

MINIMAL MODEL PROGRAM FOR ALGEBRAICALLY INTEGRABLE FOLIATIONS AND GENERALIZED PAIRS

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ABSTRACT. By systematically introducing and studying the structure of algebraically integrable generalized foliated quadruples, we establish the minimal model program for $\mathbb{Q}$ -factorial foliated dlt algebraically integrable foliations and lc generalized pairs by proving their cone theorems, contraction theorems, and the existence of flips. We also provide numerous applications on their birational geometry and resolve a conjecture of Cascini and Spicer.

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1. INTRODUCTION

We work over the field of complex numbers $\mathbb{C}$.

1.1. Main theorems. Algebraically integrable foliations and generalized pairs are two structures that play important roles in modern birational geometry, specifically in the minimal model program. For more details on their backgrounds, we refer the reader to Subsections 2.1 and 2.2. The primary objective of this paper is to develop the minimal model program for both structures. The main theorems of the paper are as follows:

Theorem A. *Let $(X, \mathcal{F}, B)$ be a $\mathbb{Q}$ -factorial projective F -dlt foliated triple such that $\mathcal{F}$ is algebraically integrable. Let A be an ample $\mathbb{R}$ -divisor on X . Then:*

- (1) *The cone theorem, contraction theorem, and the existence of flips hold for $(X, \mathcal{F}, B)$. In particular, we can run a $(K_{\mathcal{F}} + B)$ -MMP.*
- (2) *If $K_{\mathcal{F}} + B + A$ is nef, then $K_{\mathcal{F}} + B + A$ is semi-ample¹.*
- (3) *If $B \geq A \geq 0$, then $(X, \mathcal{F}, B)$ has a good minimal model or a Mori fiber space.*
- (4) *If $K_{\mathcal{F}} + B + A$ is $\mathbb{Q}$ -Cartier, then the canonical ring of $K_{\mathcal{F}} + B + A$,*

$$R(X, K_{\mathcal{F}} + B + A) = \bigoplus_{m=0}^{+\infty} H^0(X, \mathcal{O}_X(\lfloor m(K_{\mathcal{F}} + B + A) \rfloor)),$$

is finitely generated.

Theorem B. *Let $(X, B, \mathbf{M})$ be a $\mathbb{Q}$ -factorial projective lc generalized pair. Then:*

- (1) *The cone theorem, contraction theorem, and the existence of flips hold for $(X, B, \mathbf{M})$. In particular, we can run a $(K_X + B + \mathbf{M}_X)$ -MMP.*
- (2) *If $K_X + B + A + \mathbf{M}_X$ is nef for some ample $\mathbb{R}$ -divisor A , then $K_X + B + A + \mathbf{M}_X$ is semi-ample.*

We refer the reader to Section 2 for stronger versions of Theorems A and B and other main results of this paper.

As explained in [CS21, SS22], F -dlt foliated triples play the same role as dlt pairs in the classical minimal model program, making it a natural class of singularities to study in the theory of foliations. Roughly speaking, Theorem A is an establishment of the minimal model program for algebraically integrable foliations with “klt” singularities in any dimension. In fact, when $\mathcal{F} = T_X$ and $\lfloor B \rfloor = 0$, Theorem A becomes the classical result of the existence of good minimal models of varieties of general type and the finite generation of the canonical ring [BCHM10, Theorem 1.2]. We remark that Theorem A does not hold in general without the polarization of the ample $\mathbb{R}$ -divisor A (cf. [ACSS21, Example 5.4]).

In parallel, Theorem B is a full establishment of the minimal model program for generalized pairs, providing a complete answer to a fundamental question posed by Birkar and Zhang when they first introduced the concept of generalized pairs [BZ16, Before Lemma 4.4] (see [Bir21, 6.1] and [HL22, 3.1, 3.3] for other variations). Hacon suggested us that Theorem B might have

¹[CD23, Theorem 1.2] provided a proof of some special cases of Theorem A(2) and other results that are similar to some results in this paper. However, the current proofs in [CD23] seem to be incomplete, mainly because of the failure of [CD23, Lemma 2.4] and some gaps of the proof of [CD23, Theorem 3.5]. In this paper, we will avoid using any results in [CD23].

essential implications on the cone theorem, contraction theorem, and the existence of flips for Kähler varieties in higher dimensions; see Scenario 2.6.3 for details.

We recall some previous results related to Theorems A and B:

- [ACSS21] proved the cone theorem part of Theorem A(1) without the $\mathbb{Q}$ -factorial F-dlt condition.
- When $\dim X = 3$, [CS20, CS21] (see also [SS22]) proved Theorem A(1) when $\dim X \leq 3$ without the algebraically integrable condition. If we further assume that $\text{rank } \mathcal{F} = 2$, then Theorem A(3) was implicitly proven in [SS22, Proof of Theorem 2.6].
- When $\dim X \leq 4$ or assuming the termination of klt flips in dimension $\leq \text{rank } \mathcal{F}$, [CS23a] proved Theorem A(1) without the F-dlt condition, but required (X, B) to be klt.
- When $(X, B, \mathbf{M})$ satisfies the “NQC” condition (see Definition 3.4.1 for details), [HL21a] proved Theorem B, and [HL21a, Xie22, CLX23, LX23b] together proved Theorem B for NQC generalized pairs without the $\mathbb{Q}$ -factorial condition.

1.2. Ideas of the proofs of Theorems A and B. The proofs of Theorems A and B are crucially relied on a larger framework: the theory of *generalized foliated quadruples*.

Definition 1.2.1 (cf. [LLM23, Definition 1.2]). A *generalized foliated quadruple* (gfq for short) $(X, \mathcal{F}, B, \mathbf{M})/U$ consists of a normal quasi-projective variety X , a foliation $\mathcal{F}$ on X , an $\mathbb{R}$ -divisor $B \geq 0$ on X , a projective morphism $X \rightarrow U$, and a nef/ U $\mathbb{R}$ -divisor $\mathbf{M}_{X'}$ on a high model X' of X , such that $K_{\mathcal{F}} + B + \mathbf{M}_X$ is $\mathbb{R}$ -Cartier. Here $\mathbf{M}_X$ is the image of $\mathbf{M}_{X'}$ on X .

The notation $\mathbf{M}$ in Definition 1.2.1 is considered as a $\mathbf{b}$ -divisor on X . We refer the reader to Definition 3.4.1 for the definition of $\mathbf{b}$ -divisors, and to Definition 3.4.3 for a more detailed definition of generalized foliated quadruples. It is clear that when $\mathbf{M} = \mathbf{0}$ is the trivial $\mathbf{b}$ -divisor, a generalized foliated quadruple is just a foliated triple $(X, \mathcal{F}, B)/U$; on the other hand, when $\mathcal{F} = T_X$, a generalized foliated quadruple is a generalized pair $(X, B, \mathbf{M})/U$ ([BZ16, Definition 1.4]). Therefore, generalized foliated quadruples can be considered as a mixture of foliated triples and generalized pairs. We refer the reader to Subsection 2.6 for a detailed explanation of why this new structure is vital not only for this paper but also for future studies of foliations and generalized pairs.

Under the framework of generalized foliated quadruples, the proofs of Theorems A and B proceed simultaneously.

The first results to prove are the cone theorems for Theorem A(1) and Theorem B(1). As a positive beginning, the cone theorem for projective algebraically integrable foliations is already known [ACSS21]. With some adjustments to the details of the proofs, the same approach used in [ACSS21] also works for algebraically integrable generalized foliated quadruples (Theorem 2.3.1). In particular, the cone theorem for algebraically integrable generalized foliated quadruples implies the cone theorem for generalized pairs by letting $\mathcal{F} = T_X$.

Now we move on to prove the rest of Theorem A(1). We only need to show that each step of a $(K_{\mathcal{F}} + B)$ -MMP is also a $(K_X + \Delta)$ -MMP for some lc pair (X, Δ) . To do this, we first show that $(X, \mathcal{F}, B)$ satisfies a property called “ACSS” and that this property is preserved under each step of the MMP (Lemma 9.1.4). The property “ACSS”, named in honor of Ambro-Cascini-Shokurov-Spicer, can be viewed as the analogue of the concept of qdlt (cf. [dFKX17]) for algebraically integrable foliations; see Definition 7.2.3 for details. With this, we prove the termination of MMP with scaling for algebraically integrable foliations satisfying the property “ACSS” and with very exceptional foliated log canonical divisor (Theorem 9.4.1), which implies that F-dlt foliated triples are always ACSS. This implies the rest of Theorem A(1). Note that the same approach to the proof also works for $\mathbb{Q}$ -factorial generalized foliated quadruples with F-dlt singularities.

Our next goal is to establish the rest of Theorem B. First, by considering a class of structures larger than the category of generalized pairs (see Lemma 15.0.1) and applying some arguments,

we can reduce the existence of flips for generalized pairs to the contraction theorem for generalized pairs. The contraction theorem for generalized pairs is an immediate corollary of the base-point-freeness theorem for generalized pairs, so we only need to prove the latter, which is Theorem B(2).

A crucial observation is that the base-point-freeness theorem for generalized pairs relies only on the subadjunction formula (Theorem 2.2.8), which in turn, only depends on the fact that the moduli part of the canonical bundle formula for a generalized pair $f : (X, B, \mathbf{M}) \rightarrow Z$ is nef (Theorem 11.4.4). A key observation is that the moduli part corresponds to the foliated canonical divisor $K_{\mathcal{F}} + B + \mathbf{M}_X$ (Proposition 7.3.6), where $\mathcal{F}$ is the foliation induced by f . With this, the canonical bundle formula for generalized pairs follows from the existence of log minimal models for generalized foliated quadruples with numerical dimension zero (Propositions 11.2.1, 11.2.3). The latter follows from Theorem A(1) which we have already established. This concludes the proof of Theorem B. It is worth mentioning that some previous literature addresses the canonical bundle formula for generalized pairs. However, these papers only consider generalized pairs with the additional “NQC” condition (cf. [Fil19, Fil20, J LX22, FS23]) and cannot be applied to our scenario.

Finally, we turn to the proof of Theorem A(2-4). Although the Bertini-type theorem fails for foliations, by employing the structure of generalized foliated quadruples, we can, roughly speaking, reduce Theorem A(3) and Theorem A(4) to Theorem A(2) (see Lemma 16.1.1 and Theorem 16.1.4). The proof of Theorem A(2) is divided into three steps:

In the first step, we use the already-proven contraction theorem for generalized pairs from Theorem A(2) to construct a contraction $X \rightarrow T$, where the general fibers of $X \rightarrow Z$ are tangent to $\mathcal{F}$.

In the second step, we apply the canonical bundle formula for generalized foliated quadruples to derive a generalized pair structure polarized with an ample divisor on T . This canonical bundle formula (Definition-Theorem 12.1.4) can be derived from the canonical bundle formula for lc-trivial fibrations of generalized pairs. The latter can be deduced using our approach via the theory of foliations. It is worth noting that the existing literature on the canonical bundle formula for generalized pairs ([Fil19, Fil20, J LX22, FS23]) cannot handle arbitrary lc-trivial fibrations $f : (X, B, \mathbf{M}) \rightarrow Z$ as they required that $B \geq 0$ over the generic point of Z or that $\mathbf{M}$ is $\mathbf{b}$ -semi-ample. Therefore, those works cannot be applied to deduce the canonical bundle formula for generalized foliated quadruples, which is essential for our purposes.

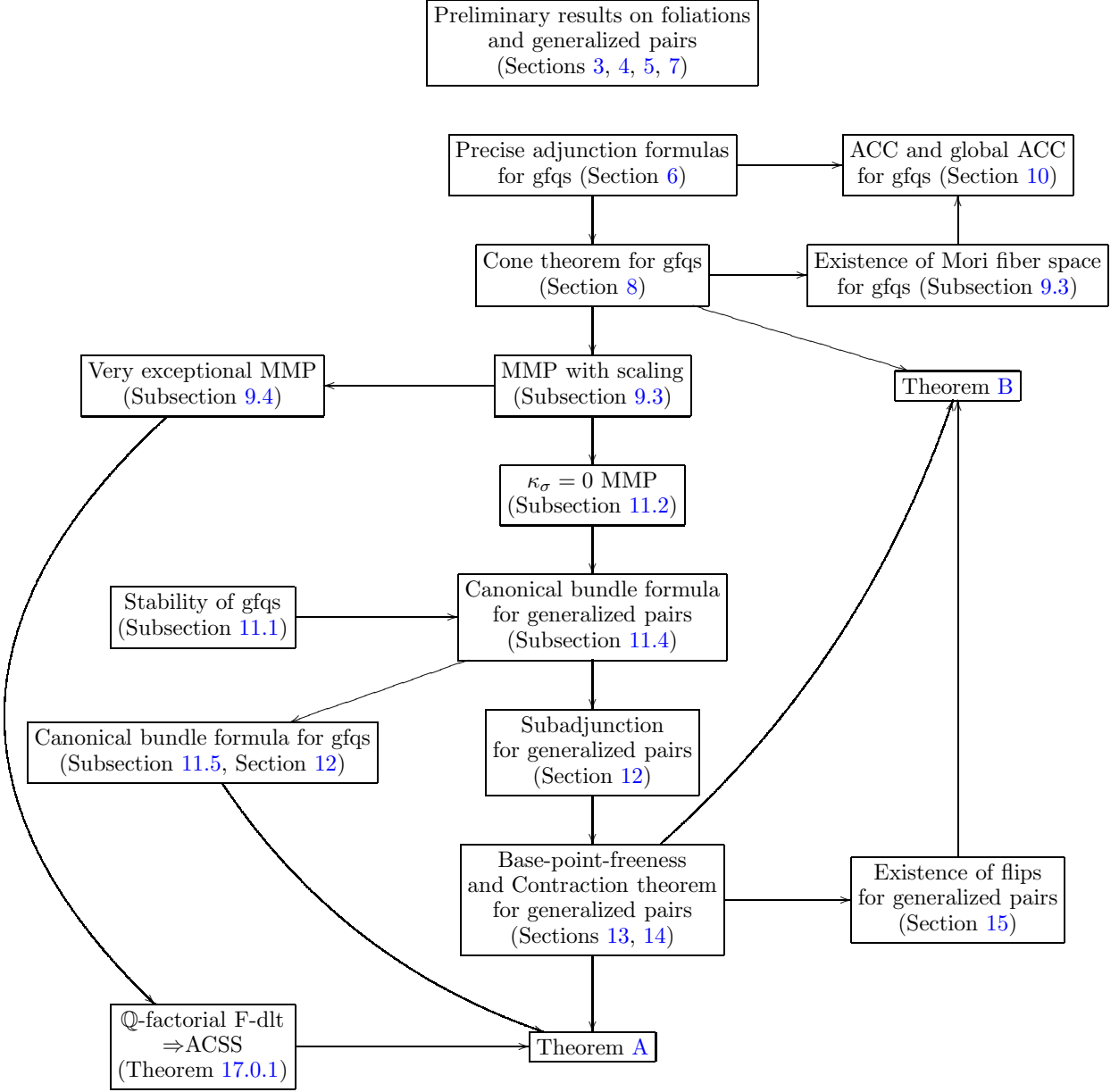
In the last step, we apply the cone theorem for generalized foliated quadruples to show that the generalized foliated log canonical divisor on T is ample. Hence, the foliated log canonical divisor on X is semi-ample, completing the proof of Theorem A(2). This concludes the proof of Theorem A.

1.3. Structure of the Paper. In Section 2, we list the main results of this paper and explain the importance of the structure of generalized foliated quadruples. The rest of the paper is divided into four parts. Part I states some preliminary results, Part II establishes the cone theorem and the minimal model program for algebraically integrable foliations, Part III establishes the canonical bundle formula and the minimal model program for generalized pairs, and Part IV proves the existence of good minimal models.

For the convenience of the reader, we have prepared the following flowchart (Table 1) to illustrate the streamlined process involved in the proofs of our main theorems.

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TABLE 1. Structure of the paper



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2. STATEMENT OF MAIN RESULTS

In this section, we provide the statements of the main results of this paper.

2.1. Minimal model program for algebraically integrable foliations. The theory of foliations holds a significant place in birational geometry. Most notably, it has played a critical role in Miyaoka’s proof of several key cases of the abundance conjecture in dimension three [Miy87]. In recent developments, foliations have been used by Bogomolov and McQuillan to analyze projective varieties which admit a non-trivial fibration with rationally connected fibers [BM16]. Furthermore, foliation theory has strong connections with other areas of algebraic geometry, such as the algebraic geometry in characteristic $p > 0$ and number theory as highlighted by the Grothendieck-Katz conjecture and the Ekedahl-Shepherd-Barron-Taylor conjecture. Its importance is also highlighted in hyperbolicity theory, where it was essential in McQuillan’s proof of a specific case of the Green-Griffiths-Lang conjecture [McQ98].

In recent years, it has been discovered that many structures and results in classical birational geometry can be extended to foliations, especially, within the context of the minimal model program. Instead of examining the structures associated with the canonical divisor of the ambient variety K_X , the foliations theory concentrates on the structures connected to the foliated canonical divisor $K_{\mathcal{F}}$. This approach offers greater flexibility in practice. Specifically, when $\mathcal{F} = T_X$, we find that $K_{\mathcal{F}} = K_X$, bringing us back to the classical setting.

The foundational work for the minimal model program for foliations has been established for foliated surfaces (cf. [McQ08, Bru15]) and foliated threefolds (cf. [CS20, Spi20, CS21, SS22]). Moreover, several classic questions from the minimal model programs, such as the ascending chain condition (ACC) conjecture for minimal log discrepancies, the ACC conjecture for lc thresholds, the global ACC, and the index theorems, have been adapted to foliations and verified in dimensions 2 and/or 3, as indicated in [Che22, Che23, LLM23, LMX23a, LMX23b].

Given these developments, it is natural to ask whether the minimal model program for foliations could extend to higher dimensions. Unfortunately, this seems to be a challenging question, with limited information available, even in dimension 4. However, from the perspective of the minimal model program, it seems sufficient to focus on a subset of foliations that have an additional structure: algebraically integrable foliations.

Algebraically integrable foliations are foliations where the general leaves are algebraic varieties; in other words, they are induced by dominant rational maps. These foliations naturally come into play when a fibration structure is established. Notably, Miyaoka’s study of the abundance conjecture in dimension 3 primarily utilized algebraically integrable foliations [Miy87], as opposed to arbitrary ones. This approach has been reflected in recent research into the abundance conjecture for Kähler threefolds [DO23a, DO23b] and threefolds over fields of characteristic $p > 3$ with numerical dimension 2 [Xu23]. In these studies, the algebraic integrability of foliations is guaranteed; indeed, all the foliations addressed in these papers are induced by MRC fibrations, making them automatically algebraically integrable. Given this, algebraically integrable foliations are expected to be crucial in future research of the minimal model program, particularly in questions related to the abundance conjecture.

The first objective of this paper is to develop the minimal model program for algebraically integrable foliations of arbitrary rank with “mild” singularities in arbitrary dimensions. Here “mild” singularity is usually referred to as “F-dlt” (see Definition 6.2.6). As explained in [CS21, SS22], F-dlt foliated triples play the same role as dlt pairs in the classical MMP and is a natural class of singularities to study. Moreover, any terminal foliated singularity is F-dlt.

Recall that a foliated triple $(X, \mathcal{F}, B)/U$ consists of a normal quasi-projective variety X associated with a projective surjective morphism $X \rightarrow U$, a foliation $\mathcal{F}$ on X , and an $\mathbb{R}$ -divisor $B \geq 0$ on X , such that $K_{\mathcal{F}} + B$ is $\mathbb{R}$ -Cartier. The first result of this paper shows that we can run a $(K_{\mathcal{F}} + B)$ -MMP/ U provided that it is $\mathbb{Q}$ -factorial F-dlt:

Theorem 2.1.1 (Minimal model program). *Let $(X, \mathcal{F}, B)/U$ be a $\mathbb{Q}$ -factorial foliated triple. Assume that $\mathcal{F}$ is algebraically integrable and $(X, \mathcal{F}, B)$ is F-dlt. Then we may run a $(K_{\mathcal{F}} + B)$ -MMP/ U .*

We remark that when $\dim X \leq 3$, Theorem 2.1.1 is known when $\text{rank } \mathcal{F} = 2$ ([SS22, Corollary 2.3]; [CS21, Theorem 1.1] when $U = \{pt\}$) and when $\text{rank } \mathcal{F} = 1$ and $U = \{pt\}$ ([CS21, Theorems 1.1, 2.36, Section 6]), even without the algebraically integrable condition. When assuming the termination of klt flips in dimension $\leq \text{rank } \mathcal{F}$, [CS23a, Theorem 1.1] proves Theorem 2.1.1 even without the F-dlt condition, but requires that (X, B) is klt. In particular, when (X, B) is klt and $\dim X \leq 4$, Theorem 2.1.1 can be deduced from [CS23a, Theorem 1.1].

Proceeding further, we demonstrate the termination of MMP with scaling as well as the existence of good minimal models for algebraically integrable foliations polarized with an ample divisor. The polarization of the ample divisor is a natural condition to add, as can be seen in [CS21, Theorem 1.2] and [CS20, Theorem 1.3]. It is worth noting that, even within the framework of the classical MMP, the existence of good minimal models in higher dimensions is only known when polarized with an ample divisor ([BCHM10, Theorem C], [HH20, Theorem 1.5]), while the general case remains an open conjecture.

Theorem 2.1.2 (Good minimal model). *Let $(X, \mathcal{F}, B)/U$ be a $\mathbb{Q}$ -factorial foliated triple. Assume that $\mathcal{F}$ is algebraically integrable, $B \geq A \geq 0$ for some ample/ U $\mathbb{R}$ -divisor A , and $(X, \mathcal{F}, B)$ is F-dlt. Then we may run a $(K_{\mathcal{F}} + B)$ -MMP/ U with scaling of an ample/ U $\mathbb{R}$ -divisor H , and any such MMP terminates*

- (1) *with a Mori fiber space of $(X, \mathcal{F}, B)/U$ if $K_{\mathcal{F}} + B$ is not pseudo-effective/ U , and*
- (2) *with a good minimal model of $(X, \mathcal{F}, B)/U$ if $K_{\mathcal{F}} + B$ is pseudo-effective/ U .*

We also have the following result on the abundance of algebraically integrable foliations polarized with an ample divisor.

Theorem 2.1.3 (Abundance). *Let $(X, \mathcal{F}, B)/U$ be a $\mathbb{Q}$ -factorial foliated triple and A an ample/ U $\mathbb{R}$ -divisor on X . Assume that $\mathcal{F}$ is algebraically integrable and $(X, \mathcal{F}, B)$ is F-dlt. Then*

$$\kappa_{\sigma}(X/U, K_{\mathcal{F}} + B + A) = \kappa_{\iota}(X/U, K_{\mathcal{F}} + B + A).$$

It is important to note that Theorem 2.1.3 is not a direct consequence of Theorem 2.1.2. This is because Bertini type theorems fail for foliations, and it is possible that $(X, \mathcal{F}, B + H)$ is not lc for any $H \in |A/U|_{\mathbb{R}}$ (see [DLM23, Example 3.4]).

We also prove a base-point-freeness theorem for algebraically integrable foliations.

Theorem 2.1.4 (Base-point-freeness). *Let $(X, \mathcal{F}, B)/U$ be a $\mathbb{Q}$ -factorial foliated triple. Assume that $\mathcal{F}$ is algebraically integrable and $(X, \mathcal{F}, B)$ is F-dlt. Let A be an ample/ U $\mathbb{R}$ -divisor on X such that $K_{\mathcal{F}} + B + A$ is nef/ U . Then:*

- (1) *$K_{\mathcal{F}} + B + A$ is semi-ample/ U .*
- (2) *Suppose that there exists a positive integer m such that $m(K_{\mathcal{F}} + B + A)$ is Cartier. Then*

$$\mathcal{O}_X(mn(K_{\mathcal{F}} + B + A))$$

is globally generated/ U for any integer $n \gg 0$.

In the literature, the semi-ampleness of $K_{\mathcal{F}} + B + A$ is known when $(X, \mathcal{F}, B + A)$ is $\mathbb{Q}$ -factorial F-dlt, $U = \{pt\}$, and $\dim X \leq 3$, even without the algebraically integrable condition (see [CS20, Theorem 1.3], [CS21, Theorem 1.3]). However, there was no result on the base-point-freeness theorem of foliations in dimensions ≥ 3 . It is worth mentioning that the base-point-freeness theorem Theorem 2.1.4(2) is crucial for us to prove a special case of the Prokhorov-Shokurov's $\mathbf{b}$ -semi-ampleness conjecture later in this paper (Theorem 2.5.3).

An important application of Theorem 2.1.2 is the existence of Mori fiber spaces for foliated triples, even with, at worst, lc singularities. We note that in this paper, Mori fiber spaces and log minimal models are in the sense of Birkar-Shokurov; that is, we allow the extraction of lc centers. See Definitions 3.4.7 and 9.1.1 for details.

Theorem 2.1.5. *Let $(X, \mathcal{F}, B)/U$ be an lc foliated triple. Assume that $\mathcal{F}$ is algebraically integrable and $K_{\mathcal{F}} + B$ is not pseudo-effective/ U . Then $(X, \mathcal{F}, B)/U$ has a Mori fiber space.*

Another interesting type of foliations is the class of foliations with numerical dimension zero. For example, based on [CS20, Theorem 1.4] and [CS21, Theorem 1.7], [LLM23, Theorem 1.9] has shown the existence of good minimal models for numerical dimension zero foliations in dimension ≤ 3 . In this paper, we obtain the existence of good minimal models for algebraically integrable foliations with numerical dimension zero:

Theorem 2.1.6. *Let $(X, \mathcal{F}, B)$ be a projective lc foliated triple. Assume that $\mathcal{F}$ is algebraically integrable and $\kappa_{\sigma}(K_{\mathcal{F}} + B) = 0$. Then:*

- (1) $(X, \mathcal{F}, B)$ has a good minimal model.
- (2) $\kappa_l(K_{\mathcal{F}} + B) = 0$.
- (3) If $(X, \mathcal{F}, B)$ is $\mathbb{Q}$ -factorial dlt, then we may run a $(K_{\mathcal{F}} + B)$ -MMP with scaling of an ample $\mathbb{R}$ -divisor, and any such MMP terminates with a good minimal model of $(X, \mathcal{F}, B)$.

Siu [Siu10] has used Eckl's construction of numerically trivial foliations [Eck04] to sketch a plan to solve the abundance conjecture. One step of Siu's approach, [Siu10, (4.1)], focuses on the abundance conjecture for smooth projective varieties associated with an "algebraically integrable numerically trivial foliation". Though the concept of "numerically trivial foliation" in [Siu10], which was defined analytically in [Eck04], seems to differ from the concept of "foliations whose canonical divisor has numerical dimension zero", these two types of foliations are closely connected. Hence, studying the abundance properties of numerical dimension zero algebraically integrable foliations (potentially with singularities that are worse than lc) on smooth projective varieties becomes intriguing, as it may have implications for the abundance conjecture. With this in mind, we prove the following theorem in this paper:

Theorem 2.1.7. *Let $(X, \mathcal{F}, B)$ be a projective algebraically integrable f-triple such that $\kappa_{\sigma}(K_{\mathcal{F}} + B) = 0$. Assume that $K_X + B$ is pseudo-effective and (X, B) is lc. Then $\kappa_l(K_{\mathcal{F}} + B) = 0$.*

Finally, we recall the following conjecture of Cascini and Spicer:

Conjecture 2.1.8 ([CS23a, Conjecture 4.2]). *Let $(X, \mathcal{F}, B)$ be a $\mathbb{Q}$ -factorial projective foliated triple, such that $\mathcal{F}$ is algebraically integrable, B is a $\mathbb{Q}$ -divisor, (X, B) is klt, and one of the following cases hold:*

- (1) $(X, \mathcal{F}, B)$ is F-dlt.
- (2) $(X, \mathcal{F}, B)$ is canonical.

Then there exists a morphism $f : X \rightarrow Y$ which induces $\mathcal{F}$.

In this paper, we provide a positive answer to Conjecture 2.1.8(1) with weaker assumptions and stronger results:

Theorem 2.1.9. *Let $(X, \mathcal{F}, B)$ be a $\mathbb{Q}$ -factorial foliated triple such that $\mathcal{F}$ is algebraically integrable and $(X, \mathcal{F}, B)$ is F-dlt. Then:*

- (1) (X, B) is qdlt (cf. Definition 7.1.1). In particular, if $\lfloor B \rfloor = 0$, then (X, B) is klt.
- (2) There exists a morphism $f : X \rightarrow Y$ to a smooth variety which induces $\mathcal{F}$.

We also prove a weaker form of Conjecture 2.1.8(2) without assuming that (X, B) is klt.

Theorem 2.1.10. *Let $(X, \mathcal{F}, B)$ be a $\mathbb{Q}$ -factorial canonical foliated triple such that $\mathcal{F}$ is algebraically integrable. Then $\mathcal{F}$ is induced by an almost holomorphic map.*

A very recent result [CS23b, Theorem 1.4] shows that the algebraic part of a foliation $\mathcal{F}$ on a projective variety X is induced by an almost holomorphic map, provided that X is $\mathbb{Q}$ -factorial klt and $\mathcal{F}$ is canonical. In particular, [CS23b, Theorem 1.4] implies Theorem 2.1.10 when X is projective klt and $B = 0$.

We would like to mention that the results in this subsection are not expected to work over fields of characteristic $p > 0$ due to counterexamples in [Ber23].

2.2. Minimal model program for generalized pairs. In the past few years, there has been significant advancement in the minimal model program for NQC generalized pairs. The cone theorem, as well as the $\mathbb{Q}$ -factorial cases of the contraction theorem and the proof of the existence of flips, were established in [HL21a]. Later, the existence of flips for (potentially non- $\mathbb{Q}$ -factorial) NQC generalized pairs was verified in [LX23a], while the contraction theorem for these pairs was proven in [Xie22]. Additionally, [CLX23] confirmed the Kodaira and the Kawamata-Viehweg vanishing theorems for NQC generalized pairs, offering an alternative proof for the contraction theorem. These developments form the foundation of the minimal model program for NQC generalized pairs, with numerous corollaries and applications already provided in [LT22, TX23].

The structure of NQC generalized pairs has naturally arisen in the study of the canonical bundle formulas, making them a fundamental structure in the study in the minimal model program. For a considerable amount of time, it has been presumed that the realm of NQC generalized pairs would be the most extensive category necessary to establish in the minimal model program. This is because the structure of NQC generalized pairs is maintained under the canonical bundle formula, adjunction formula, and each stage of the minimal model program, thereby eliminating the need to consider the minimal model program for non-NQC generalized pairs or other larger categories.

However, recent studies on the minimal model program for Kähler varieties [DH23, DHY23] have emphasized the critical role the structure of non-NQC generalized pairs plays in the minimal model program for Kähler varieties. In the case of Kähler varieties, the selection of ample divisors is restricted, preventing many procedures, such as the minimal model program with scaling and general hyperplane section cuttings. Nevertheless, the associated Kähler class ω on a Kähler variety serves as a substitute for ample divisors. Although ω cannot be categorized as an $\mathbb{R}$ -divisor, it can be considered as a nef $\mathbb{R}$ -class and is suitable for the nef part of a generalized pair. As explained in [DHY23], it is now possible to formally define “running MMP with scaling of the nef $\mathbb{R}$ -(1,1)-class ϖ ”. Given that ω is only an $\mathbb{R}$ -class and NQC cannot be assured, the study of the structure of non-NQC generalized pairs immediately becomes vital for the minimal model program on Kähler varieties.

