boldsymbol{\beta}$ is a positive closed b-(1,1) current that descends to X and $X \dashrightarrow X'$ is bimeromorphic (and X' is normal), then $\boldsymbol{\beta}_{X'}$ may not have local potentials, but if it does, then it has psh local potentials. To see this, first note that in this case $[\boldsymbol{\beta}_{X'}] \in H^{1,1}_{\mathrm{BC}}(X')$. Let $p : X'' \to X$ and $q : X'' \to X'$ be a common resolution and $U' := X' \setminus (X'_{\mathrm{sing}} \cup q(\mathrm{Ex}(q)))$ so that $U'' := q^{-1}U' \to U'$ is an isomorphism. Then $\boldsymbol{\beta}_{X'}|_{U'} = \boldsymbol{\beta}_{X''}|_{U''}$, and since $\boldsymbol{\beta}_{X''}$ is a positive current, from [BG13, Proposition 4.6.3(i)] it follows that $\boldsymbol{\beta}_{X'}|_{U'}$ has a unique extension $\widehat{\boldsymbol{\beta}_{X'}|_{U'}}$ to a closed positive (1,1) current on X' such that $\left[\widehat{\boldsymbol{\beta}_{X'}|_{U'}}\right] = [\boldsymbol{\beta}_{X'}]$.

Definition 2.7. Let $f : X \to S$ be a proper morphism of normal Kähler varieties, where S is relatively compact, $\nu : X' \to X$ a resolution, B' an $\mathbb{R}$-divisor on X' with simple normal crossings support such that $B := \nu_* B' \geq 0$, and $\boldsymbol{\beta}$ a closed b-(1,1) current. We say that $(X, B + \boldsymbol{\beta})$ is a *generalized pair* if

(1) $\boldsymbol{\beta}$ is a positive closed b-(1,1) current that descends to X',
(2) $[\boldsymbol{\beta}_{X'}] \in H^{1,1}_{\mathrm{BC}}(X')$ is nef over S, and
(3) $[K_{X'} + B' + \boldsymbol{\beta}_{X'}] = \nu^*\gamma$ for some $\gamma \in H^{1,1}_{\mathrm{BC}}(X)$.

Note that we are abusing notation as we are implicitly assuming the existence of (X', B') as above. We will say that $\nu : (X', B') \to (X, B)$ is a *structure morphism* or a *log resolution* of $(X, B + \boldsymbol{\beta})$.

Remark 2.8. We make the following observations:

(i) Given $(X, B + \boldsymbol{\beta})$ and $\beta' = \boldsymbol{\beta}_{X'}$ with the above properties, then B' is uniquely determined (by the negativity lemma applied to $\nu : X' \to X$).
(ii) If S is a point so that X is compact, then we drop S and say that $(X, B + \boldsymbol{\beta})$ is a compact generalized pair.
(iii) If $U \subset X$ is a relatively compact subset, then $(U/U, B|_U + \boldsymbol{\beta}|_U)$ is a generalized pair over U.

(iv) If $S = X$ and $\pi : X \to S$ is the identity (and in particular X is relatively compact), then we also drop S and we often abuse notation and say that $(X, B + \boldsymbol{\beta})$ is a generalized pair.

Definition 2.9. (1) Let P be a prime Weil divisor over X. We define the *generalized discrepancy* $a(P, X, B + \boldsymbol{\beta})$ as follows: Let $\nu : X' \to X$ be a log resolution of $(X, B + \boldsymbol{\beta})$ such that $P \subset X'$ is prime Weil divisor on X'. Then $a(P, X, B + \boldsymbol{\beta}) := -\mathrm{mult}_P(B')$. Note that these can be computed locally over X and hence S plays no role here (and hence we drop it from the notation).

(2) We say that $(X, B + \boldsymbol{\beta})$ is *generalized klt* or gklt or generalized Kawamata log terminal (resp. *generalized lc* or glc or generalized log canonical) if for some log resolution $\nu' : X' \to X$, we have $\lfloor B' \rfloor \leq 0$, i.e. $a(P, X, B + \boldsymbol{\beta}) > -1$ for all prime divisors $P \subset X'$ (resp. $a(P, X, B + \boldsymbol{\beta}) \geq -1$ for all prime divisors $P \subset X'$).

(3) We say that $(X, B + \boldsymbol{\beta})$ is *generalized dlt* or gdlt or generalized divisorially log terminal if there is an open subset $U \subset X$ such that $(U, (B + \boldsymbol{\beta})|_U)$ is a log resolution (of itself) and $-1 \leq a(P, X, B + \boldsymbol{\beta}) \leq 0$ for any prime divisor P on U and $-1 < a(P, X, B + \boldsymbol{\beta}) \leq 0$ for any prime divisor P over X with center contained in $X \setminus U$.

Remark 2.10. By abuse of notation we will often say β is a $(1,1)$ class in $H^{1,1}_{\mathrm{BC}}(X)$ when we actually mean β is a closed positive bi-degree $(1,1)$ current on X with local (psh) potentials. Especially, we will often add a Kähler form ω to a generalized pair $(X, B + \boldsymbol{\beta})$ while calling it a Kähler class; however, this doesn't create any problem as a Kähler form ω on X defines a b-(1,1) current as $\boldsymbol{\omega} := \bar{\omega}$ which descends to X and $[\boldsymbol{\omega}_X] \in H^{1,1}_{\mathrm{BC}}(X)$ is nef (in fact Kähler), so the singularities of $(X, B + \boldsymbol{\beta} + \boldsymbol{\omega})$ are the same as those of $(X, B + \boldsymbol{\beta})$.

2.2. **Generalized Models.**

Definition 2.11. If $(X/S, B + \boldsymbol{\beta})$ is a generalized dlt pair over S, then we say that a bimeromorphic map $\phi : X \dashrightarrow X^m$ (proper over S) is a *log minimal model over* S (resp. a *log terminal model* over S) if (1-3) below hold (resp. (1-4) below hold).

(1) $(X^m, B^m + \boldsymbol{\beta})$ is $\mathbb{Q}$-factorial generalized dlt pair, where $B^m = \phi_* B + E$, and E is the reduced sum of all ϕ^{-1}-exceptional divisors,
(2) $K_{X^m} + B^m + \boldsymbol{\beta}_{X^m}$ is nef over S,
(3) $a(P, X, B, \boldsymbol{\beta}) < a(P, X^m, B^m, \boldsymbol{\beta})$ for every ϕ-exceptional divisor P, and
(4) there are no ϕ^{-1}-exceptional divisors i.e. $E = 0$.

If $(X/S, B+\boldsymbol{\beta})$ is a generalized dlt pair over S, then we say that a bimeromorphic map $\phi : X \dashrightarrow X^m$ (proper over S) is a *weak log canonical model over* S (resp. a *log canonical model over* S) if (1-3) below hold (resp. (1-4) below hold).

(1) $(X^m, B^m+\boldsymbol{\beta})$ is generalized lc pair, where $B^m = \phi_*B + E$, and E is the reduced sum of all ϕ^{-1}-exceptional divisors,
(2) $K_{X^m} + B^m + \boldsymbol{\beta}_{X^m}$ is nef over S,
(3) $a(P, X, B, \boldsymbol{\beta}) \leq a(P, X^m, B^m, \boldsymbol{\beta})$ for every ϕ-exceptional divisor P, and
(4) $[K_{X'^m} + B^m + \boldsymbol{\beta}_{X^m}] \in H^{1,1}_{\rm BC}(X^m)$ is a Kähler class.

If X is proper and S is a point, then we drop "over S" and simply say that we have a log minimal model, log terminal model etc.

Lemma 2.12. *Suppose that $(X/S, B+\boldsymbol{\beta})$ is generalized dlt over S.*

(1) If $\phi : X \dashrightarrow X^m$ is a weak log canonical model over S, then $a(P, X, B, \boldsymbol{\beta}) \leq a(P, X^m, B^m, \boldsymbol{\beta})$ for every divisor P over X and $a(P, X, B, \boldsymbol{\beta}) = a(P, X^m, B^m, \boldsymbol{\beta})$ for every divisor P on X^m.
(2) If $X \dashrightarrow X^m$ and $X \dashrightarrow X^w$ are weak log canonical models of $(X/S, B+\beta)$ over S, then $(X^m, B^m+\boldsymbol{\beta})$ and $(X^w, B^w+\boldsymbol{\beta})$ are crepant equivalent, i.e. if $p : Z \to X^m$ and $q : Z \to X^w$ is a resolution of the induced map $X^m \dashrightarrow X^w$, then $p^*(K_{X^m} + B^m + \boldsymbol{\beta}_{X^m}) \equiv_S q^*(K_{X^w} + B^w + \boldsymbol{\beta}_{X^w})$.
(3) If $X \dashrightarrow X^m$ and $X \dashrightarrow X^w$ are log canonical models of $(X/S, B+\boldsymbol{\beta})$ over S, then (X^m, B^m) and (X^w, B^w) are isomorphic.
(4) If $(X, B+\boldsymbol{\beta})$ is generalized klt, then every log minimal model over S is a log terminal model over S.
(5) If $f : X' \to X$ is a log resolution of $(X/S, B+\boldsymbol{\beta})$ and $K_{X'}+B^*+\boldsymbol{\beta}_{X'} = f^*(K_X + B + \boldsymbol{\beta}_X) + F$, where $B^* \geq 0, f_*B^* = B$ and $F \geq 0$ is f-exceptional such that for every f-exceptional divisor P with $a(P, X, B+\boldsymbol{\beta}) > 0$ we have $P \subset \mathrm{Supp}(F)$. Then any log minimal model (resp. (weak) log canonical model) of $(X'/S, B^*+\boldsymbol{\beta}')$ over S is a log minimal model (resp. (weak) log canonical model) of $(X/S, B+\boldsymbol{\beta})$ over S.

Proof. (1) Let $p : Z \to X$ and $q : Z \to X^m$ be a resolution of ϕ. Then we can write $F = \sum(a(P, X^m, B^m, \boldsymbol{\beta}) - a(P, X, B, \boldsymbol{\beta}))P$, where the sum runs over all prime divisors $P \subset Z$. Then from the definition above it follows that $p_*F \geq 0$. Note that $F \equiv_S p^*(K_X+B+\boldsymbol{\beta}_X) - q^*(K_{X^m}+B^m+\boldsymbol{\beta}_{X^m})$, and hence $F \geq 0$ by the negativity lemma, as $K_{X^m} + B^m + \boldsymbol{\beta}_{X^m}$ is nef over S. We also claim that $q_*F = 0$. Observe that, here $B^m = \phi_*B + E$, where E is the reduced sum of ϕ^{-1}-exceptional divisors on X^m. Thus it is enough to show that $\mathrm{mult}_P(F) = 0$ for any prime divisor P in the support of E (i.e. any p-exceptional divisor which is not q-exceptional). We have

$$-1 = a(P, X^m, B^m+\boldsymbol{\beta}) \geq a(P, X, B+\boldsymbol{\beta}) \geq -1,$$

where the second inequality holds because $F \geq 0$. In particular, we have

$$a(P, X, B + \boldsymbol{\beta}) = a(P, X^m, B^m + \boldsymbol{\beta})$$

for all prime Weil divisors P on X^m.

(2) Follows easily by what we have seen in (1).

(3) Let W be the normalization of the graph of $X^m \dashrightarrow X^w$, and $p : W \to X^m$, $q : W \to X^w$ the induced morphisms. Then by part (2) we have $p^*(K_{X^m} + B^m + \boldsymbol{\beta}_{X^m}) \equiv q^*(K_{X^w} + B^w + \boldsymbol{\beta}_{X^w})$. Since p, q are bimeromorphic, and hence Moishezon morphisms, if $X^m \dashrightarrow X^w$ is not an isomorphism, we may assume that there is a curve $C \subset W$ such that $p_*C = 0$ and $q_*C \neq 0$ (or $p_*C \neq 0$ and $q_*C = 0$). But then

$$\begin{aligned} 0 = p_*C \cdot (K_{X^m} + B^m + \boldsymbol{\beta}_{X^m}) &= C \cdot p^*(K_{X^m} + B^m + \boldsymbol{\beta}_{X^m}) \\ &= C \cdot q^*(K_{X^w} + B^w + \boldsymbol{\beta}_{X^w}) \\ &= q_*C \cdot (K_{X^w} + B^w + \boldsymbol{\beta}_{X^w}) > 0, \end{aligned}$$

which is a contradiction.

(4) Suppose that $\phi : X \dashrightarrow X^m$ is a log minimal model and P is a ϕ^{-1} exceptional divisor. Then as (X, B) is klt and P is contained in the support of B^m with multiplicity 1 (as $B^m = \phi_*B + \mathrm{Ex}(\phi^{-1})$), from Part (1) we have

$$-1 < a(P, X, B) = a(P, X^m, B^m) = -1,$$

which is impossible, and so there are no ϕ^{-1}-exceptional divisors, i.e. ϕ is a log terminal model.

(5) See the proof of [HL21, Lemma 3.10]. □

Lemma 2.13. *Let $(X, B + \boldsymbol{\beta})$ be a generalized klt (resp. dlt) pair. If $K_X + B$ is $\mathbb{Q}$-Cartier, then (X, B) is klt (resp. dlt).*

Proof. Since the statement is local on X, we may assume that X is Stein and relatively compact. Let $f : X' \to X$ be a log resolution and $K_{X'} + B' + \boldsymbol{\beta}_{X'} = f^*(K_X + B + \boldsymbol{\beta}_X)$, where $\lfloor B' \rfloor \leq 0$, as $(X, B + \boldsymbol{\beta})$ is generalized klt. Let $K_{X'} + B^\sharp := f^*(K_X + B)$. Then

$$f^*\boldsymbol{\beta}_X - \boldsymbol{\beta}_{X'} \equiv K_{X'} + B' - f^*(K_X + B) =: E,$$

where $E \geq 0$ by the negativity lemma. But then

$$B' = E + f^*(K_X + B) - K_{X'} = B^\sharp + E$$

and so $\lfloor B^\sharp \rfloor \leq 0$, i.e. (X, B) is klt. The statement about dlt singularities follows similarly. □

Lemma 2.14. *Let $\phi : X \dashrightarrow Y$ be a bimeromorphic map of normal compact Kähler 3-folds with $\mathbb{Q}$-factorial klt singularities that extracts no divisors. Then the linear map $\phi_* : H^{1,1}_{\mathrm{BC}}(X) \to H^{1,1}_{\mathrm{BC}}(Y)$ is well defined and surjective.*

Proof. From [DH20, Corollary 2.28] it follows that ϕ_* is well defined. So it only remains to show that ϕ_* is surjective. To that end, let $p : W \to X$ and $q : W \to Y$ be a resolution of the graph of ϕ. Replacing W by a higher resolution if necessary and then using the relative Chow lemma of Hironaka we may assume that $p : W \to X$ is a projective morphism. Choose $\beta \in H^{1,1}_{\mathrm{BC}}(Y)$; then by [DH20, Lemma 2.27] there is an $\mathbb{R}$-divisor E supported on the exceptional locus of p such that $q^*\beta + [E] = 0$ in $H^{1,1}_{\mathrm{BC}}(W)/p^*H^{1,1}_{\mathrm{BC}}(X)$. In particular, there is a $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ such that $q^*\beta + [E] = p^*\alpha$ for some $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$. Since ϕ does not extract any divisor, E is also q-exceptional. Therefore $\phi_*\alpha = q_*p^*\alpha = q_*(q^*\beta + [E]) = \beta$, and hence ϕ_* is surjective. □

Lemma 2.15. *Let $f : X' \to X$ be a proper bimeromorphic morphism of normal compact Kähler varieties. Assume further that X has $\mathbb{Q}$-factorial klt singularities and X' has rational singularities. Then $\mathrm{Ex}(f)$ is a pure codimension 1 subset of X'.*

Proof. We claim that we can apply [DH20, Lemma 2.27] here to show that $\mathrm{Ex}(\phi)$ has pure codimension 1. Indeed, if we assume this lemma in our case, then for any Kähler class ω_X on X, we can write $-\omega_X \equiv \phi^*\alpha_{X'} + F$ for some $\alpha_{X'} \in H^{1,1}_{\mathrm{BC}}(X')$ and F a ϕ-exceptional $\mathbb{R}$-divisor. Then the negativity lemma implies that F is effective and $\mathrm{Supp}(F) = \mathrm{Ex}(\phi)$, and we are done.
Now observe that in [DH20, Lemma 2.27] it is assumed that the morphism ϕ is projective and dimension of the varieties are 3, however, the projectivity of ϕ was never used in the proof and the dimension argument was only necessary to run the relative MMP which can be achieved in arbitrary dimension by [DHP22, Theorem 1.4]. □

Lemma 2.16. *Let $\phi : X \dashrightarrow X'$ be a small bimeromorphic map over Y of normal compact Kähler varieties such that X and X' both have klt singularities and X' is $\mathbb{Q}$-factorial. Let $\omega \in H^{1,1}_{\mathrm{BC}}(X)$ be nef over Y such that $\phi_*\omega \in H^{1,1}_{\mathrm{BC}}(X')$ is Kähler over Y . Then ϕ is an isomorphism.*

Proof. Let W be the normalization of the graph of ϕ and $p : W \to X$ and $q : W \to X'$ are the induced bimeromorphic morphisms. Write $p^*\omega = q^*\omega' + E$, where E is an $\mathbb{R}$-divisor. Since ϕ is small, from the negativity lemma it follows that $E = 0$, i.e. $p^*\omega = q^*\omega'$. If ϕ is a not a morphism, then there is a curve $C \subset W$ such that $p_*(C) = 0$ but $q_*(C) \neq 0$ and $(f' \circ q)_*(C) = 0$, where $f' : X' \to Y$ is the induced morphism. In particular, $0 = p^*\omega \cdot C = q^*\omega' \cdot C = \omega' \cdot q_*(C) > 0$, a contraction. Thus ϕ is a morphism. Then we arrive at a contradiction by Lemma 2.15 unless ϕ is an isomorphism.

□

Definition 2.17. [Fuj22, Page 3] Let X be a normal analytic variety and $W \subset X$ a fixed compact subset. We say that $W \subset X$ satisfies *Property* **P** if the following hold:

(P1) X is a Stein space.
(P2) W is a Stein compact subset of X.
(P3) $\Gamma(W, \mathcal{O}_X)$ is noetherian (or equivalently, for any open subset $U \subset X$ and any analytic subset Z of U, $W \cap Z$ has finitely many connected components).

A projective morphism $g : S \to T$ between analytic varieties is said to satisfy *Property* **Q** if S and T are both compact.

Remark 2.18. Let X be a normal analytic variety and for each point $x \in X$, let $x \in U$ be a Stein open neighborhood. Since U is locally compact, there is a compact neighborhood $x \in K \subset U$ of x. Then by [Fuj22, Lemma 2.5], its holomorphically convex hull $\widehat{K}$ in U is Stein compact. Note that from [Fuj22, Theorem 2.10] it follows that $\widehat{K} \subset U$ satisfies Property **P** if and only if $\Gamma(\widehat{K}, \mathcal{O}_U) = \varinjlim_{\widehat{K} \subset V} \Gamma(V, \mathcal{O}_V)$, where V is an open subset of U, is a noetherian ring. But then from [Fuj22, Lemma 2.16] we see that there is a Stein compact subset L such that $x \in \widehat{K} \subset L \subset U$ such that $\Gamma(L, \mathcal{O}_U)$ is noetharian. In particular, every point $x \in X$ has a Stein open neighborhood U and a Stein compact subset $x \in L \subset U$ such that U satisfies Property **P**.

Theorem 2.19. *Let $(X, B + \boldsymbol{\beta})$ be a generalized klt pair, where X relatively compact analytic variety. Then the following hold locally over X:*

(1) X has rational singularities,
(2) there exists a small bimeromorphic morphism $\mu : X^\sharp \to X$ such that $X^\sharp$ is $\mathbb{Q}$-factorial,
(3) if $K_{X^\sharp} + B^\sharp + \boldsymbol{\beta}_{X^\sharp} = \mu^*(K_X + B + \boldsymbol{\beta}_X)$, then $\boldsymbol{\beta}_{X^\sharp} \equiv_X \Delta^\sharp$ so that $(X^\sharp, B^\sharp + \Delta^\sharp)$ is klt, and
(4) if $\Delta = \mu_*\Delta^\sharp$, then $(X, B + \Delta)$ is klt.

Proof. (1) immediately follows from (4) and [Kol97, Corollary 11.14].

(2-3) From Remark 2.18, it follows that for any $x \in X$ there is a Stein compact subset $x \in W \subset X$ such that X satisfies Property **P**. In what follows we work locally around W i.e. we repeatedly shrink X to a neighborhood of W (without further mention). Let $\nu : X' \to X$ be a projective log resolution of $(X, B + \boldsymbol{\beta})$ and write $K_{X'} + B' + \boldsymbol{\beta}_{X'} = \nu^*(K_X + B + \boldsymbol{\beta}_X)$. Let $E = \mathrm{Ex}(\nu)$, and for $0 < \epsilon \ll 1$ define $B^* := (B')^{>0} + \epsilon E$ and $F := (B')^{<0} + \epsilon E$. Then $K_{X'} + B^* + \boldsymbol{\beta}_{X'} \equiv \nu^*(K_X + B + \beta_X) + F$, where the support of F equals the set of all ν-exceptional divisors, and $(X', B^* + \boldsymbol{\beta}_{X'})$ is generalized klt. In particular, $\boldsymbol{\beta}_{X'} \equiv_X F - (K_{X'} + B^*)$ where $F - (K_{X'} + B^*)$ is an $\mathbb{R}$-divisor, nef over X. As ν is projective and X is Stein, we may assume that $F-(K_{X'}+B^*)$ is big and nef (over X). But then $\boldsymbol{\beta}_{X'} \equiv_X \Delta'$, where $\Delta' \geq 0$ is an effective $\mathbb{R}$-divisor such that $(X', B^* + \Delta')$ is klt.

We may therefore run the relative $K_{X'} + B^* + \Delta'$-MMP (see [DHP22, Theorem 1.4] and [Fuj22, Theorem 1.8]) and hence we may assume that we have a bimeromorphic map $\psi : X' \dashrightarrow X^\sharp$ such that if $F^\sharp = \psi_* F$, $B^\sharp = \psi_* B^*$, $\boldsymbol{\beta}_{X^\sharp} = \psi_* \boldsymbol{\beta}_{X'}$ and $\Delta^\sharp = \psi_* \Delta'$, then

$$F^\sharp \equiv_X K_{X^\sharp} + B^\sharp + \boldsymbol{\beta}_{X^\sharp} \equiv_X K_{X^\sharp} + B^\sharp + \Delta^\sharp$$

is nef over X so that $F^\sharp = 0$ by the negativity lemma. Therefore $\mu : X^\sharp \to X$ is a small bimeromorphic morphism, $B^\sharp = \mu_*^{-1} B$ and $X^\sharp$ is $\mathbb{Q}$-factorial. Clearly $(X^\sharp, B^\sharp + \Delta^\sharp)$ is klt. Note that each step of the above MMP preserves the numerical equivalence $\boldsymbol{\beta}_{X^\sharp} \equiv_X \Delta^\sharp$, and in particular $K_{X^\sharp} + B^\sharp + \boldsymbol{\beta}_{X^\sharp} = \mu_*^{-1}(K_X + B + \boldsymbol{\beta}_X)$.

(4) By the Base-point free theorem [Fuj22, Theorem 8.1], we have (locally over X) that $K_{X^\sharp} + B^\sharp + \Delta^\sharp \sim_{\mathbb{Q},X} 0$ and the claim follows.

□

We have following immediate corollary.

Lemma 2.20. *Let $(X, B + \boldsymbol{\beta})$ be a generalized klt (resp. dlt) pair, where X is compact analytic surface. Then X is $\mathbb{Q}$-factorial with rational singularities, and (X, B) is klt (resp. dlt).*

Proof. By Theorem 2.19, X has rational singularities. Then from [Fuj21, Lemma 3.10] it follows that X is $\mathbb{Q}$-factorial. □

2.3. **Existence of Flips for Generalized Pairs.** In this subsection we prove the existence of flips for generalized klt pairs in dimension 3.

Theorem 2.21. *Let $(X/S, B + \boldsymbol{\beta})$ be a generalized klt Kähler 3-fold pair, such that $K_X + B$ is $\mathbb{R}$-Cartier, and $f : X \to Z$ is a $K_X + B + \boldsymbol{\beta}_X$-negative small bimeromorphic morphism over S. Then f is locally projective, the log* *canonical model $f^+ : X^+ \to Z$ for $(X, B+\boldsymbol{\beta})$ over Z exists and there is an f-exceptional rational curve C such that $0 > (K_X + B + \boldsymbol{\beta}_X)\cdot C \geq -6$.*

Proof. Let $C = \cup C_i$ be the set of curves contracted by f. Assume for simplicity that C is connected. It suffices to construct the flip locally around $z = f(C) \subset Z$. Let $z \in W \subset Z$ be a relatively compact Stein open subset. Shrinking W, we may assume that for every curve C_i, there is a Cartier divisor D_i on $X_W := f^{-1}W$ that intersects C_i transversely and does not intersect C_j for $j \neq i$. To construct D_i, pick a general point x_i on C_i and a sufficiently small neighborhood $x_i \in U_i \subset X$. We identify $x_i \in U_i$ with a locally closed analytic subvariety of $\mathbb{C}^N$ and take the divisor D_i given by a general hyperplane through x_i. Shrinking W and intersecting D_i with X_W, we may assume that each D_i is a subvariety of X_W. It then follows that if $D = \sum d_i D_i$, where $d_i = [\boldsymbol{\beta}_X]\cdot C_i$, then $D \equiv_W \boldsymbol{\beta}_X$.

Now let $\nu : X' \to X$ be a log resolution of the generalized pair $(X, B+\boldsymbol{\beta})$. Since $K_X + B$ is $\mathbb{R}$-Cartier, we have $[\boldsymbol{\beta}_X] \in H^{1,1}_{\rm BC}(X)$, and so by Remark 2.8 we may write $-E \equiv \boldsymbol{\beta}_{X'} - \nu^*\boldsymbol{\beta}_X$ for some ν-exceptional $\mathbb{R}$-divisor E on X'. Let $D' := \nu^*D - E|_{X'_W} \equiv_W \boldsymbol{\beta}_{X'}|_{X'_W}$. We may assume that $\nu : X'_W \to W$ is projective (via Hironaka's Chow lemma [Hir75, Corollary 2]). Since D' is nef and big over X_W, replacing D' by an $\mathbb{R}$-linearly equivalent divisor, we may assume that $(X'_W, B'_W + D')$ is sub-klt and hence $(X_W, B_W + D)$ is klt, since $K_{X'_W} + B'_W + D' \equiv \nu^*(K_{X_W} + B_W + D)$. But then the required log canonical model X^+_W exists (see [CHP16, Theorem 4.3]). In particular, $-(K_{X_W} + B_W + D)$ is ample over W and so f is locally projective. The existence of f-exceptional rational curve $C \subset X_W$ such that $0 > (K_X + B + \boldsymbol{\beta}_X)\cdot C = (K_{X_W} + B_W + D)\cdot C \geq -6$ now follows from [DO23, Theorem 4.2]. □

As an easy corollary, we will prove the existence of flips. Recall that if $(X/S, B+\boldsymbol{\beta})$ is a $\mathbb{Q}$-factorial compact Kähler generalized klt 3-fold pair, then a $K_X + B + \boldsymbol{\beta}_X$-flipping contraction over S is a small bimeromorphic morphism $f : X \to Z$ over S such that $\rho(X/Z) = 1$, and $-(K_X + B + \boldsymbol{\beta}_X)$ is Kähler over Z. By definition, the flip of $f : X \to Z$, if it exists, is a small bimeromorphic morphism $f^+ : X^+ \to Z$ over S such that X^+ is Kähler over S, and $K_{X^+} + B^+ + \boldsymbol{\beta}_{X^+}$ is Kähler over Z. We need the following lemma first.

Lemma 2.22. *Let X be a normal $\mathbb{Q}$-factorial compact Kähler variety, $f : X \to Z$ a 3-fold $K_X + B + \boldsymbol{\beta}_X$-flipping contraction of a generalized klt pair over S, and $f^+ : X^+ \to Z$ the corresponding flip, then*

1. *$f^+ : X^+ \to Z$ is uniquely determined,*
2. *X^+ is $\mathbb{Q}$-factorial, and*
3. *$\rho(X^+/Z) = 1$.*

Proof. Suppose that $f':X'\to Z$ is another flip of $f:X\to Z$, then $X^+\dashrightarrow X'$ is a small bimeromorphic map over Z. Let Y be the normalization of the graph and $p:Y\to X^+$ and $q:Y\to X'$ are the induced morphisms, then from the negativity lemma it follows easily that $q^*(K_{X^+}+B^++\boldsymbol{\beta}_{X^+})=p^*(K_{X'}+B'+\boldsymbol{\beta}_{X'})$. Let $C\subset Y$ be a p-exceptional curve. Then $q_*C\neq 0$ and $(f'\circ q)_*C=0$. Thus we have

$$0<C\cdot q^*(K_{X^+}+B^++\boldsymbol{\beta}_{X^+})=C\cdot p^*(K_{X'}+B'+\beta_{X'})=0$$

which is a contradiction. Therefore, there are no such curves and hence $X^+\dashrightarrow X'$ is a morphism. Similarly, it follows that $X'\dashrightarrow X^+$ is a morphism and hence $X^+\cong X'$; in particular, (1) holds.

Let G^+ be a prime Weil divisor on X^+ and G its strict transform on X. Then G is $\mathbb{Q}$-Cartier, as X is $\mathbb{Q}$-factorial. For any point $p\in X^+$ we must show that there is a neighborhood of p on which G^+ is $\mathbb{Q}$-Cartier. This is clear if p is not contained in the flipped locus $\operatorname{Ex}(f^+)$, so assume that $p\in\operatorname{Ex}(f^+)$ and let $q=f^+(p)$. Working locally over a neighborhood $q\in W\subset Z$ as in the proof of Theorem 2.21, we may assume that $K_{X_W}+B_W+D$ is klt for some $\mathbb{R}$-divisor D on X_W such that $D\equiv_W\boldsymbol{\beta}_X|_{X_W}$ and that $X^+\to W$ is the relative log canonical model for $K_{X_W}+B_W+D$. Since $-(K_{X_W}+B_W+D)$ is ample over W, we may pick an effective $\mathbb{R}$-divisor $0\leq H\sim_{\mathbb{R},W}\epsilon G_W-\frac{1}{2}(K_{X_W}+B_W+D)$ for $0<\epsilon\ll 1$ such that (X_W,B_W+D+H) is klt and $-(K_{X_W}+B_W+D+H)$ is ample over W. Then $X^+\to W$ is also the relative log canonical model for $K_{X_W}+B_W+D+H$ over W and so $K_{X^+_W}+B^+_W+D^++H^+\sim_{\mathbb{R},W}\frac{1}{2}(K_{X^+_W}+B^+_W+D^+)+\epsilon G^+_W$ is $\mathbb{R}$-Cartier for $0<\epsilon\ll 1$, and hence G^+_W is $\mathbb{Q}$-Cartier and (2) is proven.

(3) now follows from [DH20, Lemma 2.27].

□

Corollary 2.23. *Let $(X/S,B+\boldsymbol{\beta})$ be a $\mathbb{Q}$-factorial compact Kähler generalized klt 3-fold pair, and $f:X\to Z$ is a $K_X+B+\boldsymbol{\beta}_X$-flipping contraction over S. Then f is locally projective, the flip $f^+:X\to Z$ for $K_X+B+\boldsymbol{\beta}_X$ over Z exists (and unique), and there is an f-exceptional rational curve C such that $0>(K_X+B+\boldsymbol{\beta}_X)\cdot C\geq -6$.*

Proof. Follows immediate from Theorem 2.21 and Lemma 2.22. □

Proof of Theorem 1.1. This follows from Corollary 2.23. □

Lemma 2.24. *Let $\pi:X\to S$ be a proper morphism of compact complex varieties such that X is Kähler. If $(X,B+\boldsymbol{\beta})$ is a generalized dlt pair and $\phi:X\dashrightarrow X'$ is a $K_X+B+\boldsymbol{\beta}_X$ flip, flipping contraction or divisorial contraction, then X' is Kähler.*

Proof. Let ω be a Kähler form such that $\gamma = K_X + B + \boldsymbol{\beta}_X + \omega$ is a supporting hyperplane for the $K_X + B + \boldsymbol{\beta}_X$-negative extremal ray. If $f : X \to Z$ is the corresponding contraction, then $\gamma_Z = K_Z + B_Z + \boldsymbol{\beta}_Z + \omega_Z = f_*(K_X + B + \boldsymbol{\beta}_X + \omega)$ is generalized dlt and hence Z has rational singularities. But then, by the proof of [CHP16, Corollary 3.8], γ_Z is Kähler (over S). Suppose now that $f : X \to Z$ is a flipping contraction and let $f^+ : X^+ \to Z$ be the flip, then $-\omega^+ = -\phi_*\omega$ is Kähler over Z and so, for any $0 < \epsilon \ll 1$,

$$K_{X^+} + B_{X^+} + \boldsymbol{\beta}_{X^+} + (1-\epsilon)\omega^+ \equiv {f^+}^*\gamma_Z - \epsilon\omega^+$$

is Kähler on X^+. □

2.4. **Generalized Surface MMP.** We begin by recalling the following well known fact.

Lemma 2.25. *If $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ is pseudo-effective but not nef on a normal compact Kähler surface X, then $\int_C \alpha < 0$ for some curve $C \subset X$.*

Proof. Follows immediately from [DHP22, Theorem 2.36]. □

Lemma 2.26. *Let $f : X \to Y$ be a proper bimeromorphic morphism of normal compact Kähler surfaces with rational singularities. If $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ is nef and $\alpha_Y := f_*\alpha$, then $\alpha_Y \in H^{1,1}_{\mathrm{BC}}(Y)$ is nef.*

Proof. Passing to a resolution of singularities of X we may assume that X is smooth. Now recall that by the Hodge index theorem the intersection matrix of the set of all f-exceptional curves is a negative definite matrix. Therefore there is an f-exceptional $\mathbb{R}$-divisor E on X such that $\alpha + E \equiv_Y 0$. By [HP16, Lemma 3.3], $\alpha + E = f^*\alpha_Y$ for some $\alpha_Y \in H^{1,1}_{\mathrm{BC}}(Y)$, and thus $\alpha_Y = f_*(f^*\alpha_Y) = f_*(\alpha + E) = f_*\alpha$. From the the negativity lemma it follows that $E \geq 0$. Thus α_Y is pseudo-effective, and so by Lemma 2.25, it suffices to check that $\alpha_Y|_C$ is pseudo-effective, i.e. that $\int_C \alpha_Y \geq 0$ for all curves $C \subset Y$. If $C' = f^{-1}_*C$, then we have

$$\int_C \alpha_Y = \int_{C'} \alpha + (E \cdot C') \geq 0,$$

since C' is not contained in the support of E and α is nef. □

An immediate corollary of this lemma is the following.

Corollary 2.27. *If $(X, B + \boldsymbol{\beta})$ is a compact generalized lc pair such that X is a compact Kähler surface with rational singularities, then $\boldsymbol{\beta}_X$ has local potentials on X and $[\boldsymbol{\beta}_X] \in H^{1,1}_{\mathrm{BC}}(X)$ is nef.*

Definition 2.28. Let X be a compact analytic variety. The Neron-Severi $\mathbb{R}$-vector space of X is defined as:

$$\mathrm{NS}(X)_{\mathbb{R}} := \mathrm{Im}(\mathrm{Pic}(X) \to H^2(X, \mathbb{R})).$$

Lemma 2.29. *Let X be a normal compact Kähler variety with rational singularities. If $H^2(X, \mathcal{O}_X) = 0$, then X is projective and $\mathrm{NS}(X)_{\mathbb{R}} = H^{1,1}_{\mathrm{BC}}(X)$.*

Proof. Since $H^2(X, \mathcal{O}_X) = 0$, from the usual exponential sequence it follows that $\mathrm{Pic}(X) \twoheadrightarrow H^2(X, \mathbb{Z})$ is surjective. Since X is compact, all singular homology groups of X are finitely generated $\mathbb{Z}$-modules, and thus by the Universal Coefficient Theorem we have $H^2(X, Z) \otimes_{\mathbb{Z}} \mathbb{R} = H^2(X, \mathbb{R})$. Therefore $\mathrm{NS}(X)_{\mathbb{R}} = H^2(X, \mathbb{R})$. Now consider the following short exact sequence (see [HP16, page 223])

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{H}_X \longrightarrow 0. \tag{2.1}$$

The associated long exact sequence of cohomology yields

$$0 \to H^1(X, \mathbb{R}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{H}_X) \to H^2(X, \mathbb{R}) \to H^2(X, \mathcal{O}_X) \to \cdots . \tag{2.2}$$

Let $\pi : \hat{X} \to X$ be a resolution of singularities of X. Since X has rational singularities, $H^1(\hat{X}, \mathbb{R}) \cong H^1(X, \mathbb{R})$ and $H^1(\hat{X}, \mathcal{O}_{\hat{X}}) \cong H^1(X, \mathcal{O}_X)$. Since $\hat{X}$ is a compact Kähler manifold, from the Hodge decomposition it follows that $H^1(\hat{X}, \mathbb{R}) \to H^1(\hat{X}, \mathcal{O}_{\hat{X}})$ is an isomorphism, and thus $H^1(X, \mathbb{R}) \to H^1(X, \mathcal{O}_X)$ is an isomorphism. In particular, from the sequence (2.2) and the fact that $H^2(X, \mathcal{O}_X) = 0$, it follows that $H^2(X, \mathbb{R}) \cong H^1(X, \mathcal{H}_X) := H^{1,1}_{\mathrm{BC}}(X)$. This completes our proof.

□

In the next few results we will establish the cone theorem and existence of minimal models (and Mori fiber spaces) for generalized pairs in dimension 2 which will be used in rest of the articles in without reference.

Lemma 2.30. *Let (X, B) be a dlt pair, where X is a compact Kähler surface. Then there exists countably many rational curves $\{\Gamma_i\}_{i \in I}$ such that $0 < -(K_X + B) \cdot \Gamma_i \leq 4$ and*

$$\overline{\mathrm{NA}}(X) = \overline{\mathrm{NA}}(X)_{(K_X+B) \geq 0} + \sum_{i \in I} \mathbb{R}^+ \cdot [\Gamma_i].$$

Proof. From Lemma 2.20 it follows that X has $\mathbb{Q}$-factorial rational singularities. First assume that $K_X + B$ is pseudo-effective. Then from Lemma 2.25 it follows that $K_X + B$ is nef if and only if $(K_X + B) \cdot C \geq 0$ for every curve $C \subset X$. Let $K_X + B \equiv \sum_{i \in I} \lambda_i C_i + \beta$ be the Boucksom-Zariski decomposition as in [Bou04], where $\lambda_i \geq 0$ for all $i \in I \subset \mathbb{N}$ (a finite subset) and $\beta \cdot C \geq 0$ for

every curve $C \subset X$. Now if $K_X + B$ is not nef, then there is a curve $\Gamma \subset X$ such that $(K_X+B)\cdot\Gamma < 0$. This implies that $(\sum_{i\in I}\lambda_i C_i)\cdot\Gamma < 0$, in particular, $\Gamma = C_i$ for some $i \in I$ and $\Gamma^2 < 0$. Then the rest of proof works similarly as in the proof of [DO23, Theorem 6.1]. The length bound $0 > (K_X + B)\cdot\Gamma \geq -4$ follows from [DO23, Theorem 4.2].
Now assume that $K_X + B$ is not pseudo-effective. Then K_X is not pseudo-effective. Let $\nu : \widetilde{X} \to X$ be the minimal resolution of singularities of X. Then from [DH20, Lemma 2.40] it follows that $\widetilde{X}$ is an uniruled projective surface. In particular, X is Moishezon. Since X is also a compact Kähler variety with rational singularities, from [Nam02, Theorem 1.6] it follows that X is (uniruled) projective. Let $\pi : X \dashrightarrow Y$ be the MRC(C) fibration of X, where $\dim Y \leq 1$. Then from the argument of [DH20, Lemma 2.39] it follows that $H^2(X, \mathcal{O}_X) = 0$. In particular, from Lemma 2.29 it follows that $\mathrm{NS}(X)_{\mathbb{R}} = H^2(X,\mathbb{R}) = H^{1,1}_{\mathrm{BC}}(X)$, and hence $\overline{\mathrm{NE}}(X) = \overline{\mathrm{NA}}(X)$ and the cone theorem is well known in this case.

□

Lemma 2.31. *Let $(X, B+\boldsymbol{\beta})$ be a generalized klt (resp. dlt) pair, where X is a compact Kähler surface. Then we can run the $K_X + B + \boldsymbol{\beta}_X$-MMP*

$$X = X_0 \to X_1 \to \ldots \to X_n$$

so that:

1. *each $(X_i, B_i + \boldsymbol{\beta}_{X_i})$ is a generalized klt (resp. dlt) surface with $\mathbb{Q}$-factorial rational singularities (where B_i is defined by pushforward),*
2. *if $K_X + B + \boldsymbol{\beta}_X$ is pseudo-effective, then $K_{X_n} + B_n + \boldsymbol{\beta}_{X_n}$ is nef, and*
3. *if $K_X + B + \boldsymbol{\beta}_X$ is not pseudo-effective, then there is a $K_{X_n} + B_n + \boldsymbol{\beta}_{X_n}$-Mori fiber space $f : X_n \to Z$.*

Proof. By Lemma 2.20, X has $\mathbb{Q}$-factorial rational singularities. Then by Lemma 2.13, (X, B) is klt (resp. dlt). If $K_X + B$ is nef, then $K_X + B + \boldsymbol{\beta}_X$ nef by Corollary 2.27 and we are done. So assume that $K_X + B$ is not nef. Suppose that there is a $K_X + B$-negative extremal ray R which is also $K_X + B + \boldsymbol{\beta}_X$-negative (cf. Lemma 2.30). Then, by the usual MMP, there are two cases. If R defines a Mori fiber space, then we are done. Otherwise, R defines a divisorial contraction $g : X \to X'$ so that $(X', B' = g_*B)$ is klt (resp. dlt) and in particular $\mathbb{Q}$-factorial with rational singularities. From Corollary 2.27 it follows that $\boldsymbol{\beta}_{X'} := g_*\boldsymbol{\beta}_X$ has local potentials and $[\boldsymbol{\beta}_{X'}] \in H^{1,1}_{\mathrm{BC}}(X')$ is nef. We may replace $(X, B+\boldsymbol{\beta})$ by $(X', B'+\boldsymbol{\beta} = g_*(B+\boldsymbol{\beta}))$. The Kähler condition is preserved by Lemma 2.24. Repeating this procedure finitely many times we may assume that either it terminates with a Mori fiber space or every $K_X + B$-negative extremal ray R is $K_X + B + \boldsymbol{\beta}_X$-non-negative. In the latter case since

$\boldsymbol{\beta}_X$ is nef (see Corollary 2.27), it then follows that $K_X+B+\boldsymbol{\beta}_X$ is non-negative on $\overline{\mathrm{NA}}(X)$ and hence nef. This concludes our proof. □

Corollary 2.32. *Let $(X, B+\boldsymbol{\beta})$ be a generalized dlt pair, where X is a compact Kähler surface. Then the following holds:*

(1) There are at most countably many curves $\{\Gamma_i\}_{i\in I}$ such that $0>(K_X+B+\boldsymbol{\beta}_X)\cdot\Gamma_i\geq -4$ and

$$\overline{\mathrm{NA}}(X)=\overline{\mathrm{NA}}(X)_{K_X+B+\boldsymbol{\beta}_X\geq 0}+\sum_{i\in I}\mathbb{R}[\Gamma_i].$$

(2) If F is a face spanned by a set of $K_X+B+\boldsymbol{\beta}_X$-negative extremal rays, then there is a contraction $f: X\to Y$ contracting curves C with $[C]\in F$ and either Y is a point, or a smooth projective curve or a normal $\mathbb{Q}$-factorial surface with rational singularities.

(3) If $(X, B+\boldsymbol{\beta})$ is a generalized klt and $B+\boldsymbol{\beta}_X$ or $K_X+B+\boldsymbol{\beta}_X$ is big, then I is finite.

Proof. (1) By Lemma 2.13, (X, B) is dlt with rational $\mathbb{Q}$-factorial singularities. By Corollary 2.27, $\boldsymbol{\beta}_X$ is nef and so $\overline{\mathrm{NA}}(X)_{K_X+B\geq 0}\subset\overline{\mathrm{NA}}(X)_{K_X+B+\boldsymbol{\beta}_X\geq 0}$. Thus by Lemma 2.30 we have

$$\overline{\mathrm{NA}}(X)=\overline{\mathrm{NA}}(X)_{K_X+B\geq 0}+\sum_{i\in I}\mathbb{R}[\Gamma_i]=\overline{\mathrm{NA}}(X)_{K_X+B+\boldsymbol{\beta}_X\geq 0}+\sum_{i\in I}\mathbb{R}[\Gamma_i].$$

(2) Clearly F is also K_X+B-negative and hence the contraction exists by the usual contraction theorem. Since we are unable to find a reference for this fact we recall an easy proof. Let γ be the supporting hyperplane so that γ is nef and $\gamma^\perp\cap\overline{\mathrm{NA}}(X)=F$. Pick an extremal ray of F, say R. By [Fuj19] or [DO23, Theorem 6.2], we may contract R to obtain another generalized dlt Kähler surface $(X', B'+\boldsymbol{\beta}_{X'})$ with rational $\mathbb{Q}$-factorial singularities (see also Lemma 2.24). Since X' has rational singularities and hence $\gamma=\nu^*(\gamma')$, where $\gamma'\in H^{1,1}_{\mathrm{BC}}(X)$ is nef. Repeating this procedure, after finitely many steps we may assume that $\gamma'\in H^{1,1}_{\mathrm{BC}}(X)$ is Kähler, and thus we have contracted the face F.