Although little was known about the minimal model program for non-NQC generalized pairs, we have been able to establish the cone theorem and the contraction theorem for non-NQC generalized pairs, thanks to the cone theorem and the canonical bundle formula for generalized foliated quadruples. With additional effort, we also prove the existence of flips for $\mathbb{Q}$ -factorial non-NQC generalized pairs. These results collectively lay the groundwork for the minimal model program for $\mathbb{Q}$ -factorial generalized pairs. The detailed theorems are as follows:

Theorem 2.2.1 (Cone and contraction theorems). *Let $(X, B, \mathbf{M})/U$ be a generalized pair and $\pi : X \rightarrow U$ the associated morphism. Let $\{R_j\}_{j \in \Lambda}$ be the set of $(K_X + B + \mathbf{M}_X)$ -negative extremal rays in $\overline{NE}(X/U)$ that are rational. Then:*

(1)

$$\overline{NE}(X/U) = \overline{NE}(X/U)_{K_X+B+\mathbf{M}_X \geq 0} + \overline{NE}(X/U)_{\text{Nlc}(X,B,\mathbf{M})} + \sum_{j \in \Lambda} R_j.$$

In particular, any $(K_X + B + \mathbf{M}_X)$ -negative extremal ray in $\overline{NE}(X/U)$ is rational.

(2) *Each R_j is spanned by a rational curve C_j such that $\pi(C_j) = \{\text{pt}\}$ and*

$$0 < -(K_X + B + \mathbf{M}_X) \cdot C_j \leq 2 \dim X.$$

(3) *For any ample/ U $\mathbb{R}$ -divisor A on X ,*

$$\Lambda_A := \{j \in \Lambda \mid R_j \subset \overline{NE}(X/U)_{K_X+B+A+\mathbf{M}_X < 0}\}$$

is a finite set. In particular, $\{R_j\}_{j \in \Lambda}$ is countable, and is a discrete subset in $\overline{NE}(X/U)_{K_X+B+\mathbf{M}_X < 0}$. Moreover, we may write

$$\overline{NE}(X/U) = \overline{NE}(X/U)_{K_X+B+A+\mathbf{M}_X \geq 0} + \overline{NE}(X/U)_{\text{Nlc}(X,B,\mathbf{M})} + \sum_{j \in \Lambda_A} R_j.$$

(4) Let F be a $(K_X + B + \mathbf{M}_X)$ -negative extremal face in $\overline{NE}(X/U)$ that relatively ample at infinity (cf. Definition 3.1.7) with respect to $(X, B, \mathbf{M})$. Then F is a rational extremal face, and there exists a contraction $\text{cont}_F : X \rightarrow Z$ of F satisfying the following.

- (a) For any integral curve C on X such that the image of C in U is a closed point, $\text{cont}_F(C)$ is a point if and only if $[C] \in F$.
- (b) $\mathcal{O}_Y = (\text{cont}_F)_* \mathcal{O}_X$. In other words, cont_F is a contraction.
- (c) For any Cartier divisor D on Y such that $D \cdot C = 0$ for any curve C contracted by cont_F , there exists a Cartier divisor D_Y on Y such that $D = \text{cont}_F^* D_Y$.

When $\mathbf{M}$ is NQC/ U and $(X, B, \mathbf{M})$ is lc, Theorem 2.2.1(1-3) was proven in [HL21a, Theorem 1.3] and Theorem 2.2.1(4) was proven in [Xie22, Theorem 1.5] (see also [CLX23, Theorem 1.7]).

Theorem 2.2.2 (Existence of flips). *Let $(X, B, \mathbf{M})/U$ be a $\mathbb{Q}$ -factorial lc generalized pair and $f : X \rightarrow Z$ a $(K_X + B + \mathbf{M}_X)$ -flipping contraction/ U .*

Then the flip $f^+ : X^+ \rightarrow Z$ of f exists. Moreover, X^+ is $\mathbb{Q}$ -factorial, and $\rho(X) = \rho(X^+)$.

When $\mathbf{M}$ is NQC/ U , Theorem 2.2.2 was proven in [HL21a, Theorem 1.2] (see also [LX23b, Theorem 1.2]). Theorem 2.2.1 and Theorem 2.2.2 allow us to run the minimal model program for any $\mathbb{Q}$ -factorial lc generalized pair:

Theorem 2.2.3. *We may run the minimal model program for $\mathbb{Q}$ -factorial lc generalized pairs. More precisely, for any $\mathbb{Q}$ -factorial lc generalized pair $(X, B, \mathbf{M})/U$, there exists a sequence of $(K_X + B + \mathbf{M}_X)$ -flips and divisorial contractions/ U . Moreover, any such sequence ends either with a Mori fiber space of $(X, B, \mathbf{M})/U$, or a minimal model of $(X, B, \mathbf{M})/U$, or an infinite sequence of flips over U .*

There are several other important results on the structure of generalized lc pairs. The first two are the Kodaira vanishing theorem and the Kawamata-Viehweg vanishing theorem:

Theorem 2.2.4 (Kodaira vanishing theorem for lc generalized pairs). *Let $(X, B, \mathbf{M})$ be a projective lc generalized pair, and let D be a Cartier divisor on X such that $D - (K_X + B + \mathbf{M}_X)$ is ample. Then $H^i(X, \mathcal{O}_X(D)) = 0$ for any positive integer i .*

Theorem 2.2.5 (Relative Kawamata-Viehweg vanishing for lc generalized pairs). *Let $(X, B, \mathbf{M})/U$ be an lc generalized pair associated with morphism $f : X \rightarrow U$, and let D be a Cartier divisor on X such that $D - (K_X + B + \mathbf{M}_X)$ is nef/ U and log big/ U with respect to $(X, B, \mathbf{M})$. Then $R^i f_* \mathcal{O}_X(D) = 0$ for any positive integer i .*

When $\mathbf{M}$ is NQC/ U , Theorem 2.2.4 was proven in [CLX23, Theorem 1.3] while Theorem 2.2.5 was proven in [CLX23, Theorem 1.4].

Next, we have the base-point-freeness theorem and the semi-ampleness theorem for lc generalized pairs:

Theorem 2.2.6 (Base-point-freeness theorem). *Let $(X, B, \mathbf{M})/U$ be an lc generalized pair and D a nef/ U Cartier divisor on X , such that $aD - (K_X + B + \mathbf{M}_X)$ is ample/ U for some positive real number a . Then $\mathcal{O}_X(mD)$ is globally generated over U for any integer $m \gg 0$.*

Theorem 2.2.7 (Semi-ampleness theorem). *Let $(X, B, \mathbf{M})/U$ be an lc generalized pair and D a nef/ U $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor on X , such that $D - (K_X + B + \mathbf{M}_X)$ is ample/ U . Then D is semi-ample/ U .*

When $\mathbf{M}$ is NQC/ U , Theorem 2.2.6 was proven in [Xie22, Theorem 1.4], [CLX23, Theorem 1.5] while Theorem 2.2.7 was proven in [Xie22, Theorems 1.2], [CLX23, Theorem 1.6].

We also prove the canonical bundle formula and the subadjunction formula for generalized pairs. As the canonical bundle formula's statement is very technical and is a special case of Theorem 2.3.2 below (by setting $\mathcal{F} = T_X$), we will omit it here and only state the subadjunction formula.

Theorem 2.2.8 (Subadjunction formula). *Let $(X, B, \mathbf{M})/U$ be an lc generalized pair and V an lc center of $(X, B, \mathbf{M})$ such that $\dim V \geq 1$. Let W be the normalization of V . Then there exists an lc generalized pair $(W, B_W, \mathbf{M}^W)/U$ such that*

$$K_W + B_W + \mathbf{M}_W^W \sim_{\mathbb{R}} (K_X + B + \mathbf{M}_X)|_W.$$

Moreover, the image of any lc center of $(W, B_W, \mathbf{M}^W)$ in X is an lc center of $(X, B, \mathbf{M})$.

The main part of Theorem 2.2.8 was proven in [HL21b, Theorem 5.1] when $\mathbf{M}$ is NQC/ U .

Finally, we can show that lc generalized pairs are Du Bois:

Theorem 2.2.9. *Let $(X, B, \mathbf{M})$ be an lc generalized pair. Then any union of lc centers of $(X, B, \mathbf{M})$ is Du Bois. In particular, X is Du Bois.*

Theorem 2.2.9 was proven in [LX23b, Theorem 1.6] when $\mathbf{M}$ is NQC/ X .

2.3. Generalized foliated quadruples. As explained above, to establish the minimal model program for algebraically integrable foliations and generalized pairs, we need to broaden the category of objects for our study and consider the structure of *generalized foliated quadruples* $(X, \mathcal{F}, B, \mathbf{M})$, as defined in Definition 1.2.1.

Most of the main theorems of this paper on algebraically integrable foliations can also be extended to the category of algebraically integrable generalized foliated quadruples. Two results related to this structure that are particularly worth mentioning are the cone theorem and the canonical bundle formula. These two results will be essential in other main theorems of the paper, the statements of most of which do not rely on the language of generalized foliated quadruples.

2.3.1. Cone theorem. We establish the cone theorem for algebraically integrable generalized foliated quadruples in full generality.

Theorem 2.3.1 (Cone theorem for algebraically integrable generalized foliated quadruples). *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a generalized foliated quadruple and $\pi : X \rightarrow U$ the associated morphism. Let $\{R_j\}_{j \in \Lambda}$ be the set of $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -negative extremal rays in $\overline{NE}(X/U)$ that are rational. Assume that $\mathcal{F}$ is algebraically integrable. Then:*

(1)

$$\overline{NE}(X/U) = \overline{NE}(X/U)_{K_{\mathcal{F}} + B + \mathbf{M}_X \geq 0} + \overline{NE}(X/U)_{\text{Nlc}(X, \mathcal{F}, B, \mathbf{M})} + \sum_{j \in \Lambda} R_j.$$

Here $\text{Nlc}(X, \mathcal{F}, B, \mathbf{M})$ is the non-lc locus of $(X, \mathcal{F}, B, \mathbf{M})$ (cf. Definition 3.4.5). In particular, any $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -negative extremal ray in $\overline{NE}(X/U)$ is rational.

(2) Each R_j is spanned by a rational curve C_j such that $\pi(C_j) = \{pt\}$, C_j is tangent to $\mathcal{F}$, and

$$0 < -(K_{\mathcal{F}} + B + \mathbf{M}_X) \cdot C_j \leq 2 \dim X.$$

(3) For any ample/ U $\mathbb{R}$ -divisor A on X ,

$$\Lambda_A := \{j \in \Lambda \mid R_j \subset \overline{NE}(X/U)_{K_{\mathcal{F}} + B + A + \mathbf{M}_X < 0}\}$$

is a finite set. In particular, $\{R_j\}_{j \in \Lambda}$ is countable, and is a discrete subset in $\overline{NE}(X/U)_{K_{\mathcal{F}}+B+\mathbf{M}_X < 0}$. Moreover, we may write

$$\overline{NE}(X/U) = \overline{NE}(X/U)_{K_{\mathcal{F}}+B+A+\mathbf{M}_X \geq 0} + \overline{NE}(X/U)_{\text{Nlc}(X, \mathcal{F}, B, \mathbf{M})} + \sum_{j \in \Lambda_A} R_j.$$

- (4) Let F be a $(K_X + B + \mathbf{M}_X)$ -negative extremal face in $\overline{NE}(X/U)$ that relatively ample at infinity (cf. Definition 3.1.7) with respect to $(X, \mathcal{F}, B, \mathbf{M})$. Then F is a rational extremal face.

When $\mathcal{F} = T_X$ and $\mathbf{M} = \mathbf{0}$, Theorem 2.3.1 follows from [Amb03, Theorem 5.10] and [Fuj17, Theorem 4.5.2]. However, whenever either $\mathcal{F} \neq T_X$ or $\mathbf{M} \neq \mathbf{0}$, Theorem 2.3.1 becomes new. More precisely, there are two cases that worth to mention:

- (1) When $\mathcal{F} = T_X$, we get the cone theorem for generalized pairs, Theorem 2.2.1, which is new.
- (2) When $U = \{pt\}$ and $\mathbf{M} = \mathbf{0}$, (2) and a large part of (1) (the part without considering the rationality of R_j) were proven in [ACSS21, Theorem 3.9], but the rest parts are missing. Therefore, we cannot directly use [ACSS21, Theorem 3.9] to prove Theorem 2.1.1 and Theorem 2.3.1 becomes necessary.

We would like to note that the contraction theorem, the existence of flips, and the base-point-freeness theorem are still valid for generalized foliated quadruples that possess nice singularities (e.g. F-dlt). However, since these theorems do not hold the same level of importance in proving our other main theorems as the cone theorem does, we choose to omit them here.

2.3.2. Canonical bundle formula. The canonical bundle formula for foliated triples, as established in [LLM23, Theorem 1.3], plays a crucial role in proving the global ACC for foliated threefolds. However, due to technical challenges, the work presented in [LLM23] could not prove the canonical bundle formula for generalized foliated quadruples $(X, \mathcal{F}, B, \mathbf{M})$ unless the nef part $\mathbf{M}$ is $\mathbf{b}$ -semi-ample. In this study, we overcome these technical difficulties with innovative approaches, successfully proving the canonical bundle formula for generalized foliated quadruples in a more comprehensive manner.

Theorem 2.3.2. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a sub-generalized foliated quadruple and $f : X \rightarrow Z$ a contraction/ U , such that $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$ is an lc-trivial morphism (see Definition 12.1.3). Let B_Z and $\mathbf{M}^Z$ be the discriminant part and the base moduli part of $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$ (see Definition-Theorem 12.1.4). Then $\mathbf{M}^Z$ is $\mathbf{b}$ -nef/ U and*

$$K_{\mathcal{F}} + B + \mathbf{M}_X \sim_{\mathbb{R}} f^*(K_{\mathcal{F}_Z} + B_Z + \mathbf{M}_Z^Z).$$

Moreover, we have the following properties:

- (1) B_Z is uniquely determined and $\mathbf{M}^Z$ is uniquely determined up to $\mathbb{R}$ -linear equivalence.
- (2) If $B \geq 0$, then $B_Z \geq 0$.
- (3) If $(X, \mathcal{F}, B, \mathbf{M})$ is sub-lc, then $(Z, \mathcal{F}_Z, B_Z, \mathbf{M}^Z)$ is sub-lc.
- (4) If $(X, \mathcal{F}, B, \mathbf{M})$ is lc, then $(Z, \mathcal{F}_Z, B_Z, \mathbf{M}^Z)$ is lc.
- (5) If $(X, \mathcal{F}, B, \mathbf{M})$ is sub-lc or f has connected fibers, then any lc center of $(Z, \mathcal{F}_Z, B_Z, \mathbf{M}^Z)$ is the image of an lc center of $(X, \mathcal{F}, B, \mathbf{M})$ on Z .
- (6) If f has connected fibers, then the image of any lc center of $(X, \mathcal{F}, B, \mathbf{M})$ on Z is an lc center of $(Z, \mathcal{F}_Z, B_Z, \mathbf{M}^Z)$.
- (7) If f has connected fibers, then for any prime divisor D on X ,

$$\text{mult}_D B_Z = \epsilon(D) - \sup\{t \mid (X, \mathcal{F}, B + t f^* D, \mathbf{M}) \text{ is lc over the generic point of } D\}$$

where $\epsilon(D) = 0$ if D is $\mathcal{F}_Z$ -invariant, and $\epsilon(D) = 1$ otherwise (see Definition 3.3.5).

- (8) The $\mathbb{R}$ -linear equivalence class of $\mathbf{M}^Z$ only depends on $(X, B, \mathbf{M})$ over the generic point of Z .

(9) If $\mathbf{M}$ is NQC/U , then $\mathbf{M}^Z$ is NQC/U .

We want to emphasize that Theorem 2.3.2 is applicable to any foliation, not just those that are algebraically integrable. Consequently, we anticipate that Theorem 2.3.2 will play a significant role in future studies, encompassing both algebraically integrable foliations and those that are not necessarily algebraically integrable.

Next, we revisit the history of partial results that have contributed to the main part of Theorem 2.3.2, i.e. the nefness/ U of $\mathbf{M}^Z$.

- (1) When $\mathcal{F} = T_X$ and $\mathbf{M} = \mathbf{0}$, the main part of Theorem 2.3.2 is [JLX22, Theorem 1.2] (or [JLX22, Theorem 2.23] combined with [FG12, Lemma 1.1]). For other related references, see [Kod64, Kaw98, Amb05, Kol07, Flo14, FG14].
- (2) When $\mathcal{F} = T_X$ and $\mathbf{M}$ is NQC/U , previously we only knew the cases where either $B \geq 0$ at the generic point of Z or $\mathbf{M}$ is $\mathbf{b}$ -semi-ample/ Z ([JLX22, Theorem 2.23] + [HL21b, Theorem 4.5]). We direct the reader to [Fil19, Fil20, FS23] for other related references. It is worth noting that no results were known when $\mathbf{M}$ is not NQC/U .
- (3) When $\mathcal{F} \neq T_X$, we only knew the cases where f is a contraction and $\mathbf{M}$ is NQC/U and $\mathbf{b}$ -semi-ample/ Z ([LLM23, Proposition 6.14]).

In this paper, we not only prove Theorem 2.3.2 in full generality but also clarify why [Kol07] was able to address the horizontal negative coefficients, while [Fil19, Fil20, JLX22, FS23] cannot deal with this issue. Further details on this are provided in Remark 11.3.2. Following this discussion, we refine the definition of lc-trivial fibrations and lc-trivial morphisms, which are elaborated in Definition 11.3.1.

Finally, we note that the proof of Theorem 2.3.2 does not depend on the mixed Hodge structure, as opposed to what is done in [Kol07]. Remark 11.3.2 also explains why the mixed Hodge structure cannot be applied to our case. Instead, our approach is based on the structure of algebraically integrable foliations, a method similar to the one used in [ACSS21, Proof of Theorem 1.3]. However, the canonical bundle formula in that reference is not complete as the “BP stable” condition is required. Moreover, [ACSS21, Theorem 1.3] also has the additional requirement that $B \geq 0$ over the generic point of Z , a condition we aim to avoid.

2.4. Singularities of algebraically integrable generalized foliated quadruples. The cone theorem and the canonical bundle formula are primarily concerned with understanding the global behavior of algebraically integrable generalized foliated quadruples. However, it is equally important to examine the local behavior, particularly the singularities of these structures. In this paper, we will concentrate on two key aspects that are tied to the singularity structure of generalized foliated quadruples: the precise adjunction formula and the ACC for lc thresholds.

2.4.1. Adjunction formulas. [ACSS21, Proposition 3.2] proves the adjunction formula for algebraically integrable foliations provided that the ambient variety is $\mathbb{Q}$ -factorial and that the foliation is induced by a contraction. In this paper, we remove these two technical conditions and prove the adjunction formula for algebraically integrable foliations in full generality:

Theorem 2.4.1. *Let $(X, \mathcal{F}, B)$ be an foliated triple such that $\mathcal{F}$ is algebraically integrable. Let S be a prime divisor on X , such that $\text{mult}_S B = 0$ if S is $\mathcal{F}$ -invariant and $\text{mult}_S B = 1$ otherwise. Let S^ν be the normalization of S and $\mathcal{F}_S$ the restricted foliation (see Definition 6.1.5) of $\mathcal{F}$ on S^ν . Then*

$$K_{\mathcal{F}_S} + B_S = (K_{\mathcal{F}} + B)|_{S^\nu}$$

for some $\mathbb{R}$ -divisor $B_S \geq 0$. Moreover, if $(X, \mathcal{F}, B)$ is lc, then $(S^\nu, \mathcal{F}_S, B_S)$ is lc.

We remark that [CS23b, Theorem 3.16] proves the adjunction formula to non-invariant divisors for any foliation with a minor requirement that the boundary has $\mathbb{Q}$ -coefficients. In particular, when B has rational coefficients and $\text{mult}_S B = 1$, Theorem 2.4.1 is implied by [CS23b, Theorem 3.16].

Theorem 2.4.1 can be extended to the category of algebraically integrable generalized foliated quadruples:

Theorem 2.4.2. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a generalized foliated quadruple such that $\mathcal{F}$ is algebraically integrable. Let S be a prime divisor on X , such that $\text{mult}_S B = 0$ if S is $\mathcal{F}$ -invariant, and $\text{mult}_S B = 1$ otherwise. Let S^ν be the normalization of S , $\mathbf{M}^S := \mathbf{M}|_{S^\nu}$ (see Definition 3.4.2), and $\mathcal{F}_S$ the restricted foliation of $\mathcal{F}$ on S^ν . Then*

$$K_{\mathcal{F}_S} + B_S + \mathbf{M}_{S^\nu}^S := (K_{\mathcal{F}} + B + \mathbf{M}_X)|_{S^\nu}$$

for some $\mathbb{R}$ -divisor $B_S \geq 0$. Moreover, if $(X, \mathcal{F}, B, \mathbf{M})$ is lc, then $(S^\nu, \mathcal{F}_S, B_S, \mathbf{M}^S)$ is lc.

In [DLM23, Theorem 1.6], a precise adjunction formula was introduced for algebraically integrable foliated triples, playing a crucial role in proving the ACC for lc thresholds and the global ACC for algebraically integrable foliated triples. Building upon this concept, we formulate and establish a precise adjunction formula for algebraically integrable generalized foliated quadruples in this paper. We then apply it to prove the ACC for lc thresholds and the global ACC for algebraically integrable generalized foliated quadruples. We have the following theorem:

Theorem 2.4.3. *Let $\Gamma \subset [0, +\infty)$ be a set of real numbers. Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be an lc generalized foliated quadruple, S a prime divisor on X with normalization S^ν , such that $\text{mult}_S B = 0$ if S is $\mathcal{F}$ -invariant and $\text{mult}_S B = 1$ otherwise. Assume that the coefficients of B belong to Γ and $\mathbf{M}$ is a Γ -linear combination of nef/ U $\mathbf{b}$ -Cartier $\mathbf{b}$ -divisors. Let*

$$K_{\mathcal{F}_S} + B_S + \mathbf{M}_{S^\nu}^S := (K_{\mathcal{F}} + B + \mathbf{M}_X)|_{S^\nu}$$

where $\mathcal{F}_S$ is the restricted foliation of $\mathcal{F}$ on S^ν and $\mathbf{M}^S = \mathbf{M}|_{S^\nu}$.

Then the coefficients of B_S belong to $D(\Gamma)$ (see Definition 3.2.2). In particular, if Γ is a DCC set, then the coefficients of B_S belong to a DCC set.

We offer a more detailed version of Theorem 2.4.3 in Theorem 6.0.1. Given its highly technical nature, we omit it from the introduction.

2.4.2. ACC and the global ACC. Theorem 2.4.3 leads to the proof of the ACC and the global ACC for algebraically integrable generalized foliated quadruples. These results were proven in [HMX14] for usual pairs ($\mathcal{F} = T_X$ and $\mathbf{M} = \mathbf{0}$), in [BZ16] for generalized pairs ($\mathcal{F} = T_X$), and in [DLM23] for foliated triples ($\mathbf{M} = \mathbf{0}$).

Theorem 2.4.4 (ACC for lc thresholds for algebraically integrable generalized foliated quadruples). *Let r be a positive integer and $\Gamma \subset [0, +\infty)$ a DCC set. Then there exists an ACC set Γ' depending only on r and Γ satisfying the following. Let $(X, \mathcal{F}, B, \mathbf{M})/X$ be an lc generalized foliated quadruple, such that*

- (1) $\mathcal{F}$ is algebraically integrable of rank r ,
- (2) the coefficients of B belong to Γ , and
- (3) $\mathbf{M}$ is a Γ -linear combination of nef/ X $\mathbf{b}$ -Cartier $\mathbf{b}$ -divisors.

Then the lc threshold

$$\text{lct}(X, \mathcal{F}, B, \mathbf{M}; D, \mathbf{N}) := \sup\{t \mid t \geq 0, (X, \mathcal{F}, B + tD, \mathbf{M} + t\mathbf{N}) \text{ is lc}\}$$

is contained in Γ' .

Theorem 2.4.5 (Global ACC for algebraically integrable generalized foliated quadruples). *Let r be a positive integer and $\Gamma \subset [0, 1]$ a DCC set. Then there exists a finite set $\Gamma_0 \subset \Gamma$ depending only on r and Γ satisfying the following. Let $(X, \mathcal{F}, B, \mathbf{M})$ be a projective lc generalized foliated quadruple such that*

- (1) $\mathcal{F}$ is algebraically integrable of rank r ,
- (2) the coefficients of B belong to Γ ,

- (3) $\mathbf{M} = \sum \gamma_j \mathbf{M}_j$, where each $\gamma_j \in \Gamma$ and each $\mathbf{M}_j$ is a nef $\mathbf{b}$ -Cartier $\mathbf{b}$ -divisor,
- (4) $\mathbf{M}_j \not\equiv \mathbf{0}$ if $\gamma_j \neq 0$, and
- (5) $K_{\mathcal{F}} + B + \mathbf{M}_X \equiv 0$.

Then the coefficients of B belong to Γ_0 , and $\gamma_j \in \Gamma_0$ for each j .

The proof of Theorem 2.4.5 is harder than the proof of the global ACC for algebraically integrable foliated triples [DLM23, Theorem 1.2]. This is because our proof heavily relies on the existence of Mori fiber spaces (Theorem 2.1.5), unlike the proof in [DLM23, Theorem 1.2].

As a straightforward corollary of Theorem 2.4.5, we obtain the global ACC for rank one generalized foliated quadruples:

Corollary 2.4.6 (Global ACC for rank one generalized foliated quadruples). *Let $\Gamma \subset [0, 1]$ be a DCC set. Then there exists a finite set $\Gamma_0 \subset \Gamma$ depending only on Γ satisfying the following. Let $(X, \mathcal{F}, B, \mathbf{M})$ be an lc generalized foliated quadruple such that*

- (1) $\text{rank } \mathcal{F} = 1$,
- (2) the coefficients of B belong to Γ ,
- (3) $\mathbf{M} = \sum \gamma_j \mathbf{M}_j$, where each $\gamma_j \in \Gamma$ and each $\mathbf{M}_j$ is a nef $\mathbf{b}$ -Cartier $\mathbf{b}$ -divisor,
- (4) $\mathbf{M}_j \not\equiv \mathbf{0}$ if $\gamma_j \neq 0$, and
- (5) $K_{\mathcal{F}} + B + \mathbf{M}_X \equiv 0$.

Then the coefficients of B belong to Γ_0 , and $\gamma_j \in \Gamma_0$ for each j .

When $\mathbf{M} = \mathbf{0}$, Corollary 2.4.6 was proven in [DLM23, Corollary 1.3].

2.4.3. Uniform rational polytopes. As a direct consequence of Theorems 2.4.4 and 2.4.5, we establish the existence of uniform lc rational polytopes for algebraically integrable generalized foliated quadruples. Despite their complex nature, these polytopes are powerful tools in birational geometry. They are notably used in several applications on the ACC conjecture for minimal log discrepancies and the boundedness of complements. These polytopes are essential for the formal definition of KSBA moduli spaces [Kol23, 6.27.3, Theorem 11.49], and play a crucial role in proving the global ACC for foliated threefolds [LMX23b]. In this paper, we prove the existence of uniform lc rational polytopes for algebraically integrable generalized foliated quadruples:

Theorem 2.4.7. *Let r be a positive integer, $v_1^0, \dots, v_m^0, u_1^0, \dots, u_n^0$ positive real numbers, $\mathbf{v}_0 := (v_1^0, \dots, v_m^0)$, and $\mathbf{u}_0 := (u_1^0, \dots, u_n^0)$. Then there exists an open set $U \ni (\mathbf{v}_0, \mathbf{u}_0)$ of the rational envelope of $(\mathbf{v}_0, \mathbf{u}_0)$ in $\mathbb{R}^{m+n}$ depending only on r and $\mathbf{v}_0, \mathbf{u}_0$ satisfying the following. Let*

$$\left(X, \mathcal{F}, B = \sum_{j=1}^m v_j^0 B_j, \mathbf{M} = \sum_{k=1}^n u_k^0 \mathbf{M}_k \right) / X$$

be an lc generalized foliated quadruple, such that $\mathcal{F}$ is algebraically integrable, $\text{rank } \mathcal{F} = r$, $B_j \geq 0$ are distinct Weil divisors, and $\mathbf{M}_k$ are nef/ X $\mathbf{b}$ -Cartier $\mathbf{b}$ -divisors. Then

$$\left(X, \mathcal{F}, B = \sum_{j=1}^m v_j B_j, \sum_{k=1}^n u_k \mathbf{M}_k \right)$$

is lc for any $(v_1, \dots, v_m, u_1, \dots, u_n) \in U$.

2.5. Miscellaneous results on the minimal model program and foliations. We also prove several other interesting theorems that can be useful for further applications.

2.5.1. Analogues of dlt models. We establish the existence of $(*)$ -models and ACSS models for algebraically integrable generalized foliated quadruples (see Definitions 7.2.3 and 7.4.1). As detailed in [ACSS21, CS23a, DLM23], these models play the same role as dlt models in the classic MMP. Moreover, $\mathbb{Q}$ -factorial dlt algebraically integrable generalized foliated quadruples always satisfy the property “ACSS” and the property $(*)$ (see Theorem 17.0.1).

Theorem 2.5.1 (Existence of ACSS model). *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be an lc generalized foliated quadruple. Assume that $\mathcal{F}$ is algebraically integrable. Then $(X, \mathcal{F}, B, \mathbf{M})/U$ has an ACSS model which is also a $(*)$ -model.*

In particular, there exists a birational morphism $h_Y : Y \rightarrow X$ and a contraction $f_Y : Y \rightarrow Z$ satisfying the following. Let $\mathcal{F}_Y := h_Y^{-1}\mathcal{F}$ and

$$K_{\mathcal{F}_Y} + B_Y + \mathbf{M}_Y = h_Y^*(K_{\mathcal{F}} + B + \mathbf{M}_X),$$

then

- (1) $(Y, B_Y, \mathbf{M})$ is $\mathbb{Q}$ -factorial qdlt,
- (2) f_Y is equi-dimensional and $\mathcal{F}_Y$ is induced by f_Y , and
- (3) any prime f -exceptional divisor is an lc place of $(X, \mathcal{F}, B, \mathbf{M})$.

2.5.2. Minimal model program for very exceptional divisors. When running the relative minimal model program, especially the birational minimal model program, we often encounter the minimal model program for very exceptional divisors [Bir12, Theorem 1.8]. In this paper, we establish the minimal model program for algebraically integrable generalized foliated quadruples whose generalized foliated log canonical divisor is very exceptional.

Theorem 2.5.2. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a $\mathbb{Q}$ -factorial F -dlt generalized foliated quadruple and $E \geq 0$ and $\mathbb{R}$ -divisor on X , such that E is very exceptional/ U and*

$$K_{\mathcal{F}} + B + \mathbf{M}_X \sim_{\mathbb{R}, U} E.$$

Then we may run a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U with scaling of an ample/ U $\mathbb{R}$ -divisor, and any such MMP terminates with a good minimal model $(X', \mathcal{F}', B', \mathbf{M})/U$ of $(X, \mathcal{F}, B, \mathbf{M})/U$, such that $K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} \sim_{\mathbb{R}, U} 0$.

Theorem 2.5.2 is vital for proving that $\mathbb{Q}$ -factorial dlt implies ACSS (Theorem 17.0.1). This is essential for proving Theorem A.