(3) We claim that if $\psi\in H^{1,1}_{\mathrm{BC}}(X)$ is a big class, then there are at most finitely many curves $C\subset X$ such that $\int_C\psi<0$. To see this, note that for some Kähler form ω, the class $[\psi-\omega]$ is still big. Let $\psi-\omega\equiv Z+P$ be a Boucksom-Zariski decomposition such that $Z\geq 0$ is an effective $\mathbb{R}$-divisor and P is a nef class (see [Bou04, Proposition 2.4] and Lemma 2.25). But then one sees that if $\int_C\psi<0$, then C is contained in the support of Z. Thus, if $K_X+B+\boldsymbol{\beta}_X$ is big, then the claim immediately holds.

Suppose now that $B + \boldsymbol{\beta}_X$ is big, then we may write $B + \boldsymbol{\beta}_X \equiv Z + \omega + P$ as above. Thus

$$B + \boldsymbol{\beta}_X \equiv ((1-\epsilon)B + \epsilon Z) + ((1-\epsilon)\boldsymbol{\beta}_X + \epsilon(\omega + P))$$

where $(X, (1-\epsilon)B + \epsilon Z)$ is klt and $(1-\epsilon)\boldsymbol{\beta}_X + \epsilon(\omega + P)$ is Kähler for all $0 < \epsilon \ll 1$. The finiteness of $K_X + B + \boldsymbol{\beta}_X$ negative extremal rays now follows from the usual cone theorem. □

Theorem 2.33. *Let $(X, B+\boldsymbol{\beta})$ be a generalized klt pair, where X is a compact Kähler surface. If $K_X + B + \boldsymbol{\beta}_X$ is big, then $(X, B + \boldsymbol{\beta})$ has a log canonical model.*

Proof. By running a $K_X+B+\boldsymbol{\beta}_X$-MMP, we may assume that $\alpha = K_X+B+\boldsymbol{\beta}_X$ is nef and big (Lemma 2.31). We claim that $\mathrm{Null}(\alpha)$ consists of finitely many curves. To see this, choose a Kähler form ω such that $K_X + B + \boldsymbol{\beta}_X - \omega$ is also a big class. Then, by the Boucksom-Zariski decomposition [Bou04], we can write $K_X + B + \boldsymbol{\beta}_X - \omega \equiv D + \gamma$, where D is an effective $\mathbb{R}$-divisor and γ is a modified nef class (and hence a nef class by Lemma 2.25). Choose $0 < \varepsilon \ll 1$ such that $(X, B + \varepsilon D)$ is klt. Then

$$(1 + \varepsilon)\alpha = (K_X + B + \varepsilon D + \boldsymbol{\beta}_X) + \varepsilon(\gamma + \omega).$$

Now if $C \subset \mathrm{Null}(\alpha)$ is a curve, then $\alpha \cdot C = 0$ implies that $(K_X + B + \varepsilon D + \boldsymbol{\beta}_X) \cdot C < 0$. Since $K_X + B + \varepsilon D + \boldsymbol{\beta}_X$ is big, by a similar argument as in the proof of Corollary 2.32(3) it follows that there are finitely many such curves. This proves our claim. Moreover, from the above equation it also follows that if $C \subset \mathrm{Null}(\alpha)$ is a curve, then $(K_X + B + \varepsilon D) \cdot C < 0$, and thus this curve can be contracted. Repeating this process finitely many times (since $\mathrm{Null}(\alpha)$ contains finitely many curves) we obtain a projective bimeromorphic morphism $f : X \to Z$ to a normal compact surface Z with rational singularities such that $\alpha = f^*\alpha_Z$ and $\mathrm{Null}(\alpha_Z) = \emptyset$, where $\alpha_Z := f_*(K_X + B + \boldsymbol{\beta}_X) =: K_Z + B_Z + \boldsymbol{\beta}_Z$. Then from [DHP22, Theorem 2.30] it follows that α_Z is a Kähler class. Thus $(Z, B_Z + \boldsymbol{\beta}_Z)$ is the log canonical model of $(X, B + \boldsymbol{\beta}_X)$.

□

Remark 2.34. Note that by [LP20, Example 6.2], it is not the case that all generalized pairs have a good minimal model, however it is known that if β is an $\mathbb{R}$-divisor and $K_X + B$ is pseudo-effective, then good minimal models exist [LP20, Corollary C]. It would be interesting to know if good minimal models exist for generalized klt Kähler surface pairs $(X, B + \boldsymbol{\beta})$ such that $K_X + B$ is pseudo-effective and $[\boldsymbol{\beta}_X] \in H^{1,1}_{\mathrm{BC}}(X)$.

2.5. **Relative MMP for 3-Folds.** Using [DHP22, Theorem 5.2] we will show that we can run a relative MMP for *proper* morphism between Kähler varieties.

Theorem 2.35. *Let (X, B) be a $\mathbb{Q}$-factorial dlt pair, where X is a compact Kähler 3-fold. Let $f : X \to Z$ be a proper morphism to a normal compact Kähler variety. Then we can run a $K_X + B$-MMP over Z which terminates with either a log terminal model over Z or a Mori fiber space over Z.*

Proof. Let ω_Z be a Kähler class on Z. We may assume that $K_X + B$ is not nef over Z. Then $K_X + B + tf^*\omega_Z$ is not nef on X for any $t \geq 0$. From the cone theorem [DHP22, Theorem 5.2] we know that there are at most countably many rational curves $\{C_i\}_{i\in I}$ such that $0 > (K_X + B) \cdot C_i \geq -6$ for all $i \in I$ and

$$\overline{\mathrm{NA}}(X) = \overline{\mathrm{NA}}(X)_{(K_X+B)\geq 0} + \sum_{i\in I} \mathbb{R}^+ \cdot [C_i].$$

We claim that there is an $i \in I$ such that $f_*C_i = 0$. If not, then $f^*\omega_Z \cdot C_i = \omega_Z \cdot f_*C_i > 0$ for all $i \in I$, since ω_Z is a Kähler class on Z. Since the classes $[C_i]$ are contained in a discrete lattice of $H^4(X, \mathbb{Z})$, it follows that there is an $\epsilon > 0$ such that $\omega_Z \cdot f_*C_i \geq \epsilon$ for all $i \in I$. Then for some $t_0 \gg 0$ we may assume that $t_0 f^*\omega_Z \cdot C_i \geq 7$ for all $i \in I$. Thus $(K_X + B + t_0 f^*\omega_Z) \cdot C_i > 0$ for all $i \in I$, and hence $K_X + B + t_0 f^*\omega_Z$ is nef on X, a contradiction. Now we contract an extremal ray $R = \mathbb{R}^+ \cdot [C_i]$ such that $f_*C_i = 0$ using [DH20, Theorem 1.7] and obtain a morphism $g : X \to Y$ to a normal Kähler variety Y. Then from the rigidity lemma it follows that there is a unique morphism $h : Y \to Z$ such that $f = h \circ g$. Repeating this process we construct a MMP over Z. Termination of flips follow from [DO23, Theorem 3.3].

□

3. Threefold generalized MMP

3.1. **Running the MMP for $\mathbb{R}$-Cartier Divisors.** Throughout this section we will repeatedly use the results of [DH20] on the 3-fold MMP for $\mathbb{Q}$-factorial compact Kähler klt pairs (X, B). Note that in this reference, the results are stated for the case that $K_X + B$ is $\mathbb{Q}$-Cartier, however, they also hold when $K_X + B$ is an $\mathbb{R}$-Cartier divisor. This is because if $K_X + B$ is an $\mathbb{R}$-Cartier divisor, then it can be approximated by a sequence of klt $\mathbb{Q}$-Cartier divisors $K_X + B_n$ (for example, if X is $\mathbb{Q}$-factorial, let $B_n = \frac{1}{n}\lfloor nB \rfloor$). The cone theorem for $K_X + B$ is easily seen to follow from the cone theorem (cf. [DH20, Theorems 2.17, 4.6]) applied to the sequence of $\mathbb{Q}$-Cartier divisors $K_X + B_n$. If Γ is a $K_X + B$-negative extremal ray, then it is also a $K_X + B_n$-negative extremal ray for any $n \gg 0$ and so the contraction of Γ, $c_\Gamma : X \to Y$ exists by [DH20, Theorems 1.5 and 2.18]. Similarly, if $X \to Y$ is a $K_X + B$-flipping contraction, then it is also a $K_X + B_n$-flipping contraction and hence the flip $X^+ \to Y$ exists [CHP16, Theorem 4.3]. The termination of flips follows by the usual arguments (see [DO23, Theorem 3.3]).

Lemma 3.1. *Let $(X, B + \boldsymbol{\beta})$ be a compact generalized 3-fold pair with $\mathbb{Q}$-factorial rational singularities. If $\boldsymbol{\beta}_X$ is not nef, then $\boldsymbol{\beta}_X \cdot C < 0$ for some curve $C \subset X$ contained in the indeterminacy locus of f^{-1}, where $f : X' \to X$ is a structure morphism of the generalized pair.*

Proof. Since $K_X + B$ is $\mathbb{Q}$-Cartier, the current $\boldsymbol{\beta}_X$ has local potentials. Let

$$E := K_{X'} + B' - f^*(K_X + B) = f^*\boldsymbol{\beta}_X - \boldsymbol{\beta}_{X'},$$

where E is exceptional, and so $E \geq 0$ is effective by the negativity lemma as $\boldsymbol{\beta}_{X'}$ is f-nef. If $\boldsymbol{\beta}_X$ is not nef, then $\boldsymbol{\beta}_X|_V$ is not pseudo-effective for some subvariety $V \subset X$, by [DHP22, Theorem 2.36]. Since $\boldsymbol{\beta}_{X'}$ is nef, it is pseudo-effective and hence so is $\boldsymbol{\beta} = f_*\boldsymbol{\beta}_{X'}$, and hence $\dim V < 3$. If $\dim V = 2$, let $V' = f_*^{-1}V$; then $(\boldsymbol{\beta}_{X'} + E)|_{V'} = f^*\boldsymbol{\beta}_X|_{V'}$ is pseudo-effective and hence so is $\boldsymbol{\beta}_X|_V$. Thus $\dim V = 1$ and it is easy to see that V is contained in the image of E and hence in the indeterminacy locus of f^{-1}. □

Lemma 3.2. *Let X be a normal compact Kähler 3-fold and ω is a modified Kähler class on X. Then for any countable collection of non-numerically equivalent curves $\{C_i\}_{i\in I}$, there is a positive real number $b > 0$ such that $\omega \cdot C_i \geq b$ for all but finitely many curves. Moreover, if (X, B) is a log canonical pair for some $\mathbb{R}$-divisor $B \geq 0$ and $\{C_i\}_{i\in I}$ are all the rational curves generating the $K_X + B$-negative extremal rays of $\overline{\mathrm{NA}}(X)$, then there are only finitely many curves $\{C_j\}_{j\in J}$, $J \subset I$, such that $(K_X + B + \omega) \cdot C_j < 0$ for all $j \in J$.*

Proof. Let $f : X' \to X$ be a resolution of singularities of X and ω' a Kähler class on X' such that $f_*\omega' = \omega$. Then $f^*\omega = \omega' + E$, where E is a f-exceptional divisor. From the negativity lemma it follows that E is effective. Since $\dim X = 3$ and E is f-exceptional, $\dim f(\mathrm{Supp} E) \leq 1$. Therefore there can be at most finitely many curves $\{C_j\}_{j\in J}$, $J \subset I$, contained in $f(\mathrm{Supp} E)$. In particular, $\omega \cdot C_i = f^*\omega \cdot C_i' = (\omega' + E) \cdot C_i' > 0$ for all $i \in I \setminus J$, where C_i' is the strict transform of C_i. Note that these C_i' are also not numerically equivalent. Moreover, since ω' is a Kähler class, there is a positive real number $b > 0$ such that $\omega' \cdot C_i' \geq b$ for all $i \in I \setminus J$. In particular, $\omega \cdot C_i \geq \omega' \cdot C_i' \geq b$ for all $i \in I \setminus J$.

If $\{C_i\}_{i\in I}$ are generators of $K_X + B$-negative extremal rays, then from [DHP22, Corollary 5.3] it follows that $(K_X + B) \cdot C_i \geq -6$ for all $i \in I$. Therefore if $(K_X + B + \omega) \cdot C_i < 0$ for some $i \in I$, then $\omega' \cdot C_i' \leq \omega \cdot C_i < -(K_X + B) \cdot C_i \leq 6$. Since ω' is a Kähler class, it follows that there are only finitely many $K_X + B + \omega$-negative extremal rays. □

3.2. **Existence of Log Terminal Models.** In this subsection we will establish the existence of log terminal models and log canonical models, and prove Theorem 1.2.
In the following two results we will show that we can run a MMP with scaling (which terminates after finitely many steps) when $K_X + B + \boldsymbol{\beta}_X$ is pseudo-effective and $\boldsymbol{\beta}_X$ is a modified Kähler class.

Proposition 3.3. *Let (X, B) be a $\mathbb{Q}$-factorial compact Kähler 3-fold klt pair. Let $\omega \in H^{1,1}_{\mathrm{BC}}(X)$ be a modified Kähler class, $K_X+B+\omega$ is pseudo-effective and $K_X+B+(1+t)\omega$ is nef for some $t \geq 0$. Then we can run a $K_X+B+\omega$-MMP with scaling of $t\omega$ which terminates with a log terminal model.*

Proof. Let $\lambda := \inf\{t \geq 0 \ : \ K_X + B + (1 + t)\omega \text{ is nef}\}$.

Claim 3.4. There exists a $K_X + B$-negative extremal ray $\mathbb{R}^+[C]$ such that $(K_X + B + (1 + \lambda)\omega) \cdot C = 0$.

Proof. By [DHP22, Theorem 5.2], there are countably many $K_X + B$-negative extremal rays generated by curves $\{C_i\}_{i\in I}$ such that $0 > (K_X + B) \cdot C_i \geq -6$. Since ω is a modified Kähler class, by Lemma 3.2 there is a finite subset $I' \subset I$ such that $(K_X + B + \omega) \cdot C_i \geq 0$ if and only if $i \in I \setminus I'$. Let $I_0 \subset I'$ be the set of $i \in I_0$ such that $(K_X + B + (1 + \lambda)\omega) \cdot C_i = 0$.

We claim that $I_0 \neq \emptyset$. To see this, suppose that $I_0 = \emptyset$, then there is a positive real number $b > 0$ such that $(K_X + B + (1 + \lambda)\omega) \cdot C_i > b$ for any $i \in I'$, and there is a positive real number $c > 0$ such that $\omega \cdot C_i \leq c$ for all $i \in I'$. Recall that $(K_X + B) \cdot C_i \geq -6$ for all $i \in I$. Choose a positive real number $0 < \delta < \min\{\lambda, \ b/c\}$, then

$$(K_X + B + (1 + \lambda - \delta)\omega) \cdot C_i \geq b - \delta c > 0 \quad \text{for all } i \in I'. \tag{3.1}$$

Since

$$K_X + B + (1 + \lambda - \delta)\omega = \frac{\delta}{\lambda}(K_X + B + \omega) + (1 - \frac{\delta}{\lambda})(K_X + B + (1 + \lambda)\omega),$$

then $(K_X + B + (1 + \lambda - \delta)\omega) \cdot C_i \geq 0$ for all $i \in I \setminus I'$. Observe that

$$K_X + B + (1 + \lambda - \delta)\omega = \frac{\delta}{1 + \lambda}(K_X + B) + \left(1 - \frac{\delta}{1 + \lambda}\right)(K_X + B + (1 + \lambda)\omega)$$

and so $K_X + B + (1 + \lambda - \delta)\omega$ is non-negative on $\overline{\mathrm{NA}}(X)_{K_X+B\geq 0}$. Since by [DHP22, Theorem 5.2],

$$\overline{\mathrm{NA}}(X) = \overline{\mathrm{NA}}(X)_{K_X+B\geq 0} + \sum_{i\in I} \mathbb{R}^+ \cdot [C_i],$$

then $K_X+B+(1+\lambda-\delta)\omega$ is non-negative on $\overline{\mathrm{NA}}(X)$ and so $K_X+B+(1+\lambda-\delta)\omega$ is nef, which is a contradiction to the definition of λ.

□

Now, let $R = \mathbb{R}^+ \cdot [C]$ be a K_X+B-negative extremal ray such that $(K_X+B+(1+\lambda)\omega)\cdot C = 0$; in particular, $\omega \cdot C > 0$. Then, by [DH20, Theorem 1.7], we can contract this ray and obtain a morphism $f : X \to Y$ to a normal compact Kähler variety Y with rational singularities. Note that f is bimeromorphic, since it is also a contraction of a $(K_X+B+\omega)$-negative extremal ray and $K_X+B+\omega$ is pseudo-effective. If f is a flipping contraction then let $f' : X' \to Y$ be the associated flip (and if f is a divisorial contraction, let $X' = Y$), and B', ω' the pushforwards of B and ω on X'. Note that $K_{X'}+B'+(1+\lambda)\omega'$ is nef and ω' is modified Kähler. We now let $\lambda' := \inf\{t \geq 0 \ :\ K_{X'} + B' + (1+t)\omega' \text{ is nef}\}$ and repeat the process. Note that $0 \leq \lambda' \leq \lambda$ and the process terminates as there is no infinite sequence of steps for any $(K_X + B)$-MMP by [DO23, Theorem 3.3]. □

Corollary 3.5. *Let (X, B) be a $\mathbb{Q}$-factorial compact Kähler 3-fold klt pair and $\pi : X \to S$ a proper surjective morphism to a Kähler variety. Let $\omega \in H^{1,1}_{\rm BC}(X)$ be a modified Kähler class over S, $K_X + B + \omega$ is pseudo-effective over S and $K_X+B+(1+t)\omega$ is nef over S for some $t \geq 0$. Then we can run a $K_X+B+\omega$-MMP over S with scaling of $t\omega$ which terminates with a log terminal model over S.*

Proof. Replacing ω by $\omega+\pi^*\omega_S$ for some Kähler class ω_S on S, we may assume that $\omega \in H^{1,1}_{\rm BC}(X)$ is a modified Kähler class, $K_X+B+\omega$ is pseudo-effective, and $K_X+B+(1+t)\omega$ is nef. Let $\{C_i\}_{i\in I}$ be the set of curves generating all K_X+B-negative extremal rays of $\overline{\rm NA}(X)$. Then from [DHP22, Theorem 5.2] it follows that $0 < -(K_X+B)\cdot C_i \leq 6$ for all $i \in I$. Since $K_X+B+(1+t)\omega$ is nef, it follows that $\omega \cdot C_i > 0$ for all $i \in I$. In particular, we have $(K_X+B+(1+\lambda)\omega)\cdot C_i \geq -6$ for any $0 \leq \lambda \leq t$ and for all $i \in I$. Pick a Kähler class η_S on S such that $C \cdot \eta_S > 6$ for any curve C on S. Let $\omega' := \omega + \pi^*\eta_S$, then ω' is modified Kähler on X, $K_X + B + \omega'$ is pseudo-effective, and $K_X + B + (1+t)\omega'$ is nef. By Proposition 3.3, we may run the $K_X + B + \omega'$-MMP with scaling of $t\omega'$. Let

$$\lambda := \inf\{s \geq 0 \ :\ K_X + B + \omega' + s(t\omega') \text{ is nef}\}.$$

Then by Claim 3.4 there is a $K_X + B$-negative extremal ray spanned by a curve C_i such that $(K_X + B + (1+\lambda)\omega') \cdot C_i = 0$. We claim that $\pi_*C_i = 0$. If not, i.e. if $\pi_*C_i \neq 0$, then we have

$$0 = (K_X + B + (1+\lambda)\omega) \cdot C_i + (1+\lambda)\pi^*\eta \cdot C_i > -6 + (1+\lambda)6 > 0$$

which is a contradiction. Therefore $\pi_*C_i = 0$ and so the corresponding flip or divisorial contraction is a step of the $K_X + B$-MMP over S. Since there is no infinite sequence of $K_X + B$-flips (see [DO23, Theorem 3.3]), we may repeat this procedure finitely many times until we obtain a $K_X + B + \omega$ minimal model over S. □

We now prove the existence of log canonical model when $K_X + B + \boldsymbol{\beta}_X$ is big. This is result is of fundamental importance and will be used repeatedly in the rest of the article.

Theorem 3.6. *Let $(X, B+\boldsymbol{\beta})$ be a generalized klt pair, where X is a compact Kähler 3-fold. Assume that $K_X + B + \boldsymbol{\beta}_X$ is big. Then*

(1) $(X, B + \boldsymbol{\beta}_X)$ has a (unique) log canonical model,

(2) there exists a log terminal model and all such models admit a morphism to the log canonical model, and

(3) if $[K_X + B + \boldsymbol{\beta}_X] \in H^{1,1}_{\mathrm{BC}}(X)$ is very general, then there is a unique log terminal model coinciding with the log canonical model.

Proof. We begin with the following reduction.

Claim 3.7. We may assume that (X, B) is log smooth and $\boldsymbol{\beta}_X$ is a Kähler class.

Proof. Let $f : X' \to X$ be a structure morphism of the generalized pair $(X, B+\boldsymbol{\beta})$. Since $K_X + B + \boldsymbol{\beta}_X$ is big, by [Bou02, Theoreme 1.4] and passing to a higher resolution if necessary, we may assume that $f^*(K_X+B+\boldsymbol{\beta}_X) \equiv F'+\omega'$, where ω' is a Kähler class and $F' \geq 0$ is an effective $\mathbb{Q}$-divisor. Let $F + \omega := f_*(F' + \omega')$, then $F \geq 0$ and ω is modified Kähler. For any $0 < \epsilon \ll 1$, $(X, B + \epsilon F + \boldsymbol{\beta} + \epsilon\bar{\omega})$ is generalized klt and $K_X + B + \epsilon F + \boldsymbol{\beta}_X + \epsilon\omega \equiv (1 + \epsilon)(K_X + B + \boldsymbol{\beta}_X)$. Thus, replacing $(X, B + \boldsymbol{\beta})$ by $(X, \epsilon F + \boldsymbol{\beta} + \epsilon\bar{\omega}')$, we may assume that $\boldsymbol{\beta}_{X'}$ is Kähler for some log resolution $f : X' \to X$ of the generalized pair $(X, B + \boldsymbol{\beta})$.

Let $E = \mathrm{Ex}(f)$. By Lemma 2.12(5) and [BCHM10, Lemma 3.6.9], a log terminal model (resp. the log canonical model) of $K_{X'} + (B')_{\geq 0} + \epsilon E + \boldsymbol{\beta}_{X'}$, where $E \geq 0$ is an effective $\mathbb{Q}$-divisor such that $\mathrm{Supp}(E) = \mathrm{Ex}(f)$ and $0 \leq \epsilon \ll 1$ is also a log terminal model (resp. the log canonical model) of $K_X + B + \boldsymbol{\beta}_X$. Thus replacing $(X, B+\boldsymbol{\beta})$ by $(X', (B')_{\geq 0} + \epsilon E + \boldsymbol{\beta}' + \epsilon\bar{\omega}')$, we may assume that (X, B) is log smooth and $\boldsymbol{\beta}_X$ is a Kähler class. Note that if $[K_X + B + \boldsymbol{\beta}_X] \in H^{1,1}_{\mathrm{BC}}(X)$ is very general, then by [DH20], after possibly perturbing E, we may assume that $[K_{X'} + (B')_{\geq 0} + \epsilon E + \boldsymbol{\beta}_{X'} + \epsilon\omega']$ is very general in $H^{1,1}_{\mathrm{BC}}(X')$. $\square$

Then $K_X + B + (1 + t)\boldsymbol{\beta}_X$ is Kähler for $t \gg 0$ and $K_X + B + \boldsymbol{\beta}_X$ is pseudo-effective, and thus by Proposition 3.3, we can run the $K_X + B + \boldsymbol{\beta}_X$-MMP with scaling of $t\boldsymbol{\beta}_X$. We obtain a log terminal model $\phi : X \dashrightarrow X^m$ such that $\alpha^m := K_{X^m} + B^m + \boldsymbol{\beta}_{X^m} = \phi_*(K_X + B + \boldsymbol{\beta}_X)$ is nef and big, and $\boldsymbol{\beta}_{X^m}$ is a modified Kähler class. Moreover, we also have that $K_{X^m} + B^m + (1 + \epsilon)\boldsymbol{\beta}_{X^m}$ is nef (and big) for all $0 \leq \epsilon \ll 1$.

Claim 3.8. After a finite sequence of α^m-trivial steps of the $K_{X^m} + B^m$-MMP $X^m \dashrightarrow X^n$, we may assume that $(K_{X^n} + B^n) \cdot C \geq 0$ for any α^n-trivial curve $C \subset X^n$.

Proof. The proof follows exactly as in the proof of [DH20, Theorem 6.4] where it is shown that we may flip and contract all $K_{X^m}+B^m$-negative extremal rays that are α^m-trivial. Note that in [DH20] it is assumed that $\boldsymbol{\beta}_{X^m}$ is nef and big, but the arguments of the proof only use that $\boldsymbol{\beta}_{X^m}$ is modified Kähler. □

Claim 3.9. $\mathrm{Null}(\alpha^n)$ does not contain any surface.

Proof. This also follows from the proof of [DH20, Theorem 6.4]. □

Claim 3.10. There is a proper bimeromorphic contraction $\pi : X^n \to Z$ contracting $\mathrm{Null}(\alpha^n)$ such that $\mu : X^m \dashrightarrow Z$ is also a morphism.

Proof. The morphism $\pi : X^n \to Z$ contracting $\mathrm{Null}(\alpha^n)$ exists by [DH20, Proposition 6.2]. Following the proof of [DH20, Theorem 6.4], we argue that $\mu : X^m \to Z$ is also an α^m-trivial morphism. □

Recall that $K_{X^m}+B^m+(1+\epsilon)\boldsymbol{\beta}_{X^m}$ is nef. So from

$$\begin{aligned}\epsilon\boldsymbol{\beta}_{X^m} &= (K_{X^m}+B^m+(1+\epsilon)\boldsymbol{\beta}_{X^m}) - (K_{X^m}+B^m+\boldsymbol{\beta}_{X^m})\\ &= (K_{X^m}+B^m+(1+\epsilon)\boldsymbol{\beta}_{X^m}) - \alpha^m\end{aligned}$$

it follows that $\boldsymbol{\beta}_{X^m}\cdot C\geq 0$ for all curves $C\subset X^m$ contracted by $\mu : X^m\to Z$. Thus $-(K_{X^m}+B^m)$ is μ-nef-big, as $-(K_{X^m}+B^m)|_{X^m_z}\equiv \boldsymbol{\beta}_{X^m}|_{X^m_z}$ for all $z\in Z$. Then by [DHP22, Lemma 8.8], Z has rational singularities. Now since Z is in Fujiki's class $\mathcal{C}$, by [HP16, Lemma 3.3] there exists a $(1,1)$ class $\alpha_Z\in H^{1,1}_{\mathrm{BC}}(Z)$ such that $\alpha^m\equiv\mu^*\alpha_Z$. One then easily checks that $\mathrm{Null}(\alpha_Z)=\emptyset$ and so α_Z is Kähler by [DH20, Lemma 6.3]. Thus $K_Z+B_Z+\beta_Z:=\mu_*(K_{X^m}+B^m+\boldsymbol{\beta}_{X^m})$ is a log canonical model of $K_X+B+\boldsymbol{\beta}_X$. The uniqueness of log canonical models follows by (3) of Lemma 2.12; this proves (1).

(2) The fact that log terminal models admit a morphism to the log canonical model follows from the Claim 3.10 above.

(3) Finally, suppose that $[K_X+B+\boldsymbol{\beta}_X]$ is very general in $H^{1,1}_{\mathrm{BC}}(X)$ and $\pi : X^m\to Z$ is the morphism from a log terminal model X^m to the log canonical model Z. From Lemma 2.14 it follows that the induced morphism $\phi_* : H^{1,1}_{\mathrm{BC}}(X)\to H^{1,1}_{\mathrm{BC}}(X^m)$ is surjective, where $\phi : X\dashrightarrow X^m$; in particular, the class of $K_{X^m}+B^m+\boldsymbol{\beta}_{X^m}$ is very general in $H^{1,1}_{\mathrm{BC}}(X^m)$. Let C be a curve contracted by π, then $(K_{X^m}+B^m+\boldsymbol{\beta}_{X^m})\cdot C=0$, contradicting the fact that $[K_X+B+\boldsymbol{\beta}_X]$ is very general. Therefore π is a quasi-finite proper morphism with connected fibers, and hence an isomorphism.

□

We will also need the following relative version of Theorem 3.6.

Theorem 3.11. *Let $(X, B+\boldsymbol{\beta})$ be a generalized compact Kähler 3-fold klt pair, where $\pi : X \to S$ is a morphism to a compact Kähler variety and $K_X + B + \boldsymbol{\beta}_X$ is big over S. Then the following hold:*

(1) $(X, B + \boldsymbol{\beta}_X)$ has a (unique) log canonical model $X \dashrightarrow X^c$ over S.
(2) There exists a log terminal model $X \dashrightarrow X^m$ over S such that $K_{X^m} + B^m + \boldsymbol{\beta}_{X^m} + p^*\omega_S$ is nef for some Kähler form ω_S on S (where $p : X^m \to S$ is the corresponding morphism) and there is a morphism $X^m \to X^c$.
(3) If $(X^m, B^m + \boldsymbol{\beta}_{X^m})$ is a log terminal model over S, then $K_{X^m} + B^m + \boldsymbol{\beta}_{X^m} + p^*\omega_S$ is nef for some Kähler form ω_S on S, and there is a morphism $X^m \to X^c$.

Proof. Adding a sufficiently large multiple of a Kähler form ω_S on S, we may assume that $K_X + B + \boldsymbol{\beta}_X$ is big. Proceeding as in the proof of Theorem 3.6, replacing X by a higher model, we may assume that $\boldsymbol{\beta}_X$ is Kähler so that $K_X + B + (1 + t)\boldsymbol{\beta}_X$ is also Kähler for $t \gg 0$. As in the proof of Corollary 3.5, after adding the pullback of a sufficiently large multiple of a Kähler form ω_S on S, we run the $K_X + B + \boldsymbol{\beta}_X$-MMP with scaling of $t\boldsymbol{\beta}_X$ which turns out to be a MMP over S, and we obtain a log terminal model $X \dashrightarrow X'$ over S such that $K_{X^m} + B^m + \boldsymbol{\beta}_{X^m}$ is nef and hence also nef over S. By Theorem 3.6 there is a log canonical model $\psi : X^m \to X^c$ for $K_{X^m} + B^m + \boldsymbol{\beta}_{X^m} + p^*\omega_S$, where ω_S is a Kähler class on S, and $p : X^m \to S$ is the induced morphism. Note that ψ is a bimeromorphic morphism, so its fibers are covered by curves and $\psi : X^m \to X^c$ contracts $K_{X^m} + B^m + \boldsymbol{\beta}_{X^m} + p^*\omega_S$-trivial curves. Since $K_{X^m} + B^m + \boldsymbol{\beta}_{X^m}$ is nef and ω_S is Kähler, any such curve must be vertical over S and hence by the rigidity lemma (see [BS95, Lemma 4.1.13]), there is a morphism $X^c \to S$ so that $\psi : X^m \to X^c$ is the log canonical model for $K_{X^m} + B^m + \boldsymbol{\beta}_{X^m}$ over S. Thus (1) and (2) hold.

Suppose now that $X \dashrightarrow X^m$ is any log terminal model of $K_X + B + \boldsymbol{\beta}_X$ over S. We begin by showing the following.

Claim 3.12. There exists a Kähler form ω_S on S such that $K_{X^m} + B^m + \boldsymbol{\beta}_{X^m} + p^*\omega_S$ is nef.

Proof. If $K_{X^m} + B^m + \boldsymbol{\beta}_{X^m}$ is nef, then the claim is obvious. Otherwise, let $X \dashrightarrow X^n$ be the log terminal model of $K_X + B + \boldsymbol{\beta}_X$ over S constructed in (2). Then $K_{X^n} + B^n + \boldsymbol{\beta}_{X^n} + q^*\omega_S$ is nef for some Kähler class ω_S on S, where $q : X^n \to S$ is the corresponding morphism. Now, by Theorem A.11, $X^m \dashrightarrow X^n$ is an isomorphism in codimension 1 between log terminal models of $K_X + B + \boldsymbol{\beta}_X$ over S, and hence it easily follows from the negativity lemma that if $r : W \to X^m$ and $s : W \to X^n$ is a common resolution, then $r^*(K_{X^m} + B^m + \boldsymbol{\beta}_{X^m} + p^*\omega_S) = s^*(K_{X^n} + B^n + \boldsymbol{\beta}_{X^n} + q^*\omega_S)$. Since $K_{X^n} + B^n + \boldsymbol{\beta}_{X^n} + q^*\omega_S$ is nef, so is $K_{X^m} + B^m + \boldsymbol{\beta}_{X^m} + p^*\omega_S$. □

Arguing as above, it follows easily that $X^m \to X^c$ is a morphism and hence (3) also holds. □

Theorem 3.13. *Let $(X, B+\boldsymbol{\beta})$ be a generalized klt pair, where X is a compact Kähler 3-fold. Then the following hold:*

(1) X has rational singularities,
(2) there exists a small bimeromorphic morphism $\nu : X^q \to X$ such that X^q is $\mathbb{Q}$-factorial, and
(3) there exists a bimeromorphic morphism $\nu : X^t \to X$ such that X^t is $\mathbb{Q}$-factorial and $(X^t, B^t + \boldsymbol{\beta})$ is a generalized terminal pair such that $K_{X^t} + B^t + \boldsymbol{\beta}_{X^t} = \nu^*(K_X + B + \boldsymbol{\beta}_X)$.

Note that a local version of (2) was proven in Theorem 2.19.

Proof. (1) follows from Theorem 2.19.

(2) Let $f : X' \to X$ be a projective log resolution of the generalized pair $(X, B + \boldsymbol{\beta})$. Fix $0 < \epsilon \ll 1$ and let $\phi : X' \dashrightarrow X^q$ be a log terminal model of $K_{X'} + f_*^{-1}B + (1-\epsilon)\mathrm{Ex}(f)$ over X which exists by Theorem 3.11. Since $K_{X'} + f_*^{-1}B + (1-\epsilon)\mathrm{Ex}(f) + \boldsymbol{\beta}_{X'} \equiv_X F$ where $F \geq 0$ and $\mathrm{Supp}(F) = \mathrm{Ex}(f)$, it follows that $F^q = \phi_* F \geq 0$ is $f^q : X^q \to X$ exceptional and F^q is nef over X and so by the negativity lemma, $F^q = 0$. Therefore f^q is a small bimeromorphic morphism and X^q is $\mathbb{Q}$-factorial.

The proof of (3) is also standard and similar to the proof of (2) and so we omit it.

□

The following theorem is a variant of the Base-point-free theorem [DH20, Theorem 1.7].

Theorem 3.14. *Let $(X, B+\boldsymbol{\beta})$ be a generalized klt pair, where X is a compact Kähler 3-fold. Assume that $K_X + B + \boldsymbol{\beta}_X$ is nef but not big and $\boldsymbol{\beta}_{X'}$ is big. Then there is a morphism $g : X \to Z$ to a normal Kähler variety Z such that $K_X + B + \boldsymbol{\beta}_X = g^*\alpha_Z$, where α_Z is a Kähler class on Z.*

Proof. Note that if $\nu : X' \to X$ is a bimeromorphic morphism and $f' : X' \to Z$ a proper morphism (not necessarily bimeromorphic) of normal compact Kähler varieties such that $\nu^*\alpha = f'^*\alpha_Z$, where α_Z is a Kähler class on Z, then f' contracts all ν-vertical curves and so by the rigidity lemma (see [BS95, Lemma 4.1.13]) there is a morphism $f : X \to Z$ such that $f \circ \nu = f'$ and $\alpha = f^*\alpha_Z$. Therefore by passing to a small $\mathbb{Q}$-factorialization using Theorem 3.13 we may assume that X is $\mathbb{Q}$-factorial and $(X, B + \boldsymbol{\beta})$ is terminal. Since $\boldsymbol{\beta}_{X'}$ is nef and big, possibly replacing X' by a higher model, we may assume that $\boldsymbol{\beta}_{X'} = F + \omega'$ where $F \geq 0$ is an effective $\mathbb{R}$-divisor and ω' is Kähler [Bou02, Theoreme 1.4].

Pick $\epsilon > 0$ such that $(X', B' + \epsilon F)$ is sub-klt. Define $B^* := f_*(B' + \epsilon F)$ and $\beta^* := f_*((1-\epsilon)\boldsymbol{\beta}_{X'} + \epsilon\omega')$. Then $(X, B^* + \bar{\beta}^*)$ is a generalized pair and β^* is a modified Kähler class. Note that $K_X + B^* + \beta^* \equiv K_X + B + \boldsymbol{\beta}_X$; thus replacing $(X, B + \boldsymbol{\beta})$ by $(X, B^* + \bar{\beta}^*)$ we may assume that $\boldsymbol{\beta}_X$ is a modified Kähler class.

Now, if K_X is pseudo-effective, then $K_X + B + \boldsymbol{\beta}_X$ is big, which is a contradiction. Therefore K_X is not pseudo-effective, and hence X is uniruled.

Claim 3.15. Let $\pi : X \dashrightarrow T$ be the MRCC fibration. Then we may assume that $\dim T = 2$.

Proof. Since X is uniruled, $\dim T \leq 2$. If $\dim T \leq 1$, then from the proof of [DH20, Lemma 2.39] it follows that $H^2(X, \mathcal{O}_X) = 0$. Thus X is projective and every $(1,1)$ class is represented by an $\mathbb{R}$-Cartier divisor. In particular, $(X, B + \boldsymbol{\beta}_X)$ is numerically equivalent to a traditional generalized pair for projective varieties, i.e. $[\boldsymbol{\beta}_{X'}] = c_1(N')$, where N' is a nef and big $\mathbb{R}$-divisor on X' and $f : X' \to X$ is the given log resolution of $(X, B + \boldsymbol{\beta}_X)$. We then have $N' \sim_{\mathbb{R}} A' + E$, where A' is a general ample $\mathbb{R}$-divisor and E is an effective $\mathbb{R}$-divisor. Therefore

$$K_{X'} + B' + N' \sim_{\mathbb{R}} K_{X'} + B' + \epsilon E + (1-\epsilon)N' + \epsilon A' =: K_{X'} + B'' + N'',$$

where $B'' := B' + \epsilon E$, $N'' \sim_{\mathbb{R}} (1-\epsilon)N' + \epsilon A'$ is a general ample $\mathbb{R}$-divisor and $(X', B'' + N'')$ is sub klt. But then $(X, \Delta := f_*(B'' + N''))$ is klt such that $\Delta \geq 0$ is big and $K_X + B + \boldsymbol{\beta}_X \equiv K_X + \Delta$. The conclusion now follows from the base-point free theorem for $\mathbb{R}$-divisors, for example see [BCHM10, Theorem 3.9.1]. Therefore we may assume that $\dim T = 2$. □

Claim 3.16. Let F be a general fiber of $\pi : X \dashrightarrow T$, then $F \cong \mathbb{P}^1$ and $(K_X + B + \boldsymbol{\beta}_X) \cdot F = 0$.

Proof. Let $g : Y \to X$ be a log resolution of $(X, B + \boldsymbol{\beta})$ which also resolves the map $\pi : X \dashrightarrow T$. Write

$$K_Y + B_Y + \boldsymbol{\beta}_Y = g^*(K_X + B + \boldsymbol{\beta}_X) + E,$$

where $B_Y \geq 0, E \geq 0, g_* B_Y = B, g_* E = 0$, B_Y and E do not share any component, and $\boldsymbol{\beta}_Y$ is nef.
Observe that the general fibers of $\pi \circ g$ and π are isomorphic. Now since $(K_X + B + \boldsymbol{\beta}_X)$ is pseudo-effective, so is $K_Y + B_Y + \boldsymbol{\beta}_Y$, and thus $(K_Y + B_Y + \boldsymbol{\beta}_Y) \cdot F \geq 0$. If $(K_X + B + \boldsymbol{\beta}_X) \cdot F > 0$, then $(K_Y + B_Y + \boldsymbol{\beta}_Y) \cdot F = (K_X + B + \boldsymbol{\beta}_X) \cdot F > 0$, and thus $(K_Y + B_Y + t\boldsymbol{\beta}_Y) \cdot F > 0$ for some $1 > t > 0$. Then by [Gue20, Theorem], $K_Y + B_Y + t\boldsymbol{\beta}_Y$ is pseudo-effective and so $K_Y + B_Y + \boldsymbol{\beta}_Y + (1-t)\operatorname{Ex}(f)$ is big, since $\boldsymbol{\beta}_X$ is big. In particular, $K_X + B + \boldsymbol{\beta}_X = g_*(K_Y + B_Y + \boldsymbol{\beta}_Y + (1-t)\operatorname{Ex}(f))$ is big, a contradiction. □

Now, as in the proof of [DH20, Theorem 5.2] we will analyze the nef dimension of $K_X + B + \boldsymbol{\beta}_X$. Since a dense open subset of X is covered by $K_X+B+\boldsymbol{\beta}_X$-trivial curves, we see that the nef dimension $n(K_X+B+\boldsymbol{\beta}_X) \leq 2$. If $n(K_X + B + \boldsymbol{\beta}_X) = 0$, then $K_X + B + \boldsymbol{\beta}_X \equiv 0$ and we are done by choosing $Z := \operatorname{Specan}(\mathbb{C})$. If $n(K_X + B + \boldsymbol{\beta}_X) = 1$, then there is a smooth projective curve C and a morphism $g : X \to C$ such that $K_X + B + \boldsymbol{\beta}_X = g^*\alpha_C$, where $\alpha_C \in H^{1,1}_{\mathrm{BC}}(C)$, (see [BCE+02, 2.4.4] and [HP15, Theorem 3.19]). Since the nef dimension $n(g^*\alpha_C) = 1$, it follows that α_C is a Kähler class and we are done. The final case is $n(K_X + B + \boldsymbol{\beta}_X) = 2$. In this case, by an argument identical to the one in [DH20, Theorem 5.5], we find the required morphism $g : X \to Z$.

Note that in [DH20], $\boldsymbol{\beta}_X$ is assumed to be nef and big, however, in the proof it is only used to show that a nef and big class can be written as a sum of a modified Kähler class and a sufficiently small effective divisor (see the Step 3 of the proof of [DH20, Theorem 5.2]); in particular, $\boldsymbol{\beta}_X$ being modified Kähler is enough for the proof in [DH20].

□

Theorem 3.17. *Let $(X, B + \boldsymbol{\beta})$ be a $\mathbb{Q}$-factorial generalized klt pair, where X is a compact Kähler 3-fold, such that $K_X + B + \boldsymbol{\beta}_X$ is pseudo-effective but not big and $\boldsymbol{\beta}_{X'}$ is big. Then there is a log terminal model $f : X \dashrightarrow X^{\mathrm{m}}$ and a morphism $g : X^{\mathrm{m}} \to Z$ such that $K_{X^{\mathrm{m}}} + B^{\mathrm{m}} + \boldsymbol{\beta}_{X^{\mathrm{m}}} = g^*\alpha_Z$, where α_Z is a Kähler class on Z.*

Proof. If K_X is pseudo-effective, then $K_X + B + \boldsymbol{\beta}_X$ is big, contradicting our assumptions. Thus, K_X is not pseudo-effective. In particular, X is uniruled. Let $\pi : X \dashrightarrow T$ be the MRCC fibration of X. If $\dim T \leq 1$, then from the proof of [DH20, Lemma 2.39] it follows that $H^2(X, \mathcal{O}_X) = 0$. In particular, all $(1, 1)$ classes on X are represented by $\mathbb{R}$-Cartier divisors and X is projective. So we may assume that $(X, B + \boldsymbol{\beta})$ is a traditional generalized pair on a projective variety and the statement follows from known results (see [BZ16, Lemma 4.4]). Therefore we may assume that $\dim T = 2$.

Let $\nu : Y \to X$ be a log resolution of $(X, B + \boldsymbol{\beta})$ so that $K_Y + B_Y + \boldsymbol{\beta}_Y = \nu^*(K_X + B + \boldsymbol{\beta}_X)$, where (Y, B_Y) is log smooth and $\boldsymbol{\beta}_Y$ is nef and big. Passing to a higher model, we may assume that $\boldsymbol{\beta}_Y \equiv \omega' + E$, where E is an effective $\mathbb{R}$-divisor and ω' is Kähler. Therefore, for $0 < \epsilon \ll 1$,

$$K_Y + B_Y + \boldsymbol{\beta}_Y \equiv K_Y + B_Y + \epsilon E + (1 - \epsilon)\boldsymbol{\beta}_Y + \epsilon\omega',$$

where $(1 - \epsilon)\boldsymbol{\beta}_Y + \epsilon\omega'$ is Kähler and $(Y, B_Y + \epsilon E)$ is sub-klt. Let $B^* := \nu_*(B_Y + \epsilon E)$ and $\beta^* := \nu_*((1-\epsilon)\beta_Y + \omega')$. Then the generalized pair $(X, B^* + \bar{\beta}^*)$ is generalized klt and β^* is a modified Kähler class. Moreover, $K_X + B^* + \beta^* \equiv K_X + B + \boldsymbol{\beta}_X$; thus replacing $(X, B + \boldsymbol{\beta})$ by $(X, B^* + \bar{\beta}^*)$ we may assume that $\boldsymbol{\beta}_X$

is a modified Kähler class. Then from the proof of Claim 3.16 it follows that $(K_X+B+\boldsymbol{\beta}_X)\cdot F=0$ for general fibers F of the MRC fibration $\pi:X\dashrightarrow T$, $K_X+B+(1+\epsilon)\boldsymbol{\beta}_X$ is big for $0<\epsilon\ll 1$, and $(X,B+(1+\epsilon)\boldsymbol{\beta})$ is generalized klt.