2.5.3. A special case of Prokhorov-Shokurov’s base-point-freeness conjecture. Prokhorov-Shokurov’s base-point-freeness conjecture [PS09, Conjecture 7.13] is a major conjecture in birational geometry. It has been verified when the relative dimension of the fibration is 1 ([PS09, Theorem 8.1]) or 2 ([ABBDILW23, Theorem 1.4]). However, for the relative dimension of the fibration ≥ 3 , the conjecture is still largely open. Through the application of foliation theory, we prove a special case of the Prokhorov-Shokurov base-point-freeness conjecture:

Theorem 2.5.3. *Let d and m be two positive integers. Then there exists a positive integer I depending only on d and I satisfying the following.*

Let (X, B) be a projective klt pair, and $f : X \rightarrow Z$ a contraction to a smooth variety Z . Let B^h and B^v be the horizontal/ Z part of B and the vertical/ Z part of B respectively, and let $\mathbf{M}$ be the moduli part of $f : (X, B) \rightarrow Z$. Assume that the following conditions hold.

- (1) (Semi-stability) (X, B) is BP semi-stable/ Z .
- (2) (Klt-trivial) $K_X + B \sim_{\mathbb{Q}, Z} 0$.
- (3) (Coefficient control) mB is a Weil divisor.
- (4) (Fano type) There exists an ample $\mathbb{R}$ -divisor H such that $B^h \geq H \geq 0$.
- (5) (Snc condition) There exists a reduced divisor Σ_Z on Z , such that $B^v = f^{-1}(\Sigma_Z)$, (Z, Σ_Z) is log smooth, and for any reduced divisor $H \geq 0$ such that $(Z, \Sigma + H)$ is log smooth, $(X, B + f^*H)$ is lc.

Then $\mathbf{M}$ descends to X , IM_X is Cartier, and nIM_X is base-point-free for any integer $n \gg 0$.

While the conditions of Theorem 2.5.3 are quite restrictive, mainly because condition (4) is not preserved under birational transformations, there are no requirements on the dimension of the varieties or the relative dimension of f . This makes the theorem potentially useful for future applications.

2.6. Why should we care about generalized foliated quadruples? Before we move on to the main part of the paper, we would like to briefly explain why we need to consider the structure of generalized foliated quadruples and why it is essential for us to prove some main theorems of the paper, even if these theorems' statements do not explicitly mention this structure. To clarify this, we present five scenarios where generalized foliated quadruples play a crucial role. Four out of these five scenarios are unavoidable in the proofs of this paper.

Scenario 2.6.1 (Canonical bundle formula of foliations). [LLM23, Theorem 1.2] established the canonical bundle formula for foliations. More precisely, given a projective lc foliated triple $(X, \mathcal{F}, B)$ and a contraction $f : X \rightarrow Z$ such that the general fibers of f are tangent to $\mathcal{F}$ and $K_{\mathcal{F}} + B \sim_{\mathbb{R}, Z} 0$, we have

$$K_{\mathcal{F}} + B \sim_{\mathbb{R}} f^*(K_{\mathcal{F}_Z} + B_Z + \mathbf{M}_Z)$$

where $(Z, \mathcal{F}_Z, B_Z, \mathbf{M})$ is a projective lc generalized foliated quadruple. Therefore, if we want to study the behavior of $(X, \mathcal{F}, B)$, then it is necessary to study the structure of $(Z, \mathcal{F}_Z, B_Z, \mathbf{M})$. When $\mathcal{F} = T_X$ and (X, B) is klt, the classical approach is to find an $\mathbb{R}$ -divisor

$$0 \leq \Delta_Z \sim_{\mathbb{R}} B_Z + \mathbf{M}_Z$$

such that (Z, Δ_Z) is klt [Amb05, Theorem 0.2] and use the structure of (Z, Δ_Z) instead of $(Z, B_Z, \mathbf{M})$. This approach is essential for the proof of the finite generation of the canonical ring ([BCHM10, Proof of Corollary 1.1.2], [FM00]).

However, when $\mathcal{F} \neq T_X$, it is generally not possible for us to combine B_Z and $\mathbf{M}_Z$ and get an lc triple structure $(Z, \mathcal{F}_Z, \Delta_Z \sim_{\mathbb{R}} B_Z + \mathbf{M}_Z)$. This is because of the following two reasons:

- (1) “Klt” is almost an empty condition for foliations when $\mathcal{F} \neq T_X$. Actually, unless the foliation is purely transcendental, there are always (infinitely many) lc centers of $(X, \mathcal{F}, B)$ as long as $\mathcal{F} \neq T_X$. This will cause trouble when we try to perturb the coefficients of $B_Z + \mathbf{M}_Z$ and get Δ_Z . In fact, even when $\mathcal{F} = T_X$, if (X, B) is not klt, then we do not know whether there exists such Δ_Z so that (Z, Δ_Z) is lc, and we usually need the $\mathbf{b}$ -semi-ampleness of $\mathbf{M}$ to show this fact. The $\mathbf{b}$ -semi-ampleness of $\mathbf{M}$, on the other hand, is the Prokhorov-Shokurov conjecture [PS09, Conjecture 7.13] as mentioned above, which is widely open when $\dim X - \dim Z \geq 3$. Indeed, the unconfirmed status of the Prokhorov-Shokurov conjecture is one key reason why Birkar-Zhang introduced the concept of generalized pairs in [BZ16].
- (2) Even if $\mathbf{M}$ is semi-ample, the existence of such Δ_Z is also unknown as Bertini type theorems fail for foliations in general, even for surfaces (cf. [DLM23, Example 3.4]). In other words, it is possible that $(Z, \mathcal{F}_Z, B_Z + G_Z)$ is not lc for any $G_Z \in |\mathbf{M}_Z|_{\mathbb{R}}$.

Therefore, in many situations, we must analyze the structure of generalized foliated quadruples rather than foliated triples. Furthermore, due to (2), the concept of generalized foliated quadruples becomes essential for studying foliations, even in lower dimensions. This is a key reason why [LLM23, LMX23b] rely on the theory of generalized foliated quadruples to establish the global ACC for foliated triples in dimension 3.

Scenario 2.6.2 (MMP with scaling). We recall how we run the minimal model program for with scaling for usual pairs. For simplicity, we only consider the projective case. Given a projective lc pair (X, B) and an $\mathbb{R}$ -divisor $A \geq 0$ on X such that $K_X + B + A$ is nef, we consider the scaling numbers

$$\lambda := \inf\{t \mid t \geq 0, K_X + B + tA \text{ is nef}\}.$$

If $t = 0$ then we are done. Otherwise, we contract a $(K_X + B)$ -negative extremal ray R such that $(K_X + B + tA) \cdot R = 0$, and let $f : (X, B) \dashrightarrow (X', B')$ be a corresponding divisorial contraction, flip, or Mori fiber space associated to the contraction of R . We may replace (X, B) with (X', B') and A with A' and continue this process.

Although we only need $K_X + B + A$ to be nef to run the MMP, in practice, we usually also need the additional condition that $(X, B + A)$ is lc. This is helpful in many situations: since we do not know the termination of the MMP, it is likely for us to consider pairs $(X, B + \mu A)$ where μ is related to the scaling numbers λ . In this case, we usually need $(X, B + A)$ to be lc in order to guarantee that $(X, B + \mu A)$ is lc. For this reason, for the very first step of the MMP, we usually require A to be a general ample $\mathbb{R}$ -divisor, or a general base-point-free big and nef $\mathbb{R}$ -divisor when (X, B) is klt.

Now we consider the minimal model program for foliations. We definitely want to consider the minimal model program with scaling of ample divisors as well. However, as we have explained above, Bertini type theorems fail for foliations in general, even for surfaces (cf. [DLM23, Example 3.4]). Therefore, it is possible that for any ample $\mathbb{R}$ -divisor $A \geq 0$ on X , $(X, \mathcal{F}, B + A)$ is not lc. Now the minimal model program of $(X, \mathcal{F}, B)$ with scaling of A becomes weird: we can still run the minimal model program, but it will become difficult to study the intermediate outputs $(X', B' + \lambda A')$ with $\lambda > 0$ after each step of the MMP, where λ is the scaling number. This causes a lot of inconvenience for the minimal model program of foliations.

The structure of generalized foliated quadruples, however, can easily resolve this issue: if we identify $(X, \mathcal{F}, B)$ with the generalized foliated quadruple $(X, \mathcal{F}, B, \overline{0})$, then instead of running an MMP with scaling of an ample $\mathbb{R}$ -divisor A , we can let $\mathbf{A} := \overline{A}$ be the nef $\mathbf{b}$ -divisor associated to A . Now may run an MMP with scaling of $(0, \mathbf{A})$. That is, although we still consider

$$\lambda := \inf\{t \mid t \geq 0, K_{\mathcal{F}} + B + tA \text{ is nef}\},$$

the output of the first step of the MMP $\phi : X \dashrightarrow X'$ becomes

$$(X', \mathcal{F}' = \phi_* \mathcal{F}, B' = \phi_* B, \lambda \mathbf{A})$$

which is still an lc generalized foliated quadruple. Therefore, by using the theory of generalized foliated quadruples, we can bypass the failure of Bertini type theorems of foliations straightforwardly.

Scenario 2.6.3 (Minimal model program on Kähler varieties). It is well-known that foliations, especially algebraically integrable foliations, has a tight connection with the minimal model program for Kähler varieties. As we have mentioned above, Das and Ou essentially use the structure of algebraically integrable foliations to prove the abundance conjecture for Kähler manifolds in dimension 3 [DO23a, DO23b]. There is no doubt that foliations are expected to be useful in the study of Kähler minimal model program in the future.

On the other hand, generalized pairs is also known to have a tight connection with the minimal model program for Kähler varieties [DH23, DHY23]. The key reason is due to the minimal model program with scaling of Kähler classes. Kähler classes cannot be considered as divisors, but by considering Kähler classes as $\mathbf{b}$ -nef classes and move it to the nef part of the generalized pair, we can formally define the minimal model program with scaling of Kähler classes.

In summary, the study of Kähler varieties seems to be a natural place for foliations and generalized pairs to get mixed together. We therefore can expect the structure of generalized foliated quadruples, particularly the algebraically integrable ones, to play a crucial role in the study of Kähler varieties in the future.

Scenario 2.6.4 (Cone theorem and semi-ampleness theorem). In [ACSS21, Theorem 3.9], a version of the cone theorem for algebraically integrable foliated triples $(X, \mathcal{F}, B)$ is proved. When $(X, \mathcal{F}, B)$ is lc, the main part of the cone theorem, i.e. the formula

$$\overline{NE}(X) = \overline{NE}(X)_{K_{\mathcal{F}} + B \geq 0} + \sum R_j$$

was proved in [ACSS21, Theorem 3.9]. However, [ACSS21, Theorem 3.9] did not prove the countableness of R_j nor the finiteness of R_j when polarizing $(X, \mathcal{F}, B)$ with an ample divisor A . That is, the formula

$$\overline{NE}(X) = \overline{NE}(X)_{K_{\mathcal{F}}+B+A \geq 0} + \sum_{\text{finite}} R_j$$

is missing. One key reason for this seems to be the issue that $(X, \mathcal{F}, B + A)$ may no longer be lc, and this is, again, due to the failure of the Bertini type theorems for foliations. However, if we consider $(X, \mathcal{F}, B, \bar{A})$ instead of $(X, \mathcal{F}, B + A)$, then $(X, \mathcal{F}, B, \bar{A})$ becomes an lc generalized pair and we have immediately have more flexibility.

Similar issues appear when we consider the semi-ampleness theorem for foliations. For usual pairs, the semi-ampleness theorem is usually formulated in the following way:

$$(X, B) \text{ lc}, A \text{ ample}, K_X + B + A \text{ nef} \Rightarrow K_X + B + A \text{ semi-ample.}$$

However, for foliations, the semi-ampleness theorem is usually formulated in the following way: (under suitable conditions)

$$(X, \mathcal{F}, B + A) \text{ lc}, B \geq 0, A \geq 0 \text{ ample}, K_{\mathcal{F}} + B + A \text{ nef} \Rightarrow K_{\mathcal{F}} + B + A \text{ semi-ample.}$$

This is again due to the failure of Bertini-type theorems. Nevertheless, with the new concept of generalized foliated quadruples, these arguments can now be strengthened back to: (under suitable conditions)

$$(X, \mathcal{F}, B) \text{ lc}, A \text{ ample}, K_{\mathcal{F}} + B + A \text{ nef} \Rightarrow K_{\mathcal{F}} + B + A \text{ semi-ample.}$$

Scenario 2.6.5 (Canonical bundle formula for generalized pairs). The final scenario where the structure of generalized foliated quadruples plays a vital role is in obtaining the canonical bundle formula for generalized pairs. To establish this formula for lc-trivial fibrations in cases involving either non-NQC generalized pairs or NQC generalized pairs with potentially negative coefficients, we cannot rely on the structure of mixed Hodge structure (see Remark 11.3.2). Filipazzi's approach [Fil19, Fil20] is also unsuitable due to its requirements for $\mathbb{Q}$ -coefficients and its inability to handle negative coefficients. Therefore, the only viable approach to achieve such a canonical bundle formula is by utilizing the theory of foliations as in [ACSS21]. Now, since we are dealing with generalized pairs in this context, the introduction of generalized foliated quadruples becomes necessary. For more details, we refer the reader to the proof of Theorem 11.4.4.

Part I. Preliminaries

3. BASIC DEFINITIONS

Throughout the paper, we will mainly work with normal quasi-projective varieties to ensure consistency with the references. However, most results should also hold for normal varieties that are not necessarily quasi-projective. Similarly, most results in our paper should hold for any algebraically closed field of characteristic zero. We will adopt the standard notations and definitions in [KM98, BCHM10] and use them freely. For foliations, we will generally follow the notations and definitions in [CS20, ACSS21, CS21], but there may be minor differences. For generalized pairs, we will follow the notations and definitions in [HL21a].

3.1. Special notations.

Notation 3.1.1. In this paper, $\mathbb{N}$ stands for the set of non-negative integers and $\mathbb{N}^+$ stands for the set of positive integers.

Notation 3.1.2. In this paper, the notation “/” is always considered as a simplified writing of “over”. For example, “/Z” means “over Z”.

Notation 3.1.3. Let $X \rightarrow U$ be a projective morphism from a normal variety to a variety, and let A be a semi-ample/ U $\mathbb{R}$ -divisor on X . An $\mathbb{R}$ -divisor H on X is said to be *general* in $|A/U|_{\mathbb{R}}$ if there exist base-point-free/ U divisors $A_1, \dots, A_n$ and real numbers $r_1, \dots, r_n \in (0, 1)$, such that $A = \sum_{i=1}^n r_i A_i$ and $H = \sum_{i=1}^n r_i H_i$, where $H_i \in |A_i/U|$ are general elements. A *general ample/ U $\mathbb{R}$ -divisor* on X is an ample/ U $\mathbb{R}$ -divisor D on X such that D is general in $|D/U|_{\mathbb{R}}$.

Notation 3.1.4. Let Γ be a set of real numbers, X a normal variety, and B an $\mathbb{R}$ -divisor on X . We write $B \in \Gamma$ if the coefficients of B belong to Γ .

Notation 3.1.5. Let $\pi : X \rightarrow U$ be a projective morphism between varieties and D an $\mathbb{R}$ -divisor on X . We denote by $\kappa_{\sigma}(X/Z, D)$ (resp. $\kappa_t(X/Z, D)$, $\kappa(X/Z, D)$) the relative numerical dimension (resp. relative invariant Iitaka dimension, relative Iitaka dimension) of D over Z . When $Z = \{pt\}$, we may drop X/Z and use the notation $\kappa_{\sigma}(D)$ (resp. $\kappa_t(D)$, $\kappa(D)$) instead. We refer the reader to [HH20, Section 2] for the formal definitions and basic properties of $\kappa_{\sigma}(X/Z, D)$, $\kappa_t(X/Z, D)$, and $\kappa(X/Z, D)$.

Definition 3.1.6 (Log big). Let $(X, B, \mathbf{M})/U$ be a g-pair and D an $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor D on X . We say that D is *log big/ U with respect to $(X, B, \mathbf{M})$* if $D|_V$ is big/ U for any lc center V of $(X, B, \mathbf{M})$. In particular, D is big/ U .

Definition 3.1.7 (cf. [Amb03, Definition 5.3], [Fuj11, Definition 6.7.2]). Let (X, Δ) be a (not necessarily lc) pair and $\pi : X \rightarrow U$ a projective morphism. Let F be an extremal face of $\overline{NE}(X/U)$.

- (1) A *supporting function* of F is a π -nef $\mathbb{R}$ -divisor H such that $F = \overline{NE}(X/U) \cap H^{\perp}$. If H is a $\mathbb{Q}$ -divisor, we say that H is a *rational supporting function*. Since F is an extremal face of $\overline{NE}(X/U)$, F always has a supporting function.
- (2) We say that F is *rational* if F has a rational supporting function.
- (3) For any $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor D on X , we say that F is *D -negative* if

$$F \cap \overline{NE}(X/U)_{D \geq 0} = \{0\}.$$

- (4) We say that F is *relatively ample at infinity with respect to (X, Δ)* if

$$F \cap \overline{NE}(X/U)_{\text{Nlc}(X, \Delta)} = \{0\}.$$

Equivalently, $H|_{\text{Nlc}(X, \Delta)}$ is $\pi|_{\text{Nlc}(X, \Delta)}$ -ample for any supporting function H of F .

- (5) We say that F is *contractible at infinity with respect to (X, Δ)* if F has a rational supporting function H and $H|_{\text{Nlc}(X, \Delta)}$ is $\pi|_{\text{Nlc}(X, \Delta)}$ -semi-ample.

Definition-Lemma 3.1.8. Let K be a convex cone containing no lines. A ray R of K is called *exposed* if there is a hyperplane meeting K exactly along R . In particular, any exposed ray of K is extremal in K . If K does not contain any line, then K is the closure of the subcone of K spanned by exposed rays ([Roc97, Corollary 18.7.1], [Spi20, Lemma 6.2]).

Let $\pi : X \rightarrow U$ be a projective morphism from a normal quasi-projective variety to a variety. By definition, an extremal ray in $\overline{NE}(X/U)$ is exposed if and only if it has a supporting function (that is not necessarily rational). Moreover, for any sub-cone V of $\overline{NE}(X/U)$, we have

$$\overline{NE}(X/U) = \overline{V + \sum R_i}$$

where R_i are exposed rays that are not contained in V .

3.2. Sets.

Definition 3.2.1. Let $\Gamma \subset \mathbb{R}$ be a set. We say that Γ satisfies the *descending chain condition* (DCC) if any decreasing sequence in Γ stabilizes, and Γ satisfies the *ascending chain condition* (ACC) if any increasing sequence in Γ stabilizes.

Definition 3.2.2. Let $\Gamma \subset [0, +\infty)$ be a set. We define

$$\Gamma_+ := \{0\} \cup \left\{ \sum_{i=1}^l \gamma_i \mid \gamma_i \in \Gamma, l \in \mathbb{N}^+ \right\} \text{ and } D(\Gamma) := \left\{ \frac{m-1+\gamma}{m} \mid \gamma \in \Gamma_+, m \in \mathbb{N}^+ \right\}.$$

3.3. Foliations. In this subsection, we define foliations and some of its related concepts. For preliminaries regarding algebraically integrable foliations, we refer the reader to Subsection 6.1.

Definition 3.3.1 (Foliations, cf. [CS21, Section 2.1]). Let X be a normal variety. A *foliation* on X is a coherent sheaf $\mathcal{F} \subset T_X$ such that

- (1) $\mathcal{F}$ is saturated in T_X , i.e. $T_X/\mathcal{F}$ is torsion free, and
- (2) $\mathcal{F}$ is closed under the Lie bracket.

The *rank* of the foliation $\mathcal{F}$ is the rank of $\mathcal{F}$ as a sheaf and is denoted by $\text{rank } \mathcal{F}$. The *co-rank* of $\mathcal{F}$ is $\dim X - \text{rank } \mathcal{F}$. The *canonical divisor* of $\mathcal{F}$ is a divisor $K_{\mathcal{F}}$ such that $\mathcal{O}_X(-K_{\mathcal{F}}) \cong \det(\mathcal{F})$. We define $N_{\mathcal{F}} := (T_X/\mathcal{F})^{\vee\vee}$ and $N_{\mathcal{F}}^* := N_{\mathcal{F}}^{\vee}$.

If $\mathcal{F} = 0$, then we say that $\mathcal{F}$ is a *foliation by points*.

Definition 3.3.2 (Singular locus). Let X be a normal variety and $\mathcal{F}$ a rank r foliation on X . We can associate to $\mathcal{F}$ a morphism

$$\phi : \Omega_X^{[r]} \rightarrow \mathcal{O}_X(K_{\mathcal{F}})$$

defined by taking the double dual of the r -wedge product of the map $\Omega_X^1 \rightarrow \mathcal{F}^*$, induced by the inclusion $\mathcal{F} \rightarrow T_X$. This yields a map

$$\phi' : (\Omega_X^{[r]} \otimes \mathcal{O}_X(-K_{\mathcal{F}}))^{\vee\vee} \rightarrow \mathcal{O}_X$$

and we define the singular locus, denoted as $\text{Sing}\mathcal{F}$, to be the co-support of the image of ϕ' .

Definition 3.3.3 (Pullbacks and pushforwards, cf. [ACSS21, 3.1]). Let X be a normal variety, $\mathcal{F}$ a foliation on X , $f : Y \dashrightarrow X$ a dominant map, and $g : X \dashrightarrow X'$ a birational map. We denote $f^{-1}\mathcal{F}$ the *pullback* of $\mathcal{F}$ on Y as constructed in [Dru21, 3.2]. We also say that $f^{-1}\mathcal{F}$ is the *induced foliation* of $\mathcal{F}$ on Y . If $\mathcal{F} = 0$, then we say $f^{-1}\mathcal{F}$ is induced by f . In this case, we say $f^{-1}\mathcal{F}$ is *algebraically integrable*.

We define the *pushforward* of $\mathcal{F}$ on X' as $(g^{-1})^{-1}\mathcal{F}$ and denote it by $g_*\mathcal{F}$.

Definition 3.3.4 (Invariant subvarieties, cf. [ACSS21, 3.1]). Let X be a normal variety, $\mathcal{F}$ a foliation on X , and $S \subset X$ a subvariety. We say that S is *$\mathcal{F}$ -invariant* if and only if for any open subset $U \subset X$ and any section $\partial \in H^0(U, \mathcal{F})$, we have

$$\partial(\mathcal{I}_{S \cap U}) \subset \mathcal{I}_{S \cap U}$$

where $\mathcal{I}_{S \cap U}$ is the ideal sheaf of $S \cap U$. Note that if $\mathcal{F}$ is the foliation induced by a dominant map $f : X \dashrightarrow Z$, then a divisor D is $\mathcal{F}$ -invariant if and only if D is vertical with respect to f .

Definition 3.3.5 (Special divisors on foliations, cf. [CS21, Definition 2.2]). Let X be a normal variety and $\mathcal{F}$ a foliation on X . For any prime divisor C on X , we define $\epsilon_{\mathcal{F}}(C) := 1$ if C is not $\mathcal{F}$ -invariant, and $\epsilon_{\mathcal{F}}(C) := 0$ if C is $\mathcal{F}$ -invariant. If $\mathcal{F}$ is clear from the context, then we may use $\epsilon(C)$ instead of $\epsilon_{\mathcal{F}}(C)$. For any $\mathbb{R}$ -divisor D on X , we define

$$D^{\mathcal{F}} := \sum_{C|C \text{ is a component of } D} \epsilon_{\mathcal{F}}(C)C.$$

Let E be a prime divisor over X and $f : Y \rightarrow X$ a projective birational morphism such that E is on Y . We define $\epsilon_{\mathcal{F}}(E) := \epsilon_{f^{-1}\mathcal{F}}(E)$. It is clear that $\epsilon_{\mathcal{F}}(E)$ is independent of the choice of f .

3.4. Polarized foliations. In this subsection we introduce the concept of generalized foliated quadruples, which was originally introduced by the third author, Luo, and Meng in their study of the global ACC for foliated threefolds [LLM23]. The category of generalized foliated quadruples is a larger category comparing to generalized pairs, foliated triples, and usual pairs, so we shall not formally define the latter three concepts and only consider them as special generalized foliated quadruples. Since this is a very technical definition and we do not need its full power for some parts of the paper (e.g. we only need the concept of generalized pairs in Sections 4 and 5), for the reader's convenience, we refer the reader to [KM98, BCHM10] for the definition of pairs and [BZ16, HL21a] for the definition of generalized pairs. We refer the reader to [CS20, CS21] for the definition of foliated pairs $(\mathcal{F}, B)$; a foliated pair $(\mathcal{F}, B)$ together with its ambient variety X is a foliated triple $(X, \mathcal{F}, B)$.

Definition 3.4.1 (*b-divisors*). Let X be a normal quasi-projective variety. We call Y a *birational model* over X if there exists a projective birational morphism $Y \rightarrow X$.

Let $X \dashrightarrow X'$ be a birational map. For any valuation ν over X , we define $\nu_{X'}$ to be the center of ν on X' . A *b-divisor* $\mathbf{D}$ on X is a formal sum $\mathbf{D} = \sum_{\nu} r_{\nu} \nu$ where ν are valuations over X and $r_{\nu} \in \mathbb{R}$, such that ν_X is not a divisor except for finitely many ν . If in addition, $r_{\nu} \in \mathbb{Q}$ for every ν , then $\mathbf{D}$ is called a *$\mathbb{Q}$ -b-divisor*. The *trace* of $\mathbf{D}$ on X' is the $\mathbb{R}$ -divisor

$$\mathbf{D}_{X'} := \sum_{\nu_{X'} \text{ is a divisor}} r_{\nu} \nu_{X'}.$$

If $\mathbf{D}_{X'}$ is $\mathbb{R}$ -Cartier and $\mathbf{D}_Y$ is the pullback of $\mathbf{D}_{X'}$ on Y for any birational model Y over X' , we say that $\mathbf{D}$ *descends* to X' and $\mathbf{D}$ is the *closure* of $\mathbf{D}_{X'}$, and write $\mathbf{D} = \overline{\mathbf{D}_{X'}}$.

Let $X \rightarrow U$ be a projective morphism and assume that $\mathbf{D}$ is a *b-divisor* on X such that $\mathbf{D}$ descends to some birational model Y over X . If $\mathbf{D}_Y$ is nef/ U (resp. base-point-free/ U , semi-ample/ U), then we say that $\mathbf{D}$ is nef/ U (resp. base-point-free/ U , semi-ample/ U). If $\mathbf{D}_Y$ is a Cartier divisor, then we say that $\mathbf{D}$ is *b-Cartier*. If $\mathbf{D}_Y$ is a $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor, then we say that $\mathbf{D}$ is *$\mathbb{Q}$ -b-Cartier*. If $\mathbf{D}$ can be written as an $\mathbb{R}_{\geq 0}$ -linear combination of nef/ U *b-Cartier* *b-divisors*, then we say that $\mathbf{D}$ is *NQC/ U* .

Let $X \rightarrow U$ be a projective morphism and assume that $\mathbf{D}$ and $\mathbf{D}'$ are two *b-divisors* over X . We write $\mathbf{D} \sim_{\mathbb{R}, U} \mathbf{D}'$ (resp. $\mathbf{D} \sim_{\mathbb{Q}, U} \mathbf{D}'$, $\mathbf{D} \equiv_{\mathbb{Q}, U} \mathbf{D}'$) if for any birational model Y of X , $\mathbf{D}_Y \sim_{\mathbb{R}, U} \mathbf{D}'_Y$ (resp. $\mathbf{D}_Y \sim_{\mathbb{Q}, U} \mathbf{D}'_Y$, $\mathbf{D}_Y \equiv_{\mathbb{Q}, U} \mathbf{D}'_Y$).

We let $\mathbf{0}$ be the *b-divisor* $\mathbf{0}$.

Definition 3.4.2. We will use two types of restrictions of *b-divisors* in this paper. Let X be a normal variety and $\mathbf{D}$ a *b-divisor* on X .

- (1) Let V be a non-empty subset of X . We define the *restricted b-divisor* of $\mathbf{D}$ on V , which is denoted by $\mathbf{D}|_V$, in the following way.

For any birational morphism $\pi : W \rightarrow V$, there exists a birational morphism $\pi' : Y \rightarrow X$ such that $W \subset Y$ and $\pi'|_W = \pi$. We let $(\mathbf{D}|_V)_W = (\mathbf{D}_Y)|_W$. It is easy to see that this definition is independent of the choice of Y and defines a *b-divisor*.

- (2) Suppose that $\mathbf{D}$ descends to a birational model of X . Let S be a prime divisor on X and $\nu : S^\nu \rightarrow S$ the normalization of S . The *restricted b-divisor* of $\mathbf{D}$ on S^ν , which is denoted by $\mathbf{D}|_{S^\nu}$, is defined in the following way.

Let $f : Y \rightarrow X$ be a log resolution of (X, S) such that $\mathbf{D}$ descends to Y . Let $S_Y := f_*^{-1}S$. Then there exists an induced birational morphism $f_S : S_Y \rightarrow S^\nu$ such that $\nu \circ f_S = f|_{S_Y}$. We define

$$\mathbf{D}|_{S^\nu} := \overline{\mathbf{D}_Y|_{S_Y}}.$$

It is clear that $\mathbf{D}|_{S^\nu}$ is well-defined and is independent of the choice of Y .

Definition 3.4.3 (Generalized foliated quadruples). A *generalized foliated sub-quadruple* (sub-gfq for short) $(X, \mathcal{F}, B, \mathbf{M})/U$ consists of a normal quasi-projective variety X , a foliation $\mathcal{F}$ on

X , an $\mathbb{R}$ -divisor B on X , a projective morphism $X \rightarrow U$, and a nef/ U $\mathbf{b}$ -divisor $\mathbf{M}$ over X , such that $K_{\mathcal{F}} + B + \mathbf{M}_X$ is $\mathbb{R}$ -Cartier. If $\mathbf{M}$ is NQC/U , then we say that $(X, \mathcal{F}, B, \mathbf{M})/U$ is *NQC*. If $B \geq 0$, then we say that $(X, \mathcal{F}, B, \mathbf{M})/U$ is a *generalized foliated quadruple* (gfq for short). If $U = \{pt\}$, we usually drop U and say that $(X, \mathcal{F}, B, \mathbf{M})$ is *projective*.

Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a (sub-)gfq. If $\mathbf{M} = \mathbf{0}$, then we may denote $(X, \mathcal{F}, B, \mathbf{M})/U$ by $(X, \mathcal{F}, B)/U$ or $(X, \mathcal{F}, B)$, and say that $(X, \mathcal{F}, B)$ is a *foliated (sub-)triple* (*f-(sub-)triple* for short). If $\mathcal{F} = T_X$, then we may denote $(X, \mathcal{F}, B, \mathbf{M})/U$ by $(X, B, \mathbf{M})/U$, and say that $(X, B, \mathbf{M})/U$ is a *generalized (sub-)pair* (*g-(sub-)pair* for short). If $\mathbf{M} = \mathbf{0}$ and $\mathcal{F} = T_X$, then we may denote $(X, \mathcal{F}, B, \mathbf{M})/U$ by $(X, B)/U$ or (X, B) , and say that (X, B) is a *(sub-)pair*.

A (sub-)gfq (resp. f-(sub-)triple, f-(sub-)pair, g-(sub-)pair, (sub-)pair) $(X, \mathcal{F}, B, \mathbf{M})/U$ (resp. $(X, \mathcal{F}, B)/U, (X, B, \mathbf{M})/U, (X, B)/U$) is called a $\mathbb{Q}$ -(sub-)gfq (resp. $\mathbb{Q}$ -f-(sub-)triple, $\mathbb{Q}$ -g-(sub-)pair, $\mathbb{Q}$ -(sub-)pair) if B is a $\mathbb{Q}$ -divisor and $\mathbf{M}$ is a $\mathbb{Q}$ - $\mathbf{b}$ -divisor.