Claim 3.18. Let $N:=N(K_X+B+\boldsymbol{\beta}_X)$ and $N_\epsilon:=N(K_X+B+(1+\epsilon)\boldsymbol{\beta}_X)$ be the negative parts of the Boucksom-Zariski decomposition of the pseudo-effective classes $K_X+B+\boldsymbol{\beta}_X$ and $K_X+B+(1+\epsilon)\boldsymbol{\beta}_X$ for $\epsilon>0$ (see §A). We may assume that

$$\operatorname{Supp}(N_\epsilon)\subset\operatorname{Supp}(N),\qquad \text{for all } 0<\epsilon\ll 1,$$

and in particular, $\operatorname{Supp}(N_\epsilon)$ is independent of $0<\epsilon\ll 1$.

Proof of Claim 3.18. Note that $N(K_X+B+\boldsymbol{\beta}_X)$ is an effective $\mathbb{R}$-divisor. Since $\boldsymbol{\beta}_X$ is modified Kähler, from Remark A.8 it follows that if $\epsilon>\epsilon'\geq 0$, then $N_\epsilon\leq N_{\epsilon'}$. Since their support is contained in N, they must stabilize. ☐

Let $f_\epsilon:X\dashrightarrow X^{\mathrm{m}}_\epsilon$ be a log terminal model of $K_X+B+(1+\epsilon)\boldsymbol{\beta}_X$ (which exists by Theorem 3.6 as $K_X+B+(1+\epsilon)\boldsymbol{\beta}_X$ is a big class). Then for all $0<\epsilon\ll 1$, the divisors contracted by f_ϵ are just N_ϵ (see Theorem A.11) and so X^{m}_ϵ are all isomorphic in codimension 1 for all $0<\epsilon\ll 1$. We now fix an $0<\epsilon\ll 1$ satisfying the above Claim 3.18 and run the $K_{X^{\mathrm{m}}_\epsilon}+B_{X^{\mathrm{m}}_\epsilon}+\boldsymbol{\beta}_{X^{\mathrm{m}}_\epsilon}$-MMP with scaling of $\epsilon\boldsymbol{\beta}_{X^{\mathrm{m}}_\epsilon}$. This MMP terminates with a log terminal model $\psi:X^{\mathrm{m}}_\epsilon\dashrightarrow X^{\mathrm{m}}$ by Proposition 3.3. Let

$$K_{X^{\mathrm{m}}}+B_{X^{\mathrm{m}}}+(1+t)\boldsymbol{\beta}_{X^{\mathrm{m}}}:=\psi_*(K_{X^{\mathrm{m}}_\epsilon}+B_{X^{\mathrm{m}}_\epsilon}+(1+t)\boldsymbol{\beta}_{X^{\mathrm{m}}_\epsilon})$$

for $t\geq 0$. Then, by the properties of the MMP with scaling, there exists $0<\delta<\epsilon$ such that ψ is also a $K_{X^{\mathrm{m}}_\epsilon}+B_{X^{\mathrm{m}}_\epsilon}+(1+t)\boldsymbol{\beta}_{X^{\mathrm{m}}_\epsilon}$-MMP with scaling of $(\epsilon-t)\boldsymbol{\beta}_{X^{\mathrm{m}}_\epsilon}$ for every $0\leq t\leq\delta$. In particular, $K_{X^{\mathrm{m}}}+B_{X^{\mathrm{m}}}+(1+t)\boldsymbol{\beta}_{X^{\mathrm{m}}}$ is nef for all $0\leq t\leq\delta$. Note that from Claim 3.18 it follows that

$$\operatorname{Supp}N(K_{X^{\mathrm{m}}_\epsilon}+B_{X^{\mathrm{m}}_\epsilon}+(1+t)\boldsymbol{\beta}_{X^{\mathrm{m}}_\epsilon})=\operatorname{Supp}N(K_{X^{\mathrm{m}}_\epsilon}+B_{X^{\mathrm{m}}_\epsilon}+(1+\epsilon)\boldsymbol{\beta}_{X^{\mathrm{m}}_\epsilon})=0,$$

where the second equality holds because $K_{X^{\mathrm{m}}_\epsilon}+B_{X^{\mathrm{m}}_\epsilon}+(1+\epsilon)\boldsymbol{\beta}_{X^{\mathrm{m}}_\epsilon}$ is nef. Thus from Theorem A.11 it follows that ψ is a small map. Therefore $X^{\mathrm{m}}\dashrightarrow X^{\mathrm{m}}_t$ is a small bimeromorphic map for every $0<t\leq\delta$ where, as above, $f_t:X\dashrightarrow X^{\mathrm{m}}_t$ is a log terminal model of $K_X+B+(1+t)\boldsymbol{\beta}_X$. Since $K_{X^{\mathrm{m}}}+B_{X^{\mathrm{m}}}+(1+t)\boldsymbol{\beta}_{X^{\mathrm{m}}}$ and $K_{X^{\mathrm{m}}_t}+B_{X^{\mathrm{m}}_t}+(1+t)\boldsymbol{\beta}_{X^{\mathrm{m}}_t}$ are both nef, we have

$$a(P;X^{\mathrm{m}},B_{X^{\mathrm{m}}}+(1+t)\boldsymbol{\beta}_{X^{\mathrm{m}}})=a(P;X^{\mathrm{m}}_t,B_{X^{\mathrm{m}}_t}+(1+t)\boldsymbol{\beta}_{X^{\mathrm{m}}_t})\geq a(P;X,B+(1+t)\boldsymbol{\beta})$$

for any prime Weil divisor P over X.
Since X is $\mathbb{Q}$-factorial, $\boldsymbol{\beta}_X$ has local potentials. In particular,

$$a(P,X,B+(1+t)\boldsymbol{\beta})=a(P,X,B+\boldsymbol{\beta})-t\cdot\operatorname{mult}_P(f^*\boldsymbol{\beta}_X-\boldsymbol{\beta}_{X'}).$$

Therefore, taking the limit as $t \to 0^+$ we see that $a(P, X, B+\boldsymbol{\beta}) \leq a(P, X^{\mathrm{m}}, B_{X^{\mathrm{m}}}+\boldsymbol{\beta}_{X^{\mathrm{m}}})$ for every prime Weil divisor P over X, and hence $\phi : X \dashrightarrow X^{\mathrm{m}}$ is a $\mathbb{Q}$-factorial weak log canonical model. Let $\{P_i\}_{i\in I}$ be the set of all ϕ-exceptional divisors on X such that $a(P_i, X, B+\boldsymbol{\beta}) = a(P_i, X^{\mathrm{m}}, B_{X^{\mathrm{m}}}+\boldsymbol{\beta})$. To obtain a log terminal model of $(X, B+\boldsymbol{\beta})$, we need to extract P_i from X^{m}. To that end, let $h : Y \to X^{\mathrm{m}}$ be a log resolution of $(X^{\mathrm{m}}, B_{X^{\mathrm{m}}}+\boldsymbol{\beta}_{X^{\mathrm{m}}})$ which extracts the divisors $\{P_i\}_{i\in I}$. Write $K_Y + B_Y + \boldsymbol{\beta}_Y = f^*(K_{X^{\mathrm{m}}} + B_{X^{\mathrm{m}}} + \boldsymbol{\beta}_{X^{\mathrm{m}}})$, where (Y, B_Y) is log smooth and $\boldsymbol{\beta}_Y$ is nef. Note that $a(P_i, X^{\mathrm{m}}, B_{X^{\mathrm{m}}}+\boldsymbol{\beta}_{X^{\mathrm{m}}}) = a(P_i, X, B+\boldsymbol{\beta}) \leq 0$ for all $i \in I$, since $P_i \subset X$ is a divisor on X. Let $\{P_i\}_{i\in I} \cup \{Q_j\}_{j\in J}$ be the set of all h-exceptional divisors. We define

$$\tilde{B}_Y := h_*^{-1}B_{X^{\mathrm{m}}} - \sum_{i\in I} a(P_i, X^{\mathrm{m}}, B_{X^{\mathrm{m}}} + \boldsymbol{\beta}_{X^{\mathrm{m}}})P_i + \sum_{j\in J}(1-\epsilon)Q_j$$

for $0 < \epsilon \ll 1$. Then $(Y, \tilde{B}_Y + \boldsymbol{\beta}_Y)$ is a klt pair such that $K_Y + \tilde{B}_Y + \boldsymbol{\beta}_Y \equiv_{X^{\mathrm{m}}} E \geq 0$, where $\mathrm{Supp}(E) = \cup_{j\in J} Q_j$. We run a $K_Y + \tilde{B}_Y + \boldsymbol{\beta}_Y$-MMP over X^{m} as in Corollary 3.5. Replacing Y by the corresponding minimal model, we may assume that E is nef over X^{m}, and since it is exceptional, it follows from the negativity lemma that $E = 0$. We then have that $K_Y + \tilde{B}_Y + \boldsymbol{\beta}_Y = h^*(K_{X^{\mathrm{m}}} + B_{X^{\mathrm{m}}} + \boldsymbol{\beta}_{X^{\mathrm{m}}})$ is nef. Replacing $(X^{\mathrm{m}}, B_{X^{\mathrm{m}}} + \boldsymbol{\beta})$ by $(Y, \tilde{B}_Y + \boldsymbol{\beta})$ we see that $\pi : X \dashrightarrow X^{\mathrm{m}}$ is a log terminal model for $K_X + B + \boldsymbol{\beta}_X$.
The existence of the morphism $g : X^{\mathrm{m}} \to Z$ such that $K_{X^{\mathrm{m}}} + B_{X^{\mathrm{m}}} + \boldsymbol{\beta}_{X^{\mathrm{m}}} = g^*\alpha_Z$, where α_Z is Kähler on Z follows from Theorem 3.14. □

Proof of Theorem 1.2. It follows from combining Theorems 3.6 and 3.17. □

Next we will establish an analog of [BCHM10, Corollary 1.1.5] for log canonical models.

Theorem 3.19. *Let X be a normal $\mathbb{Q}$-factorial compact Kähler 3-fold and $\nu : X' \to X$ a resolution. Let (X, B) be a pair and Ω' be a compact convex polyhedral set of closed positive (1,1) currents on X' such that for every $\beta' \in \Omega'$, $(X, B + \boldsymbol{\beta})$ is a generalized klt pair, where $\boldsymbol{\beta} = \bar{\beta}'$. Assume that one of the following conditions hold:*

(i) $K_X + B + \boldsymbol{\beta}_X$ is big for every $\beta' \in \Omega'$ (and $\boldsymbol{\beta} = \bar{\beta}'$), or
(ii) there is a bimeromorphic morphism $\pi : X \to S$ of normal compact Kähler 3-folds.

Then there exists a finite polyhedral decomposition $\Omega' = \cup\Omega'_i$ and finitely many bimeromorphic maps $\psi_i : X \dashrightarrow X_i$ (resp. finitely many bimeromorphic maps $\psi_i : X \dashrightarrow X_i$ over S) such that if $\psi : X \dashrightarrow Y$ is a log canonical model for $K_X + B + \nu_*\beta'$ (resp. a log canonical model for $K_X + B + \nu_*\beta'$ over S) for some $\beta' \in \Omega'_i$, then $\psi = \psi_i$.

Note that a compact convex polyhedral set is a convex hull of finitely many vectors. Then by finite polyhedral decomposition $\Omega' = \cup \Omega'_i$ we simply mean that each Ω'_i is a subset of Ω' defined by finitely many affine linear equations and inequalities such that $\Omega'_i \cap \Omega'_j = \emptyset$ for $i \neq j$.

Proof. We will prove both cases (i) and (ii) simultaneously. We will use the convention that in case (i), $S = \operatorname{Specan}(\mathbb{C})$ and we remark that in case (ii) the condition that $K_X + B + \boldsymbol{\beta}_X$ is big over S is automatic as π is bimeromorphic. We will use induction on the dimension of Ω'. We will abuse notation and denote $\boldsymbol{\beta}_X$ by β. If $\dim \Omega' = 0$, then $\Omega' = \{\beta'_0\}$ for some β'_0 such that $(X, B + \beta_0 = B + \nu_*\beta'_0)$ is a generalized klt pair and $K_X + B + \beta_0$ is big (over S). In this case the existence of the required log canonical model follows by Theorems 3.6 and 3.11.

Since Ω' is compact, it is enough to prove the statement locally in a neighborhood of each point $\beta' \in \Omega'$. Fix a point $\beta'_0 \in \Omega'$ and let $\beta_0 = \nu_*\beta'_0 \in \Omega := \nu_*\Omega'$. By Theorems 3.6 and 3.11, there is a $K_X + B + \beta_0$-log terminal model $\phi : X \dashrightarrow X^m$ (over S) and a log canonical model $\psi : X^m \to X^c$ (over S). Since $a(E, X, B + \beta_0) < a(E, X^m, B^m + \beta_0^m)$ for all ϕ-exceptional divisors E of X (where $B^m + \beta_0^m = \phi_*(B + \beta_0)$), shrinking Ω' (to a smaller polytope containing β'_0 but without changing its dimension) around β'_0 we may assume that if $\beta = \nu_*\beta'$ and $\beta^m = \phi_*\beta$, then $a(E, X, B + \beta) < a(E, X^m, B^m + \beta^m)$ for all $\beta' \in \Omega'$ and for all ϕ-exceptional divisors E of X. In particular, if $\phi^m : X^m \to \bar{X}^m$ is a log canonical model for $K_{X^m} + B^m + \beta^m$ (over S), then $\phi^m \circ \phi : X \dashrightarrow \bar{X}^m$ is a log canonical model for $K_X + B + \beta$ (over S).

Now let $\Omega^m := \phi_*\Omega$. Note that Ω^m is a compact convex polyhedral subset of $H^{1,1}_{\mathrm{BC}}(X^m)$, since ϕ_* is a linear map by Lemma 2.14. Then, by induction, there is a finite polyhedral decomposition $\partial\Omega^m = \cup_{i=1}^k \mathcal{P}_i$ of the boundary $\partial\Omega^m$ of Ω^m and finitely many meromorphic maps $\phi_i : X^m \dashrightarrow X_i$ (over X^c) $1 \leq i \leq \ell$ such that if $f : X^m \dashrightarrow Y$ is a log canonical model of $K_{X^m} + B^m + \beta^m$ over X^c for some $\beta^m \in \mathcal{P}_i$, then $f = \phi_i$ (note that as $K_X + B + \beta_0$ is big over S, $\psi : X^m \to X^c$ is bimeromorphic). Recall that $\beta_0^m := \phi_*\beta_0 \in \Omega^m$. Choose $\beta_1^m \in \partial\Omega^m$ such that $\beta_1^m \neq \beta_0^m$. For $0 < \lambda \leq 1$ we define

$$\beta_\lambda^m := (1 - \lambda)\beta_0^m + \lambda\beta_1^m. \tag{3.2}$$

Recall that $K_{X^m} + B^m + \beta_0^m = \psi^*\omega \equiv_{X^c} 0$ for some Kähler class ω (over S) on X^c. By induction, $\phi_i : X^m \dashrightarrow X_i$ is a log canonical model of $K_{X^m} + B^m + \beta_1^m$ over X^c for some i. Thus from (3.2) we have

$$K_{X_i} + B_i + \beta_{\lambda,i} \equiv (1 - \lambda)\psi_i^*\omega + \lambda(K_{X_i} + B_i + \beta_{1,i}), \tag{3.3}$$

where $\psi_i : X_i \to X^c$ is the induced bimeromorphic morphism.
Thus from [BCHM10, Lemma 3.6.8] and our induction hypothesis it follows that ϕ_i is a log canonical model of $K_{X^m} + B^m + \beta_\lambda^m$ over X^c for all $0 < \lambda \leq 1$.

Let $\nu_i : X_i^q \to X_i$ be a small $\mathbb{Q}$-factorization as in Theorem 3.13 such that

$$K_{X_i^q} + B_i^q + \beta_i^q = \nu_i^*(K_{X_i} + B_i + \beta_i). \tag{3.4}$$

Let $\phi_i^q : X^m \dashrightarrow X_i^q$ be the induced bimeromorphic maps.

Claim 3.20. There exists a constant $\bar{\lambda} > 0$ such that for every $\beta_1^m \in \partial\Omega^m$ there exists a $\phi_i : X^m \dashrightarrow X_i$ for some $i \in \{1, 2, \ldots, \ell\}$ such that ϕ_i is a log canonical model of $K_{X^m} + B^m + \beta_\lambda^m$ (over S) for all $0 < \lambda \leq \bar{\lambda}$.

Proof of Claim 3.20. Let $p_i^q : X_i^q \to S$ be the induced morphisms for all $1 \leq i \leq \ell$. Consider the following set of curves in X_i^q for each $1 \leq i \leq \ell$:

$$\mathcal{C}_i := \{C \subset X_i^q \mid p_{i,*}^q(C) = 0 \text{ and } \beta_i^q \cdot C < 0 \text{ for some } \beta_i^q\},$$

where $\beta_i^q := \phi_{i,*}^q(\beta^m)$ for $\beta^m \in \Omega^m$.
We claim that $\mathcal{C}_i$ is a finite set. Indeed, since each $\beta' \in \Omega'$ is nef over S and descends to X' and the composite map $\phi_i^q \circ \phi \circ \nu : X' \dashrightarrow X_i^q$ does not extract any divisors, it follows that if $C \subset X_i^q$ is a curve such that $p_{i,*}^q(C) = 0$ and $C \nsubseteq \mathrm{Ex}((\phi_i^q \circ \phi)^{-1})$, then $\beta_i^q \cdot C \geq 0$. Thus $\mathcal{C}_i \subset \mathcal{D}_i := \{C \subset X \mid p_{i,*}^q(C) = 0 \text{ and } C \subset \mathrm{Ex}((\phi_i^q \circ \phi)^{-1})\}$ for all $1 \leq i \leq \ell$, and clearly each $\mathcal{D}_i$ is a finite set as $\dim X_i^q = 3$. Observe that if $C \subset X_i^q$ is a curve such that $p_{i,*}^q(C) = 0$ but $C \notin \mathcal{C}_i$, then $\beta_i^q \cdot C \geq 0$ for all β_i^q. Suppose that $K_{X_{i'}^q} + B_{i'}^q + \beta_{i'}^q$ is not nef (over S) for some $1 \leq i' \leq \ell$ and $\beta' \in \Omega'$. Pick a Kähler class $\omega_{i'}^q$ on $X_{i'}^q$ which is very general in $H^{1,1}_{\mathrm{BC}}(X_{i'}^q)$, and consider the corresponding nef threshold $\mu > 0$ so that $K_{X_{i'}^q} + B_{i'}^q + \beta_{i'}^q + \mu\omega_{i'}^q$ is nef and big (over S) but not Kähler (over S). By Theorems 3.6 and 3.11, there is a log canonical model $g_{i'} : X_{i'}^q \to Z_{i'}$ (over S) which is a bimeromorphic morphism and hence the exceptional locus is covered by curves. Since $\omega_{i'}^q$ is very general in $H^{1,1}_{\mathrm{BC}}(X_{i'}^q)$, it follows that all of the exceptional curves of $g_{i'}$ belong to a fixed ray, say $\mathbb{R}^{\geq 0}[\Gamma_{i'}]$, where $p_{i',*}^q(\Gamma_{i'}) = 0$; observe that the curve $\Gamma_{i'}$ depends on the class $\beta_{i'}^q$. Let $\mathcal{T}$ be the collection of all such curves $\Gamma_{i'}$ as β^m varies in Ω^m, where $\beta_{i'}^q := \phi_{i',*}^q\beta^m$. We note here that $\mathcal{T}$ could be an infinite collection. We claim that there is a positive real number $M > 0$ (independent of indices $1 \leq i \leq \ell$) such that if $K_{X_i^q} + B_i^q + \beta_i^q$ is not nef (over S) for some $1 \leq i \leq \ell$ and $\beta^m \in \Omega^m$, then $0 > (K_{X_i^q} + B_i^q + \beta_i^q) \cdot \Gamma_i \geq -M$ for the corresponding $\Gamma_i \in \mathcal{T}$. To see this, observe that if $\beta_i^q \cdot \Gamma_i \geq 0$, then $(K_{X_i^q} + B_i^q) \cdot \Gamma_i < 0$ and so by the usual cone theorem (see [DHP22, Corollary 5.3]) we may assume that

$$0 > (K_{X_i^q} + B_i^q + \beta_i^q) \cdot \Gamma_i \geq (K_{X_i^q} + B_i^q) \cdot \Gamma_i \geq -6.$$

If $\beta_i^q \cdot \Gamma_i < 0$, then from our construction of the sets $\mathcal{C}_i$ above it follows that $\Gamma_i \in \mathcal{C}_i$. Since $\mathcal{C}_i$ is a finite set and the indices i also vary in the finite set $\{1, \ldots, \ell\}$, by the compactness of Ω^m the claim follows.

Recall from equation (3.3) that ω is a Kähler class on X^c (over S) such that $K_{X_i^q}+B_i^q+\beta_0^q=(\psi_i^q)^*\omega$, where $\psi_i^q=\psi_i\circ\nu_i$ for all $1\leq i\leq \ell$. There exists a $\delta>0$ such that $\omega\cdot C>\delta$ for every curve $C\subset X^c$ which is vertical over S. Let $\bar\lambda:=\frac{\delta}{M+\delta}$. If $K_{X_i^q}+B_i^q+\beta_{\lambda,i}^q$ is not nef (over S) for some $0<\lambda\leq\bar\lambda$ and for some $1\leq i\leq\ell$, then from (3.3) it follows that $K_{X_i^q}+B_i^q+\beta_{1,i}^q$ is not nef (over S) for some $\beta_1^m\in\partial\Omega^m$. Then by the claim above there is a curve $\Gamma_i\in\mathcal{T}$ such that $0>(K_{X_i^q}+B_i^q+\beta_{1,i}^q)\cdot\Gamma_i\geq -M$. Moreover, since $K_{X_i^q}+B_i^q+\beta_{1,i}^q$ is nef over X^c, Γ_i is not contracted by $\psi_i^q:X_i^q\to X^c$. Let $\bar\Gamma_i:=\psi_{i,*}^q\Gamma_i$, then $\omega\cdot\bar\Gamma_i>\delta$.
Thus from (3.3) we have

$$(K_{X_i^q}+B_i^q+\beta_{\lambda,i}^q)\cdot\Gamma_i=(1-\lambda)\omega\cdot\bar\Gamma_i+\lambda(K_{X_i^q}+B_i^q+\beta_{1,i}^q)\cdot\Gamma_i>(1-\lambda)\delta-\lambda M\geq 0$$

which is a contradiction, and hence $K_{X_i^q}+B_i^q+\beta_{\lambda,i}^q=\nu_i^*(K_{X_i}+B_i+\beta_{\lambda,i})$ is nef over S and so is $K_{X_i}+B_i+\beta_{\lambda,i}$.