It is worth mentioning that our definition of generalized foliated quadruples slightly differs from [LLM23, Definition 1.2], as the latter requires $\mathbf{M}$ to be NQC/U (see Definition 3.4.1), while we only require it to be nef/ U . This will be crucial for us to use this structure to prove Theorem B.

Notation 3.4.4. In the previous definition, if U is not important, we may also drop U . This usually happens when we emphasize the structures of $(X, \mathcal{F}, B, \mathbf{M})$ which are independent of the choice of U , such as the singularities of $(X, \mathcal{F}, B, \mathbf{M})$. In addition, if $B = 0$, then we may drop B .

Definition 3.4.5 (Singularities of gfqs). Let $(X, \mathcal{F}, B, \mathbf{M})$ be a (sub-)gfq. For any prime divisor E over X , let $f : Y \rightarrow X$ be a birational morphism such that E is on Y , and suppose that

$$K_{\mathcal{F}_Y} + B_Y + \mathbf{M}_Y := f^*(K_{\mathcal{F}} + B + \mathbf{M}_X)$$

where $\mathcal{F}_Y := f^{-1}\mathcal{F}$. We define $a(E, \mathcal{F}, B, \mathbf{M}) := -\text{mult}_E B_Y$ to be the *discrepancy* of E with respect to $(X, \mathcal{F}, B, \mathbf{M})$. It is clear that $a(E, \mathcal{F}, B, \mathbf{M})$ is independent of the choice of Y . If $\mathbf{M} = \mathbf{0}$, then we let $a(E, \mathcal{F}, B) := a(E, \mathcal{F}, B, \mathbf{M})$. If $\mathcal{F} = T_X$, then we let $a(E, X, B, \mathbf{M}) := a(E, \mathcal{F}, B, \mathbf{M})$. If $\mathbf{M} = \mathbf{0}$ and $\mathcal{F} = T_X$, then we let $a(E, X, B) := a(E, \mathcal{F}, B, \mathbf{M})$.

We say that $(X, \mathcal{F}, B, \mathbf{M})$ is *(sub-)lc* (resp. *(sub-)klt*) if $a(E, \mathcal{F}, B, \mathbf{M}) \geq -\epsilon_{\mathcal{F}}(E)$ (resp. $> -\epsilon_{\mathcal{F}}(E)$) for any prime divisor E over X . We say that $(X, \mathcal{F}, B, \mathbf{M})$ is *(sub-)canonical* (resp. *(sub-)terminal*) if $a(E, \mathcal{F}, B, \mathbf{M}) \geq 0$ (resp. > 0) for any prime divisor E that is exceptional over X . An *lc place* of $(X, \mathcal{F}, B, \mathbf{M})$ is a prime divisor E over X such that $a(E, \mathcal{F}, B, \mathbf{M}) = -\epsilon_{\mathcal{F}}(E)$. An *lc center* of $(X, \mathcal{F}, B, \mathbf{M})$ is a subvariety W of X , such that either W is the center of an lc place of $(X, \mathcal{F}, B, \mathbf{M})$ on X , or $W = X$. A *non-trivial lc center* of $(X, \mathcal{F}, B, \mathbf{M})$ is an lc center of $(X, \mathcal{F}, B, \mathbf{M})$ that is not X . A *non-lc place* of $(X, \mathcal{F}, B, \mathbf{M})$ is a prime divisor E over X such that $a(E, \mathcal{F}, B, \mathbf{M}) < -\epsilon_{\mathcal{F}}(E)$. A *non-lc center* of $(X, \mathcal{F}, B, \mathbf{M})$ is the center of a non-lc place of $(X, \mathcal{F}, B, \mathbf{M})$ on X . The union of all non-lc centers of $(X, \mathcal{F}, B, \mathbf{M})$ is called the *non-lc locus* of $(X, \mathcal{F}, B, \mathbf{M})$ and is denoted by $\text{Nlc}(X, \mathcal{F}, B, \mathbf{M})$. The union of all non-lc centers and non-trivial lc centers of $(X, \mathcal{F}, B, \mathbf{M})$ is called the *non-klt locus* of $(X, \mathcal{F}, B, \mathbf{M})$ and is denoted by $\text{Nklt}(X, \mathcal{F}, B, \mathbf{M})$.

Definition 3.4.6. Let $(X, \mathcal{F}, B, \mathbf{M})$ be a sub-gfq, $D \geq 0$ an $\mathbb{R}$ -divisor on X and $\mathbf{N}$ a nef/ X $\mathbf{b}$ -divisor, such that $D + \mathbf{N}_X$ is $\mathbb{R}$ -Cartier. The *lc threshold* (*lct* for short) of $(D, \mathbf{N})$ with respect to $(X, \mathcal{F}, B, \mathbf{M})$ is defined as

$$\text{lct}(X, \mathcal{F}, B, \mathbf{M}; D, \mathbf{N}) := \sup\{+\infty, t \mid (X, \mathcal{F}, B + tD, \mathbf{M} + t\mathbf{N}) \text{ is sub-lc}\}.$$

If $\mathbf{N} = 0$, then we may drop $\mathbf{N}$ and denote $\text{lct}(X, \mathcal{F}, B, \mathbf{M}; D, \mathbf{N})$ by $\text{lct}(X, \mathcal{F}, B, \mathbf{M}; D)$.

Definition 3.4.7 (Models, I). Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be an lc gfq, $\phi : X \dashrightarrow X'$ a birational map over U , $E := \text{Exc}(\phi^{-1})$ the reduced ϕ^{-1} -exceptional divisor, $\mathcal{F}' := \phi_*\mathcal{F}$, and $B' := \phi_*B + E^{\mathcal{F}'}$.

- (1) $(X', \mathcal{F}', B', \mathbf{M})/U$ is called a *log birational model* of $(X, \mathcal{F}, B, \mathbf{M})/U$.

- (2) $(X', \mathcal{F}', B', \mathbf{M})/U$ is called a *weak lc model* of $(X, \mathcal{F}, B, \mathbf{M})/U$ if
- (a) $(X', \mathcal{F}', B', \mathbf{M})/U$ is a log birational model of $(X, \mathcal{F}, B, \mathbf{M})/U$,
 - (b) $K_{\mathcal{F}'} + B' + \mathbf{M}_{X'}$ is nef/ U , and
 - (c) for any prime divisor D on X which is exceptional over X' ,
$$a(D, \mathcal{F}, B, \mathbf{M}) \leq a(D, \mathcal{F}', B', \mathbf{M}).$$
- (3) $(X', \mathcal{F}', B', \mathbf{M})/U$ is called a *semi-good minimal model* of $(X, \mathcal{F}, B, \mathbf{M})/U$ if
- (a) $(X', \mathcal{F}', B', \mathbf{M})/U$ is a weak lc model of $(X, \mathcal{F}, B, \mathbf{M})/U$, and
 - (b) $K_{\mathcal{F}'} + B' + \mathbf{M}_{X'}$ is semi-ample/ U .
- (4) Suppose that there exists a contraction/ U : $X' \rightarrow Z$. $(X', \mathcal{F}', B', \mathbf{M}) \rightarrow Z$ is called a *Mori fiber space* of $(X, \mathcal{F}, B, \mathbf{M})/U$ if
- (a) $(X', \mathcal{F}', B', \mathbf{M})/U$ is a log birational model of $(X, \mathcal{F}, B, \mathbf{M})/U$,
 - (b) X' is $\mathbb{Q}$ -factorial,
 - (c) $X' \rightarrow Z$ is a $(K_{\mathcal{F}'} + B' + \mathbf{M}_{X'})$ -Mori fiber space/ U ,
 - (d) for any prime divisor D on X which is exceptional over X' ,
$$a(D, \mathcal{F}, B, \mathbf{M}) < a(D, \mathcal{F}', B', \mathbf{M}).$$

We shall not define “good minimal models” until Definition 9.1.1.

Notation 3.4.8. Let $(X_0, \mathcal{F}_0, B_0, \mathbf{M})/U$ be a gfg. When we say the following

$$(X_0, \mathcal{F}_0, B_0, \mathbf{M}) - \xrightarrow{f_0} (X_1, \mathcal{F}_1, B_1, \mathbf{M}) - \xrightarrow{f_1} \dots - \xrightarrow{f_n} (X_n, \mathcal{F}_n, B_n, \mathbf{M}) - \xrightarrow{f_n} \dots$$

is a (possibly infinite) sequence of steps of a $(K_{\mathcal{F}_0} + B_0 + \mathbf{M}_{X_0})$ -MMP/ U , we mean the following: for any i , $f_i : X_i \dashrightarrow X_{i+1}$ is a step of a $(K_{\mathcal{F}_i} + B_i + \mathbf{M}_{X_i})$ -MMP/ U that is not a Mori fiber space, $\mathcal{F}_{i+1} := (f_i)_* \mathcal{F}_i$, and $B_{i+1} := (f_i)_* B_i$.

Construction 3.4.9 (MMP with scaling). Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be an lc gfg. Let $D \geq 0$ be an $\mathbb{R}$ -divisor on X and $\mathbf{N}$ a nef/ U $\mathbf{b}$ -divisor on X such that $D + \mathbf{N}_X$ is $\mathbb{R}$ -Cartier and $K_{\mathcal{F}} + B + \mathbf{M}_X + t(D + \mathbf{N}_X)$ is nef/ U for some positive real number t . A step of a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U with scaling of $(D, \mathbf{N})$ is defined in the following way. Let

$$\lambda := \inf\{s \geq 0 \mid K_{\mathcal{F}} + B + sD + \mathbf{M}_X + s\mathbf{N}_X \text{ is nef}/U\}.$$

Assume that the following conditions hold:

- There exists an extremal ray R in $\overline{NE}(X/U)$ such that $(K_{\mathcal{F}} + B + \lambda D + \mathbf{M}_X + \lambda \mathbf{N}_X) \cdot C = 0$ and $(D + \mathbf{N}_X) \cdot C > 0$. In particular, R is a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -negative extremal ray.
- The contraction associated to R exists, and if it is a small contraction, then the corresponding $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -flip exists.

Then for any such R , we call the divisorial contraction or the Mori fiber space associated to R , or the $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -flip associated to R , as a step of a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U with scaling of $(D, \mathbf{N})$.

A sequence of steps of a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U with scaling of $(D, \mathbf{N})$ is a sequence of steps of a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U

$$(X_0, \mathcal{F}_0, B_0, \mathbf{M}) - \xrightarrow{f_0} (X_1, \mathcal{F}_1, B_1, \mathbf{M}) - \xrightarrow{f_1} \dots - \xrightarrow{f_n} (X_n, \mathcal{F}_n, B_n, \mathbf{M}) - \xrightarrow{f_n} \dots$$

where $(X_0, \mathcal{F}_0, B_0, \mathbf{M}) = (X, \mathcal{F}, B, \mathbf{M})$, each f_i is a step of a $(K_{\mathcal{F}_i} + B_i + \mathbf{M}_{X_i})$ -MMP/ U with scaling of $(D_i, \mathbf{N})$, where D_i is the image of D on X_i .

$$\lambda_i := \inf\{t, s \geq 0 \mid K_{\mathcal{F}_i} + B_i + sD_i + \mathbf{M}_{X_i} + s\mathbf{N}_{X_i} \text{ is nef}/U\}$$

are called the *scaling numbers* (of this MMP with scaling of $(D, \mathbf{N})$). Note that $D_i + \mathbf{N}_{X_i}$ is $\mathbb{R}$ -Cartier for any i by our construction, so λ_i is well-defined. If this MMP does not terminate, then we call $\lim_{i \rightarrow +\infty} \lambda_i$ the *limit of the scaling numbers*. By definition, we have $\lambda_i \geq \lambda_{i+1}$ for any i (of this MMP with scaling of $(D, \mathbf{N})$), so the limit of the scaling numbers is well-defined.

If $\mathbf{N} = \mathbf{0}$, then a (sequence of) step(s) of a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U with scaling of $(D, \mathbf{N})$ is called a (sequence of) step(s) of a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U with scaling of D .

We remark that Construction 3.4.9 does not require the condition that $(X, \mathcal{F}, B + D, \mathbf{M} + \mathbf{N})$ is lc.

Definition 3.4.10. Let $(X, \mathcal{F}, B, \mathbf{M})$ and $(X', \mathcal{F}', B', \mathbf{M}')$ be two sub-gfqs. We say that $(X, \mathcal{F}, B, \mathbf{M})$ and $(X', \mathcal{F}', B', \mathbf{M}')$ are *crepant* to each other if $\mathbf{M} = \mathbf{M}'$, and there exist two birational morphisms $p : W \rightarrow X$ and $q : W \rightarrow X'$ and a foliation $\mathcal{F}_W$ on W , such that $\mathcal{F}_W = p^{-1}\mathcal{F} = q^{-1}\mathcal{F}'$, $\mathbf{M} = \mathbf{M}'$, and

$$p^*(K_{\mathcal{F}} + B + \mathbf{M}_X) = q^*(K_{\mathcal{F}'} + B' + \mathbf{M}'_{X'}).$$

4. BASIC PROPERTIES OF GENERALIZED PAIRS

In this section, we present several results concerning the structure of generalized pairs. Although most of these results, or their analogous forms, have already been established in existing literature, it is somewhat surprising to note that many fundamental results for non-NQC generalized pairs remain unaddressed, despite the significant progress in the field of generalized pairs in recent years. Additionally, there are relatively few references available on this subject. For clarity, for the reader's convenience, and to provide a solid reference for future work, we will present detailed proofs for all the results in this section.

4.1. Dlt modification. First we recall the definition of dlt for generalized pairs.

Definition 4.1.1 (Dlt, [HL22, Definition 2.3]). Let $(X, B, \mathbf{M})/U$ be an lc g-pair. We say that $(X, B, \mathbf{M})$ is *dlt* if there exists an open subset $V \subset X$ satisfying the following.

- (1) $(V, B|_V)$ is log smooth. In particular, $B|_V$ is an snc Weil $\mathbb{Q}$ -divisor.
- (2) V contains the generic point of any lc center of $(X, B, \mathbf{M})$.
- (3) The generic point of any lc center of $(X, B, \mathbf{M})$ is the generic point of an lc center of $(V, B|_V)$.

If $(X, B, \mathbf{M})$ is dlt and $[B]$ is normal, then we say that $(X, B, \mathbf{M})$ is plt.

The following lemma indicates that the definition of dlt in [HL22, Definition 2.3] is the same as the definition of dlt in [Bir20, FS23].

Lemma 4.1.2. *Let $(X, B, \mathbf{M})/U$ be an lc g-pair. Then the following two conditions are equivalent:*

- (1) $(X, B, \mathbf{M})$ is dlt.
- (2) *For any lc center of $(X, B, \mathbf{M})$ with generic point η , over a neighborhood of η , $(V, B|_V)$ is log smooth and $\mathbf{M}$ descends to X .*

Proof. Since dlt, the property in (2), and log smooth are local properties, we may work over a neighborhood of a generic point η of an lc center of $(X, B, \mathbf{M})$. (2) $\Rightarrow$ (1) immediately becomes obvious, so we only need to prove (1) $\Rightarrow$ (2).

By Definition 4.1.1, there exists a neighborhood V of η such that $(V, B|_V)$ is log smooth and η is an lc center of $(V, B|_V)$. Since $(V, B|_V)$ is log smooth, $\mathbf{M}_X|_V$ is $\mathbb{R}$ -Cartier. We let $\mathbf{M}^V := \mathbf{M}|_V$ be the restricted $\mathbf{b}$ -divisor of $\mathbf{M}$ on V , then $\mathbf{M}^V$ is nef/ X and $\mathbf{M}_V^V = \mathbf{M}_X|_V$. Suppose that $h : V' \rightarrow V$ is a resolution of V such that $\mathbf{M}^V$ descends to V' and there exists a prime divisor E on V' such that $\text{center}_V E = \bar{\eta}$ and E is an lc place of $(V, B|_V)$. By the negativity lemma,

$$\mathbf{M}_{V'}^V = h^*\mathbf{M}_V^V - F$$

for some $F \geq 0$, such that either $F = 0$ over $\bar{\eta}$ or $\text{Supp } F = \text{Supp } h^{-1}(\bar{\eta})$. Since $(X, B, \mathbf{M})$ is lc, $(V, B|_V, \mathbf{M}^V)$ is lc. Thus $F = 0$ over $\bar{\eta}$. Possibly shrinking V , we may assume that $\mathbf{M}$ descends to V . The lemma follows. $\square$

Lemma 4.1.2 implies the following result:

Definition-Lemma 4.1.3 (Dlt modification, [FS23, Theorem 2.9]). Let $(X, B, \mathbf{M})/U$ be a g-pair. Then there exists a birational morphism $f : Y \rightarrow X$ satisfying the following. Let $E_1, \dots, E_n$ be the prime f -exceptional divisors and $B_Y := f_*^{-1}(B \wedge \text{Supp } B) + \sum_{i=1}^n E_i$, then:

- (1) $(Y, B_Y, \mathbf{M})$ is $\mathbb{Q}$ -factorial dlt.
- (2) $a(E_i, X, B, \mathbf{M}) \leq 0$ for any i .

In particular, if $(X, B, \mathbf{M})$ is lc, then $a(E_i, X, B, \mathbf{M}) = 0$ for any i , and

$$K_Y + B_Y + \mathbf{M}_Y = f^*(K_X + B + \mathbf{M}_X).$$

For any such f , we call f a *dlt modification* of $(X, B, \mathbf{M})$, and say that $(Y, B_Y, \mathbf{M})$ is a *dlt model* of $(X, B, \mathbf{M})$.

We conjecture that dlt has another equivalent definition:

Conjecture 4.1.4. *Let $(X, B, \mathbf{M})/U$ be an lc g-pair. Then $(X, B, \mathbf{M})$ is dlt if and only if there exists a log resolution $f : Y \rightarrow X$ of $(X, \text{Supp } B)$ and an open subset $V \subset X$, such that $\mathbf{M}$ descends to Y , f is an isomorphism over V , and V contains the generic point of any lc center of $(X, B, \mathbf{M})$.*

When $(X, B, \mathbf{M})/U$ is NQC, Conjecture 4.1.4 was proven in [Has22, Theorem 6.1].

4.2. Perturbation and MMP.

Lemma 4.2.1. *Let $(X, B + A, \mathbf{M})/U$ be a $\mathbb{Q}$ -factorial lc g-pair such that X is klt, $A \geq 0$ is ample/ U , and $B \geq 0$. Then any $(K_X + B + A + \mathbf{M}_X)$ -MMP/ U with scaling of an ample/ U $\mathbb{R}$ -divisor terminates with either a semi-good minimal model of $(X, B + A, \mathbf{M})/U$ or a Mori fiber space of $(X, B + A, \mathbf{M})/U$.*

Proof. By [HL22, Lemma 3.4], there exists $0 \leq \Delta \sim_{\mathbb{R}, U} B + A + \mathbf{M}_X$ such that (X, Δ) is klt. By [BCHM10, Corollary 1.4.2], any $(K_X + \Delta)$ -MMP/ U with scaling of an ample/ U $\mathbb{R}$ -divisor terminates with either a Mori fiber space of $(X, \Delta)/U$ or a semi-good minimal model of $(X, \Delta)/U$. The lemma follows. $\square$

Lemma 4.2.2. *Let $(X, B, \mathbf{M})/U$ be a $\mathbb{Q}$ -factorial lc g-pair such that X is klt, and $A \geq 0$ an ample/ U $\mathbb{R}$ -divisor on X . Then we may run a $(K_X + B + \mathbf{M}_X)$ -MMP/ U with scaling of A . Moreover, let*

$$(X, B, \mathbf{M}) := (X_1, B_1, \mathbf{M}) \dashrightarrow (X_2, B_2, \mathbf{M}) \dashrightarrow \dots \dashrightarrow (X_i, B_i, \mathbf{M}) \dashrightarrow \dots$$

be any $(K_X + B + \mathbf{M}_X)$ -MMP/ U with scaling of A , and let λ_i be the i -th scaling number of this MMP for each i , i.e.

$$\lambda_i := \inf\{t \mid t \geq 0, K_{X_i} + B_i + tA_i + \mathbf{M}_{X_i} \text{ is nef}/U\},$$

where A_i is the strict transform of A on X_i for each i . Then one of the followings holds:

- (1) *This MMP terminates after finitely many steps.*
- (2) $\lim_{i \rightarrow +\infty} \lambda_i = 0$.

Proof. Possibly rescaling A we may assume that $K_X + B + A + \mathbf{M}_X$ is nef/ U . We first prove that we may run this MMP by induction on i . Let $\lambda_0 := 1$ and suppose that there is already a sequence of steps of a $(K_X + B + \mathbf{M}_X)$ -MMP/ U with scaling of A

$$(X, B, \mathbf{M}) := (X_1, B_1, \mathbf{M}) \dashrightarrow (X_2, B_2, \mathbf{M}) \dashrightarrow \dots \dashrightarrow (X_k, B_k, \mathbf{M})$$

for some $k \geq 1$, such that $\lambda_i \geq \lambda_{i+1}$ for any $i \leq k-2$. If $K_{X_k} + B_k + \mathbf{M}_{X_k}$ is nef/ U , then we are done, so we may assume that $K_{X_k} + B_k + \mathbf{M}_{X_k}$ is not nef/ U . Since nef/ U is a closed condition, $\lambda_k > 0$. By construction, $\lambda_{k-1} \geq \lambda_k$.

By [HL22, Lemma 3.4], there exists a klt pair (X, Δ) such that

$$K_X + \Delta \sim_{\mathbb{R}, U} K_X + B + \mathbf{M}_X + \frac{\lambda_k}{2} A.$$

Possibly replacing A , we may assume that $(X, \Delta + (1 - \frac{\lambda_k}{2})A)$ is lc. Then we have an induced sequence of steps of a $(K_X + \Delta)$ -MMP/ U with scaling of $(1 - \frac{\lambda_k}{2})A$

$$(X, \Delta) := (X_1, \Delta_1) \dashrightarrow (X_2, \Delta_2) \dashrightarrow \cdots \dashrightarrow (X_k, \Delta_k),$$

such that $K_{X_k} + \Delta_k$ is not nef. Let $(X_k, \Delta_k) \dashrightarrow (X_{k+1}, \Delta_{k+1})$ be the next step of the $(K_X + \Delta)$ -MMP/ U with scaling of $(1 - \frac{\lambda_k}{2})A$. Then the induced birational map $X_k \dashrightarrow X_{k+1}$ is a step of a $(K_X + B + \mathbf{M}_X)$ -MMP/ U with scaling of A .

We left to prove that if this MMP does not terminate, then $\lim_{i \rightarrow +\infty} \lambda_i = 0$. Suppose that $\lambda := \lim_{i \rightarrow +\infty} \lambda_i > 0$. Then

$$\begin{aligned} \left(X, B + \frac{\lambda}{2} A, \mathbf{M} \right) &:= \left(X_1, B_1 + \frac{\lambda}{2} A_1, \mathbf{M} \right) \dashrightarrow \left(X_2, B_2 + \frac{\lambda}{2} A_2, \mathbf{M} \right) \dashrightarrow \\ &\cdots \dashrightarrow \left(X_i, B_i + \frac{\lambda}{2} A_i, \mathbf{M} \right) \dashrightarrow \cdots \end{aligned}$$

is an infinite sequence of steps of a $(K_X + B + \frac{\lambda}{2} A + \mathbf{M}_X)$ -MMP/ U , which contradicts Lemma 4.2.1. $\square$

The following result seems to be missed in known literature.

Proposition 4.2.3. *Let $(X, B + A, \mathbf{M})/U$ be a $\mathbb{Q}$ -factorial NQC lc g -pair such that $A \geq 0$ is ample/ U and $B \geq 0$. Then any $(K_X + B + A + \mathbf{M}_X)$ -MMP/ U with scaling of an ample/ U $\mathbb{R}$ -divisor terminates with either a semi-good minimal model of $(X, B + A, \mathbf{M})/U$ or a Mori fiber space of $(X, B + A, \mathbf{M})/U$.*

Proof. By [LT22, Lemma A.5], possibly replacing A with a general element in $|A/U|_{\mathbb{R}}$, there exists an lc pair $(X, \Delta + \frac{1}{2}A)$ such that $0 \leq \Delta \sim_{\mathbb{R}} B + \frac{1}{2}A + \mathbf{M}_X$. By [HH20, Theorem 1.5] and [Bir12, Theorem 1.9], any $(K_X + \Delta + \frac{1}{2}A)$ -MMP/ U with scaling of an ample/ U $\mathbb{R}$ -divisor terminates. Thus any $(K_X + B + A + \mathbf{M}_X)$ -MMP/ U with scaling of an ample/ U $\mathbb{R}$ -divisor terminates. The rest part of the proposition follows from [LX23a, Theorem 1.3] and [HL21a, Lemma 3.9(1)]. $\square$

Lemma 4.2.4 (cf. [Gon11, Lemma 2.6(3)]). *Let X be a normal projective variety and D a movable $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor on X such that $\kappa_{\sigma}(D) = 0$. Then $D \equiv 0$.*

Proof. We let $f : Y \rightarrow X$ be a resolution of X . Let $P_Y := P(Y, f^*D)$ and $N_Y := N(Y, f^*D)$ be the positive and negative part of the Nakayama-Zariski decomposition of f^*D respectively, and let $P := P(X, D)$ and $N := N(X, D)$ be the positive and negative part of the Nakayama-Zariski decomposition of D respectively. Since D is movable, by [LX23a, Lemma 3.7(3)], $N = 0$. By [LX23a, Lemma 3.4(3)], $f_*N_Y = N$, so N_Y is exceptional/ X . Since $\kappa_{\sigma}(f^*D) = \kappa_{\sigma}(D) = 0$, by [Nak04, V 2.7 Proposition(8)], $P_Y \equiv 0$. Thus $D = f_*D_Y = f_*(P_Y + N_Y) \equiv 0$. $\square$

Proposition 4.2.5. *Let $(X, B, \mathbf{M})$ be a projective $\mathbb{Q}$ -factorial lc g -pair such that $\kappa_{\sigma}(K_X + B + \mathbf{M}_X) = 0$ and X is klt. Let A be an ample $\mathbb{R}$ -divisor. Then we may run a $(K_X + B + \mathbf{M}_X)$ -MMP with scaling of A , and any such MMP terminates with a model $(X', B', \mathbf{M})$ of $(X, B, \mathbf{M})$ such that $K_{X'} + B' + \mathbf{M}_{X'} \equiv 0$. Moreover, if $\kappa_i(K_X + B + \mathbf{M}_X) = 0$, then $K_{X'} + B' + \mathbf{M}_{X'} \sim_{\mathbb{R}} 0$.*

Proof. By Lemma 4.2.2, we may run a $(K_X + B + \mathbf{M}_X)$ -MMP with scaling of A . Let

$$(X, B, \mathbf{M}) := (X_1, B_1, \mathbf{M}) \dashrightarrow (X_2, B_2, \mathbf{M}) \dashrightarrow \cdots \dashrightarrow (X_i, B_i, \mathbf{M}) \dashrightarrow \cdots$$

be any such MMP with scaling numbers $\lambda_i \geq \lambda_{i+1}$. If this MMP does not terminate, then $\lim_{i \rightarrow +\infty} \lambda_i = 0$ by Lemma 4.2.2. There exists a positive integer m such that $X_i \dashrightarrow X_{i+1}$ is a

flip for any $i \geq m$. We may denote by $\phi_i : X_m \dashrightarrow X_i$ the induced birational contraction and A_i the strict transform of A on X_i for any $i > m$. Since $K_{X_i} + B_i + \lambda_i A_i + \mathbf{M}_{X_i}$ is nef for each i ,

$$K_{X_m} + B_m + \mathbf{M}_{X_m} = \lim_{i \rightarrow +\infty} (\phi_i^{-1})_*(K_{X_i} + B_i + \lambda_i A_i + \mathbf{M}_{X_i})$$

is movable. Moreover, since $\kappa_\sigma(K_X + B + \mathbf{M}_X) = 0$, $\kappa_\sigma(K_{X_m} + B_m + \mathbf{M}_{X_m}) = 0$. By Lemma 4.2.4, $K_{X_m} + B_m + \mathbf{M}_{X_m} \equiv 0$, a contradiction. Thus this MMP terminates with a model $(X', B', \mathbf{M})$ such that $K_{X'} + B' + \mathbf{M}_{X'}$ is nef and $\kappa_\sigma(K_{X'} + B' + \mathbf{M}_{X'}) = 0$. By Lemma 4.2.4 again, $K_{X'} + B' + \mathbf{M}_{X'} \equiv 0$. Moreover, if $\kappa_\ell(K_X + B + \mathbf{M}_X) = 0$, then $\kappa_\ell(K_{X'} + B' + \mathbf{M}_{X'}) = 0$, hence $K_{X'} + B' + \mathbf{M}_{X'} \sim_{\mathbb{R}} 0$. $\square$

4.3. Lc centers of generalized pairs. We will discuss the structure of lc centers of lc g-pairs in this section. For NQC generalized pairs, the structure of their lc centers is well-studied in [LX23a] based on the connected principle established in [Bir20, FS23] and the canonical bundle formula [Fil20, HL21b, J LX22, FS23] but little was known for the non-NQC case.

Definition 4.3.1 (Adjunction for generalized pairs to divisors, cf. [BZ16, Definition 4.7]). Let $(X, B, \mathbf{M})/U$ be a g-(sub-)pair and S a component of B^1 . Let S^ν be the normalization of S . The g-(sub-)pair $(S^\nu, B_S, \mathbf{M}^S)/U$ induced by the adjunction

$$K_{S^\nu} + B_S + \mathbf{M}_S^S := (K_X + B + \mathbf{M}_X)|_{S^\nu}$$

is given in the following way. Let $f : Y \rightarrow X$ be a log resolution of $(X, \text{Supp } B)$ such that $\mathbf{M}$ descends to Y , S_Y the strict transform of S on Y , and

$$K_Y + B_Y + \mathbf{M}_Y := f^*(K_X + B + \mathbf{M}_X).$$

We define $\mathbf{M}^S := \mathbf{M}|_{S^\nu}$ and $B_{S_Y} := (B_Y - S_Y)|_{S_Y}$. We let $f|_{S_Y} : S_Y \rightarrow S^\nu$ be the induced birational morphism and define $B_S := (f|_{S_Y})_* B_{S_Y}$.

Lemma 4.3.2 (cf. [LX23b, Lemma 3.18(2)]). *Let $(X, B, \mathbf{M})/U$ be a dlt g-pair, S a component of $[B]$, and $(S, B_S, \mathbf{M}^S)/U$ the g-pair induced by the adjunction*

$$K_S + B_S + \mathbf{M}_S^S := (K_X + B + \mathbf{M}_X)|_S.$$

Then $(S, B_S, \mathbf{M}^S)$ is dlt. Moreover:

- (1) *Any lc center of $(S, B_S, \mathbf{M}^S)$ is an lc center of $(X, B, \mathbf{M})$.*
- (2) *Any lc center of $(X, B, \mathbf{M})$ that is contained in S is an lc center of $(S, B_S, \mathbf{M}^S)$.*

Proof. By [HL22, Lemma 2.9], $(S, B_S, \mathbf{M}^S)$ is dlt.