Since $\phi_i:X^m\dashrightarrow X_i$ is a log canonical model of $K_{X^m}+B^m+\beta_1^m$ over X^c, $K_{X_i}+B_i+\beta_{1,i}$ is Kähler over X^c. Recall again that $K_{X_i}+B_i+\beta_{0,i}=\psi_i^*\omega$, where ω is a Kähler class over S and $\psi_i:X_i\to X^c$ is the induced morphism. In the argument above we saw that $K_{X_i}+B_i+\beta_{\lambda,i}$ is nef over S for all $0\leq\lambda\leq\bar\lambda$. By contradiction assume that there is an $0<\lambda\leq\bar\lambda/2$ such that $K_{X_i}+B_i+\beta_{\lambda,i}$ is not Kähler over S. Then, by Theorem 3.11 there is a proper bimeromorphic morphism $X_i\to Z$ over S, where Z is the log canonical model for $K_{X_i}+B_i+\beta_{\lambda,i}$ over S. So there is a curve $C\subset X_i$ (over S) such that $(K_{X_i}+B_i+\beta_{\lambda,i})\cdot C=0$. Also, note that there is a $0<\mu<1$ such that

$$\beta_{\lambda,i}=(1-\mu)\beta_{0,i}+\mu\beta_{\bar\lambda,i}. \tag{3.5}$$

As observed above, $K_{X_i}+B_i+\beta_{0,i}$ and $K_{X_i}+B_i+\beta_{\bar\lambda,i}$ are both nef over S, and thus from (3.5) it follows that $(K_{X_i}+B_i+\beta_{0,i})\cdot C=(K_{X_i}+B_i+\beta_{\bar\lambda,i})\cdot C=0$. Then again from (3.3) (with λ replaced by $\bar\lambda$) it follows that $(K_{X_i}+B_i+\beta_{1,i})\cdot C=0$. In particular, C is not vertical over X^c (as $K_{X_i}+B_i+\beta_{1,i}$ is Kähler over X^c). However, since $K_{X_i}+B_i+\beta_{0,i}=\psi_i^*\omega$ where ω is Kähler over S, it follows that $(K_{X_i}+B_i+\beta_{0,i})\cdot C>0$; this is a contradiction to (3.3).

Finally replacing $\bar\lambda$ by $\bar\lambda/2$ completes the proof of Claim 3.20.

□

Note that as observed above, $X\dashrightarrow X_i$ is a log canonical model for $K_X+B+\beta_\lambda$ (over S) for all $\beta_1\in\mathcal{P}_i$ and $0<\lambda\leq\bar\lambda$. The decomposition $\partial\Omega^m=\cup\mathcal{P}_i$ induces a corresponding decomposition of $\Omega^m-\{\beta_0^m\}=\cup\Omega_i^m$ where each Ω_i^m is the polytope spanned by β^0 and $\mathcal{P}_i$ excluding β_0^m, and $\Omega_0^m:=\{\beta_0^m\}$ is a 0-dimensional polytope. We then obtain a decomposition $\Omega'=\cup\Omega_i'$, where Ω_i is the inverse image of Ω_i^m. Finally we replace Ω' by $\Omega' \cap \{\beta' \in \Omega' \ : \ ||\beta' - \beta'_0|| \leq \bar{\lambda}\}$ for some fixed norm $||\cdot||$. This completes the proof. ☐

3.3. **Existence of Mori Fiber Space.** In this subsection we will show that if $K_X + B + \boldsymbol{\beta}_X$ is not pseudo-effective, then we can run an MMP which ends with a Mori fiber space.
First we will show that if $K_X + B + \boldsymbol{\beta}_X$ is big, then we can run a terminating MMP with scaling of a very general Kähler class. Using this result, we will then show that we can also obtain a Mori fiber in the non pseudo-effective case.

Theorem 3.21. *Let $(X, B + \boldsymbol{\beta})$ be a $\mathbb{Q}$-factorial generalized dlt pair such that $K_X + B + \boldsymbol{\beta}_X$ is big, where X is a compact Kähler 3-fold. Let ω be a very general Kähler class in $H^{1,1}_{\rm BC}(X)$ such that $K_X + B + \boldsymbol{\beta}_X + \omega$ is a Kähler class. Then we can run a terminating $K_X + B + \boldsymbol{\beta}_X$-MMP with scaling of ω.*

Proof. To run the $K_X + B + \boldsymbol{\beta}_X$-MMP with scaling of ω, we will inductively construct a sequence of bimeromorphic maps $\phi_i : X_i \dashrightarrow X_{i+1}$ and real numbers $t_i > t_{i+1}$ for $i \geq 0$ such that $X_0 = X$ and $t_0 = 1$ and the following conditions are now satisfied

1. $(X_i, B_i + \boldsymbol{\beta}_{X_i} + t_i\omega_i)$ is a generalized dlt pair,
2. $K_{X_i} + B_i + \boldsymbol{\beta}_{X_i} + t_i\omega_i$ is nef,
3. $K_{X_i} + B_i + \boldsymbol{\beta}_{X_i} + (t_i - \epsilon)\omega_i$ is Kähler for $0 < \epsilon \ll 1$,
4. $K_{X_i} + B_i + \boldsymbol{\beta}_{X_i}$ is big,
5. X_i is $\mathbb{Q}$-factorial and $\omega_i \in H^{1,1}_{\rm BC}(X_i)$ is very general.

The base of the induction is clear. Assume that we have constructed $(X_i, B_i + \boldsymbol{\beta}_{X_i} + t_i\omega_i)$ as above. Let

$$t_{i+1} := \inf\{s \geq 0 \mid K_{X_i} + B_i + \boldsymbol{\beta}_{X_i} + s\omega_i \text{ is nef}\}.$$

If $t_{i+1} = 0$, then the MMP terminates and $X \dashrightarrow X_i$ is a log terminal model for $(X, B + \boldsymbol{\beta}_X)$. Thus we may assume that $t_{i+1} > 0$.

Let $\psi : X_i \to Z_i$ be the log canonical model for $K_{X_i} + B_i + \boldsymbol{\beta}_{X_i} + t_{i+1}\omega_i$ (which exists by Theorem 3.6; note that $(X_i, B_i + \boldsymbol{\beta}_{X_i} + t_{i+1}\omega_i)$ is generalized dlt but not necessarily generalized klt, however if $\omega' = t_{i+1}\omega + \epsilon B_i$ for $0 < \epsilon \ll 1$, then $(X_i, (1-\epsilon)B_i + \boldsymbol{\beta}_{X_i} + \omega'_i)$ is generalized klt and so Theorem 3.6 applies).

Since $\psi_i : X_i \to Z_i$ is bimeromorphic, the fibers of ψ_i are covered by curves. Since ω_i is very general in $H^{1,1}_{\rm BC}(X_i)$ and $(K_{X_i} + B_i + \boldsymbol{\beta}_{X_i} + t_{i+1}\omega_i) \cdot C = 0$ for any ψ_i-exceptional curve $C \subset X_i$, it follows that $\rho(X_i/Z_i) = 1$ and ψ_i is a contraction of a $K_{X_i} + B_i + \boldsymbol{\beta}_{X_i}$-negative extremal ray R_i spanned by (any) one of these curves, i.e. $R_i = \mathbb{R}^{\geq 0}[C]$. If ψ_i is a divisorial contraction, then we let $\phi_i = \psi_i$ and

$$K_{X_{i+1}} + B_{i+1} + \boldsymbol{\beta}_{X_{i+1}} + t_{i+1}\omega_{i+1} := \psi_{i,*}(K_{X_i} + B_i + \boldsymbol{\beta}_{X_i} + t_{i+1}\omega_i).$$

If ψ_i is a small contraction, then it is a $K_{X_i} + B_i + \boldsymbol{\beta}_{X_i}$ flipping contraction (as it is ω_i-positive).

Claim 3.22. Let $X \dashrightarrow X_{i+1}$ be the log canonical model of $K_X + B + \boldsymbol{\beta}_X + (t_{i+1} - \epsilon)\omega$ (for any $0 < \epsilon \ll 1$). Then $\phi_i : X_i \dashrightarrow X_{i+1}$ is the flip of ψ_i.

Proof. By Theorem 3.19, we may assume that there is an $\epsilon_0 > 0$ such that $X \dashrightarrow X_{i+1}$ is the log canonical model of $K_X + B + \boldsymbol{\beta}_X + (t_{i+1} - \epsilon)\omega$ for any $0 < \epsilon \leq \epsilon_0$. In particular, $K_{X_{i+1}} + B_{i+1} + \boldsymbol{\beta}_{X_{i+1}} + t_{i+1}\omega_{i+1}$ is nef and hence admits a morphism $\psi_i^+ : X_{i+1} \to \bar{Z}$ to the log canonical model of $(X_{i+1}, B_{i+1} + \boldsymbol{\beta}_{X_{i+1}} + t_{i+1}\omega_{i+1})$ (which exists by Theorem 3.6). Since $X \dashrightarrow X_{i+1}$ is $K_X+B+\boldsymbol{\beta}_X+t_{i+1}\omega$ non-positive, then $X \dashrightarrow \bar{Z}$ is also the log canonical model of $K_X + B + \boldsymbol{\beta}_X + t_{i+1}\omega$ and hence $\bar{Z} = Z_i$. Note that $-(K_{X_i} + B_i + \boldsymbol{\beta}_{X_i} + (t_i - \epsilon)\omega_i)$ is Kähler over Z_i and $K_{X_{i+1}} + B_{i+1} + \boldsymbol{\beta}_{Xi+1} + (t_{i+1} - \epsilon)\omega_{i+1}$ is Kähler over Z_i and so $X_i \dashrightarrow X_{i+1}$ is a $K_{X_i} + B_i + \boldsymbol{\beta}_{X_i} + (t_{i+1} - \epsilon)\omega_i$-flip. Since $K_{X_i} + B_i + \boldsymbol{\beta}_{X_i} + t_{i+1}\omega_i \equiv_{Z_i} 0$, it follows that $X_i \dashrightarrow X_{i+1}$ is a $K_{X_i} + B_i + \boldsymbol{\beta}_{X_i}$-flip. □

It is easy to check that properties (1-5) hold for $(X_{i+1}, B_{i+1} + \boldsymbol{\beta}_{X_{i+1}} + t_{i+1}\omega_{i+1})$. Repeating the above procedure we obtain a sequence of $K_X + B + \boldsymbol{\beta}_X + (t_i - \epsilon_i)\omega$ distinct log canonical models where $0 \leq t_i - \epsilon_i \leq 1$. By Theorem 3.19, this sequence can not be infinite and so the above minimal model program with scaling terminates and the proof is complete. □

The next result shows that if $K_X + B + \boldsymbol{\beta}_X$ is not pseudo-effective, then we can run a terminating $K_X + B + \boldsymbol{\beta}_X$-MMP with scaling of a very general Kähler class and end with a Mori fiber space.

Theorem 3.23. *Let $(X, B+\boldsymbol{\beta})$ be a $\mathbb{Q}$-factorial generalized klt pair, where X is a compact Kähler 3-fold. Assume that $K_X+B+\boldsymbol{\beta}_X$ is not pseudo-effective, and let ω be a very general Kähler class in $H^{1,1}_{\mathrm{BC}}(X)$ such that $K_X+B+\boldsymbol{\beta}_X+\omega$ is Kähler. Then we can run the $K_X + B + \boldsymbol{\beta}_X$-MMP with scaling of ω and obtain $\phi : X \dashrightarrow X'$ such that $K_{X'} + B' + \boldsymbol{\beta}_{X'} + \tau\omega'$ is pseudo-effective but not big for some $0 < \tau < 1$, and there is a Mori-fiber space $g : X' \to W$.*

Proof. We define

$$\tau := \inf\{s \geq 0 \mid K_X + B + \boldsymbol{\beta}_X + s\omega \text{ is pseudo-effective}\}$$

and

$$t_1 := \inf\{s \geq 0 \ : \ K_X + B + \boldsymbol{\beta}_X + s\omega \text{ is nef}\}.$$

Then $K_X + B + \boldsymbol{\beta}_X + \tau\omega$ is pseudo-effective but not big. By Theorem 3.17, there is a log terminal model $\phi : X \dashrightarrow X'$ and a morphism $g : X' \to Z$ of normal Kähler varieties such that $K_{X'} + B' + \boldsymbol{\beta}_{X'} + \tau\omega' = g^*\alpha_Z$, where α_Z is a Kähler class. Since $K_X + B + \boldsymbol{\beta}_X + \tau\omega$ is not big, g is not bimeromorphic.

We begin by proving that we can run a $K_X + B + \boldsymbol{\beta}_X$-MMP with scaling of ω terminating with a log terminal model of $K_X + B + \boldsymbol{\beta}_X + \tau\omega$. Indeed, if X_i is a step of this MMP, then let $t_{i+1} := \inf\{s \geq 0 : K_{X_i} + B_i + \boldsymbol{\beta}_{X_i} + s\omega_i \text{ is nef}\}$. If $t_{i+1} > \tau$, then $K_{X_i} + B_i + \boldsymbol{\beta}_{X_i} + t_{i+1}\omega_i$ is big and by Theorem 3.21 we can run this MMP. Thus as long as $t_i > \tau$, we can continue running this MMP and it will stop once we have $t_i = \tau$ for some i (note that every step of this MMP is also a step of $K_X + B + \boldsymbol{\beta}_X$-MMP with the scaling of ω). However, it is not clear whether this MMP will terminate after finitely many steps. Assume by contradiction that this MMP does not terminate. We claim that $\lim t_i = \tau$. If not, then let $\lim t_i = \tau_0 > \tau$; note that $\tau_0 = \inf\{t_i : i \geq 0\}$. Then every step of the above MMP is also a step of $K_X + B + \boldsymbol{\beta}_X + \tau_0\omega$-MMP, but since $K_X + B + \boldsymbol{\beta}_X + \tau_0\omega$ big (as $\tau_0 > \tau$), this MMP terminates by Theorem 3.21, a contradiction. Now from Claim 3.18 we observe that

(3.6)
$$N(K_X+B+\boldsymbol{\beta}_X+t\omega) = N(K_X+B+\boldsymbol{\beta}_X+\tau\omega) \quad \text{for all } t > 0 \text{ s.t. } 0 < t-\tau \ll 1.$$

Thus by Theorem A.11 we may assume that $X_i \dashrightarrow X'$ is a small bimeromorphic map for $i \gg 0$. Then from the proof of Theorem 3.21, it follows that $K_{X_i} + B_i + \boldsymbol{\beta}_{X_i} + t\omega_i$ is Kähler for any $t > 0$ satisfying $t_i > t > t_{i+1}$. We may also assume that if $0 < t_0 - \tau \ll 1$, then $a(E, X, B + \boldsymbol{\beta}_X + t_0\omega) < a(E, X', B' + \boldsymbol{\beta}_{X'} + t_0\omega')$ for all ϕ-exceptional divisors and that $(X', B' + \boldsymbol{\beta}_{X'} + t_0\omega')$ is generalized klt. Fix t_0 as above. By Theorem 3.17, there is a morphism $g : X' \to Z$ such that $K_{X'} + B' + \boldsymbol{\beta}_{X'} + \tau\omega' \equiv g^*\alpha_Z$ where α_Z is Kähler on Z. Let $b > 0$ be a constant such that $\alpha_Z \cdot C > b$ for any curve C on Z and fix $\tau < t < \frac{bt_0+6\tau}{b+6}$. By Theorem 3.21 there is a sequence of $K_{X'} + B' + \boldsymbol{\beta}_{X'} + t\omega'$-flips $X'_j \dashrightarrow X'_{j+1}$ with $0 \leq j \leq \bar{j} - 1$ ending with $X' \dashrightarrow X'_{\bar{j}}$, a log terminal model of $(X', B' + \boldsymbol{\beta}_{X'} + t\omega')$. Then $X \dashrightarrow X'_{\bar{j}}$ is a log terminal model of $(X, B + \boldsymbol{\beta}_X + t\omega)$. Since ω is very general in $H^{1,1}_{\mathrm{BC}}(X)$, we may assume that $t_i > t > t_{i+1}$ for some $i \gg 0$; then $K_{X_i} + B_i + \boldsymbol{\beta}_{X_i} + t\omega_i$ is Kähler as argued above. Since $X'_{\bar{j}} \dashrightarrow X_i$ is a small bimeromorphic map, by Lemma 2.16 it's an isomorphism, i.e. $X'_{\bar{j}} \cong X_i$.

We claim that each flip $\psi_j : X'_j \dashrightarrow X'_{j+1}$ for $0 \leq j \leq \bar{j} - 1$ is $K_{X'} + B' + \boldsymbol{\beta}_{X'} + \tau\omega'$-trivial. We prove this by induction on i. Suppose that $X'_0 = X'$ and the claim holds for the first $k - 1$ flips, then each flip is a flip over Z and so there is a morphism $g'_k : X'_k \to Z$ such that $K_{X'_k} + B'_k + \boldsymbol{\beta}_{X'_k} + \tau\omega'_k = (g'_k)^*\alpha_Z$. Observe that $\psi_0, \ldots, \psi_{k-1}$ are $K_{X'} + B' + \boldsymbol{\beta}_{X'} + \lambda\omega'$-flips for any $\tau < \lambda \leq t_0$, as each of the them are $K_{X'}+B'+\boldsymbol{\beta}_{X'}+\tau\omega'$-trivial. Recall that $\alpha_Z\cdot C\geq b$ for every curve $C\subset Z$. Let $X'_k \dashrightarrow X'_{k+1}$ be the next $K_{X'}+B'+\boldsymbol{\beta}_{X'}+t\omega'$-flip and C_k a corresponding flipping curve. Since $K_{X'_k}+B'_k+\boldsymbol{\beta}_{X'_k}+\tau\omega'_k$ is nef, we may assume that this curve is also a $K_{X'}+B'+\boldsymbol{\beta}_{X'}+t_0\omega'$-flipping curve, and hence by Corollary 2.23 we may assume that $-(K_{X'_k}+B'_k+\boldsymbol{\beta}_{X'_k}+t_0\omega'_k)\cdot C_k\leq 6$. Moreover, if $(K_{X'_k}+B'_k+\boldsymbol{\beta}_{X'_k}+\tau\omega'_k)\cdot C_k>0$, then $(K_{X'_k}+B'_k+\boldsymbol{\beta}_{X'_k}+\tau\omega'_k)\cdot C_k\geq b$. Observe that

$$K_{X'_k}+B'_k+\boldsymbol{\beta}_{X'_k}+t\omega'_k=\frac{t_0-t}{t_0-\tau}(K_{X'_k}+B'_k+\boldsymbol{\beta}_{X'_k}+\tau\omega'_k)+\frac{t-\tau}{t_0-\tau}(K_{X'_k}+B'_k+\boldsymbol{\beta}_{X'_k}+t_0\omega'_k).$$

Since $\tau<t<\frac{bt_0+6\tau}{b+6}$ and hence $b(t_0-t)+6(\tau-t)>0$, we then have

$$0>(K_{X'_k}+B'_k+\boldsymbol{\beta}_{X'_k}+t\omega'_k)\cdot C_k\geq\frac{b(t_0-t)}{t_0-\tau}-\frac{6(t-\tau)}{t_0-\tau}>0.$$

Since this is impossible, we have $(K_{X'_k}+B'_k+\boldsymbol{\beta}_{X'_k}+\tau\omega'_k)\cdot C_k=0$ and hence ψ_k is $K_{X'_k}+B'_k+\boldsymbol{\beta}_{X'_k}+\tau\omega'_k$-trivial and the induction is complete.

Since $X'_j\cong X_i$ for some $i\gg 0$, we may assume that there is a morphism $g_i:X_i\to Z$ such that $K_{X_i}+B_i+\boldsymbol{\beta}_{X_i}+\tau\omega_i=g_i^*\alpha_Z$. But this leads to an immediate contradiction, since if $X_i\dashrightarrow X_{i+1}$ is a flip and Σ_i is a flipping curve for the $K_X+B+\boldsymbol{\beta}_X$-MMP with scaling of ω, then $(K_{X_i}+B_i+\boldsymbol{\beta}_{X_i}+t_{i+1}\omega_i)\cdot\Sigma_i=0$ and $\omega_i\cdot\Sigma_i>0$ so that $(K_{X_i}+B_i+\boldsymbol{\beta}_{X_i}+\tau\omega_i)\cdot\Sigma_i<0$, but $(K_{X_i}+B_i+\tau\omega_i)\cdot\Sigma_i=\alpha_Z\cdot g_{i,*}(\Sigma_i)\geq 0$.
This shows that our $K_X+B+\boldsymbol{\beta}_X$-MMP with scaling of ω terminates after finitely many steps producing a log terminal model of $K_X+B+\boldsymbol{\beta}_X+\tau\omega$. Let $\phi:X\dashrightarrow X'$ be the composite maps of this MMP so that $K_{X'}+B'+\boldsymbol{\beta}_{X'}+\tau\omega':=\phi_*(K_X+B+\boldsymbol{\beta}_X+\tau\omega)$ is nef, and by Theorem 3.14 there is a morphism $g:X'\to Z$ to a normal compact Kähler variety Z such that $K_{X'}+B'+\boldsymbol{\beta}_{X'}+\tau\omega=g^*\alpha_Z$, where α_Z is a Kähler class on Z.

We will now show that we have a Mori fiber space. Observe that $-(K_{X'}+B')|_{X_z}\equiv(\boldsymbol{\beta}_{X'}+\tau\omega')|_{X_z}$ is big for general points $z\in Z$; in particular X_z is Moishezon and $K_{X'}+B'$ not pseudo-effective over Z. Thus by Theorem 2.35, we can run a $K_{X'}+B'$-MMP over Z which terminates with a Mori fiber space $h:X''\to W$ over Z. Note that each step of this MMP is $K_{X'}+B'+\boldsymbol{\beta}_{X'}+\tau\omega'$-trivial.
Now we will show that the induced map $\psi:X'\dashrightarrow X''$ is an isomorphism. To see this, let $g':X'\to Y$ be the first contraction of the above MMP over Z, and Σ is a curve contracted by g'. Let C be a curve contained in a general fiber of $g:X'\to Z$. Then Σ and C are linearly independent in $N_1(X')$, however they are both $K_{X'}+B'+\boldsymbol{\beta}_{X'}+\tau\omega'$-trivial, contradicting the fact that ω is very general in $H^{1,1}_{\rm BC}(X)$. Thus $\psi:X'\dashrightarrow X''$ is an isomorphism and $Z=W$. In particular, $\rho(X'/Z)=1$ and $-(K_{X'}+B')$ is g-ample. We will show that

$-(K_{X'}+B'+\boldsymbol{\beta}_{X'})$ is g-Kähler. To that end, let F be a general fiber of g. Now if $\dim Z=2$, then $\dim F=1$, and since ω' is a modified Kähler class, $\omega'\cdot F>0$. This implies that $-(K_{X'}+B'+\boldsymbol{\beta}_{X'})\cdot F>0$, and hence $-(K_{X'}+B'+\boldsymbol{\beta}_{X'})$ is g-Kähler, as $\rho(X'/Z)=1$. If $\dim Z\leq 1$, then $\omega'|_F$ is a big class on F. Let $\{C_t\}_{t\in T}$ be a covering family of curves in F. Then $\omega'\cdot C_t=\omega'|_F\cdot C_t>0$, and thus $-(K_{X'}+B'+\boldsymbol{\beta}_{X'})\cdot C_t=-(K_{X'}+B'+\boldsymbol{\beta}_{X'})|_F\cdot C_t>0$. This shows that $-(K_{X'}+B'+\boldsymbol{\beta}_{X'})$ is g-Kähler, since $\rho(X'/Z)=1$. This completes our proof. □

Proof of Theorem 1.5. This follows from Theorem 3.23. □

3.4. **Cone Theorem.** In this section we will prove the cone theorem for generalized pairs in dimension 3. We start with the following lemma.