(1) Let $f : \tilde{X} \rightarrow X$ be a log resolution of $(X, \text{Supp } B)$ such that $\mathbf{M}$ descends to $\tilde{X}$. Let $K_{\tilde{X}} + \tilde{B} + \mathbf{M}_{\tilde{X}} := f^*(K_X + B + \mathbf{M}_X)$ and let $\tilde{S}$ be the strict transform of S on $\tilde{X}$, then $f|_{\tilde{S}}$ is a log resolution of $(S, \text{Supp } B_S)$ such that $\mathbf{M}^S$ descends to $\tilde{S}$. We have

$$f|_{\tilde{S}}^*(K_S + B_S + \mathbf{M}_S^S) = K_{\tilde{S}} + B_{\tilde{S}} + \mathbf{M}_{\tilde{S}}^S := (K_{\tilde{X}} + \tilde{B} + \mathbf{M}_{\tilde{X}})|_{\tilde{S}}.$$

Let W_S be an lc center of $(S, B_S, \mathbf{M}^S)$. Then W_S is the image of an lc center $W_{\tilde{S}}$ of $(\tilde{S}, B_{\tilde{S}}, \mathbf{M}_{\tilde{S}}^S)$ in S . Since $(\tilde{X}, \tilde{B})$ is log smooth and $\mathbf{M}$ descends to $\tilde{X}$, $W_{\tilde{S}}$ is also an lc center of $(\tilde{X}, \tilde{B}, \mathbf{M})$ which is contained in $\tilde{S}$, so $W := f(W_{\tilde{S}})$ is an lc center of $(X, B, \mathbf{M})$ which is contained in S . It is clear that $W_S = W$ under the natural inclusion $S \rightarrow X$. This implies (1).

(2) Let W be an lc center of $(X, B, \mathbf{M})$ that is contained in S . Since $(X, B, \mathbf{M})$ is dlt, by Lemma 4.1.2, possibly shrinking X to a neighborhood of the generic point of W , we may assume that (X, B) is log smooth and $\mathbf{M}$ descends to X . Thus W is an lc center of (X, B) , $K_S + B_S = (K_X + B)|_S$, and $\mathbf{M}^S$ descends to S . Since (X, B) is log smooth, W is an lc center of (S, B_S) , hence an lc center of $(S, B_S, \mathbf{M}^S)$. This implies (2). $\square$

Definition-Lemma 4.3.3. Let $(X, B, \mathbf{M})/U$ be a dlt g-pair and V an lc center of $(X, B, \mathbf{M})$ such that $\dim V \geq 1$. Then we may construct a dlt g-pair $(V, B_V, \mathbf{M}^V)/U$ on V inductively the following way. If $V = X$ then we let $(V, B_V, \mathbf{M}^V) := (X, B, \mathbf{M})$. Otherwise, let S be a codimension 1 lc center of $(X, B, \mathbf{M})$ such that $V \subset S$. By [HL22, Lemma 2.9], there exists an dlt g-pair $(S, B_S, \mathbf{M}^S)$ induced by adjunction

$$K_S + B_S + \mathbf{M}_S^S = (K_X + B + \mathbf{M}_X)|_S.$$

By Lemma 4.3.2, V is an lc center of $(S, B_S, \mathbf{M}^S)$. By repeating this process and applying induction on dimension, we get a dlt g-pair $(V, B_V, \mathbf{M}^V)/U$ on V . $(V, B_V, \mathbf{M}^V)/U$ is called the dlt g-pair *induced by repeatedly applying adjunction to codimension 1 lc centers*:

$$K_V + B_V + \mathbf{M}_V^V := (K_X + B + \mathbf{M}_X)|_V.$$

Definition 4.3.4. An *lc crepant log structure* is of the form $f : (X, B, \mathbf{M}) \rightarrow Z$, where

- (1) $(X, B, \mathbf{M})/Z$ is an lc g-pair,
- (2) $K_X + B + \mathbf{M}_X \sim_{\mathbb{R}, Z} 0$, and
- (3) f is a contraction. In particular, $f_*\mathcal{O}_X = \mathcal{O}_Z$.

In addition, if

- (4) $(X, B, \mathbf{M})$ is dlt,

then we say that $f : (X, B, \mathbf{M}) \rightarrow Z$ is a *dlt crepant log structure*.

For any irreducible subvariety $W \subset Z$, we say that W is an *lc center* of an lc crepant log structure $f : (X, B, \mathbf{M}) \rightarrow Z$, if there exists an lc center W_X of $(X, B, \mathbf{M})$ such that $W = f(W_X)$. For any (not necessarily closed) point $z \in Z$, we say that $\bar{z}$ is an *lc center* of $f : (X, B, \mathbf{M}) \rightarrow Z$ if $\bar{z}$ is an lc center of $f : (X, B, \mathbf{M}) \rightarrow Z$.

Remark 4.3.5. In Section 11 below we will introduce the concept of lc-trivial fibrations. We will see that an lc crepant log structure is an lc-trivial fibration $f : (X, B, \mathbf{M}) \rightarrow Z$ such that $B \geq 0$ (see Definition 11.3.1 below). We will also see that an lc center of an lc crepant log structure $f : (X, B, \mathbf{M}) \rightarrow Z$ is indeed an lc center of the induced g-pair $(Z, B_Z, \mathbf{M}^Z)$ via the canonical bundle formula (see Theorem 11.4.4 below).

Definition 4.3.6 (Standard $\mathbb{P}^1$ -link, cf. [FS23, Definition 2.21]). Let $X \rightarrow U$ be a projective morphism from a normal quasi-projective variety to a variety. A *standard $\mathbb{P}^1$ -link* $f : (X, B, \mathbf{M}) \rightarrow T$ is an lc g-pair $(X, B, \mathbf{M})/U$ with a projective morphism $f : X \rightarrow T$ over U satisfying the following properties.

- (1) $K_X + B + \mathbf{M}_X \sim_{\mathbb{R}, T} 0$,
- (2) there exists a birational morphism $X' \rightarrow X$ such that $\mathbf{M}_{X'} \sim_{\mathbb{R}, T} 0$,
- (3) $[B] = D_1 + D_2$, where D_1, D_2 are prime divisors and $f|_{D_i} : D_i \rightarrow T$ are isomorphisms,
- (4) $(X, B, \mathbf{M})$ is plt, and
- (5) every reduced fiber of f is isomorphic to $\mathbb{P}^1$.

We call D_1 and D_2 the *horizontal sections* of $(X, B, \mathbf{M})/T$.

Definition 4.3.7 ($\mathbb{P}^1$ -link, cf. [FS23, Definition 2.23]). Let $(X, B, \mathbf{M})/U$ be a dlt g-pair associated with a projective morphism $f : X \rightarrow U$, such that $K_X + B + \mathbf{M}_X \sim_{\mathbb{R}, Z} 0$. Let Z_1, Z_2 be two lc centers of $(X, B, \mathbf{M})$. We say that Z_1 and Z_2 are *directly $\mathbb{P}^1$ -linked* $/U$ if there exists an lc center W of $(X, B, \mathbf{M})$ satisfying the following.

- (1) $Z_i \subset W$ for each i .
- (2) $f(W) = f(Z_1) = f(Z_2)$.
- (3) Let $(W, B_W, \mathbf{M}^W)/U$ be the dlt g-pair induced by repeatedly applying adjunction to codimension 1 lc centers

$$K_W + B_W + \mathbf{M}_W^W := (K_X + B + \mathbf{M}_X)|_W.$$

Then there exists a standard $\mathbb{P}^1$ -link/ U $h : (W', B_{W'}, \mathbf{M}^W) \rightarrow T$ such that $(W', B_{W'}, \mathbf{M}^W)$ is crepant to $(W, B_W, \mathbf{M}^W)$, and $Z_1|_{W'}$ and $Z_2|_{W'}$ are the horizontal sections of $(W', B_{W'}, \mathbf{M}^W)/T$.

We say that Z_1 and Z_2 are $\mathbb{P}^1$ -linked/ U if either $Z_1 = Z_2$, or there exists an integer $n \geq 2$ and lc centers $Z'_1, \dots, Z'_n$ of $(X, B, \mathbf{M})$, such that $Z'_1 = Z_1, Z'_n = Z_2$, and Z'_i and Z'_{i+1} are directly $\mathbb{P}^1$ -linked/ U for any $1 \leq i \leq n-1$.

The following theorem is important when characterizing the structure of lc centers of g-pairs. We emphasize that, in the following theorem, we do not require $(X, B, \mathbf{M})$ to be NQC.

Theorem 4.3.8 ([Bir20, Theorem 3.5]; [FS23, Theorem 1.4] for the $\mathbb{Q}$ -coefficient case). *Let $(X, B, \mathbf{M})/U$ be a dlt g-pair associated with a projective morphism $f : X \rightarrow U$, such that $K_X + B + \mathbf{M}_X \sim_{\mathbb{R}, U} 0$. Let $s \in U$ be a (not necessarily closed) point such that $f^{-1}(s)$ is connected (as a $k(s)$ -scheme). Let*

$$\mathcal{S} := \{V \mid V \text{ is an lc center of } (X, B, \mathbf{M}), s \in f(V)\}$$

and $Z, W \in \mathcal{S}$ be two elements such that Z is minimal in $\mathcal{S}$ with respect to the inclusion. Then there exists $Z_W \in \mathcal{S}$ such that $Z_W \subset W$, and Z and Z_W are $\mathbb{P}^1$ -linked/ U . In particular, any minimal elements in $\mathcal{S}$ with respect to inclusion are $\mathbb{P}^1$ -linked/ U to each other.

Proof. Step 1. In this step, we show that the theorem holds over an étale neighborhood $(s' \in U') \rightarrow (s \in U)$ such that $k(s) = k(s')$. We use induction on $\dim X$ and on $\dim U$.

If $f^{-1}(s) \cap [B]$ is disconnected, then by [Bir20, Theorem 3.5], after an étale base change, there are exactly two non-trivial lc centers of $(X, B, \mathbf{M})$ intersecting $f^{-1}(s)$, and they are $\mathbb{P}^1$ -linked with each other. We are done in this case.

If $f^{-1}(s) \cap [B]$ is connected, then we let $D_1, \dots, D_r$ be the irreducible components of $[B]$. By passing to an étale neighborhood of $s \in S$ without changing $k(s)$, we may assume that each D_i has connected fiber over s , and every lc center of $(X, B, \mathbf{M})$ intersects $f^{-1}(s)$ (cf. [Kol13, Claim 4.38.1]). Possibly reordering indices, we may assume that $Z \subset D_1, W \subset D_r$, and

$$f^{-1}(s) \cap D_i \cap D_{i+1} \neq \emptyset$$

for any $1 \leq i \leq r-1$. Let $(D_i, B_{D_i}, \mathbf{M}^{D_i})$ be the g-pair induced by adjunction

$$K_{D_i} + B_{D_i} + \mathbf{M}_{D_i}^{D_i} := (K_X + B + \mathbf{M}_X)|_{D_i}$$

for each i .

Claim 4.3.9. *Let $Z_1 := Z$. For any $2 \leq i \leq r$, there exists an lc center $Z_i \subset D_{i-1} \cap D_i$ in $\mathcal{S}$ such that*

- (1) Z_i and Z_{i-1} are $\mathbb{P}^1$ -linked/ U with each other,
- (2) Z_i is minimal in $\mathcal{S}$, and
- (3) Z_i is an lc center of $(D_i, B_{D_i}, \mathbf{M}^{D_i})$ and $(D_{i-1}, B_{D_{i-1}}, \mathbf{M}^{D_{i-1}})$.

Proof. Suppose we have already constructed Z_{i-1} . By Lemma 4.3.2, Z_{i-1} and $D_{i-1} \cap D_i$ are lc centers of $(D_{i-1}, B_{D_{i-1}}, \mathbf{M}^{D_{i-1}})$, and Z_{i-1} is minimal among all lc centers of $(D_{i-1}, B_{D_{i-1}}, \mathbf{M}^{D_{i-1}})$ which dominate s . By induction hypothesis of $\dim X$ and $\dim U$, there exists an lc center $Z_i \subset D_{i-1} \cap D_i$ that is minimal in among all lc centers of $(D_{i-1}, B_{D_{i-1}}, \mathbf{M}^{D_{i-1}})$ which dominate s , and Z_i and Z_{i-1} are $\mathbb{P}^1$ -linked with each other. By Lemma 4.3.2, Z_i is an lc center of $(X, B, \mathbf{M})$, is minimal in $\mathcal{S}$, and is an lc center of $(D_i, B_{D_i}, \mathbf{M}^{D_i})$. The claim follows by induction on i . $\square$

Proof of Theorem 4.3.8 continued. By Claim 4.3.9 applied to $i = r$, the theorem holds over an étale neighborhood $(s' \in U') \rightarrow (s \in U)$ such that $k(s) = k(s')$ under the induction hypothesis of $\dim X$ and $\dim U$.

Step 2. We show that the étale base change was not necessary and conclude the proof of the theorem. Let

$$X \xrightarrow{\tilde{f}} \tilde{U} \rightarrow U$$

be the Stein factorization of f . Since $f^{-1}(s)$ is connected, there exists a unique pre-image $\tilde{s} \in \tilde{U}$ of s . Let Z_i be the minimal elements of $\mathcal{S}$. Since lc centers commute with étale base change, we see that there is a unique irreducible subvariety

$$\tilde{s} \in \tilde{V} \subset U$$

such that $\tilde{V} = \tilde{f}(Z_i)$ for each i .

Let $\tilde{v}$ be the generic point of $\tilde{V}$. By **Step 1** and induction hypothesis, the theorem holds after an étale base change

$$\tilde{\pi} : (\tilde{v}' \in \tilde{U}') \rightarrow (\tilde{v} \in \tilde{U}).$$

Since $\tilde{f}$ has connected fibers, $\tilde{\pi}$ induces an isomorphism of the fibers

$$\tilde{\pi} : (\tilde{f}')^{-1}(\tilde{v}') \cong \tilde{f}^{-1}(\tilde{v}).$$

Thus Z_i canonically lift to $Z'_i \cong Z_i$ and the $\mathbb{P}^1$ -links/ U between the Z'_i descend to $\mathbb{P}^1$ -links/ U between the Z_i . $\square$

Lemma 4.3.10. *Let $f : (X, B, \mathbf{M}) \rightarrow Z$ be an lc crepant log structure and $z \in Z$ a (not necessarily closed) point. Let*

$$\mathcal{S}_z := \{V \mid V \text{ is an lc center of } f : (X, B, \mathbf{M}) \rightarrow Z, z \in V\}.$$

Then:

- (1) *There exists a unique element $W \in \mathcal{S}_z$ that is minimal with respect to inclusion.*
- (2) *W is unibranch ([Kol13, Definition 1.44]) at z , i.e. the completion $\widehat{W}_z$ is irreducible.*
- (3) *Any intersection of lc centers of $f : (X, B, \mathbf{M}) \rightarrow Z$ is a union of lc centers.*

Proof. By Definition-Lemma 4.1.3, possibly replacing $(X, B, \mathbf{M})$ with a dlt model, we may assume that $(X, B, \mathbf{M})$ is dlt. Since f is a contraction, $f^{-1}(z)$ is connected. For any element $W \in \mathcal{S}_z$ that is minimal with respect to inclusion, there exists an lc center Z_W of $(X, B, \mathbf{M})$ that is minimal among all lc centers whose image on Z is equal to W with respect to inclusion. By Theorem 4.3.8, all such Z_W are $\mathbb{P}^1$ -linked/ Z to each other, hence their images on Z are the same. This proves (1). (2) follows from (1) by considering every étale neighborhood of z .

For any lc centers W_1, W_2 on Z , let $z \in W_1 \cap W_2$ be any point. By (1), there exists a unique element W_z of $\mathcal{S}_z$. Then

$$z \in W_z \subset W_1 \cap W_2,$$

so

$$W_1 \cap W_2 = \cup_{z \in W_1 \cap W_2} z \subset \cup_{z \in W_1 \cap W_2} W_z \subset W_1 \cap W_2.$$

Therefore,

$$W_1 \cap W_2 = \cup_{z \in W_1 \cap W_2} W_z$$

is a union of lc centers. We get (3). $\square$

Lemma 4.3.11. *Let $f : (X, B, \mathbf{M}) \rightarrow Z$ be a dlt crepant log structure and $Y \subset X$ an lc center. Let*

$$f|_Y : Y \xrightarrow{f_Y} Z_Y \xrightarrow{\pi} Z$$

be the Stein factorization of $f|_Y$, and $(Y, B_Y, \mathbf{M}^Y)/Z$ the dlt g -pair induced by repeatedly applying adjunction to codimension 1 lc centers

$$K_Y + B_Y + \mathbf{M}_Y^Y := (K_X + B + \mathbf{M}_X)|_Y.$$

Then:

- (1) *$f_Y : (Y, B_Y, \mathbf{M}^Y) \rightarrow Z_Y$ is a dlt crepant log structure.*

- (2) For any lc center $W_Y \subset Z_Y$ of $f_Y : (Y, B_Y, \mathbf{M}^Y) \rightarrow Z_Y$, $\pi(W_Y)$ is an lc center of $f : (X, B, \mathbf{M}) \rightarrow Z$.
- (3) For any lc center $W \subset Z$ of $f : (X, B, \mathbf{M}) \rightarrow Z$, every irreducible component of $\pi^{-1}(W)$ is an lc center of $f_Y : (Y, B_Y, \mathbf{M}^Y) \rightarrow Z_Y$.

Proof. (1) We only need to show that $(Y, B_Y, \mathbf{M}^Y)$ is dlt, which follows from [HL22, Lemma 2.9].

(2) There exists an lc center V_Y of $(Y, B_Y, \mathbf{M}^Y)$ such that $f_Y(V_Y) = W_Y$. By Lemma 4.3.2, V_Y is also an lc center of $(X, B, \mathbf{M})$. Thus $\pi(W_Y) = f(V_Y)$ is an lc center of $f : (X, B, \mathbf{M}) \rightarrow Z$.

(3) Let z be the generic point of W . Since the question is étale local, possibly replacing Z by an étale neighborhood of z and replacing Y with its irreducible components, we may assume that $f^{-1}(z) \cap Y$ is connected, and we only need to show that there exists an lc center V_Y of $f_Y : (Y, B_Y, \mathbf{M}^Y) \rightarrow Z_Y$ such that $f_Y(V_Y)$ is an irreducible component of $\pi^{-1}(W)$.

Let V_X be a minimal lc center of $(X, B, \mathbf{M})$ which dominates W , i.e. V_X is minimal in

$$\{V \mid V \text{ is an lc center of } (X, B, \mathbf{M}), V \text{ dominates } W\}$$

with respect to inclusion. Then $f(V_X) = W$. By Theorem 4.3.8, there exists an lc center $V_Y \subset Y$ of $(X, B, \mathbf{M})$ that is $\mathbb{P}^1$ -linked/ Z to V_X . By Lemma 4.3.2, V_Y is also an lc center of $(Y, B_Y, \mathbf{M}^Y)$. Thus $f_Y(V_Y) \subset Z_Y$ is an lc center of $f_Y : (Y, B_Y, \mathbf{M}^Y) \rightarrow Z_Y$. Moreover, since V_Y is $\mathbb{P}^1$ -linked/ Z to V_X , $(f|_Y)(V_Y) = f(V_X) = W$. Thus $f_Y(V_Y)$ is an irreducible component of $\pi^{-1}(W)$ and we are done. $\square$

4.4. Inversion of adjunction. In this subsection, we prove the following canonical bundle formula for NQC generalized pairs:

Theorem 4.4.1. *Let $(X, B, \mathbf{M})$ be an NQC g-pair and S a component of $B^{\perp=1}$. Let S^ν be the normalization of S , and let $(S^\nu, B_S, \mathbf{M}^S)/U$ be the g-pair induced by the adjunction*

$$K_{S^\nu} + B_S + \mathbf{M}_S^S := (K_X + B + \mathbf{M}_X)|_{S^\nu}.$$

Then $(S^\nu, B_S, \mathbf{M}^S)$ is lc if and only if $(X, B, \mathbf{M})$ is lc near S .

Proof. The if part of the theorem follows from [BZ16, Definition 4.7] so we only need to prove the only if part.

First we prove the case when $(X, B, \mathbf{M})$ is a $\mathbb{Q}$ -g-pair.

By Definition-Lemma 4.1.3, there exists a birational morphism $f : Y \rightarrow X$ satisfying the following. Let E be the reduced f -exceptional divisor and $B_Y := f_*^{-1}(B \wedge \text{Supp } B) + E$, then

- (1) $(Y, B_Y, \mathbf{M})$ is $\mathbb{Q}$ -factorial dlt,
- (2) $a(F, X, B, \mathbf{M}) \leq 0$ for any prime f -exceptional divisor F .

We let

$$K_Y + \bar{B}_Y + \mathbf{M}_Y := f^*(K_X + B + \mathbf{M}_X)$$

and let S_Y be the strict transform of S on Y . Let $(S_Y, B_{S_Y}, \mathbf{M}^S)/U$ and $(S_Y, \bar{B}_{S_Y}, \mathbf{M}^S)/U$ be the g-pairs induced by adjunction

$$K_{S_Y} + B_{S_Y} + \mathbf{M}_{S_Y}^S = (K_Y + B_Y + \mathbf{M}_Y)|_{S_Y}$$

and

$$K_{S_Y} + \bar{B}_{S_Y} + \mathbf{M}_{S_Y}^S = (K_Y + \bar{B}_Y + \mathbf{M}_Y)|_{S_Y}$$

respectively. We let $Q := \bar{B}_Y - B_Y$.

Let A be an ample divisor on Y such that $K_Y + B_Y + \mathbf{M}_Y + A$ is nef. We may run a $(K_Y + B_Y + \mathbf{M}_Y)$ -MMP/ X with scaling of A

$$(Y, B_Y, \mathbf{M}) := (X_0, B_0, \mathbf{M}) \dashrightarrow (X_1, B_1, \mathbf{M}) \dashrightarrow \cdots \dashrightarrow (X_n, B_n, \mathbf{M}) \dashrightarrow \cdots$$

Let $S_i, A_i, Q_i, \bar{B}_i$ be the image of $S_Y, A, Q, \bar{B}_Y$ on X_i for each i , $f_i : X_i \rightarrow X$ the induced birational morphism, and

$$\lambda_i := \inf\{t \geq 0 \mid K_{X_i} + B_i + tA_i + \mathbf{M}_{X_i} \text{ is nef}/X\}$$

the scaling numbers. Then $K_{X_i} + \bar{B}_i + \mathbf{M}_{X_i} = f_i^*(K_X + B + \mathbf{M}_X)$ and $\bar{B}_i = B_i + Q_i$ for any i . Let

$$K_{S_i} + B_{S_i} + \mathbf{M}_{S_i}^S := (K_{X_i} + B_i + \mathbf{M}_{X_i})|_{S_i}$$

and

$$K_{S_i} + \bar{B}_{S_i} + \mathbf{M}_{S_i}^S := (K_{X_i} + \bar{B}_i + \mathbf{M}_{X_i})|_{S_i}$$

for any i . Then $\bar{B}_{S_i} = B_{S_i} + Q_i|_{S_i}$. Moreover, there exists a birational morphism $g_i : S_i \rightarrow S$ such that

$$K_{S_i} + \bar{B}_{S_i} + \mathbf{M}_{S_i}^S = g_i^*(K_S + B_S + \mathbf{M}_S^S).$$

Thus $(S_i, \bar{B}_{S_i}, \mathbf{M}^S)$ is lc. Since $(Y, B_Y, \mathbf{M})$ is dlt, $(X_i, B_i, \mathbf{M})$ is dlt. By Lemma 4.3.2, $(S_i, B_{S_i}, \mathbf{M}^S)$ is dlt. By Lemma 4.3.2, any lc center of $(X_i, B_i, \mathbf{M})$ is an lc center of $(S_i, B_{S_i}, \mathbf{M}^S)$. Since all components of Q_i are lc centers of $(X_i, B_i, \mathbf{M})$ and $(S_i, \bar{B}_{S_i}, \mathbf{M}^S)$ is lc, $\text{Supp } Q_i$ does not intersect S_i for any i .

We pick a non-negative integer m in the following way. If the $(K_Y + B_Y + \mathbf{M}_Y)$ -MMP/ X terminates, then we let m be the index so that $(X_m, B_m, \mathbf{M})/X$ is a log minimal model of $(Y, B_Y, \mathbf{M})/X$ for some non-negative integer m . If the $(K_Y + B_Y + \mathbf{M}_Y)$ -MMP/ X does not terminate, then by Lemma 4.2.2, $\lim_{i \rightarrow +\infty} \lambda_i = 0$, so by special termination (cf. [LX23a, Lemma 2.18]), we may pick a positive integer m , such that $S_i \dashrightarrow S_{i+1}$ is an isomorphism in codimension 1 for any $i \geq m$. We let $I \geq 2$ be any sufficiently divisible positive integer satisfying the following.

- IQ is a Weil divisor.
-

$$(f_m)_* \mathcal{O}_{X_m}(A_m - IQ_m) \subset (f_m)_* \mathcal{O}_{X_m}(A_m)$$

are contained in

$$\mathcal{I}_{f_m(\text{Supp } Q)} \cdot (f_m)_* \mathcal{O}_{X_m}(A_m).$$

- If $(X_m, B_m, \mathbf{M})/X$ is a log minimal model of $(Y, B_Y, \mathbf{M})/X$ and $m \geq 2$, then $\lambda_{m-1} > \frac{1}{I}$.
- If the $(K_Y + B_Y + \mathbf{M}_Y)$ -MMP/ X does not terminate, then $\lambda_m > \frac{1}{I}$.

Since $S_i \dashrightarrow S_{i+1}$ is an isomorphism in codimension 1 for any $i \geq m$, for any $j \geq m$, we have

$$(f_j)_* \mathcal{O}_{X_j}(A_j - IQ_j) = (f_m)_* \mathcal{O}_{X_m}(A_m - IQ_m).$$

Since $\text{Supp } Q_i$ does not intersect S_i for any i , we have an induced homomorphism.

$$(f_i)_* \mathcal{O}_{X_i}(A_i - IQ_i) \rightarrow (f_m|_{S_m})_* \mathcal{O}_{S_m}(A_i) = (f_i|_{S_i})_* \mathcal{O}_{S_i}(A_i)$$

which is not surjective. Therefore,

$$R^1(f_i)_* \mathcal{O}_{X_i}(A_i - IQ_i - S_i) \neq 0$$

for any $i \geq m$.

We let $l := m$ if $(X_m, B_m, \mathbf{M})/X$ is a log minimal model of $(Y, B_Y, \mathbf{M})/X$, and let l be the unique positive integer such that $\lambda_{l-1} > \frac{1}{I} \geq \lambda_l$ if the $(K_Y + B_Y + \mathbf{M}_Y)$ -MMP/ X does not terminate. Then $l \geq m$,

$$X_0 \dashrightarrow X_1 \dashrightarrow \dots \dashrightarrow X_l$$

is also a sequence of steps of a $(K_{X_0} + B_0 + \mathbf{M}_{X_0} + \frac{1}{I}A)$ -MMP/ X with scaling of A , and

$$K_{X_l} + B_l + \mathbf{M}_{X_l} + \frac{1}{I}A_l$$

is nef/ X . Since X_0 is $\mathbb{Q}$ -factorial klt, we may pick

$$0 \leq \Delta_0 \sim_{\mathbb{Q}} B_0 - S_0 + \mathbf{M}_{X_0} + \frac{1}{I}A$$

such that (X_0, Δ_0) is klt and $(X_0, S_0 + \Delta_0)$ is plt. We let Δ_l be the image of Δ_0 on X_l , then $(X_l, S_l + \Delta_l)$ is plt, so (X_l, Δ_l) is klt. Then

$$A_l - IQ_l - S_l \sim_{\mathbb{Q}, X} K_{X_l} + \Delta_l + (I - 1) \left(K_{X_l} + B_l + \mathbf{M}_{X_l} + \frac{1}{I} A_l \right),$$

so by the relative Kawamata-Viehweg vanishing [KMM87, Theorem 1-2-5],

$$R^1(f_l)_* \mathcal{O}_{X_l}(A_l - IQ_l - S_l) \neq 0,$$

a contradiction. We are done with the case when $(X, B, \mathbf{M})$ is a $\mathbb{Q}$ -g-pair.

Now we prove the case when $(X, B, \mathbf{M})$ is not necessarily a $\mathbb{Q}$ -g-pair, hence conclude the proof of the theorem. There exist real numbers $r_1, \dots, r_c$ such that $1, r_1, \dots, r_c$ are linearly independent over $\mathbb{Q}$, $\mathbf{r} := (r_1, \dots, r_c) \in \mathbb{R}^c$, and $\mathbb{Q}$ -linear functions $s_1, \dots, s_p, t_1, \dots, t_q$, such that

$$B = \sum_{i=1}^p s_i(1, \mathbf{r}) B_i, \mathbf{M} = \sum_{i=1}^q t_i(1, \mathbf{r}) \mathbf{M}_i,$$

where $B_i \geq 0$ are distinct Weil divisors and $\mathbf{M}_i$ are nef/ X $\mathbf{b}$ -Cartier $\mathbf{b}$ -divisors. Let

$$B(\mathbf{v}) := \sum_{i=1}^p s_i(1, \mathbf{v}) B_i \text{ and } \mathbf{M}(\mathbf{v}) := \sum_{i=1}^q t_i(1, \mathbf{v}) \mathbf{M}_i,$$

for any $\mathbf{v} \in \mathbb{R}^c$.

Since the coefficients of divisors under adjunction are transformed via $\mathbb{Q}$ -linear functions, there are $\mathbb{Q}$ -linear functions $s'_1, \dots, s'_p, t'_1, \dots, t'_q$, distinct Weil divisors $B_{S_i} \geq 0$, and nef/ X $\mathbf{b}$ -Cartier $\mathbf{b}$ -divisors $\mathbf{M}_i^S$,

$$B_S(\mathbf{v}) := \sum_{i=1}^p s_i(1, \mathbf{v}) B_{S,i}, \text{ and } \mathbf{M}^S(\mathbf{v}) := \sum_{i=1}^q t_i(1, \mathbf{v}) \mathbf{M}_i^S,$$

such that

$$K_{S^\nu} + B_S(\mathbf{v}) + \mathbf{M}^S(\mathbf{v})_{S^\nu} = (K_X + B(\mathbf{v}) + \mathbf{M}(\mathbf{v})_X)|_{S^\nu}$$

for any $\mathbf{v} \in \mathbb{R}^c$. Since

$$(S^\nu, B_S = B_S(\mathbf{r}), \mathbf{M}^S = \mathbf{M}^S(\mathbf{r}))$$

is lc, there exists an open neighborhood $U \ni \mathbf{r}$ of $\mathbb{R}^c$ such that

$$(S^\nu, B_S(\mathbf{v}), \mathbf{M}^S(\mathbf{v}))$$

is lc for any $\mathbf{v} \in U$. By the $\mathbb{Q}$ -g-pair case,

$$(X, B(\mathbf{v}), \mathbf{M}(\mathbf{v}))$$

is lc for any $\mathbf{v} \in U \cap \mathbb{Q}$. Thus

$$(X, B = B(\mathbf{r}), \mathbf{M} = \mathbf{M}(\mathbf{r}))$$

is lc by continuity of log discrepancies. $\square$

Remark 4.4.2. We do not need Theorem 4.4.1 in the rest of the paper but we expect it to be useful for future works. We remark that several alternative versions of Theorem 4.4.1 can be found in [Fil20, Theorems 1.5, 1.6, 6.7] but we cannot apply them directly to prove Theorem 4.4.1 because of the following reasons:

- (1) All these theorems require that $(X, B, \mathbf{M})$ is a $\mathbb{Q}$ -g-pair.
- (2) [Fil20, Theorems 1.5] requires S to be a minimal lc center and S is projective.
- (3) [Fil20, Theorems 1.6] requires that X is projective and $(X, B, \mathbf{M})$ is a $\mathbb{Q}$ -g-pair. Moreover, the potential g-pair structure constructed on W^ν [Fil20, Theorems 1.6] is not known to be identical to the g-pair structure constructed in [HL21b, Theorem 4.5].
- (4) [Fil20, Theorem 6.7] requires that X is $\mathbb{Q}$ -factorial projective klt.

4.5. Boundedness on the number of components.

Proposition 4.5.1. *Let $\gamma_0 \leq 1$ be a positive real number, and $b_1, \dots, b_n \in [\gamma_0, 1]$ positive real numbers. Let $(X, B = \sum_{i=1}^n b_i B_i + D, \mathbf{M})/X$ be an lc g-pair and $x \in X$ a point, such that $B_i \geq 0$ is a non-zero $\mathbb{Q}$ -Cartier Weil divisor for each i , and $D \geq 0$. Suppose that $\bar{x} \subset \text{Supp } B_i$ for each i . Moreover, assume that one of the followings hold:*

- (1) $\mathbf{M}$ is NQC/ X .
- (2) There exist a klt g-pair $(X, B', \mathbf{M}')/X$.
- (3) $\gamma_0 = 1$ and each B_i is Cartier.

Then

$$n \leq \frac{\dim X - \dim \bar{x}}{\gamma_0}.$$

Proof. When $\dim X = 1$ the proposition is trivial, so we may assume that $\dim X \geq 2$. We may also assume that $n \geq 1$, otherwise there is nothing left to prove.

Let $B_{n+1}, \dots, B_{n+\dim \bar{x}}$ be general hyperplane sections on X and let $b_i := 1$ when $i \geq n+1$. Possibly replacing x with $\bar{x} \cap \bigcap_{i=1}^{\dim X} H_i$ and B with $\sum_{i=1}^{n+\dim \bar{x}} b_i B_i + D$, we may assume that x is a closed point.

First we prove the proposition under conditions (1) or (2). Possibly adding general hyperplane sections which passes through x , we may assume that x is an lc center of $(X, B, \mathbf{M})$. Let E be an lc place of $(X, B, \mathbf{M})$ such that $\text{center}_X E = x$.

Claim 4.5.2. *There exists a contraction $f : Y \rightarrow X$ of E , such that $-E$ is ample/ X .*

Proof. If $\mathbf{M}$ is NQC/ X , then the claim follows from [LX23b, Theorem 1.7]. Otherwise, the claim follows from [Bir20, Lemma 2.11]. $\square$

Proof of Proposition 4.5.1 continued. By Claim 4.5.2, there exists a contraction $f : Y \rightarrow X$ of E , such that $-E$ is ample/ X . We let $B_{i,Y}, D_Y, B_Y$ be the strict transforms of B_i, D, B on Y respectively. Since $x \in \text{Supp } B_i$ for each i , $\text{mult}_E B_i > 0$ for each i , so $B_{i,Y}$ is ample/ X for each i . We let E^ν be the normalization of E , $\mathbf{M}^E := \mathbf{M}|_{E^\nu}$, and let

$$K_{E^\nu} + B_E + \mathbf{M}_{E^\nu}^E = (K_Y + B_Y + E + \mathbf{M}_Y)|_{E^\nu}.$$

We let $B_{i,E} := \text{Supp}(B_{i,Y}|_{E^\nu})$ for each i . Then for any component $D_{i,j}$ of $B_{i,E}$, we have

$$\text{mult}_{D_{i,j}} B_E = \frac{n_{i,j} - 1 + \sum_{k=1}^n b_k m_{k,i,j} + \gamma_{i,j}}{n_{i,j}}$$

for some real number $\gamma_{i,j} \geq 0$ and non-negative integers $m_{k,i,j}$, such that $m_{i,i,j} \neq 0$. Since $(E, B_E, \mathbf{M}^E)$ is lc, $\text{mult}_{D_{i,j}} B_E \leq 1$, so

$$B_E \geq \sum_{i=1}^n b_i B_{i,E}.$$

Since $B_{i,Y}$ is ample/ X , $B_{i,Y}|_{E^\nu}$ is ample, so $B_{i,E}$ is big. The proposition under conditions (1) or (2) follows from [BZ16, Proposition 5.1].

Now we prove the proposition under condition (3). Let S be the normalization of an irreducible component of B_1 such that $x \in S_1$, and let $(S, B_S, \mathbf{M}^S)$ be the g-pair induced by the adjunction

$$K_S + B_S + \mathbf{M}_S^S := (K_X + B + D + \mathbf{M}_X)|_S.$$

Since $x \in \text{Supp } B_i$ for each i , $B_i|_S \neq 0$ for any $i \geq 2$. Since B_i is Cartier and $(S, B_S, \mathbf{M}^S)$ is lc, $B_i|_S = \text{Supp}(B_i|_S)$ for any $i \geq 2$, and

$$B_S \geq \sum_{i=2}^n B_i|_S.$$

Since each $B_i|_S$ is Cartier, by induction on $\dim X$, we have $n \leq \dim X$ and the proposition follows. $\square$

5. STABILITY OF GENERALIZED PAIRS

In this section, we discuss the stability properties of g-pairs. We will define the concepts of generically lc, Property (*) BP (semi-)stable, and log stable for g-pairs, and then study the basic properties of g-pairs satisfying these properties. This section is parallel to [ACSS21, Section 2].

5.1. Toroidal generalized pairs.

Definition 5.1.1 (cf. [ACSS21, Definition 2.1]). Let $(X, \Sigma_X, \mathbf{M})/U$ be a g-pair. We say that $(X, \Sigma_X, \mathbf{M})$ is *toroidal* if Σ_X is a reduced divisor, $\mathbf{M}$ descends to X , and for any closed point $x \in X$, there exists a toric variety X_σ , a closed point $t \in X_\sigma$, and an isomorphism of complete local algebras

$$\phi_x : \widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{X_\sigma,t}$$

such that the ideal of Σ_X maps to the invariant ideal of $X_\sigma \setminus T_\sigma$, where $T_\sigma \subset X_\sigma$ is the maximal torus of X_σ . Any such (X_σ, t) will be called as a *local model* of $(X, \Sigma_X, \mathbf{M})$ at $x \in X$.

Let $(X, \Sigma_X, \mathbf{M})/U$ and $(Z, \Sigma_Z, \mathbf{M}^Z)/U$ be toroidal g-pairs and $f : X \rightarrow Z$ a surjective morphism/ U . We say that $f : (X, \Sigma_X, \mathbf{M}) \rightarrow (Z, \Sigma_Z, \mathbf{M}^Z)$ is *toroidal*, if for every closed point $x \in X$, there exist a local model (X_σ, t) of $(X, \Sigma_X, \mathbf{M})$ at x , a local model (Z_τ, s) of $(Z, \Sigma_Z, \mathbf{M}^Z)$ at $z := f(x)$, and a toric morphism $g : X_\sigma \rightarrow Z_\tau$, so that the diagram of algebras commutes.

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{X,x} & \xrightarrow{\cong} & \widehat{\mathcal{O}}_{X_\sigma,t} \\ \uparrow & & \uparrow \\ \widehat{\mathcal{O}}_{Z,z} & \xrightarrow{\cong} & \widehat{\mathcal{O}}_{Z_\tau,s} \end{array}$$

Here the vertical maps are the algebra homomorphisms induced by f and g respectively.

Definition-Theorem 5.1.2 ([LLM23, Definition-Theorem 6.5], [ACSS21, Theorem 2.2]). Let X be a normal quasi-projective variety, $X \rightarrow U$ a projective morphism, $X \rightarrow Z$ a contraction, B an $\mathbb{R}$ -divisor on X , $\mathbf{M}$ a nef/ U $\mathbf{b}$ -divisor on X , $D_1, \dots, D_m$ prime divisors over X , and $D_{Z,1}, \dots, D_{Z,n}$ prime divisors over Z . Then there exist a toroidal g-pair $(X', \Sigma_{X'}, \mathbf{M})/U$, a log smooth pair $(Z', \Sigma_{Z'})$, and a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ f' \downarrow & & \downarrow f \\ Z' & \xrightarrow{h_Z} & Z \end{array}$$

satisfying the following.

- (1) h and h_Z are projective birational morphisms.
- (2) $f' : (X', \Sigma_{X'}, \mathbf{M}) \rightarrow (Z', \Sigma_{Z'})$ is a toroidal contraction.
- (3) $\text{Supp}(h_*^{-1}B) \cup \text{Supp Exc}(h)$ is contained in $\text{Supp } \Sigma_{X'}$.
- (4) X' has at most toric quotient singularities.
- (5) f' is equi-dimensional.
- (6) $\mathbf{M}$ descends to X' .
- (7) X' is $\mathbb{Q}$ -factorial klt.
- (8) The center of each D_i on X' and the center of each $D_{Z,i}$ on Z' are divisors.

We call any such $f' : (X', \Sigma_{X'}, \mathbf{M}) \rightarrow (Z', \Sigma_{Z'})$ (associated with h and h_Z) which satisfies (1-7) an *equi-dimensional model* of $f : (X, B, \mathbf{M}) \rightarrow Z$.

Proof. Possibly replacing X and Z with high models, we may assume that $\mathbf{M}$ descends to X , each D_i is a divisor on X , and each $D_{Z,i}$ is a divisor on Z . Now the theorem follows from [ACSS21, Theorem 2.2], which in turn follows from [AK00, Theorem 2.1 and Proposition 4.4]. We also refer the reader to [Hu20, Theorem B.6] for a more detailed explanation. $\square$

Remark 5.1.3. In Definition-Theorem 5.1.2, it is important to note that the contraction $X \rightarrow Z$ may not necessarily be over U . This kind of phenomenon will appear throughout the rest of the paper.

5.2. Discriminant and moduli parts of generalized pairs.

Definition 5.2.1 (Birationally equivalent morphisms, cf. [ACSS21, Page 4, Paragraph 2]). Let $f : X \rightarrow Z$ and $f' : X' \rightarrow Z'$ be surjective morphisms between normal varieties. We say that f and f' are *birationally equivalent* if there exist birational maps $h : X \dashrightarrow X'$ and $h_Z : Z \dashrightarrow Z'$ such that $f' \circ h = h_Z \circ f$.

Definition 5.2.2 (Generically lc, cf. [ACSS21, 2.2. Discriminant and Moduli Part]). Let $(X, B, \mathbf{M})/U$ be a g-sub-pair and $f : X \rightarrow Z$ a contraction. We say that $(X, B, \mathbf{M})$ is *generically (sub-)lc/ Z* if $(X, B, \mathbf{M})$ is (sub-)lc over the generic point of Z . Note that f may not be a contraction/ U . We remark that we will not use the notation “GLC” for “generically lc” as in [ACSS21] since GLC also stands for “generalized lc” in many references.

Definition 5.2.3 (Crepant generalized pairs, cf. [ACSS21, Definition 2.3]). Let $(X, B, \mathbf{M})/U$ and $(X', B', \mathbf{M}')/U$ be two g-sub-pairs and $f : X \rightarrow Z$, $f' : X' \rightarrow Z'$ two contractions. We say that $(X, B, \mathbf{M})$ and $(X', B', \mathbf{M}')$ are *crepant over the generic point of Z* if we have the following commutative diagram

$$\begin{array}{ccc}
 & W & \\
 p \swarrow & & \searrow q \\
 X & \xrightarrow{h} & X' \\
 f \downarrow & & \downarrow f' \\
 Z & \xrightarrow{h_Z} & Z'
 \end{array}$$

satisfying the following. Let

$$K_W + B_W + \mathbf{M}_W := p^*(K_X + B + \mathbf{M}_X)$$

and

$$K_W + B'_W + \mathbf{M}'_W := q^*(K_{X'} + B' + \mathbf{M}'_X).$$

Then:

- (1) h and h_Z are birational maps. In particular, f and f' are birationally equivalent.
- (2) $\mathbf{M}$ and $\mathbf{M}'$ descends to W .
- (3) $B_W - B'_W$ and $\mathbf{M}_W - \mathbf{M}'_W$ are vertical/ Z .

Definition 5.2.4 (Discriminant and moduli parts, cf. [ACSS21, Definition 2.3]). Let $(X, B, \mathbf{M})/U$ be a g-sub-pair and $f : X \rightarrow Z$ a contraction such that $(X, B, \mathbf{M})$ is generically sub-lc/ Z . In the following, we fix a choice of K_X and a choice of K_Z , and suppose that for any birational morphism $g : \bar{X} \rightarrow X$ and $g_Z : \bar{Z} \rightarrow Z$, $K_{\bar{X}}$ and $K_{\bar{Z}}$ are chosen as the Weil divisors such that $g_*K_{\bar{X}} = K_X$ and $(g_Z)_*K_{\bar{Z}} = K_Z$.

Let $f' : X' \rightarrow Z'$ be any contraction that is birationally equivalent to f such that the induced birational maps $h : X' \dashrightarrow X$ and $h_Z : Z' \dashrightarrow Z$ are morphisms and Z' is $\mathbb{Q}$ -factorial. We let

$$K_{X'} + B' + \mathbf{M}_{X'} := h^*(K_X + B + \mathbf{M}_X).$$

For any prime divisor D on Z' , we define

$$b_D(X', B', \mathbf{M}; f) := 1 - \sup \{t \mid (X', B' + t f'^* D, \mathbf{M}) \text{ is sub-lc over the generic point of } D\}.$$

Since being sub-lc is a property that is preserved under crepant transformations, $b_D(X, B, \mathbf{M}; f)$ is independent of the choices of X' and Z' and is also independent of U .

Since $(X, B, \mathbf{M})$ is generically sub-lc/ Z , $(X', B', \mathbf{M})$ is generically sub-lc/ Z , so we may define

$$B_{Z'} := \sum_{D \text{ is a prime divisor on } Z'} b_D(X, B, \mathbf{M}; f) D.$$

and

$$N_{X'} := K_{X'} + B' + \mathbf{M}_{X'} - f'^*(K_{Z'} + B_{Z'}).$$

We call $B_{Z'}$ and $N_{X'}$ the *discriminant part* and *trace moduli part* of $f' : (X', B', \mathbf{M}) \rightarrow Z'$ respectively, and call $B_Z := (h_Z)_* B_{Z'}$ and $N_X := h_* B$ the *discriminant part* and *trace moduli part* of $f : (X, B, \mathbf{M}) \rightarrow Z$ respectively.

By construction, there exist two $\mathbf{b}$ -divisors $\mathbf{B}$ on Z and $\mathbf{N}$ on X , such that for any contraction $f'' : X'' \rightarrow Z''$ that is birationally equivalent to f such that the induced birational maps $h' : X'' \dashrightarrow X'$ and $h_{Z'} : Z'' \dashrightarrow Z'$ are morphisms and Z'' is $\mathbb{Q}$ -factorial, $\mathbf{B}_{Z''}$ is the discriminant part of $f'' : (X'', B'', \mathbf{M}) \rightarrow Z''$, and $\mathbf{N}_{X''}$ is the trace moduli part of $f'' : (X'', B'', \mathbf{M}) \rightarrow Z''$, where

$$K_{X''} + B'' + \mathbf{M}_{X''} := h'^*(K_{X'} + B' + \mathbf{M}_{X'}).$$

We call $\mathbf{N}$ the *moduli part* of $f : (X, B, \mathbf{M}) \rightarrow Z$ and $\mathbf{B}$ the *discriminant $\mathbf{b}$ -divisor* of $f : (X, B, \mathbf{M}) \rightarrow Z$. By construction, $\mathbf{B}$ is uniquely determined and $\mathbf{N}$ is uniquely determined for any fixed choices of K_X and K_Z .

5.3. BP stability of generalized pairs.

Definition 5.3.1 (BP (semi-)stable, boundary property, cf. [ACSS21, Definition 2.5]). Let $(X, B, \mathbf{M})/U$ be a g -sub-pair and $f : X \rightarrow Z$ a contraction, such that $(X, B, \mathbf{M})$ is generically sub-lc/ Z . Let $\mathbf{B}$ be the discriminant $\mathbf{b}$ -divisor of $f : (X, B, \mathbf{M}) \rightarrow Z$.

We say that $f : (X, B, \mathbf{M}) \rightarrow Z$ is BP stable (resp. BP semi-stable) if $K_Z + \mathbf{B}_Z$ is $\mathbb{R}$ -Cartier, and for any birational morphism $h_Z : Z' \rightarrow Z$,

$$h_Z^*(K_Z + \mathbf{B}_Z) = (\text{resp. } \geq) K_{Z'} + \mathbf{B}_{Z'}.$$

If $f : (X, B, \mathbf{M}) \rightarrow Z$ is BP stable (resp. BP semi-stable), then we say that $(X, B, \mathbf{M})$ is BP stable (resp. BP semi-stable) over Z .

Lemma 5.3.2 (cf. [ACSS21, Remark 2.6(2)]). Let $(X, B, \mathbf{M})/U$ be a g -sub-pair and $f : X \rightarrow Z$ a contraction, such that $f : (X, B, \mathbf{M}) \rightarrow Z$ is BP stable. Let B_Z and $\mathbf{N}$ be the discriminant part and the moduli part of $f : (X, B, \mathbf{M}) \rightarrow Z$ respectively. Then:

- (1) $\mathbf{N}_X = K_X + B + \mathbf{M}_X - f^*(K_Z + B_Z)$.
- (2) $\mathbf{N}$ descends to X .

Proof. For any $f' : X' \rightarrow Z'$ that is birationally equivalent to f , such that the induced birational maps $h : X' \dashrightarrow X$ and $h_Z : Z' \dashrightarrow Z$ are morphisms, we have $K_{Z'} + B_{Z'} = h_Z^*(K_Z + B_Z)$. Thus

$$\mathbf{N}_{X'} = K_{X'} + B' + \mathbf{M}_{X'} - f'^*(K_{Z'} + B_{Z'}) = h^*(K_X + B + \mathbf{M}_X - f^*(K_Z + B_Z)),$$

where $K_{X'} + B' + \mathbf{M}_{X'} := h^*(K_X + B + \mathbf{M}_X)$, and $B_{Z'}$ is the discriminant part of $f' : (X', B', \mathbf{M}) \rightarrow Z'$. The lemma immediately follows. $\square$

5.4. Property (*) generalized pairs.

Lemma 5.4.1 (cf. [ACSS21, Lemma 2.12]). Let $(X, B, \mathbf{M})/U$ be a g -pair and $f : X \rightarrow Z$ a contraction. Let $d := \dim X$ and $m := \dim Z$. Let $z \in Z$ be a closed point, $D_1, \dots, D_m \geq 0$ Cartier divisors on Z , such that $z \in \text{Supp } D_i$ for each i and $(X, B + \sum_{i=1}^m f^* D_i, \mathbf{M})$ is lc over $f^{-1}(z)$. Then the dimension of any irreducible component of $f^{-1}(z)$ is $d - m$.

Proof. For any irreducible component G of $f^{-1}(z)$, let $H_1, \dots, H_{\dim G}$ be general very ample divisors on X , $V := \cap_{i=1}^{\dim G} H_i$, and $(V, B_V, \mathbf{M}^V)/U$ the g-pair induced by the adjunction

$$K_V + B_V + \mathbf{M}_V^V := (K_X + B + \mathbf{M}_X)|_V.$$

Then $(V, B_V + \sum_{i=1}^m f^* D_i|_V, \mathbf{M}^V)$ is lc, $G \cap V$ is a closed point, and $A_i := f^* D_i|_V$ is Cartier and contains $G \cap V$ for any i . By Proposition 4.5.1, $m \leq \dim V = d - \dim G$. Thus $\dim G \leq d - m$. Therefore, the dimension of any irreducible component of $f^{-1}(z)$ is $\leq d - m$. By [Har77, Exercise II 3.22 (a)], the dimension of any irreducible component of $f^{-1}(z)$ is $\geq d - m$. The lemma immediately follows. $\square$

Definition 5.4.2 (Property $(*)$ generalized pairs, cf. [ACSS21, Definition 2.13]). Let $(X, B, \mathbf{M})/U$ be a g-sub-pair and $f : X \rightarrow Z$ a contraction. We say that $f : (X, B, \mathbf{M}) \rightarrow Z$ satisfies *Property $(*)$* if there exists a reduced divisor Σ_Z on Z satisfying the following.

- (1) (Z, Σ_Z) is log smooth. In particular, Z is smooth.
- (2) The vertical/ Z part B^v of B is equal to $f^{-1}(\Sigma_Z)$. In particular, B^v is reduced and Σ_Z is the image of B^v on Z .
- (3) For any closed point $z \in Z$ and any reduced divisor $\Sigma \geq \Sigma_Z$ on Z such that (Z, Σ) is log smooth near z , $(X, B + f^*(\Sigma - \Sigma_Z), \mathbf{M})$ is sub-lc over a neighborhood of z .

By (2), Σ_Z is uniquely determined by $f : (X, B, \mathbf{M}) \rightarrow Z$. We will temporarily call Σ_Z the *base divisor* associated to $f : (X, B, \mathbf{M}) \rightarrow Z$. In Lemma 5.4.3 below, we will show that Σ_Z is actually the discriminant part of $f : (X, B, \mathbf{M}) \rightarrow Z$.

Lemma 5.4.3 (cf. [ACSS21, Lemma 2.14]). *Let $(X, B, \mathbf{M})/U$ be a g-sub-pair and $f : X \rightarrow Z$ a contraction such that $f : (X, B, \mathbf{M}) \rightarrow Z$ satisfies Property $(*)$. Let Σ_Z be the base divisor associated to $f : (X, B, \mathbf{M}) \rightarrow Z$. Then:*

- (1) $(X, B, \mathbf{M})$ is sub-lc.
- (2) Σ_Z is the discriminant part of $f : (X, B, \mathbf{M}) \rightarrow Z$.
- (3) If $B \geq 0$, then f is equi-dimensional over $Z \setminus \text{Supp } \Sigma_Z$.

Proof. (1) For any closed point $z \in Z$, we pick $\Sigma := \Sigma_Z$. By Definition 5.4.2(3), $(X, B, \mathbf{M})$ is sub-lc over a neighborhood of z . Thus $(X, B, \mathbf{M})$ is sub-lc.

(2) Let B_Z be the discriminant part of $f : (X, B, \mathbf{M}) \rightarrow Z$. Since the vertical part of B coincides with $f^{-1}(\Sigma_Z)$, $B_Z \geq \Sigma_Z$.

Let P be a prime divisor on Z such that $P \not\subset \text{Supp } \Sigma_Z$, and let z be a general closed point in P . Then $(Z, \Sigma_Z + P)$ is log smooth at z . By Definition 5.4.2(3), $(X, B + f^*P, \mathbf{M})$ is sub-lc over a neighborhood of z . Thus

$$\sup\{t \mid (X, B + tf^*P, \mathbf{M}) \text{ is sub-lc over the generic point of } P\} = 1,$$

so $P \not\subset \text{Supp } B_Z$. Thus $\Sigma_Z = \text{Supp } \Sigma_Z \geq \text{Supp } B_Z$. Since $(X, B, \mathbf{M})$ is sub-lc, $\text{Supp } B_Z \geq B_Z$. This implies (2).

(3) Let $d := \dim X$ and $m := \dim Z$. Let $z \in Z \setminus \text{Supp } \Sigma_Z$ be a closed point, and let $\Sigma_1, \dots, \Sigma_m$ be general hyperplane sections on Z such that $z \in \Sigma_i$ for any i . Then $(Z, \Sigma_Z + \sum_{i=1}^m \Sigma_i)$ is log smooth at z . By Definition 5.4.2(3), $(X, B + \sum_{i=1}^m f^* \Sigma_i, \mathbf{M})$ is lc over a neighborhood of z . By Lemma 5.4.1, the dimension of any irreducible component of $f^{-1}(z)$ is $d - m$. This implies (3). $\square$

Lemma 5.4.4 (cf. [ACSS21, Lemma 2.15]). *Let $(X, B, \mathbf{M})/U$ be a g-pair and $f : X \rightarrow Z$ a contraction such that $f : (X, B, \mathbf{M}) \rightarrow Z$ satisfies Property $(*)$. Let Σ_Z be the discriminant part of $f : (X, B, \mathbf{M}) \rightarrow Z$, and let $\Sigma \geq \Sigma_Z$ be a reduced divisor on Z , such that (Z, Σ) is log smooth.*

Consider Σ as a reduced subscheme of Z . Then for any irreducible stratum V of Σ , any irreducible component of $f^{-1}(V)$ is an lc center of $(X, B + f^{-1}(\Sigma - \Sigma_Z), \mathbf{M})$.

Proof. Let $k := \dim Z - \dim V$. Since (Z, Σ) is log smooth, there exist irreducible components $\Sigma_1, \dots, \Sigma_k$ of Σ such that $V = \bigcap_{i=1}^k \Sigma_i$. By Definition 5.4.2(3), for any i and any general closed point $z \in \text{Supp } \Sigma_i$, $(X, B + f^*(\Sigma - \Sigma_Z), \mathbf{M})$ is sub-lc over a neighborhood of z . Thus any irreducible component of $f^{-1}(\Sigma_i)$ is an lc center of $(X, B + f^*(\Sigma - \Sigma_Z), \mathbf{M})$. Therefore, any irreducible component of $f^{-1}(V)$ is an intersection of lc centers of $(X, B + f^*(\Sigma - \Sigma_Z), \mathbf{M})$. The lemma follows from Lemma 4.3.10. $\square$

Proposition 5.4.5 (cf. [ACSS21, Proposition 2.16]). *Let $(X, \Sigma_X, \mathbf{M})/U$ be a toroidal g-pair, (Z, Σ_Z) a log smooth pair, and $f : (X, \Sigma_X, \mathbf{M}) \rightarrow (Z, \Sigma_Z)$ a toroidal morphism. Let $(X, B, \mathbf{M})/U$ be a g-sub-pair such that $\text{Supp } B \subset \text{Supp } \Sigma_X$, $(X, B, \mathbf{M})$ is generically sub-lc/ Z , and the vertical/ Z part of B is equal to $f^{-1}(\Sigma_Z)$. Then $f : (X, B, \mathbf{M}) \rightarrow Z$ satisfies Property (*).*

Proof. Since $(X, B, \mathbf{M})$ is generically sub-lc/ Z , $\text{Supp } B \subset \text{Supp } \Sigma_X$, and the vertical/ Z part of B is equal to $f^{-1}(\Sigma_Z)$, $(X, B, \mathbf{M})$ is sub-lc. Since $\mathbf{M}$ descends to X , by [ACSS21, Proposition 2.16], $f : (X, B) \rightarrow Z$ satisfies Property (*). By Definition 5.4.2, $f : (X, B, \mathbf{M}) \rightarrow Z$ satisfies Property (*). $\square$

The following result indicates that we can always get Property (*) g-pairs by taking equi-dimensional models.

Proposition 5.4.6 (cf. [ACSS21, Proposition 2.17]). *Let $(X, B, \mathbf{M})/U$ be a g-sub-pair and $f : X \rightarrow Z$ a contraction, such that $(X, B, \mathbf{M})$ is generically sub-lc/ Z . Let $f' : (X', \Sigma_{X'}, \mathbf{M}) \rightarrow (Z', \Sigma_{Z'})$ be an equi-dimensional model of $f : (X, B, \mathbf{M}) \rightarrow Z$, associated with $h : X' \rightarrow X$ and $h_Z : Z' \rightarrow Z$. Then there exist two $\mathbb{R}$ -divisors B' and F on X' satisfying the following.*

- (1) $\text{Supp } B' \subset \Sigma_{X'}$ and $\text{Supp } F \subset \Sigma_{X'}$.
- (2) F is vertical/ Z' and

$$K_{X'} + B' + \mathbf{M}_{X'} = h^*(K_X + B + \mathbf{M}_X) + F.$$

- (3) $(X', B', \mathbf{M})$ and $(X, B, \mathbf{M})$ are crepant over the generic point of Z .
- (4) If $(X, B, \mathbf{M})$ is sub-lc, then $F \geq 0$.
- (5) If $(X, B, \mathbf{M})$ is generically sub-lc/ Z , then $f' : (X', B', \mathbf{M}) \rightarrow Z'$ satisfies Property (*).

Proof. Possibly adding components to $\Sigma_{Z'}$, we may assume that $\Sigma_{Z'}$ coincides with the image of the vertical/ Z' part of $\Sigma_{X'}$. We let $G := f'^{-1}(\Sigma_{Z'})$ and

$$K_{X'} + \tilde{B}' + \mathbf{M}_{X'} := h^*(K_X + B + \mathbf{M}_X),$$

then $G \subset \text{Supp } \Sigma_{X'}$ and $\text{Supp } \tilde{B}' \subset \text{Supp } \Sigma_{X'}$. We define B' to be the unique $\mathbb{R}$ -divisor on X' satisfying the following: for any prime divisor D on X' ,

- if D is not a component of $\text{Supp } \tilde{B}'$ nor G , then $\text{mult}_D B' = 0$,
- if D is a component of G , then $\text{mult}_D B' = 1$, and
- if D is a component of $\tilde{B}'$ but is not a component of G , then $\text{mult}_D B' = \text{mult}_D \tilde{B}'$.

Since G is the vertical/ Z' part of $\Sigma_{X'}$ and

$$\text{Supp } B' \subset \text{Supp } G \cup \text{Supp } \tilde{B}' \subset \text{Supp } \Sigma_{X'},$$

the vertical/ Z' part of B' is equal to $G = f'^{-1}(\Sigma_{Z'})$.

We define $F := B' - \tilde{B}'$. We show that B' and F satisfy our requirements.

- (1) holds immediately by our construction.
- (2) For any component D of $\text{Supp } F$, by construction, $\text{mult}_D F \neq 0$ only if D is a component of G . Thus F is vertical/ Z' .
- (3) By (2), $(X', B', \mathbf{M})$ and $(X, B, \mathbf{M})$ are crepant over the generic point of Z .
- (4) For any component D of G , $\text{mult}_D F = 1 - \text{mult}_D \tilde{B}'$. Therefore, if $(X, B, \mathbf{M})$ is sub-lc, then $\text{mult}_D \tilde{B}' \leq 1$, so $\text{mult}_D F \geq 0$. Thus $F \geq 0$.

(5) Since $(X, B, \mathbf{M})$ is sub-lc over the generic point of Z , $(X', B', \mathbf{M})$ is sub-lc over the generic point of Z . By Proposition 5.4.5, $f' : (X', B', \mathbf{M}) \rightarrow Z'$ satisfies Property (*). $\square$

The following proposition shows that Property (*) is preserved under any sequence of steps of an MMP.

Proposition 5.4.7 (cf. [ACSS21, Proposition 2.18]). *Let $(X, B, \mathbf{M})/U$ be an lc g-pair and $f : X \rightarrow Z$ a contraction, such that $f : (X, B, \mathbf{M}) \rightarrow Z$ satisfies Property (*). Let $\phi : (X, B, \mathbf{M}) \dashrightarrow (Y, B_Y, \mathbf{M})$ be a sequence of steps of a $(K_X + B + \mathbf{M}_X)$ -MMP/ Z and $f_Y : Y \rightarrow Z$ the induced morphism. Assume that ϕ is also a sequence of steps of a $(K_X + B + \mathbf{M}_X)$ -MMP/ U . Then:*

- (1) $f_Y : (Y, B_Y, \mathbf{M}) \rightarrow Z$ satisfies Property (*), and the discriminant part of $f_Y : (Y, B_Y, \mathbf{M}) \rightarrow Z$ is equal to the discriminant part of $f : (X, B, \mathbf{M}) \rightarrow Z$.
- (2) For any closed point $z \in Z$, ϕ^{-1} is an isomorphism near the generic point of any irreducible component of $f_Y^{-1}(z)$.
- (3) If f is equi-dimensional, then f_Y is equi-dimensional.