Lemma 3.24. *Let $(X,B+\boldsymbol{\beta})$ be a $\mathbb{Q}$-factorial generalized klt pair, where X is a compact Kähler 3-fold. Let ω be a Kähler class such that $\alpha:=K_X+B+\boldsymbol{\beta}_X+\omega$ is nef but not Kähler. Then there is a rational curve $C\subset X$ such that $\alpha\cdot C=0$ and $0>(K_X+B+\boldsymbol{\beta}_X)\cdot C\geq -6$.*

Proof. If $K_X+B+\boldsymbol{\beta}_X$ is nef, then α is Kähler, which is a contradiction. So we may assume that $K_X+B+\boldsymbol{\beta}_X$ is not nef. We may write $\omega=\eta+\omega'$, where η and ω' are very general Kähler classes in $H^{1,1}_{\rm BC}(X)$. Replacing $\boldsymbol{\beta}$ by $\boldsymbol{\beta}+\epsilon\bar{\eta}$ and ω by $\omega-\epsilon\eta\equiv(1-\epsilon)\omega+\epsilon\omega'$, we may assume that $K_X+B+\boldsymbol{\beta}_X$ is not nef, $\boldsymbol{\beta}_X$ is big, $K_X+B+\boldsymbol{\beta}_X$ is either big or not pseudo-effective, and ω is a very general class in $H^{1,1}_{\rm BC}(X)$. By Theorems 3.21 and 3.23 we can run the $K_X+B+\boldsymbol{\beta}_X$-MMP with scaling of ω. Let $f:X\to Z$ be the first flipping or divisorial contraction, or fiber type contraction, and C the curve spanning the corresponding extremal ray; then $\alpha\cdot C=0$. If f is a flipping contraction, then the result follows from Theorem 2.21.
So now on assume that f is either a divisorial contraction or a fiber type contraction. Then there is a family of f-vertical curves $\{\Gamma_t\}_{t\in T}$ in X such that either $\cup_{t\in T}\Gamma_t=E$ is the exceptional divisor of f or $\cup_{t\in T}\Gamma_t=X$, respectively. Then in the former case $\boldsymbol{\beta}_X|_E$ is pseudo-effective, since $\boldsymbol{\beta}_X$ is modified nef (see [Bou04, Proposition 2.4]). Therefore $\boldsymbol{\beta}_X\cdot\Gamma_t=\boldsymbol{\beta}_X|_E\cdot\Gamma_t\geq 0$ (as $\{\Gamma_t\}_{t\in T}$ is a moving family of curves in E); in the latter case $\boldsymbol{\beta}_X\cdot\Gamma_t\geq 0$, since modified nef implies pseudo-effective. In particular, $\boldsymbol{\beta}_X\cdot C\geq 0$ for all f-exceptional curves in either case, and so $0>(K_X+B+\boldsymbol{\beta}_X)\cdot C\geq (K_X+B)\cdot C$. But then f is a K_X+B-negative contraction and $-(K_X+B)$ is f-ample (as $\rho(X/Z)=1$). Then by [DO23, Theorem 4.2] there is a rational curve Γ such that $f_*\Gamma=0$ and $0>(K_X+B+\boldsymbol{\beta}_X)\cdot\Gamma\geq(K_X+B)\cdot\Gamma\geq -6$.

□

Now we are ready to prove the Cone Theorem 1.6.

Proof of Theorem 1.6. By a Douady space argument (see [Tom16, Lemma 4.4]), there are at most countably many families of curves $\{\Gamma_i\}_{i\in I}$ such that $(K_X+B+\boldsymbol{\beta}_X)\cdot\Gamma_i<0$ and $\Gamma_i\cdot\alpha_i=0$ for some nef class α_i. Let $R=\mathbb{R}_{\geq 0}[\Gamma_i]$ be a $K_X+B+\boldsymbol{\beta}_X$-negative extremal ray. We make the following claim.

Claim 3.25. There is a Kähler class ω such that $\alpha:=K_X+B+\boldsymbol{\beta}_X+\omega$ is nef but not Kähler and $\alpha^\perp\cap\overline{\mathrm{NA}}(X)=R$.

Proof. Fix a norm $||\cdot||$ on $N_1(X)$ and let $\mathcal{S}$ be the unit sphere in $N_1(X)$, i.e. $\mathcal{S}:=\{\gamma\in N_1(X):||\gamma||=1\}$. Let $S:=\mathcal{S}\cap\overline{\mathrm{NA}}(X)$; then S is a compact subset of $\overline{\mathrm{NA}}(X)$ such that for any $\gamma\in\overline{\mathrm{NA}}(X)\setminus\{0\}$, $\frac{\gamma}{||\gamma||}\in S$. Moreover, from [HP16, Corollary 3.16] it follows that a class $\alpha\in H^{1,1}_{\mathrm{BC}}(X)$ is Kähler if and only if $\alpha\cdot\gamma>0$ for all $\gamma\in S$.

There is a unique point $r\in R$ such that $R\cap S=\{r\}$. Let η be a $(1,1)$ nef supporting class of R; then $\eta^\perp\cap\overline{\mathrm{NA}}(X)=R$. For $\epsilon>0$, let $B_\epsilon:=\{s\in S\ :\ ||s-r||<\epsilon\}$. Choosing $0<\epsilon\ll 1$ we may assume that $B_\epsilon\subset\overline{\mathrm{NA}}(X)_{(K_X+B+\boldsymbol{\beta}_X)<0}$. Then clearly $\eta-(K_X+B+\boldsymbol{\beta}_X)$ is positive on B_ϵ, i.e.

$$(\eta-(K_X+B+\boldsymbol{\beta}_X))\cdot s>0 \text{ for all } s\in B_\epsilon. \tag{3.7}$$

Now define $S_\epsilon:=S\setminus B_\epsilon$. Observe that $\eta\cdot s>0$ for all $s\in S_\epsilon$. Since S_ϵ compact, there exist positive real numbers $\delta>0$ and $M>0$ such that $\eta\cdot s\geq\delta$ and $-(K_X+B+\boldsymbol{\beta}_X)\cdot s\geq -M$ for all $s\in S_\epsilon$. Then for $t\gg 0$, $t\delta-M>0$, and thus

$$(t\eta-(K_X+B+\boldsymbol{\beta}_X))\cdot s\geq(t\delta-M)>0\quad\text{for all } s\in S_\epsilon. \tag{3.8}$$

Since η is nef, from (3.7) we have $(t\eta-(K_X+B+\boldsymbol{\beta}_X))\cdot s>0$ for all $s\in B_\epsilon$ and $t\geq 1$. Recall that $S=B_\epsilon\cup S_\epsilon$, and thus we have $(t\eta-(K_X+B+\boldsymbol{\beta}_X))\cdot s>0$ for all $s\in S$, and hence $t\eta-(K_X+B+\eta)$ is Kähler for $t\gg 0$. Let $\omega:=t\eta-(K_X+B+\boldsymbol{\beta}_X)$ for some $t\gg 0$. Then $\alpha:=t\eta=K_X+B+\boldsymbol{\beta}_X+\omega$ proves our claim. □

It then follows from Lemma 3.24 that we may assume $0>(K_X+B+\boldsymbol{\beta}_X)\cdot\Gamma_i\geq -6$ for all such Γ_i.

Let $V=\overline{\mathrm{NA}}(X)_{K_X+B+\boldsymbol{\beta}_X\geq 0}+\sum_{i\in I}\mathbb{R}^+[\Gamma_i]$. By [HP16, Lemma 6.1] it suffices to show that $\overline{\mathrm{NA}}(X)=\overline{V}$ (note that [HP16, Lemma 6.1] is only stated for K_X, but the same proof works for $K_X+B+\boldsymbol{\beta}_X$). Since $\overline{\mathrm{NA}}(X)$ is a closed strongly convex cone, it is the convex hull of its extremal rays. Thus if the inclusion $\bar{V}\subset\overline{\mathrm{NA}}(X)$ is strict, then there is a $K_X+B+\boldsymbol{\beta}_X$-negative extremal ray $R\in\overline{\mathrm{NA}}(X)$ not contained in $\bar{V}$.

Then by Lemma 3.24 and Claim 3.25 it follows that there is a rational curve $C\subset X$ such that $\alpha\cdot C=0$, where α is the nef supporting class of R. But then $[C]\in\alpha^\perp\cap\bar{V}$ is an immediate contradiction.

Finally, if $\boldsymbol{\beta}_X$ is big, then by [Bou04, Def. 3.7 and Pro. 3.8], we may write $\boldsymbol{\beta}_X \equiv N + \eta$, where $N \geq 0$ is an effective $\mathbb{R}$-divisor and η is a modified Kähler class, i.e. $\eta = f_*\eta'$ where $f : X' \to X$ is bimeromorphic and η' is Kähler on X'. Let γ be a Kähler class on X, then $\eta - \epsilon\gamma$ is modified Kähler for $0 < \epsilon \ll 1$. Replacing η by $\eta - \epsilon\gamma$ and $\epsilon\gamma$ by γ we may write $\boldsymbol{\beta}_X \equiv N + \eta + \gamma$, where $N \geq 0$ is an effective $\mathbb{R}$-divisor and η is a modified Kähler class and γ is Kähler. We then have

$$K_X + B + \boldsymbol{\beta}_X = K_X + (B + \epsilon N) + (1 - \epsilon)\boldsymbol{\beta}_X + \epsilon\eta + \epsilon\gamma,$$

where $(X, (B + \epsilon N) + (1 - \epsilon)\boldsymbol{\beta}_X + \epsilon\eta + \epsilon\gamma)$ is generalized klt for any $0 < \epsilon \ll 1$. Let $B^\epsilon := B + \epsilon N$ and $\boldsymbol{\beta}\epsilon_X := (1 - \epsilon)\boldsymbol{\beta}_X + \epsilon\eta$. Since $\boldsymbol{\beta}\epsilon_X$ is modified Kähler, by the proof of Lemma 3.2, $\boldsymbol{\beta}\epsilon_X \cdot \Gamma_i \geq 0$ for Γ_i not contained in $f(\mathrm{Ex}(f))$ and hence for all but finitely many i. Then taking limit as $\epsilon \to 0^+$, we see that $\boldsymbol{\beta}_X \cdot \Gamma_i \geq 0$ for all but finitely many i . Therefore $\beta\epsilon_X \cdot \Gamma_i \geq \epsilon\eta \cdot \Gamma_i > 0$ for all but finitely many i. So if $(K_X + B + \boldsymbol{\beta}_X) \cdot \Gamma_i = (K_X + B\epsilon + \boldsymbol{\beta}\epsilon_X + \epsilon\gamma) \cdot \Gamma_i < 0$, then arguing as above we get

$$\epsilon\gamma \cdot \Gamma_i \leq -(K_X + B\epsilon + \boldsymbol{\beta}\epsilon_X) \cdot \Gamma_i \leq -(K_X + B\epsilon) \cdot \Gamma_i \leq 6$$

for all but finitely many i, and hence, by a Douady space argument, such curves belong to finitely many families.

□

Next we will establish an analog of [BCHM10, Corollary 1.1.5] for log terminal models.

3.5. **Geography of Minimal Models.**

Theorem 3.26. *Let X be a normal compact Kähler 3-fold, $\nu : X' \to X$ a log resolution of a klt pair (X, B), and Ω' a compact convex polyhedral set of closed positive (1,1) currents on X' such that for every $\beta' \in \Omega'$, $(X, B + \boldsymbol{\beta})$ is a generalized klt pair, where $\boldsymbol{\beta} = \bar{\beta}'$. Assume that one of the following conditions hold:*

(i) $K_X + B + \boldsymbol{\beta}_X$ is big for every $\beta' \in \Omega'$ (and $\boldsymbol{\beta} = \bar{\beta}'$), or
(ii) there is a bimeromorphic morphism $\pi : X \to S$.

Then there exists a finite polyhedral decomposition $\Omega' = \cup\Omega'_i$ and finitely many bimeromorphic maps $\psi_{ij} : X \dashrightarrow X_{ij}$ (resp. finitely many bimeromorphic maps $\psi_{ij} : X \dashrightarrow X_{ij}$ over S) such that if $\psi : X \dashrightarrow Y$ is a weak log canonical model for $K_X + B + \boldsymbol{\beta}_X$ (resp. a weak log canonical model for $K_X + B + \boldsymbol{\beta}_X$ over S) for some $\beta' \in \Omega'_i$ (with $\boldsymbol{\beta} = \bar{\beta}'$), then $\psi = \psi_{ij}$ for some i, j.

Proof. Arguing as in the proof of Theorem 3.19, we will prove both cases (i) and (ii) simultaneously. We will use the convention that in case (i), S = Specan(C) and we remark that in case (ii) the condition that $K_X + B + \boldsymbol{\beta}_X$ is big over S is automatic as π is bimeromorphic. By compactness, it suffices to prove the result on a neighborhood of any $\beta_0' \in \Omega'$. For simplicity of notation, from now on we will write β on X to denote $\boldsymbol{\beta}_X$ and so on. Note that $\nu: X' \to X$ is a log resolution of $(X, B+\beta)$ for any $\beta \in \Omega = \nu_*\Omega'$. By [Bou02, Theorem 1.4], we may assume that $\nu^*(K_X+B+\beta_0) \equiv \omega' + F$ where ω' is Kähler and $F \geq 0$ has simple normal crossings support. Let $B' := \nu_*^{-1}B + (1-\delta)\mathrm{Ex}(\nu)$ for $0 < \delta \ll 1$. Then the weak log canonical models of $K_{X'}+B'+\beta'$ and $K_X+B+\nu_*\beta'$ coincide for every $\beta' \in \Omega'$. Replacing (X, B) by (X', B') and Ω by Ω' we may assume that all $\beta \in \Omega$ are nef and descend to X, and $K_X+B+\beta_0 \equiv \omega + F$, where ω is Kähler, $F \geq 0$ and $B+F$ has simple normal crossings support.

Pick $\delta > 0$ such that $(X, B+\delta F)$ is klt and consider the linear map $L(\beta) = \frac{1}{1+\delta}(\beta + \delta\beta_0)$. Note that $L(\beta_0) = \beta_0$ and $L(\Omega) \subset \Omega$ contains a neighborhood of β_0. Since

$$K_X+B+\delta F+\beta+\delta\omega \equiv K_X+B+\beta+\delta(K_X+B+\beta_0) \equiv (1+\delta)(K_X+B+L(\beta)),$$

replacing B by $B+\delta F$ and β by $\beta+\delta\omega$ we may assume that $\beta = \eta+\gamma$, where γ is a fixed Kähler class and $\eta := \beta - \gamma$ is nef for any $\beta \in \Omega$. Let $\{\gamma_1, \ldots, \gamma_\rho\}$ be Kähler forms whose classes in $H^{1,1}_{\mathrm{BC}}(X)$ form a basis of $H^{1,1}_{\mathrm{BC}}(X)$. For $\epsilon > 0$ define

$$\Omega^\epsilon := \{\beta + \sum_{i=1}^{\rho} t_i\gamma_i \;:\; \beta \in \Omega,\; |t_i| \leq \epsilon,\; 1 \leq i \leq \rho\}.$$

For $0 < \epsilon \ll 1$ we may assume that $K_X+B+\beta'$ is generalized klt and big (over S), and β' is Kähler for any $\beta' \in \Omega^\epsilon$. By Theorem 3.19, there exists a finite polyhedral decomposition $\Omega^\epsilon = \cup_{j\in J}\Omega^\epsilon_j$ and finitely many bimeromorphic maps $\psi^\epsilon_j: X \dashrightarrow X^\epsilon_j$ (over S) such that if $\psi: X \dashrightarrow Z$ is a log canonical model for $K_X+B+\beta'$ (over S) for some $\beta' \in \Omega^\epsilon_j$, then $\psi = \psi^\epsilon_j$. Suppose now that $\phi: X \dashrightarrow Y$ is a weak log canonical model of $K_X+B+\beta$, where $\beta \in \Omega$ and let η be a Kähler class on Y. Since $\{\gamma_1, \ldots, \gamma_\rho\}$ spans $H^{1,1}_{\mathrm{BC}}(X)$ and $\phi_*: H^{1,1}_{\mathrm{BC}}(X) \to H^{1,1}_{\mathrm{BC}}(Y)$ is surjective by Lemma 2.14, we may pick $t_1, \ldots, t_\rho$ such that $\phi_*(\sum_{i=1}^\rho t_i\gamma_i) \equiv \eta$. For any $0 < \delta \ll 1$, it follows that ϕ is a log canonical model for $K_X+B+\beta+\sum_{i=1}^\rho(\delta t_i)\gamma_i$ and that $\beta' := \beta+\sum_{i=1}^\rho(\delta t_i)\gamma_i \in \Omega^\epsilon$. But then, there exists $j \in J$ such that $\beta' \in \Omega^\epsilon_j$ and hence $\phi = \psi^\epsilon_j$.

We now let $\{\Omega_i\}_{i\in I}$ be the finite polyhedral decomposition induced by refining the finite polyhedral cover of Ω given by $\{\Omega^\epsilon_j \cap \Omega\}_{j\in J}$. For each $i \in I$ we let $\{\psi_{i,j}\} = \{\psi^\epsilon_j\}_{j\in J}$. Suppose now that $\psi: X \dashrightarrow Y$ is a weak log canonical model for $K_X+B+\beta$ where $\beta \in \Omega_i$, then as observed above ψ is a log canonical model for $K_X+B+\beta+\sum_{i=1}^\rho(\delta t_i)\gamma_i$ and $\beta' := \beta+\sum_{i=1}^\rho(\delta t_i)\gamma_i \in \Omega^\epsilon$. Thus $\beta' \in \Omega^\epsilon_j$ for an appropriate j and hence $\psi = \psi^\epsilon_j \in \{\psi_{i,j}\}$ as required.

□

Proof of Theorem 1.3. Immediate from Theorem 3.26. □

Corollary 3.27. *Let $(X, B+\beta)$ be a generalized klt 3-fold and $\pi : X \to S$ a proper morphism such that either π is bi-meromorphic or $S = \operatorname{Specan}(\mathbb{C})$ and $K_X + B + \beta$ is big, then $(X, B+\beta)$ has finitely many minimal models.*

3.6. **Minimal Models are Connected by Flops.** In this section we will prove that minimal models are connected by flops. Recall that if $(X, B+\beta)$ is a compact generalized klt pair and $f_i : X \to X_i$ are log terminal models for $i = 1, 2$, then X_i are $\mathbb{Q}$-factorial, $K_{X_i} + B_i + \beta_i = f_{i*}(K_X + B + \beta)$ is nef and $X_1 \dashrightarrow X_2$ is an isomorphism in codimension 1. We will show that 3-fold log terminal models are connected by flips, flops and inverse flips, and in particular two generalized klt Calabi-Yau pairs are connected by flops, which generalizes a result of Kollár for terminal varieties, see [Kol89, Theorem 4.9].

First we define the inverse flip.

Definition 3.28. Let $(X, B+\boldsymbol{\beta})$ be a $\mathbb{Q}$-factorial compact Kähler generalized klt pair and $\phi : X \dashrightarrow X'$ a small bimeromorphic map. If ϕ is a $K_X + B + \boldsymbol{\beta}_X$-flip, then we call $\phi^{-1} : X' \dashrightarrow X$ an *inverse flip* (or anti-flip) of $K_X + B + \boldsymbol{\beta}_X$.

Theorem 3.29. *Let $(X_i, B_i + \boldsymbol{\beta}_{X_i})$ be compact $\mathbb{Q}$-factorial generalized klt Kähler 3-folds, where $K_{X_i} + B_i + \boldsymbol{\beta}_{X_i}$ is nef for $i = 1, 2$ and $\phi : X_1 \dashrightarrow X_2$ a bimeromorphic map which is an isomorphism in codimension 1. Then the following hold:*

(1) ϕ decomposes as a finite sequence of flips, flops and inverse flips.

(2) Suppose that there is a positive constant $b > 0$ such that following holds: whenever $(K_{X_1} + B_1 + \boldsymbol{\beta}_{X_1}) \cdot C > 0$ for some curve $C \subset X_1$, then $(K_{X_1} + B_1 + \boldsymbol{\beta}_{X_1}) \cdot C \geq b$ holds. Then ϕ decomposes as a finite sequence of flops.

Remark 3.30. Note that if $(X_1, B_1 + \boldsymbol{\beta}_{X_1})$ is a good minimal model, then there is a morphism $f : X_1 \to Z_1$ and a Kähler form ω_1 on Z_1 such that $K_{X_1} + B_1 + \boldsymbol{\beta}_{X_1} \equiv f^*\omega_1$. Let $b := \inf\{\Sigma \cdot \omega_1 \mid \Sigma \subset Z_1 \text{ is a curve}\}$. So if $(K_{X_1} + B_1 + \boldsymbol{\beta}_{X_1}) \cdot C > 0$ for some curve $C \subset X_1$, then $\Sigma = f_{1*}C \neq 0$ and $(K_{X_1} + B_1 + \boldsymbol{\beta}_{X_1}) \cdot C = \omega_1 \cdot \Sigma \geq b$. If instead $K_{X_1} + B_1$ is $\mathbb{Q}$-Cartier (and $\beta_i = 0$), then $k(K_{X_1} + B_1)$ is Cartier for some $k > 0$ and let $b = \frac{1}{k}$. Thus, in both of these cases, the hypothesis of (2) are satisfied.

Proof. Let ω_2 be a Kähler form on X_2 and and $\omega_1 := \phi^{-1}_*\omega_2$, then by Lemma 3.31, ω_1 has local potentials, it is modified nef and $(X_1, B_1 + \boldsymbol{\beta} + \epsilon_0\bar{\omega}_1)$ is generalized klt for some $0 < \epsilon_0 \ll 1$. We now run a $K_{X_1} + B_1 + \boldsymbol{\beta}_{X_1} + \epsilon\omega_1$-MMP with scaling of a sufficiently big multiple of a very general Kähler class, where $0 < \epsilon \leq \epsilon_0$ is any fixed real number.

By Theorem 3.21, this MMP terminates with a minimal model $(X^m, B^m + \boldsymbol{\beta}_{X^m} + \epsilon\omega^m)$, $\psi : X_1 \dashrightarrow X^m$. In particular, $K_{X^m} + B^m + \boldsymbol{\beta}_{X^m} + \epsilon\omega^m$ is nef and big. Since $(X_2, B_2 + \boldsymbol{\beta}_{X_2} + \epsilon\omega_2)$ is a generalized log canonical model of $(X_1, B_1 + \boldsymbol{\beta}_{X_1} + \epsilon\omega_1)$, there is a morphism $X^m \to X_2$. Since this is a small bimeromorphic map of $\mathbb{Q}$-factorial varieties, it is in fact an isomorphism by Lemma 2.15. Now observe that from Lemma A.9 it follows that $N(K_{X_1}+B_1+\boldsymbol{\beta}_{X_1}+\epsilon\omega_1) = 0$, and thus by Theorem A.11 there are no divisorial contractions in the above MMP. So every step of this MMP is a $K_{X_1} + B_1 + \boldsymbol{\beta}_{X_1} + \epsilon\omega_1$-flip, which are in particular either flips, flops or inverse flips with respect to $K_{X_1} + B_1 + \boldsymbol{\beta}_{X_1}$ (depending on whether the $K_{X_1} + B_1 + \boldsymbol{\beta}_{X_1} + \epsilon\omega_1$ flipping contraction is $K_{X_1} + B_1 + \boldsymbol{\beta}_{X_1}$-negative, trivial or positive respectively).