Proof. Without loss of generality, we may assume that ϕ is a step of a $(K_X + B + \mathbf{M}_X)$ -MMP/ Z .

(1) Let Σ_Z be the discriminant part of $f : (X, B, \mathbf{M}) \rightarrow Z$. By definition, (Z, Σ_Z) is log smooth.

Since the vertical/ Z part of B is equal to $f^{-1}(\Sigma_Z)$ and ϕ does not extract any divisor, the vertical/ Z part of B_Y is equal to $\phi \circ f^{-1}(\Sigma_Z) = f_Y^{-1}(\Sigma_Z)$.

For any reduced divisor $\Sigma \geq \Sigma_Z$ on Z , $(X, B + f^*(\Sigma - \Sigma_Z), \mathbf{M})/U$ is lc. Since ϕ is a step of a $(K_X + B + \mathbf{M}_X)$ -MMP/ Z , ϕ is also a step of a $(K_X + B + f^*(\Sigma - \Sigma_Z) + \mathbf{M}_X)$ -MMP/ Z . Thus

$$(Y, B_Y + \phi_* f^*(\Sigma - \Sigma_Z) = B_Y + f_Y^*(\Sigma - \Sigma_Z), \mathbf{M})$$

is lc.

Therefore, $f_Y : (Y, B_Y, \mathbf{M}) \rightarrow Z$ satisfies Property (*). By Lemma 5.4.3(2), Σ_Z is the discriminant part of $f_Y : (Y, B_Y, \mathbf{M}) \rightarrow Z$.

(2) Possibly shrinking Z to a neighborhood of z , there exists a reduced divisor $\Sigma \geq \Sigma_Z$ on Z , such that (Z, Σ) is log smooth and z is a stratum of Σ . By Lemma 5.4.4, any irreducible component of $f_Y^{-1}(z)$ is an lc center of $(Y, B_Y + f_Y^*(\Sigma - \Sigma_Z), \mathbf{M})$. By Definition 5.4.2(3), $(X, B + f^*(\Sigma - \Sigma_Z), \mathbf{M})$ is lc. For any irreducible component G of $f^{-1}(z)$, let D_G be an lc place of $(Y, B_Y + f_Y^*(\Sigma - \Sigma_Z), \mathbf{M})$ over the generic point of G . Then

$$0 \leq a(D_G, X, B + f^*(\Sigma - \Sigma_Z), \mathbf{M}) \leq a(D_G, Y, B_Y + f_Y^*(\Sigma - \Sigma_Z), \mathbf{M}) = 0.$$

Thus

$$a(D_G, X, B + f^*(\Sigma - \Sigma_Z), \mathbf{M}) = a(D_G, Y, B_Y + f_Y^*(\Sigma - \Sigma_Z), \mathbf{M}) = 0,$$

so ϕ^{-1} is an isomorphism near the generic point of G .

(3) It immediately follows from (2). $\square$

Part II. Cone theorem and MMP for algebraically integrable foliations

6. PRECISE ADJUNCTION FORMULA FOR ALGEBRAICALLY INTEGRABLE FOLIATIONS

In this section, we will establish a *precise* adjunction formula for foliations that are induced by a morphism. By saying “precise”, we mean that the adjunction formulas we provide not only preserve the log canonicity of the the generalized foliated quadruple, but also give a nice characterization of the coefficients of the boundary. More precisely, in this section we will prove the following theorem under the additional assumption that $\mathcal{F}$ is induced by a contraction:

Theorem 6.0.1 (Precise adjunction formula for generalized foliated quadruples). *Let m and n be two non-negative integers, and $b_1, \dots, b_m, r_1, \dots, r_n$ non-negative real numbers. Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a generalized foliated quadruple such that $\mathcal{F}$ is algebraically integrable. Let*

$S, B_1, \dots, B_m$ be distinct prime divisors on X , and let $\mathbf{M}_1, \dots, \mathbf{M}_n$ be nef/ U $\mathbf{b}$ -Cartier $\mathbf{b}$ -divisors on X . Suppose that

$$B = \epsilon_{\mathcal{F}}(S)S + \sum_{j=1}^m b_j B_j \text{ and } \mathbf{M} = \sum_{k=1}^n r_k \mathbf{M}_k.$$

Let $S^\nu \rightarrow S$ the normalization of S , $\mathcal{F}_S$ the restricted foliation of $\mathcal{F}$ on S^ν (see Definition 6.1.5), and $\mathbf{M}_k^S := \mathbf{M}_k|_{S^\nu}$ for any k . Then there exist prime divisors $T_1, \dots, T_l, C_1, \dots, C_q$ on S^ν , positive integers $w_1, \dots, w_q$, and non-negative integers $\{w_{i,j}\}_{1 \leq i \leq q, 1 \leq j \leq m}$ and $\{v_{i,k}\}_{1 \leq i \leq q, 1 \leq k \leq n}$, such that for any real numbers $b'_1, \dots, b'_m$ and $r'_1, \dots, r'_n$, we have the following.

Let $B' := \epsilon_{\mathcal{F}}(S)S + \sum_{j=1}^m b'_j B_j$ and $\mathbf{M}' := \sum_{k=1}^n r'_k \mathbf{M}_k$. Then:

(1)

$$K_{\mathcal{F}_S} + B'_S + \mathbf{M}'^S_{S^\nu} = (K_{\mathcal{F}} + B' + \mathbf{M}'_X)|_{S^\nu},$$

where

$$B'_S := \sum_{i=1}^l T_i + \sum_{i=1}^q \frac{w_i - 1 + \sum_{j=1}^m w_{i,j} b'_j + \sum_{k=1}^n v_{i,k} r'_k}{w_i} C_i$$

and

$$\mathbf{M}'^S := \sum_{k=1}^n r'_k \mathbf{M}_k^S = \mathbf{M}'|_{S^\nu}.$$

(2) If $(X, \mathcal{F}, B', \mathbf{M}')$ is lc near S , then $(S^\nu, \mathcal{F}_S, B'_S, \mathbf{M}'^S)$ is lc.

The complete proof of Theorem 6.0.1 will be provided in Section 8 as a consequence of the cone theorem and the existence of ACSS modifications.

6.1. Preliminaries for algebraically integrable foliations. In this subsection, we recall some basic knowledge of the theory of algebraically integrable foliations that will be used in the rest part of the paper.

Definition 6.1.1 (Algebraically integrable foliations, cf. [ACSS21, 3.1]). Let X be a normal quasi-projective variety and $\mathcal{F}$ a foliation on X . We say that $\mathcal{F}$ is an *algebraically integrable foliation* if there exists a dominant map $f : X \dashrightarrow Y$ to a quasi-projective variety Y such that $\mathcal{F} = f^{-1}\mathcal{F}_Y$, where $\mathcal{F}_Y$ is a foliation by points. In this case, we say that $\mathcal{F}$ is *induced by f* .

Definition 6.1.2 (Transverse). Let X be a normal variety, $\mathcal{F}$ a foliation on X , and $V \subset X$ a subvariety. For any point $x \in V$, we say that V is *transverse* to $\mathcal{F}$ at x if $x \notin \text{Sing}(X) \cup \text{Sing}(\mathcal{F}) \cup \text{Sing}(V)$, and for any analytic neighborhood U of x , $T_V|_U \rightarrow T_X|_U$ does not factor through $T_{\mathcal{F}}|_U$. We say that V is *everywhere transverse* to $\mathcal{F}$ if V is transverse to $\mathcal{F}$ at x for any $x \in V$ (in particular, V is smooth and V does not intersect $\text{Sing}(X)$ or $\text{Sing}(\mathcal{F})$). We say that V is *generically transverse* to $\mathcal{F}$ if V is transverse to $\mathcal{F}$ at the generic point η_V of V .

Definition 6.1.3 (Tangent, cf. [ACSS21, Section 3.4]). Let X be a normal variety, $\mathcal{F}$ a foliation on X , and $V \subset X$ a subvariety. Suppose that $\mathcal{F}$ is a foliation induced by a dominant rational map $X \dashrightarrow Z$. We say that V is *tangent* to $\mathcal{F}$ if there exists a birational morphism $\mu : X' \rightarrow X$, an equi-dimensional contraction $f' : X' \rightarrow Z$, and a subvariety $V' \subset X'$, such that

- (1) $\mu^{-1}\mathcal{F}$ is induced by f' , and
- (2) V' is contained in a fiber of f' and $\mu(V') = V$.

Definition 6.1.4 (Tangency of general fibers). Let X be a normal variety, $\mathcal{F}$ a foliation on X , and $f : X \dashrightarrow Z$ a dominant map. We say that the general fibers of f are *tangent to $\mathcal{F}$* if for any general closed point x on a general fiber F of f , the linear subspace $\mathcal{F}_x \subset T_{X,x}$ determined by the inclusion $\mathcal{F} \subset T_X$ contains $T_{F,x}$.

Definition 6.1.5 (Restricted foliation). Let X be a normal variety, $\mathcal{F}$ a foliation on X , S a prime divisor on X , and $\nu : S^\nu \rightarrow S$ the normalization of S . The *restricted foliation* of $\mathcal{F}$ on S^ν is defined in the following way.

- (1) If S is $\mathcal{F}$ -invariant, then we let $U \subset X$ be the largest open subset which does not contain $\text{Sing}(\mathcal{F}) \cup \text{Sing}(X) \cup \text{Sing}(S)$ and let $S' := S \cap U$. The natural inclusion of sheaves

$$\mathcal{F}|_{S'} \rightarrow T_X|_{S'}$$

factors through $T_{S'}$ over U , which defines a foliation $\mathcal{F}_{S'}$ on S' . $\mathcal{F}_{S'}$ extends to a foliation $\mathcal{F}_S$ on S^ν (cf. [CS23b, Lemma 2.2]), and we call $\mathcal{F}_S$ the *restricted foliation* of $\mathcal{F}$ on S^ν .

- (2) If S is not $\mathcal{F}$ -invariant, then we let $U \subset X$ be the largest open subset which does not contain $\text{Sing}(\mathcal{F}) \cup \text{Sing}(X) \cup \text{Sing}(S)$ and S is transverse to $\mathcal{F}$ everywhere in U . We let $S' := S \cap U$. Then natural inclusion of sheaves

$$\mathcal{F}|_{S'} \rightarrow T_X|_{S'}$$

induces an inclusion of sheaves $\mathcal{F}|_{S'} \cap T_{S'} \rightarrow T_{S'}$. Since $\mathcal{F}$ is saturated in T_X , $\mathcal{F}|_{S'} \cap T_{S'}$ is saturated in $T_{S'}$. Since $\mathcal{F}$ is closed under the Lie bracket, $\mathcal{F}|_{S'} \cap T_{S'} \subset \mathcal{F}$ is closed under the Lie bracket. Thus $\mathcal{F}_{S'} := \mathcal{F}|_{S'} \cap T_{S'}$ is a foliation on S' . $\mathcal{F}_{S'}$ extends to a foliation $\mathcal{F}_S$ on S^ν (cf. [CS23b, Lemma 2.2]), and we call $\mathcal{F}_S$ the *restricted foliation* of $\mathcal{F}$ on S^ν .

Definition 6.1.6 (Almost holomorphic). Let $f : X \dashrightarrow Z$ be a dominant rational map. We say that f is *almost holomorphic* if there exist non-empty open subsets $U \subset X$ and $V \subset Z$ such that $f|_U : U \rightarrow V$ is a morphism.

The following several results are useful when applying the canonical bundle formula and adjunction formula for algebraically integrable foliations.

Lemma 6.1.7 (cf. [DLM23, Lemma 2.7]). *Let $f : X' \rightarrow X$ be birational morphism between normal varieties, $\mathcal{F}$ is a foliation on X , and $\mathcal{F}' := f^{-1}\mathcal{F}$ the pullback foliation on X' . Then $\mathcal{F}'$ is algebraically integrable if and only if $\mathcal{F}$ is algebraically integrable.*

Lemma 6.1.8. *Let X be a normal quasi-projective variety, $\mathcal{F}$ a foliation on X , and $f : X \rightarrow Z$ a contraction. Suppose that the general fibers of f are tangent to $\mathcal{F}$. Then there exists a foliation $\mathcal{F}_Z$ on Z , such that $\mathcal{F} = f^{-1}\mathcal{F}_Z$.*

Proof. By Definition-Lemma 5.1.2, there exists an equi-dimensional model $f' : (X', \Sigma_{X'}, \mathbf{M}) \rightarrow (Z', \Sigma_{Z'})$ of $f : X \rightarrow Z$ associated with $h : X' \rightarrow X$ and $h_Z : Z' \rightarrow Z$. By [AD13, Lemma 6.7], there exists a foliation $\mathcal{F}_{Z'}$ on Z' such that $(f')^{-1}\mathcal{F}_{Z'} = h^{-1}\mathcal{F}$. We may let $\mathcal{F}_Z := (h_Z)_*\mathcal{F}_{Z'}$. $\square$

Lemma 6.1.9. *Let $f : X \rightarrow Z$ be a projective surjective morphism from a normal variety to a variety and let $X \xrightarrow{\sigma} Y \xrightarrow{\tau} Z$ be the Stein factorization of f . Let $\mathcal{F}$ be the foliation on X induced by f . Then $\mathcal{F}$ is also induced by σ .*

Proof. Let $\mathcal{F}_Z$ be the foliation by points on Z . Then $\mathcal{F}_Y := \tau^{-1}\mathcal{F}_Z$ is the foliation by points on Y . Since

$$\mathcal{F} = (\tau \circ \sigma)^{-1}\mathcal{F}_Z = \sigma^{-1}\mathcal{F}_Y,$$

$\mathcal{F}$ is induced by σ . $\square$

Proposition 6.1.10 (cf. [DLM23, Proposition 3.2]). *Let $\mathcal{F}$ be an algebraically integrable foliation on a normal variety X , S a prime divisor on X , and $S^\nu \rightarrow S$ the normalization of S . Let $\mathcal{F}_S$ be the restricted foliation of $\mathcal{F}$ on S^ν . Then $\mathcal{F}_S$ is algebraically integrable and $\text{rank } \mathcal{F}_S = \text{rank } \mathcal{F} - \epsilon_{\mathcal{F}}(S)$.*

Finally, we recall the following theorem, which was essentially proven in [CP19, Theorem 1.1].

Theorem 6.1.11 ([LLM23, Theorem 3.1], [CP19, Theorem 1.1]). *Let $\mathcal{F}$ be a foliation on a normal projective variety X such that $K_{\mathcal{F}}$ is not pseudo-effective. Then there exists an algebraically integrable foliation $\mathcal{E}$ such that $0 \neq \mathcal{E} \subset \mathcal{F}$.*

6.2. Foliated log resolution and adjunction formula.

Definition 6.2.1 (cf. [ACSS21, §3.2]). Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a sub-gfq such that $\mathcal{F}$ is algebraically integrable. We say that $(X, \mathcal{F}, B, \mathbf{M})$ is *foliated log smooth* if there exists a contraction $f : X \rightarrow Z$ satisfying the following.

- (1) X has at most quotient toric singularities.
- (2) $\mathcal{F}$ is induced by f .
- (3) (X, Σ_X) is toroidal for some reduced divisor Σ_X such that $\text{Supp } B \subset \Sigma_X$. In particular, $(X, \text{Supp } B)$ is toroidal, and X is $\mathbb{Q}$ -factorial klt.
- (4) There exists a log smooth pair (Z, Σ_Z) such that

$$f : (X, \Sigma_X, \mathbf{M}) \rightarrow (Z, \Sigma_Z)$$

is an equi-dimensional toroidal contraction.

- (5) $\mathbf{M}$ descends to X .

We say that $f : (X, \Sigma_X, \mathbf{M}) \rightarrow (Z, \Sigma_Z)$ is *associated with* $(X, \mathcal{F}, B, \mathbf{M})$, and also say that f is *associated with* $(X, \mathcal{F}, B, \mathbf{M})$. It is important to remark that f may not be a contraction/ U . In particular, $\mathbf{M}$ may not be nef/ Z .

Lemma 6.2.2 (cf. [ACSS21, Lemma 3.1]). *Let $(X, \mathcal{F}, B, \mathbf{M})$ be a sub-gfq such that $\mathcal{F}$ is algebraically integrable and $(X, \mathcal{F}, B, \mathbf{M})$ is foliated log smooth. Then $(X, \mathcal{F}, B^\mathcal{F}, \mathbf{M})$ is lc.*

Proof. By [ACSS21, Lemma 3.1], $(X, \mathcal{F}, B^\mathcal{F})$ is lc. Since $\mathbf{M}$ descends to X , $(X, \mathcal{F}, B^\mathcal{F}, \mathbf{M})$ is lc. $\square$

Definition 6.2.3. Let X be a normal quasi-projective variety, B an $\mathbb{R}$ -divisor on X , $\mathbf{M}$ a nef/ X $\mathbf{b}$ -divisor on X , and $\mathcal{F}$ an algebraically integrable foliation on X . A *foliated log resolution* of $(X, \mathcal{F}, B, \mathbf{M})$ is a birational morphism $h : X' \rightarrow X$ such that

$$(X', \mathcal{F}' := h^{-1}\mathcal{F}, B' := h_*^{-1}B + \text{Exc}(h), \mathbf{M})$$

is foliated log smooth, where $\text{Exc}(h)$ is the reduced h -exceptional divisor.

We remark that we do not require $K_{\mathcal{F}} + B + \mathbf{M}_X$ to be $\mathbb{R}$ -Cartier.

Lemma 6.2.4. *Let X be a normal quasi-projective variety, B an $\mathbb{R}$ -divisor on X , $\mathbf{M}$ a nef/ X $\mathbf{b}$ -divisor on X , and $\mathcal{F}$ a foliation on X that is induced by a dominant map $f : X \dashrightarrow Z$. Then:*

- (1) *If f is a contraction, then for any equi-dimensional model $f' : (X', \Sigma_{X'}, \mathbf{M}) \rightarrow (Z', \Sigma_{Z'})$ of $f : (X, B, \mathbf{M}) \rightarrow Z$ associated with $h : X' \rightarrow X$ and $h_Z : Z' \rightarrow Z$, h is a foliated log resolution of $(X, \mathcal{F}, B, \mathbf{M})$ and $h^{-1}\mathcal{F}$ is induced by f' .*
- (2) *$(X, \mathcal{F}, B, \mathbf{M})$ has a foliated log resolution.*

Proof. (1) It immediately follows from the definition of equi-dimensional models.

(2) Possibly compactifying X and Z and applying [CS23b, Lemma 2.2], we may assume that X and Z are projective. Let $g : X'' \rightarrow X$ be a birational morphism such that $f \circ g : X'' \rightarrow Z$ is a morphism, $\mathcal{F}'' := g^{-1}\mathcal{F}$, and $B'' := g_*^{-1}B + \text{Exc}(g)$, where $\text{Exc}(g)$ is the reduced g -exceptional divisor. Possibly replacing $(X, \mathcal{F}, B, \mathbf{M})$ with $(X', \mathcal{F}', B', \mathbf{M})$, we may assume that f is a morphism. Since X and Z are projective, f is a projective surjective morphism. By Lemma 6.1.9, we may assume that f is a contraction. (2) follows from (1) and Definition-Theorem 5.1.2. $\square$

Next, we prove a simple version adjunction formula for algebraically integrable generalized foliated quadruples. The detailed version of this formula, with specific coefficient control, will be discussed later. In particular, we cannot show that the boundary coefficient after adjunction is non-negative, so we can only get “sub-lc” instead of “lc”.

Theorem 6.2.5. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be an lc gfq such that $\mathcal{F}$ is algebraically integrable. Let S be a prime divisor on X such that $\text{mult}_S B = \epsilon_{\mathcal{F}}(S)$, $\nu : S^\nu \rightarrow S$ the normalization of S , $\mathbf{M}^S := \mathbf{M}|_S$, $\mathcal{F}_S$ the restricted foliation of $\mathcal{F}$ on S^ν , and*

$$K_{\mathcal{F}_S} + B_S + \mathbf{M}_{S^\nu}^S := (K_X + B + \mathbf{M}_X)|_{S^\nu}.$$

Then $(S^\nu, \mathcal{F}_S, B_S, \mathbf{M}^S)$ is sub-lc.

Proof. By Lemma 6.2.4, there exists a foliated log resolution $h : X' \rightarrow X$ of $(X, \mathcal{F}, B + S, \mathbf{M})$. By Lemma 6.2.2,

$$(X', \mathcal{F}' := h^{-1}\mathcal{F}, \tilde{B}' := (B')^{\mathcal{F}'}, \mathbf{M})$$

is lc. Let

$$K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} := h^*(K_{\mathcal{F}} + B + \mathbf{M}_X),$$

and let $\tilde{B}' := B'^{\geq 0}$. Since $(X, \mathcal{F}, B, \mathbf{M})$ is lc,

$$B'^{\mathcal{F}'} \geq \tilde{B}' \geq B'.$$

Therefore, $(X', \mathcal{F}', \tilde{B}', \mathbf{M})$ is lc. In particular, $(X', \mathcal{F}', \tilde{B}')$ is lc.

Let $S' := h_*^{-1}S$. Then there a birational morphism $h_S : S' \rightarrow S^\nu$ such that $\nu \circ h_S = h|_{S'}$. Let $\mathcal{F}_{S'}$ be the restricted foliation of $\mathcal{F}$ on S' , then $\mathcal{F}_{S'} = h_S^{-1}\mathcal{F}_S$. Let

$$K_{\mathcal{F}_{S'}} + \tilde{B}_{S'} := (K_{X'} + \tilde{B}')|_{S'}$$

and

$$K_{\mathcal{F}_{S'}} + B_{S'} + \mathbf{M}_{S'}^S := (K_{\mathcal{F}'} + B' + \mathbf{M}_{X'})|_{S'}.$$

By [ACSS21, Proposition 3.2], $(S', \mathcal{F}_{S'}, \tilde{B}_{S'})$ is lc. Since $\mathbf{M}$ descends to X' , $\mathbf{M}^S$ descends to S' , so $(S', \mathcal{F}_{S'}, \tilde{B}_{S'}, \mathbf{M}^S)$ is lc. Since $\tilde{B}' \geq B'$, $\tilde{B}_{S'} \geq B_{S'}$. Thus

$$(S', \mathcal{F}_{S'}, \tilde{B}_{S'}, \mathbf{M}^S)$$

is sub-lc. Since

$$\begin{aligned} K_{\mathcal{F}_{S'}} + B_{S'} + \mathbf{M}_{S'}^S &= (K_{\mathcal{F}'} + B' + \mathbf{M}_{X'})|_{S'} = h^*(K_{\mathcal{F}} + B + \mathbf{M}_X)|_{S'} \\ &= h_S^*((K_{\mathcal{F}} + B + \mathbf{M}_X)|_{S^\nu}) = h_S^*(K_{\mathcal{F}_S} + B_S + \mathbf{M}_{S^\nu}^S), \end{aligned}$$

$(S^\nu, \mathcal{F}_S, B_S, \mathbf{M}^S)$ is sub-lc and we are done. $\square$

Finally, we recall the following definition of F-dlt.

Definition 6.2.6 (F-dlt). Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be an lc gfq such that $\mathcal{F}$ is algebraically integrable. We say that $(X, \mathcal{F}, B, \mathbf{M})$ is *F-dlt* if there exists a foliated log resolution $f : Y \rightarrow X$ of $(X, \mathcal{F}, B, \mathbf{M})$ such that $a(D, \mathcal{F}, B, \mathbf{M}) > -\epsilon_{\mathcal{F}}(D)$ for any prime f -exceptional divisor D .

6.3. Cutting foliations by general hyperplane sections. By Theorem 6.2.5, to prove the precise adjunction formulas, we need to control the coefficients of the boundary divisors on the restricted foliation. We achieve this by cutting the foliations using general hyperplane sections until we reach the surface case. Then, we use the structure of surface singularities to achieve our result. In this subsection, we tackle the first issue: cutting foliations by general hyperplane sections. It is important to note that general hyperplane sections for foliations behave very differently comparing to usual varieties. For example, log canonicity is often not preserved [ACSS21, Example 3.4]. On the other hand, we can use the methods introduced in [DLM23, Section 3.2] to resolve this issue.

Lemma 6.3.1. *Let X be a normal quasi-projective variety and H a prime divisor on X , such that H is base-point-free and is a general member of $|H|$. Let $\mathbf{M}$ be a $\mathbf{b}$ -divisor on X such that $\mathbf{M}$ descends to a birational model X' of X and $\mathbf{M}^H := \mathbf{M}|_H$. Then $\mathbf{M}_H^H = \mathbf{M}_X|_H$.*

Proof. We may assume that the induced birational map $f : X' \dashrightarrow X$ is a morphism. We let

$$V := f(\text{Supp}(\mathbf{M}_{X'} - f_*^{-1}\mathbf{M}_X)),$$

then $\dim X - \dim V \geq 2$. Since H is general, $\dim H - \dim(V \cap H) \geq 2$. Therefore, for any prime divisor D on H and identifying D with its image in X , we have that $\mathbf{M}$ descends to X near the generic point of D . The lemma follows immediately. $\square$

6.3.1. Cutting by invariant hyperplane sections. First, we show that we can cut foliations by invariant base-point-free linear systems freely.

Proposition 6.3.2. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a sub-gfq and W a proper subvariety of X . Suppose that $\mathcal{F}$ is induced by a morphism $f : X \rightarrow Z$, $\dim Z > 0$, and W is transverse to $\mathcal{F}$. Let $H_Z \subset Z$ be a general hyperplane section. Let $H := f^*H_Z$, $\mathbf{M}^H := \mathbf{M}|_H$, and*

$$K_{\mathcal{F}_H} + B_H + \mathbf{M}_H^H := (K_{\mathcal{F}} + B + \mathbf{M}_X)|_H,$$

where $\mathcal{F}_H$ is the restricted foliation of $\mathcal{F}$ on H . Then:

- (1) H intersects W .
- (2) For any component D of $\text{Supp } B$ such that D intersects H and any component C of $D \cap H$, $\text{mult}_C B_H = \text{mult}_D B$.
- (3) If $(X, \mathcal{F}, B, \mathbf{M})$ is (sub-)lc, then $(H, \mathcal{F}_H, B_H, \mathbf{M}^H)$ is (sub-)lc.
- (4) $\mathcal{F}_H$ is induced by $f|_H : H \rightarrow H_Z$.

Proof. By Definition-Theorem 5.1.2 and Lemma 6.2.4, there exists an equi-dimensional model $f' : (X', \Sigma_{X'}, \mathbf{M}) \rightarrow (Z', \Sigma_{Z'})$ of $f : (X, B, \mathbf{M}) \rightarrow Z$ associated with $h : X' \rightarrow X$ and $h_Z : Z' \rightarrow Z$, such that h is a foliated log resolution of $(X, \mathcal{F}, B, \mathbf{M})$ and $\mathcal{F}' := h^{-1}\mathcal{F}$ is induced by f' . We let $H' := h^*H$,

$$K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} := h^*(K_{\mathcal{F}} + B + \mathbf{M}_X),$$

and

$$K_{\mathcal{F}_{H'}} + B_{H'} + \mathbf{M}_{H'}^H := (K_{\mathcal{F}'} + B' + \mathbf{M}_{X'})|_{H'}.$$

First we show that $B_{H'} = B'|_{H'}$. Let

$$R(f') := \sum_{D|D \text{ is a prime divisor on } Z'} (f'^*D - f'^{-1}(D))$$

be the ramification divisor of f' , then

$$R(f') = \sum_{D \subset \text{Supp } \Sigma_{Z'}} (f'^*D - f'^{-1}(D)).$$

Since H' and $H_{Z'}$ are general, by [AK00, Proposition 3.2], $f'|_{H'} : (H', \Sigma_{X'}|_{H'}, \mathbf{M}|_{H'}) \rightarrow (H_{Z'}, \Sigma_{Z'}|_{H_{Z'}})$ is an equi-dimensional toroidal contraction. Therefore, for any prime divisor D_Z on $H_{Z'}$, $(f'|_{H'})^*D \neq f'^{-1}(D)$ only if D_Z is a component of $\Sigma_{Z'}|_{H_{Z'}}$. Therefore,

$$\begin{aligned} R(f')|_{H'} &= R(f')|_{f'^*H_{Z'}} = \sum_{D \subset \text{Supp } \Sigma_{Z'}} ((f'|_{H'})^*D|_{H_{Z'}} - (f'|_{H'})^{-1}(D|_{H_{Z'}})) \\ &= \sum_{D_Z \subset \Sigma_{Z'}|_{H_{Z'}}} ((f'|_{H'})^*D_Z - (f'|_{H'})^{-1}(D_Z)) \\ &= \sum_{D_Z|D_Z \text{ is a prime divisor on } H_{Z'}} ((f'|_{H'})^*D_Z - (f'|_{H'})^{-1}(D_Z)) := R(f'|_{H'}) \end{aligned}$$

is the ramification divisor of $f'|_{H'}$. Thus

$$K_{\mathcal{F}'}|_{H'} = (K_{X'/Z'} - R(f'))|_{H'} = K_{H'/H_{Z'}} - R(f'|_{H'}) = K_{\mathcal{F}_{H'}}.$$

Since $\mathbf{M}_H^H = \mathbf{M}_X|_H$, we have $B_{H'} = B'|_{H'}$.

(1) Since W is transverse to $\mathcal{F}$, $W' := h^{-1}(W)$ is not tangent to $\mathcal{F}'$. Thus $\dim g(W') \geq 1$, so $H_{Z'} := h_Z^* H_Z$ intersects $g(W')$ and H_Z intersects $h_Z(g(W')) = f(W)$. Hence H intersects W .

(2) Since H is general, near the generic point η_C of C , h is an isomorphism. Since $B_{H'} = B'|_{H'}$, $B|_H = B_H$ near η_C . We may write $B = \sum b_i B_i$ where B_i are the irreducible components of B , then

$$B_H = B|_H = \sum b_i (B_i \cap H)$$

near η_C . Since H is general, there exists a unique index i such that $B_i \cap H \neq 0$ at η_C . Then $B_i \cap H = C$, $B_i = D$, and hence $\text{mult}_C B_H = b_i = \text{mult}_D B$.

(3) By Lemma 6.2.2, $(X', \mathcal{F}', \tilde{B}' := (B')^{\geq 0}, \mathbf{M})$ is lc. Let $K_{\mathcal{F}_{H'}} + \tilde{B}_{H'} := (K_{\mathcal{F}'} + \tilde{B}')|_{H'}$. By [ACSS21, Proposition 3.2], $(H', \mathcal{F}_{H'}, \tilde{B}_{H'})$ is lc. Since $\mathbf{M}$ descends to X' ,

$$K_{\mathcal{F}_{H'}} + \tilde{B}_{H'} + \mathbf{M}_{H'}^H = (K_{\mathcal{F}'} + B + \mathbf{M}_X)|_H,$$

and $\mathbf{M}^H$ descends to H' . Thus $(H', \mathcal{F}_{H'}, \tilde{B}_{H'}, \mathbf{M}^H)$ is lc. Since $\tilde{B}' \geq B'$, $\tilde{B}_{H'} \geq B_{H'}$, so $(H', \mathcal{F}_{H'}, B_{H'}, \mathbf{M}^H)$ is sub-lc. Since

$$K_{\mathcal{F}_{H'}} + B_{H'} + \mathbf{M}_{H'}^H = (h|_{H'})^* (K_{\mathcal{F}_H} + B_H + \mathbf{M}_H^H),$$

$(H, \mathcal{F}_H, B_H, \mathbf{M}^H)$ is sub-lc.