Suppose now that we are in case (2) and so there is a positive constant $b > 0$ such that $(K_{X_1} + B_1 + \boldsymbol{\beta}_{X_1}) \cdot C \geq b$ for all curves $C \subset X_1$ such that $(K_{X_1} + B_1 + \boldsymbol{\beta}_{X_1}) \cdot C > 0$. We now run a $K_{X_1} + B_1 + \boldsymbol{\beta}_{X_1} + \epsilon\omega_1$-MMP with scaling of a sufficiently big multiple of a very general Kähler class, where $0 < \epsilon < b\epsilon_0/(b+6)$ is any fixed real number; this MMP terminates by Theorem 3.21. Now let $\mathcal{K}_t := K_{X_1} + B_1 + \boldsymbol{\beta}_{X_1} + t\omega_1$. Suppose that $C_1 \subset X_1$ is a $\mathcal{K}_\epsilon$-flipping curve for $t = \epsilon$. Then $C_1 \cdot \omega_1 < 0$ as $K_{X_1} + B_1 + \boldsymbol{\beta}_{X_1}$ is nef, and hence C_1 is a $\mathcal{K}_{\epsilon_0}$ flipping curve and so we may assume that $0 > \mathcal{K}_{\epsilon_0} \cdot C_1 \geq -6$ by Lemma 3.24. If $\mathcal{K}_0 \cdot C_1 > 0$, then

$$0 > \mathcal{K}_\epsilon \cdot C_1 = \left(1 - \frac{\epsilon}{\epsilon_0}\right) \mathcal{K}_0 \cdot C_1 + \frac{\epsilon}{\epsilon_0}\mathcal{K}_{\epsilon_0} \cdot C_1 \geq \left(1 - \frac{\epsilon}{\epsilon_0}\right) b - 6\frac{\epsilon}{\epsilon_0} > 0$$

which is impossible. Therefore $\mathcal{K}_0 \cdot C_1 = 0$ and the first flip $X_1 \dashrightarrow X_1^+$ is a $K_{X_1} + B_1 + \boldsymbol{\beta}_{X_1}$-flop. It follows that $K_{X_1^+} + B_1^+ + \boldsymbol{\beta}_{X_1^+}$ is nef. Suppose that $C \subset X_1^+$ is a curve such that $(K_{X_1^+} + B_1^+ + \boldsymbol{\beta}_{X_1^+}) \cdot C > 0$, then we claim that in fact $(K_{X_1^+} + B_1^+ + \boldsymbol{\beta}_{X_1^+}) \cdot C \geq b$ and hence we may continue the procedure inductively. Thus we obtain a sequence of flips for the $(X_1, B_1 + \boldsymbol{\beta}_{X_1} + \epsilon\omega_1)$ MMP with scaling which are also $K_{X_1} + B_1 + \boldsymbol{\beta}_{X_1}$-flops connecting X_1 and X_2.

To see the claim, let $p : Y \to X_1$ and $q : Y \to X_1^+$ be a common resolution. Then by the negativity lemma $p^*(K_{X_1} + B_1 + \boldsymbol{\beta}_{X_1}) = q^*(K_{X_1^+} + B_1^+ + \boldsymbol{\beta}_{X_1^+})$. Since $(K_{X_1^+} + B_1^+ + \boldsymbol{\beta}_{X_1^+}) \cdot C > 0$, then C is not contained on the indeterminacy locus of $X_1^+ \dashrightarrow X_1$ (i.e. it is not contained in the flipped locus). Let $\bar{C} \subset X$ be the strict transform of C, then $(K_{X_1} + B_1 + \boldsymbol{\beta}_{X_1}) \cdot \bar{C} = (K_{X_1^+} + B_1^+ + \boldsymbol{\beta}_{X_1^+}) \cdot C > 0$ and so $(K_{X_1^+} + B_1^+ + \boldsymbol{\beta}_{X_1^+}) \cdot C = (K_{X_1} + B_1 + \boldsymbol{\beta}_{X_1}) \cdot \bar{C} \geq b$.

□

Lemma 3.31. *Let $\phi : X \dashrightarrow X'$ be a bimeromorphic map between two normal compact Kähler 3-folds. Let ω' be a Kähler form on X'. If X has $\mathbb{Q}$-factorial klt singularities, then $\omega := \phi^*\omega'$ is a closed positive $(1,1)$ current on X with local potentials.*

Proof. Let W be a resolution of the graph of ϕ, and $p : W \to X$ and $q : W \to X'$ are the induced bimeromorphic morphisms such that p is projective.

Then $\omega = \phi^*\omega' = p_*q^*\omega'$. Since p is projective and X has $\mathbb{Q}$-factorial klt singularitites, by [DH20, Lemma 2.27], there is an $\mathbb{R}$-divisor E and an $(1,1)$ class $\alpha \in H^{1,1}_{\rm BC}(X)$ such that $[q^*\omega' + E] = p^*\alpha$. Then by the negativity lemma, $E \geq 0$ is an effective divisor; in particular, $q^*\omega' + E$ is a positive current. Thus from [HP16, Lemma 3.4] it follows that $\omega = \phi^*\omega' = p_*q^*\omega' = p_*(q^*\omega' + E)$ has local potentials.

□

Proof of Theorem 1.4. This is (1) of Theorem 3.29.

□

Appendix A. Boucksom-Zariski Decomposition

We will use the definition of Boucksom-Zariski decomposition of a $(1,1)$ pseudo-effective class $\alpha \in H^{1,1}_{\rm BC}(X)$ on a compact complex manifold as in [Bou04, Definition 3.7]. We will also define the Lelong number of a pseudo-effective $(1,1)$ class α (on a manifold) as in [Bou04, Definition 3.1]. The main result of this section is Theorem A.11.

We recall Boucksom's definition of the negative part of a pseudo-effective $(1,1)$ class.

Definition A.1. [Bou04, Definition 3.7] Let X be a compact complex manifold and α is a pseudo-effective $(1,1)$ class on X. Then we define the *negative part* $N(\alpha)$ of α as follows:

$$N(\alpha) := \sum_{P\subset X} \nu(\alpha, P)P,$$

where P is a prime Weil divisor on X.
From [Bou04, Theorem 3.12(i)] it follows that $N(\alpha)$ is an effective $\mathbb{R}$-divisor.

Remark A.2. Let X be a compact Kähler manifold and α is a pseudo-effective $(1,1)$ class. If $N(\alpha)$ is the negative part of the Boucksom-Zariski decomposition and if $\alpha = \beta + D$, where β is a modified nef class, and D is an effective $\mathbb{R}$-divisor, then $N(\alpha) = N(\beta + D) \leq N(\beta) + N(D) \leq D$ by [Bou04, Pro. 3.2(ii) and Pro. 3.11(ii)]. In particular, for any prime Weil divisor Q on X, $\nu(\alpha, Q) \leq \mathrm{mult}_Q(D)$.

The following result will be useful for the proof of our main theorem in this section.

Lemma A.3. *Let $f : Y \to X$ be a proper bimeromorphic morphism of analytic varieties where X is relatively compact. Let E be an effective f-exceptional $\mathbb{R}$-Cartier divisor on Y. Then there is a component E' of E such that E' is* *covered by an analytic family of curves* $\{C_t\}_{t\in T}$ *such that* $E\cdot C_t<0$ *and* $f_*C_t=0$ *for all* $t\in T$.

Proof. Let $\nu:\tilde{Y}\to Y$ be a resolution of singularities and $\tilde{E}=\nu^*E$. We may assume that $\tilde{f}=f\circ\nu$ is a projective morphism. Let $m:=\dim f(\mathrm{Supp}E)$. Replacing X by a Stein open neighborhood we may assume that X is a Stein space. Now we cut X by m general hyperplanes of X, and replace Y and $\tilde{Y}$ by the corresponding inverse images. Then $f(\mathrm{Supp}E)$ is a finite set of points on X. Next since $\tilde{f}$ is projective, possibly shrinking X further we may assume that there is a very ample divisor on $\tilde{Y}$. Thus cutting $\tilde{Y}$ by $n-2$ hyperplanes ($n=\dim Y$), we may assume that that $\tilde{Y}$ is a smooth surface. Next we replace X by the Stein factorization of $\tilde{f}:\tilde{Y}\to X$ and thus assume that X is a normal surface and $\tilde{f}$ is a projective bimeromorphic morphism from a smooth surface to a normal surface and $\tilde{E}$ is an effective $\tilde{f}$-exceptional divisor on Y. Let $\tilde{E}=\sum_{i=1}^{\ell}a_iC_i$. Since the intersection matrix of the exceptional curves of $\tilde{f}$ form a negative definite matrix by [KM98, Lemma 3.40], we have $0>\tilde{E}^2=\sum_{i=1}^{\ell}a_i(E\cdot C_i)$, and thus $\tilde{E}\cdot C_i<0$ for some $1\leq i\leq\ell$. Note that $E\cdot\nu_*C_i=\tilde{E}\cdot C_i<0$ and hence C_i is not ν-exceptional.
Since X is relatively compact, it can be covered by finitely many Stein open sets, and thus the lemma follows. □

Definition A.4. Let X be a normal analytic variety and $D=\sum a_iD_i$ and $D'=\sum a_i'D_i$ are two $\mathbb{R}$-divisors on X. Then we define the $\mathbb{R}$-divisor $D\wedge D'$ as

$$D\wedge D':=\sum_i\min\{a_i,a_i'\}D_i.$$

Lemma A.5. *Let* $f:Y\to X$ *be a proper bimeromorphic morphism from a compact complex manifold* Y *to a normal compact analytic variety* X *and* α *is a pseudo-effective* $(1,1)$ *class on* X. *If* $E\geq 0$ *is an effective* f*-exceptional* $\mathbb{R}$*-divisor, then* $\nu(f^*\alpha+E,P)=\nu(f^*\alpha,P)+\mathrm{mult}_P(E)$ *for every prime Weil divisor* P *on* Y. *In particular,* $N(f^*\alpha+E)=N(f^*\alpha)+E$.

Proof. Let $E=\sum a_iE_i$. By remark A.2, we have $N(f^*\alpha+E)\leq N(f^*\alpha)+E$ and so $\nu(f^*\alpha+E,E)\leq\nu(f^*\alpha,E)+a_i$. To see the reverse inequality, suppose that $f^*\alpha+E=\beta+N$ is the Boucksom-Zariski decomposition of $f^*\alpha+E$ so that $N=\sum\nu(f^*\alpha+E,Q)Q$. We claim that $E\leq N$. To see this, define $N':=N-N\wedge E$ and $E':=E-N\wedge E$, so that $f^*\alpha+E'=\beta+N'$. We must show that $E'=0$. If this is not the case, then, by Lemma A.3, there is a component E_i of E' which is covered by curves $\{C_t\}_{t\in T}$ such that $E'\cdot C_t<0$ and $f_*C_t=0$ for all $t\in T$. But then the family of curves $\{C_t\}_{t\in T}$ is not contained in the support of N' and so

$$0>E'\cdot C_t=(f^*\alpha+E')\cdot C_t\geq N'\cdot C_t\geq 0.$$

This is a contradiction, and hence $E \leq N$. Then $f^*\alpha = \beta + N'$, where β is modified nef and $N' = N - E \geq 0$. This implies that

$$\nu(f^*\alpha, E_i) \leq \text{mult}_{E_i}(N') = \text{mult}_{E_i}(N) - \text{mult}_{E_i}(E) = \nu(f^*\alpha + E, E_i) - a_i.$$

Putting all of these together, we have that $\nu(f^*\alpha + E, E_i) = \nu(f^*\alpha, E_i) + a_i$ and hence $N(f^*\alpha + E) = N(f^*\alpha) + E$.

□

Now we are ready to define the negative part of a pseudo-effective $(1, 1)$ class on a normal variety and prove the main result of this appendix.

Definition A.6. Let X be a normal compact Kähler variety and $\alpha \in H^{1,1}_{\text{BC}}(X)$ a pseudo-effective class. Let $f : Y \to X$ be a resolution of singularities X. Then we define the negative part $N(\alpha)$ as follows:

$$N(\alpha) := f_*(N(f^*\alpha)).$$

The following Lemma A.7 guarantees that this definition is independent of the choice of resolution f.

Lemma A.7. *Let X be a normal compact Kähler variety and $\alpha \in H^{1,1}_{\text{BC}}(X)$ a pseudo-effective class. Let $f : Y \to X$ and $g : Z \to X$ be two resolutions of singularities of X. Then*

$$f_*(N(f^*\alpha)) = g_*(N(g^*\alpha)).$$

Proof. It is easy to see from the definition of Lelong numbers that if $f : Y \to X$ is a proper bimeromorphic map of compact Kähler manifolds and P is a prime divisor on Y and $Q = f_*P \neq 0$, then $\nu(\alpha, Q) = \nu(f^*\alpha, P)$ and therefore $f_*(N(f^*\alpha)) = g_*(N(g^*\alpha))$.

Passing to the general situation, let W be a common resolution of Y and Z, and $p : W \to Y$ and $q : W \to Z$ are the projections. Then $p^*(f^*\alpha) = q^*(g^*\alpha)$, and thus $(f \circ p)_*N(p^*(f^*\alpha)) = (g \circ q)_*N(q^*(g^*\alpha))$. Then from what we observed above it follows that $f_*N(f^*\alpha) = g_*N(g^*\alpha)$ and we are done.

□

Remark A.8. From our definition above and Remark A.2 it follows that if α is a pseudo-effective class on a normal compact analytic variety X, then $N(\alpha)$ is an effective $\mathbb{R}$-divisor on X. Moreover, if α and β are two pseudo-effective classes on a normal compact analytic variety X, then $N(\alpha+\beta) \leq N(\alpha)+N(\beta)$.

Lemma A.9. *Let $f : X' \to X$ be a proper bimeromorphic morphism of normal compact Kähler varieties. Let $\alpha' \in H^{1,1}_{\text{BC}}(X')$ be a nef class such that $\alpha := f_*\alpha'$ is contained in $H^{1,1}_{\text{BC}}(X)$. Then $N(\alpha) = 0$.*

Proof. Let $g : X'' \to X'$ be a resolution of singularities of X'. Then we have $(f \circ g)^*\alpha = g^*\alpha' + E$, where E is $f \circ g$-exceptional $\mathbb{R}$-divisor. By the negativity lemma it follows that E is an effective divisor. Therefore by Lemma A.5, $N((f\circ g)^*\alpha)=N(g^*\alpha')+E=E$, since $g^*\alpha'$ is nef. Then from the definition we have $N(\alpha)=(f\circ g)_*(E)=0$. $\square$

Definition A.10. Let $\phi: X \dashrightarrow X'$ be a bimeromorphic contraction of normal compact analytic varieties. Let $\alpha\in H^{1,1}_{\mathrm{BC}}(X)$ and assume that $\alpha':=\phi_*\alpha\in H^{1,1}_{\mathrm{BC}}(X')$. We say that ϕ is α-negative, if for any common resolution $p: W\to X$ and $q: W\to X'$, we may write

$$p^*\alpha=q^*\alpha'+E,$$

where $E\geq 0$ is an effective $\mathbb{R}$-divisor such that it is q-exceptional and $\mathrm{Supp}(p_*E)$ consists precisely the ϕ-exceptional divisors on X.

The following theorem is the main result of this section.

Theorem A.11. *Let $\phi: X\dashrightarrow X'$ be a bimeromorphic contraction of normal compact Kähler varieties. Let $(X, B+\boldsymbol{\beta}_X)$ and $(X', B'+\boldsymbol{\beta}_{X'})$ be generalized dlt pairs such that $K_X+B+\boldsymbol{\beta}_X$ is pseudo-effective and $B'+\boldsymbol{\beta}_{X'}=\phi_*(B+\boldsymbol{\beta}_X)$. If ϕ is $K_X+B+\boldsymbol{\beta}_X$-negative, then the divisors contracted by ϕ are contained in the support of $N(K_X+B+\boldsymbol{\beta}_X)$. In particular, if $K_{X'}+B'+\boldsymbol{\beta}_{X'}$ is nef, then the divisors contracted by ϕ are precisely the divisors in the support of $N(K_X+B+\boldsymbol{\beta}_X)$.*

Proof. Let W be a compact Kähler manifold resolving the map ϕ, and $p: W\to X$ and $q: W\to Y$ are the projections. Since ϕ is $K_X+B+\boldsymbol{\beta}_X$-negative, we have

$$p^*(K_X+B+\boldsymbol{\beta}_X)=q^*(K_{X'}+B'+\boldsymbol{\beta}_{X'})+E, \tag{A.1}$$

where $E\geq 0$ is an effective q-exceptional divisor and the support of p_*E is the set of divisors contracted by ϕ.
Then by Lemma A.5

$$N(p^*(K_X+B+\boldsymbol{\beta}_X))=N(q^*(K_{X'}+B'+\boldsymbol{\beta}_{X'})+E)=N(q^*(K_{X'}+B'+\boldsymbol{\beta}_{X'}))+E,$$

and by Definition A.6, $N(K_X+B+\boldsymbol{\beta}_X)=p_*(N(q^*(K_{X'}+B'+\boldsymbol{\beta}_{X'}))+E)$. In particular, the ϕ-exceptional divisors are contained in the support of $N(K_X+B+\boldsymbol{\beta}_X)$.
Moreover, if $K_{X'}+B'+\boldsymbol{\beta}_{X'}$ is nef, then $N(q^*(K_{X'}+B'+\boldsymbol{\beta}_{X'}))=0$, and so $N(K_X+B+\boldsymbol{\beta}_X)=p_*E$ and we are done. $\square$

References

[BCE+02] T. Bauer, F. Campana, T. Eckl, S. Kebekus, T. Peternell, S. a. Rams, T. Szemberg and L. Wotzlaw, A reduction map for nef line bundles, in *Complex geometry (Göttingen, 2000)*, pages 27–36, Springer, Berlin, 2002.

[BCHM10] C. Birkar, P. Cascini, C. D. Hacon and J. McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23**(2), 405–468 (2010).

[BG13] S. Boucksom and V. Guedj, Regularizing properties of the Kähler-Ricci flow, in *An introduction to the Kähler-Ricci flow*, volume 2086 of *Lecture Notes in Math.*, pages 189–237, Springer, Cham, 2013.

[Bir21] C. Birkar, *Generalised pairs in birational geometry*, EMS Surv. Math. Sci. **8**(1-2), 5–24 (2021).

[Bou02] S. Boucksom, *On the volume of a line bundle*, Internat. J. Math. **13**(10), 1043–1063 (2002).

[Bou04] S. Boucksom, *Divisorial Zariski decompositions on compact complex manifolds*, Ann. Sci. École Norm. Sup. (4) **37**(1), 45–76 (2004).

[BS95] M. C. Beltrametti and A. J. Sommese, *The adjunction theory of complex projective varieties*, volume 16 of *De Gruyter Expositions in Mathematics*, Walter de Gruyter & Co., Berlin, 1995.

[BZ16] C. Birkar and D.-Q. Zhang, *Effectivity of Iitaka fibrations and pluricanonical systems of polarized pairs*, Pub. Math. IHES. **123**, 283–331 (2016).

[CHP16] F. Campana, A. Höring and T. Peternell, *Abundance for Kähler threefolds*, Ann. Sci. Éc. Norm. Supér. (4) **49**(4), 971–1025 (2016).

[DH20] O. Das and C. Hacon, *The log minimal model program for Kähler* 3*-folds*, arXiv e-prints , arXiv:2009.05924 (September 2020), 2009.05924.

[DH23] O. Das and C. Hacon, *The minimal model program for pseudo-effective Kähler 4-fold pairs*, In progress (2023).

[DHP22] O. Das, C. Hacon and M. Păun, *On the* 4*-dimensional minimal model program for Kähler varieties*, arXiv e-prints , arXiv:2205.12205 (May 2022), 2205.12205.

[DO23] O. Das and W. Ou, *On the Log Abundance for Compact Kähler threefolds*, manuscripta math. (January 2023).

[FM00] O. Fujino and S. Mori, *A canonical bundle formula*, J. Differential Geom. **56**(1), 167–188 (2000).

[Fuj19] O. Fujino, *Minimal model theory for log surfaces in Fujiki's class $\mathcal{C}$*, arXiv preprint arXiv:1908.00170 (2019).

[Fuj21] O. Fujino, *Minimal model theory for log surfaces in Fujiki's class $\mathcal{C}$*, Nagoya Math. J. **244**, 256–282 (2021).

[Fuj22] O. Fujino, *Minimal model program for projective morphisms between complex analytic spaces*, (2022).

[Gue20] H. Guenancia, *Families of conic Kähler-Einstein metrics*, Math. Ann. **376**(1-2), 1–37 (2020).

[Hir75] H. Hironaka, *Flattening theorem in complex-analytic geometry*, Amer. J. Math. **97**, 503–547 (1975).

[HL21] C. Hacon and J. Liu, *Existence of flips for generailzed lc pairs*, arXiv e-prints , arXiv:2105.13590 (2021), 2105.13590.

[HP15] A. Höring and T. Peternell, *Mori fibre spaces for Kähler threefolds*, J. Math. Sci. Univ. Tokyo **22**(1), 219–246 (2015).

[HP16] A. Höring and T. Peternell, *Minimal models for Kähler threefolds*, Invent. Math. **203**(1), 217–264 (2016).

[Kaw98] Y. Kawamata, *Subadjunction of log canonical divisors. II*, Amer. J. Math. **120**(5), 893–899 (1998).

[KM98] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[Kol89] J. Kollár, *Flops*, Nagoya Math. J. **113**, 15–36 (1989).

[Kol97] J. Kollár, Singularities of pairs, in *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 221–287, Amer. Math. Soc., Providence, RI, 1997.

[LP20] V. Lazić and T. Peternell, *On generalised abundance, I*, Publ. Res. Inst. Math. Sci. **56**(2), 353–389 (2020).

[Nam02] Y. Namikawa, *Projectivity criterion of Moishezon spaces and density of projective symplectic varieties*, Internat. J. Math. **13**(2), 125–135 (2002).

[Tom16] M. Toma, *Bounded sets of sheaves on Kähler manifolds*, J. Reine Angew. Math. **710**, 77–93 (2016).

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Navy Nagar, Colaba, Mumbai 400005

Email address: omdas@math.tifr.res.in

Email address: omprokash@gmail.com

Department of Mathematics, University of Utah, 155 S 1400 E, Salt Lake City, Utah 84112

Email address: hacon@math.utah.edu

Department of Mathematics, University of Utah, 155 S 1400 E, Salt Lake City, Utah 84112

Email address: yanez@math.utah.edu