If $(X, \mathcal{F}, B, \mathbf{M})$ is lc, then $B \geq 0$. By (2), $B_H \geq 0$. Thus $(H, \mathcal{F}_H, B_H, \mathbf{M}^H)$ is lc.

(4) It immediately follows from the definition of restricted foliations and the condition that H_Z is a general hyperplane section of Z . $\square$

6.3.2. Cutting by non-invariant hyperplanes. Next we show that, if we only consider the local property of foliations, then we can cut foliation by non-invariant hyperplane sections.

Lemma 6.3.3. *Let $f : (X, \Sigma, \mathbf{M}) \rightarrow (Z, \Sigma_Z)$ be a toroidal morphism and $z \in Z$ a closed point. Let $\mathcal{F}$ be the foliation induced by f and let B be the horizontal/ Z part of Σ . Let H be a general member of a base-point-free linear system on X , such that H dominates Z . Then $(X, \mathcal{F}, B + H, \mathbf{M})$ is lc over a neighborhood of z .*

Proof. By [DLM23, Lemma 3.6], $(X, \mathcal{F}, B + H)$ is lc over a neighborhood of z . Since $\mathbf{M}$ descends to X , $(X, \mathcal{F}, B + H, \mathbf{M})$ is lc over a neighborhood of z . $\square$

Proposition 6.3.4. *Let $(X, \mathcal{F}, B, \mathbf{M})$ be a sub-gfq and W a proper subvariety of X . Suppose that $\mathcal{F}$ is algebraically integrable, W is tangent to $\mathcal{F}$, and $\dim W \geq 1$. Let $H \subset X$ be a general hyperplane section. Let $\mathbf{M}^H := \mathbf{M}|_H$ and*

$$K_{\mathcal{F}_H} + B_H + \mathbf{M}_H^H := (K_{\mathcal{F}} + B + H + \mathbf{M}_X)|_H,$$

where $\mathcal{F}_H$ is the restricted foliation of $\mathcal{F}$ on H . Then:

- (1) H intersects W .
- (2) For any component D of $\text{Supp } B$ such that D intersects H and any component C of $D \cap H$, $\text{mult}_C B_H = \text{mult}_D B$.
- (3) If $(X, \mathcal{F}, B, \mathbf{M})$ is (sub-)lc near W , then $(H, \mathcal{F}_H, B_H, \mathbf{M}^H)$ is (sub-)lc near $W|_H$.
- (4) If f is induced by a morphism $f : X \rightarrow Z$, then $\mathcal{F}_H$ is induced by $f|_H : H \rightarrow Z$.

Proof. (1) is obvious.

(2) By [Dru21, Proposition 3.6], $K_{\mathcal{F}_H} = (K_{\mathcal{F}} + H)|_H$, so $B_H + \mathbf{M}_H^H = B|_H + \mathbf{M}_X|_H$. We remark that [Dru21, Proposition 3.6] requires that $\text{rank } \mathcal{F} \geq 2$, but the same lines of the proof works for the case when $\text{rank } \mathcal{F} = 1$ as well. We may write $B = \sum b_i B_i$ where B_i are the irreducible components of B . Since H is general,

$$B_H = B|_H = \sum b_i (B_i \cap H),$$

and there exists a unique index i such that $B_i \cap H \neq 0$ at the generic point of C . Then $B_i \cap H = C$, $B_i = D$, and hence $\text{mult}_C B_H = b_i = \text{mult}_D B$.

(3) By Definition-Theorem 5.1.2 and Lemma 6.2.4, there exists an equi-dimensional model $f' : (X', \Sigma_{X'}, \mathbf{M}) \rightarrow (Z, \Sigma_Z)$ of $f : (X, B, \mathbf{M}) \rightarrow Z$ associated with $h : X' \rightarrow X$ and $h_Z : Z' \rightarrow Z$, such that h is a foliated log resolution of $(X, \mathcal{F}, B, \mathbf{M})$ and $\mathcal{F}' := h^{-1}\mathcal{F}$ is induced by f' . We let

$$K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} := h^*(K_{\mathcal{F}} + B + \mathbf{M}_X),$$

$H' := h^*H$, $W' := h^{-1}(W)$, and $\tilde{B}' := (B')^{\geq 0}$. We let z be the image of W' on Z' . Since $(X, \mathcal{F}, B, \mathbf{M})$ is lc, by Lemma 6.2.2, $(X', \mathcal{F}', \tilde{B}', \mathbf{M})$ is lc. Moreover, all components of $\tilde{B}'$ are horizontal/ Z . By Lemma 6.3.3, $(X', \mathcal{F}', \tilde{B}' + H', \mathbf{M})$ is lc over a neighborhood of z' . In particular, $(X', \mathcal{F}', \tilde{B}', \mathbf{M})$ is lc near $W'|_{H'}$. Let

$$K_{\mathcal{F}_{H'}} + \tilde{B}_{H'} := (K_{\mathcal{F}'} + \tilde{B}')|_{H'}$$

and

$$K_{\mathcal{F}_{H'}} + B_{H'} := (K_{\mathcal{F}'} + B')|_{H'}.$$

By [ACSS21, Proposition 3.2], $(H', \mathcal{F}_{H'}, \tilde{B}_{H'})$ is lc near $W'|_{H'}$. Since $\tilde{B}' \geq B'$, $\tilde{B}_{H'} \geq B_{H'}$. Thus $(H', \mathcal{F}_{H'}, B_{H'})$ is sub-lc near $W'|_{H'}$. Since $\mathbf{M}$ descends to X' , $\mathbf{M}^H$ descends to H' , so $(H', \mathcal{F}_{H'}, B_{H'}, \mathbf{M}^H)$ is sub-lc near $W'|_{H'}$. Since

$$K_{\mathcal{F}_{H'}} + B_{H'} + \mathbf{M}_{H'}^H = h|_{H'}^*(K_{\mathcal{F}_H} + B_H + \mathbf{M}_H^H),$$

$(H, \mathcal{F}_H, B_H, \mathbf{M}^H)$ is sub-lc near $W|_H$.

If $(X, \mathcal{F}, B, \mathbf{M})$ is lc, then $B \geq 0$. By (2), $B_H \geq 0$. Thus $(H, \mathcal{F}_H, B_H, \mathbf{M}^H)$ is lc near $W|_H$.

(4) It immediately follows from the definition of restricted foliations and the condition that H is a general hyperplane section of X . $\square$

6.4. Basic properties of foliated surfaces. In this subsection, we recall some basic properties of foliated surfaces. Moreover, we introduce the concept of *surface numerical gfs* and study their basic properties. This is crucial for the proof of adjunction formulas.

Definition 6.4.1. Let X be a normal surface, $\mathcal{F}$ a foliation on X , and $x \in X$ a closed point such that $x \notin \text{Sing}(X)$ and $x \in \text{Sing}(\mathcal{F})$. Let v be a vector field generating $\mathcal{F}$ near x . By [Bru15, Page 2, Line 17-18], $v(x) = 0$ and $(Dv)|_x$ has exactly two eigenvalues λ_1 and λ_2 .

We say that x is a *reduced singularity* of $\mathcal{F}$ if at least one of λ_1 and λ_2 is not 0 (say, λ_2) and $\frac{\lambda_1}{\lambda_2} \notin \mathbb{Q}^+$. We say that $\mathcal{F}$ has *at most reduced singularities* if for any closed point $p \in X$, $\mathcal{F}$ is either non-singular at p or p is a reduced singularity of $\mathcal{F}$.

Definition 6.4.2 (Minimal resolution). Let X be a normal surface, $\mathcal{F}$ a foliation on X , $f : Y \rightarrow X$ a projective birational morphism, and $\mathcal{F}_Y := f^{-1}\mathcal{F}$.

We say that f is a *resolution* of $\mathcal{F}$ if Y is smooth and $\mathcal{F}_Y$ has at most reduced singularities. By [Sei68] (we refer to [Can04, Pages 908–912] for a detailed explanation), resolution of $\mathcal{F}$ always exists.

We say that f is the *minimal resolution* of $\mathcal{F}$ if for any resolution $g : W \rightarrow X$ of $\mathcal{F}$, g factors through f , i.e. there exists a projective birational morphism $h : W \rightarrow Y$ such that $g = f \circ h$. By definition, the minimal resolution of $\mathcal{F}$ is unique, and by [Che23, Proposition 1.17], the minimal resolution of $\mathcal{F}$ exists.

Definition 6.4.3. Let X be a normal surface with at most cyclic quotient singularities, $\mathcal{F}$ a foliation on X , and C a reduced curve on X such that no component of C is $\mathcal{F}$ -invariant. For any closed point $x \in X$, we define $\text{tang}(\mathcal{F}, C, x)$ in the following way.

- If $x \notin \text{Sing}(X)$, then we let v be a vector field generating $\mathcal{F}$ around x , and f a holomorphic function defining C around x . We define

$$\text{tang}(\mathcal{F}, C, x) := \dim_{\mathbb{C}} \frac{\mathcal{O}_{X,x}}{\langle f, v(f) \rangle}.$$

- If $x \in \text{Sing}(X)$, then x is a cyclic quotient singularity of index r for some integer $r \geq 2$. Let $\rho : \tilde{X} \rightarrow X$ be an index 1 cover of $X \ni x$, $\tilde{x} := \rho^{-1}(x)$, $\tilde{C} := \rho^*C$, and $\tilde{\mathcal{F}}$ the foliation induced by the sheaf $\rho^*\mathcal{F}$ near $\tilde{x}$. Then $\tilde{x}$ is a smooth point of $\tilde{X}$, and we define

$$\text{tang}(\mathcal{F}, C, x) := \frac{1}{r} \text{tang}(\tilde{\mathcal{F}}, \tilde{C}, \tilde{x}).$$

We define

$$\text{tang}(\mathcal{F}, C) := \sum_{x \in X} \text{tang}(\mathcal{F}, C, x).$$

By [Bru02, Section 2], $\text{tang}(\mathcal{F}, C)$ is well-defined.

Definition 6.4.4. Let X be a normal surface with at most cyclic quotient singularities, $\mathcal{F}$ a foliation on X , and C a reduced curve on X such that all components of C are $\mathcal{F}$ -invariant. For any closed point $x \in X$, we define $Z(\mathcal{F}, C, x)$ in the following way.

- If $x \notin \text{Sing}(X)$, then we let ω be a 1-form generating $\mathcal{F}$ around x , and f a holomorphic function generating C around x . Then there are uniquely determined holomorphic functions g, h and a holomorphic 1-form η on X near x , such that $g\omega = hdf + f\eta$ and f, h are coprime. We define

$$Z(\mathcal{F}, C, x) := \text{the vanishing order of } \frac{h}{g} \Big|_C \text{ at } x.$$

By [Chapter 2, Page 15]Bru15, $Z(\mathcal{F}, C, x)$ is independent of the choice of ω .

- If $x \in C \cap \text{Sing}(X)$, then we define $Z(\mathcal{F}, C, x) := 0$.

We define

$$Z(\mathcal{F}, C) := \sum_{x \in C} Z(\mathcal{F}, C, x).$$

By [Bru02, Section 2], $Z(\mathcal{F}, C)$ is well-defined.

Definition 6.4.5 (Dual graph). Let n be a positive integer, and $C = \cup_{i=1}^n C_i$ be a collection of irreducible curves contained in the non-singular locus of a normal surface X . We define the *dual graph* $\mathcal{D}(C)$ of C as follows.

- (1) The vertices $v_i = v_i(C_i)$ of $\mathcal{D}(C)$ correspond to the curves C_i .
- (2) For any $i \neq j$, the vertices v_i and v_j are connected by $C_i \cdot C_j$ edges.
- (3) Each vertex v_i is labeled by $w(C_i) := -C_i^2$. The integer $w(C_i)$ is called the *weight* of C_i .

For any projective birational morphism $f : Y \rightarrow X$ between surfaces, let $E = \cup_{i=1}^n E_i$ be the reduced exceptional divisor for some non-negative integer n . Suppose that E is not contained in the non-singular locus of Y . Then we define $\mathcal{D}(f) := \mathcal{D}(E)$.

Definition 6.4.6. A *surface numerical sub-gfq* (*surface num-sub-gfq* for short) $(X, \mathcal{F}, B, \mathbf{M})/U$ consists of a normal surface X , a rank 1 foliation $\mathcal{F}$ on X , an $\mathbb{R}$ -divisor B on X , and a nef/ U $\mathbf{b}$ -divisor $\mathbf{M}$. We say that $(X, \mathcal{F}, B, \mathbf{M})$ is a *surface numerical gfq* (*surface num-gfq* for short) if $(X, \mathcal{F}, B)$ is a surface num-sub-gfq and $B \geq 0$.

Let $(X, \mathcal{F}, B, \mathbf{M})$ be a surface num-sub-gfq. Let $f : Y \rightarrow X$ be a resolution of X with prime f -exceptional divisors $E_1, \dots, E_n$ for some non-negative integer n . Since $\{(E_i \cdot E_j)\}_{n \times n}$ is negative definite, the equation

$$\begin{pmatrix} (E_1 \cdot E_1) & \cdots & (E_1 \cdot E_n) \\ \vdots & \ddots & \vdots \\ (E_n \cdot E_1) & \cdots & (E_n \cdot E_n) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} -(K_{\mathcal{F}_Y} + B_Y + \mathbf{M}_Y) \cdot E_1 \\ \vdots \\ -(K_{\mathcal{F}_Y} + B_Y + \mathbf{M}_Y) \cdot E_n \end{pmatrix}$$

has a unique solution $(a_1, \dots, a_n)$, where $\mathcal{F}_Y := f^{-1}\mathcal{F}$ and $B_Y := f_*^{-1}B$. For any prime divisor E on Y , we define

$$a_{\text{num},f}(E, \mathcal{F}, B, \mathbf{M}) := -\text{mult}_E \left(B_Y + \sum_{i=1}^n a_i E_i \right).$$

Lemma 6.4.7. *Let $(X, \mathcal{F}, B, \mathbf{M})$ be a sub-gfq such that $\dim X = 2$ and $\text{rank } \mathcal{F} = 1$. Let $f : Y \rightarrow X$ be a resolution of X and E a prime divisor on Y . Then $a_{\text{num},f}(E, \mathcal{F}, B, \mathbf{M}) = a(E, \mathcal{F}, B, \mathbf{M})$.*

Proof. If E is not exceptional over X , then

$$a_{\text{num},f}(E, \mathcal{F}, B, \mathbf{M}) = -\text{mult}_E B = a(E, \mathcal{F}, B, \mathbf{M})$$

and we are done. Thus we may assume that E is exceptional over X . Let $E_1, \dots, E_n$ be all the f -exceptional prime divisors and let

$$K_{\mathcal{F}_Y} + \sum_{j=1}^n a_j E_j + B_Y + \mathbf{M}_Y = f^*(K_{\mathcal{F}} + B + \mathbf{M}_X),$$

where $\mathcal{F}_Y := f^{-1}\mathcal{F}$ and $B_Y := f_*^{-1}B$. Then

$$\left(K_{\mathcal{F}_Y} + \sum_{j=1}^n a_j E_j + B_Y + \mathbf{M}_Y \right) \cdot E_i = 0$$

for any i . Therefore,

$$a_{\text{num},f}(E_i, \mathcal{F}, B, \mathbf{M}) = -a_i = a(E_i, \mathcal{F}, B, \mathbf{M})$$

for any i . Since $E = E_j$ for some j ,

$$a_{\text{num},f}(E, \mathcal{F}, B, \mathbf{M}) = a(E, \mathcal{F}, B, \mathbf{M})$$

and we are done. $\square$

Lemma 6.4.8. *Let $(X, \mathcal{F}, B, \mathbf{M})$ be a surface num-sub-gfq and $f : Y \rightarrow X$, $f' : Y' \rightarrow X$ two resolutions of X . Let E be a prime divisor over X such that $\text{center}_Y E$ and $\text{center}_{Y'} E$ are divisors. Then*

$$a_{\text{num},f}(E, \mathcal{F}, B, \mathbf{M}) = a_{\text{num},f'}(E, \mathcal{F}, B, \mathbf{M}).$$

Proof. If E is on X then

$$a_{\text{num},f}(E, \mathcal{F}, B, \mathbf{M}) = -\text{mult}_E B = a_{\text{num},f'}(E, \mathcal{F}, B, \mathbf{M}),$$

so we may assume that E is exceptional over X .

Let $g : W \rightarrow Y$ and $g' : W \rightarrow Y'$ be a common resolution, and $h : W \rightarrow X$ the induced birational morphism. Possibly replacing f' with h , we may assume that there exists a morphism $g : Y' \rightarrow Y$. Let E_i be the prime f' -exceptional divisors,

$$B_{Y'} := f'^{-1}B - \sum_i a_{\text{num},f'}(E_i, \mathcal{F}, B, \mathbf{M}) E_i,$$

and $B_Y := g_* B_{Y'}$. Then $(K_{\mathcal{F}_{Y'}} + B_{Y'} + \mathbf{M}_{Y'}) \cdot E_i = 0$ for any E_i . Since Y is smooth, $K_{\mathcal{F}_Y} + B_Y + \mathbf{M}_Y$ is $\mathbb{R}$ -Cartier. By applying the negativity lemma twice, we have

$$K_{\mathcal{F}_{Y'}} + B_{Y'} + \mathbf{M}_{Y'} = g^*(K_{\mathcal{F}_Y} + B_Y + \mathbf{M}_Y).$$

Thus $(K_{\mathcal{F}_Y} + B_Y + \mathbf{M}_Y) \cdot g_* E_i = 0$ for any E_i , so

$$a_{\text{num},f}(E_i, \mathcal{F}, B, \mathbf{M}) = -\text{mult}_{g_* E_i} B_Y = \text{mult}_{E_i} B_{Y'} = a_{\text{num},f'}(E_i, \mathcal{F}, B, \mathbf{M})$$

for any E_i such that $g_* E_i \neq 0$. In particular, $a_{\text{num},f}(E, \mathcal{F}, B, \mathbf{M}) = a_{\text{num},f'}(E, \mathcal{F}, B, \mathbf{M})$. $\square$

Definition 6.4.9. Let $(X, \mathcal{F}, B)$ be a surface num-sub-gfq. We define $a(E, \mathcal{F}, B, \mathbf{M}) := a_{\text{num}, f}(E, \mathcal{F}, B, \mathbf{M})$ for an arbitrary resolution $f : Y \rightarrow X$ of X such that E is a divisor on Y . Lemmas 6.4.7 and 6.4.8 guarantee that there is no abuse of notations.

Let $(X, \mathcal{F}, B, \mathbf{M})$ be a surface num-gfq. We say that $(X, \mathcal{F}, B, \mathbf{M})$ is *num-lc* if $a(E, \mathcal{F}, B, \mathbf{M}) \geq -\epsilon_{\mathcal{F}}(E)$ for any prime divisor E over X .

Lemma 6.4.10. *Let $(X, \mathcal{F}, B, \mathbf{M})$ be a surface num-gfq and $x \in X$ a closed point. Then for any prime divisor E over $X \ni x$,*

$$a(E, \mathcal{F}, B, \mathbf{M}) \leq a(E, \mathcal{F}, B),$$

and

$$a(E, \mathcal{F}, B, \mathbf{M}) = a(E, \mathcal{F}, B)$$

if and only if $\mathbf{M}$ descends to X over a neighborhood of x . In particular, if $(X, \mathcal{F}, B, \mathbf{M})$ is num-lc, then $(X, \mathcal{F}, B)$ is num-lc.

Proof. It follows from [KM98, Lemma 3.41]. $\square$

Lemma 6.4.11. *Let $(X, \mathcal{F}, B, \mathbf{M})$ be an lc gfq such that $\dim X = 2$ and $\text{rank } \mathcal{F} = 1$. Then $K_{\mathcal{F}}$, $\mathbf{M}_X$, and all components of B are $\mathbb{R}$ -Cartier.*

Proof. We only need to show that $\mathbf{M}_X$ and all components of B are $\mathbb{R}$ -Cartier near x for any closed point $x \in X$. If $\mathcal{F}$ is num-terminal near x , then by [LMX23a, Theorem 3.19], $\mathcal{F}$ is terminal near x , and X is $\mathbb{Q}$ -factorial klt near x . Therefore, $\mathbf{M}_X$ and all components of B are $\mathbb{R}$ -Cartier near x . If $\mathcal{F}$ is not num-terminal near x , then by Lemma 6.4.10, $\mathbf{M}$ descends to X over a neighborhood of x , and $(X, \mathcal{F}, B)$ is num-lc. By [LMX23a, Theorem 3.19], $x \notin \text{Supp } B$. In particular, $\mathbf{M}_X$ and all components of B are $\mathbb{R}$ -Cartier near x . $\square$

6.5. Adjunction formula for surface generalized foliated quadruples. In this subsection we establish the adjunction formula for surface generalized foliated quadruples based on the classification of foliated surface singularities. Depending on whether the foliation itself is terminal, we establish two adjunction formulas.

Lemma 6.5.1. *Let $(X, \mathcal{F}, B, \mathbf{M})$ be an lc gfq such that $\dim X = 2$, $\text{rank } \mathcal{F} = 1$, and B_j are the irreducible components of B . Let C be an $\mathcal{F}$ -invariant curve with normalization $\nu : C^\nu \rightarrow C$. Let $x \in C$ be a closed point, such that $\mathcal{F}$ is not terminal near x . Then:*

- (1) $x \notin \text{Supp } B$ and $\mathbf{M}$ descends to X over a neighborhood of x .
- (2) For any closed point $y \in \nu^{-1}(x)$, the vanishing order of $K_{\mathcal{F}}|_{C^\nu}$ at y is a non-negative integer.

Proof. (1) Since $\mathcal{F}$ is not terminal near x and $(X, \mathcal{F}, B, \mathbf{M})$ is lc, by Lemma 6.4.11 and [LMX23a, Theorem 3.19], $B = 0$ near x . By Lemma 6.4.10, $\mathbf{M}$ descends to X over a neighborhood of x .

(2) By considering a local analytic neighborhood of x and separate C into different analytic irreducible components, we may assume that $y = \nu^{-1}(x)$. (2) follows from [LMX23a, Theorem 3.19]. More precisely, we let $h : Y \rightarrow X$ be the minimal resolution of $\mathcal{F}$ near x and let $C_Y := h_*^{-1}C$, then we only need to show that

$$K_{\mathcal{F}} \cdot C - K_{C^\nu} = h^* K_{\mathcal{F}} \cdot C_Y - K_{C_Y}$$

is a positive integer over a neighborhood of x . (2) follows by checking all cases of [LMX23a, Theorem 3.19] and apply [CS20, Proposition 2.16(3)] to C_Y for each case. $\square$

Lemma 6.5.2. *Let $(X, \mathcal{F}, B = \sum_{j=1}^m b_j B_j, \mathbf{M})$ be a gfq such that $\dim X = 2$, $\text{rank } \mathcal{F} = 1$, and B_j are the irreducible components of B . Let C be an $\mathcal{F}$ -invariant curve with normalization C^ν . Let $x \in C$ be a closed point such that $\mathcal{F}$ is terminal near x , I the order of the local fundamental group $\pi_1(X \ni x)$, and $\mathbf{M}^C := \mathbf{M}|_{C^\nu}$. Then there exists a positive integer I and non-negative integers $w_1, \dots, w_m$ satisfying the following.*

- (1) X is $\mathbb{Q}$ -factorial klt near x and C is non-singular near x .
 (2) $\mu := \text{mult}_x(\mathbf{M}_X|_{C^\nu} - \mathbf{M}_{C^\nu}^C) \geq 0$.
 (3) For any real numbers $b'_1, \dots, b'_m$, the vanishing order of

$$\left(K_{\mathcal{F}} + \sum_{j=1}^m b'_j B_j + \mathbf{M}_X \right) \Big|_{C^\nu} - \mathbf{M}_{C^\nu}^C$$

at x is

$$\frac{I - 1 + \sum_{j=1}^m w_j b'_j}{I} + \mu.$$

Moreover, if $(X, \mathcal{F}, \sum_{j=1}^m b'_j B_j, \mathbf{M})$ is lc, then

$$0 \leq \frac{I - 1 + \sum_{j=1}^m w_j b'_j}{I} + \mu \leq 1.$$

- (4) Suppose that $\mathbf{M} = \sum_{k=1}^m r_k \mathbf{M}_k$ where each $\mathbf{M}_k$ is a nef/ X $\mathbf{b}$ -Cartier $\mathbf{b}$ -divisor. Let $\mathbf{M}_k^C := \mathbf{M}_k|_{C^\nu}$ for each k . Then there exist non-negative integers $v_1, \dots, v_n$, such that for any real numbers $b'_1, \dots, b'_m, r'_1, \dots, r'_n$, the vanishing order of

$$\left(K_{\mathcal{F}} + \sum_{j=1}^m b'_j B_j + \sum_{k=1}^n r'_k \mathbf{M}_{i,X} \right) \Big|_{C^\nu} - \sum_{k=1}^n r'_k \mathbf{M}_{k,C^\nu}^C$$

at x is

$$\frac{I - 1 + \sum_{j=1}^m w_j b'_j + \sum_{k=1}^n v_k r'_k}{I}.$$

Moreover, if $(X, \mathcal{F}, \sum_{j=1}^m b'_j B_j, \sum_{k=1}^n r'_k \mathbf{M}_k)$ is lc, then

$$0 \leq \frac{I - 1 + \sum_{j=1}^m w_j b'_j + \sum_{k=1}^n v_k r'_k}{I} \leq 1.$$

We note that $\mathbf{M}_X|_{C^\nu}$ in (2),

$$\left(K_{\mathcal{F}} + \sum_{j=1}^m b'_j B_j + \mathbf{M}_X \right) \Big|_{C^\nu}$$

in (3), and

$$\left(K_{\mathcal{F}} + \sum_{j=1}^m b'_j B_j + \sum_{k=1}^n r'_k \mathbf{M}_{k,X} \right) \Big|_{C^\nu}$$

in (4) may not be well-defined, but they are at least well-defined near x so there is no confusion for the statements of the lemma.

Proof. (1) It follows from [LMX23a, Theorem 3.19].

Since all statements in the lemma are local near x , possibly shrinking X to a neighborhood of x , in the following, we may assume that X is $\mathbb{Q}$ -factorial klt, $\mathcal{F}$ is terminal, and C is non-singular. In particular, we will identify C with C^ν in the following arguments.

(2) Let $h : Y \rightarrow X$ be a birational morphism such that $\mathbf{M}$ descends to Y . Let $C_Y := h_*^{-1}C$. Since $\mathbf{M}$ is nef/ X and $\mathbf{M}_X$ is $\mathbb{R}$ -Cartier, by the negativity lemma, $h^* \mathbf{M}_X - \mathbf{M}_Y \geq 0$. Thus

$$\mu = \text{mult}_x((h|_{C_Y})_*(h^* \mathbf{M}_X - \mathbf{M}_Y)|_{C_Y}) \geq 0.$$

(3) Since $\mathcal{F}$ is terminal, by [LMX23b, Theorem 3.2], there exist non-negative integers $w_1, \dots, w_m$, such that the vanishing order of

$$\left(K_{\mathcal{F}} + \sum_{j=1}^m b'_j B_j + \mathbf{M}_X \right) \Big|_C - \mathbf{M}_C^C$$

at x is

$$q := \frac{I - 1 + \sum_{j=1}^m w_j b'_j}{I} + \mu$$

for any real numbers $b'_1, \dots, b'_m$. If $(X, \mathcal{F}, \sum_{j=1}^m b'_j B_j, \mathbf{M})$ is lc, then $b'_j \geq 0$ for each j and $\mu \geq 0$ by (2). Thus $q \geq 0$. By Theorem 6.2.5, $q \leq 1$. (3) follows.

(5) Since $\mathbf{M}_{k,X}$ is an integral divisor for each k , $I \operatorname{mult}_x \mathbf{M}_{k,X}|_C$ is an integer. We let

$$v_k := I(\operatorname{mult}_x \mathbf{M}_{k,X}|_C - \operatorname{mult}_x \mathbf{M}_{k,C}^C),$$

then each v_k is an integer. Possibly replacing Y with a high model, we may assume that $\mathbf{M}_k$ descends to Y for each k . By the negativity lemma, $h^* \mathbf{M}_{k,X} - \mathbf{M}_{k,Y} \geq 0$. Thus

$$v_k = I \operatorname{mult}_x((h|_{C_Y})_*(h^* \mathbf{M}_{k,X} - \mathbf{M}_{k,Y})|_{C_Y}) \geq 0,$$

so each v_k is a non-negative integer. By (4), the vanishing order of

$$\left(K_{\mathcal{F}} + \sum_{j=1}^m b'_j B_j + \sum_{k=1}^n r'_k \mathbf{M}_{k,X} \right) \Big|_C - \sum_{k=1}^n r'_k \mathbf{M}_{k,C}^C$$

at x is

$$l := \frac{I - 1 + \sum_{j=1}^m w_j b'_j + \sum_{k=1}^n v_k r'_k}{I}$$

for any real numbers $b_1, \dots, b'_m, \dots, r'_1, \dots, r'_n$. If $(X, \mathcal{F}, \sum_{j=1}^m b'_j B_j, \sum_{k=1}^n r'_k \mathbf{M}_{k,X})$ is lc, then $b'_j \geq 0$ for each j , and $\sum_{k=1}^n v_k r'_k \geq 0$ by (2). Thus $l \geq 0$. By Theorem 6.2.5, $l \leq 1$. (5) follows. $\square$

6.6. Precise adjunction formula when the foliation is induced by a morphism.

Theorem 6.6.1. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be an lc gfq such that $\mathcal{F}$ is induced by a contraction $X \rightarrow Z$. Let S be a prime divisor on X such that $\operatorname{mult}_S B = \epsilon_{\mathcal{F}}(S)$, $\nu : S^\nu \rightarrow S$ the normalization of S , $\mathbf{M}^S := \mathbf{M}|_S$, $\mathcal{F}_S$ the restricted foliation of $\mathcal{F}$ on S^ν , and*

$$K_{\mathcal{F}_S} + B_S + \mathbf{M}_{S^\nu}^S := (K_X + B + \mathbf{M}_X)|_{S^\nu}.$$

Then $(S^\nu, \mathcal{F}_S, B_S, \mathbf{M}^S)$ is lc.

Proof. By Theorem 6.2.5, $(S^\nu, \mathcal{F}_S, B_S, \mathbf{M}^S)$ is sub-lc. The rest part of Theorem 6.6.1 is only about the coefficients of divisors on S^ν , which is a codimension 2 property on X . Since $\mathcal{F}$ is induced by a contraction $X \rightarrow Z$, by Propositions 6.3.2 and 6.3.4, we may cut X by general elements in base-point-free linear systems and assume that $\dim X = 2$.

If $\operatorname{rank} \mathcal{F} = 0$, then since $(X, \mathcal{F}, B, \mathbf{M})$ is lc, $B = 0$ and $\mathbf{M}$ descends to X , and the theorem is trivial. If $\operatorname{rank} \mathcal{F} = 2$ then the theorem follows from [BZ16, Definition 4.7]. Thus we may assume that $\operatorname{rank} \mathcal{F} = 1$.

Let $\mathcal{F}_S$ be the restricted foliation of $\mathcal{F}$ on S^ν . If S is not $\mathcal{F}$ -invariant, then $\mathcal{F}_S$ is a foliation by points. By Lemma 6.4.11, $K_{\mathcal{F}} + S + B$ is $\mathbb{R}$ -Cartier. By [Spi20, Proposition 3.4], there exists an $\mathbb{R}$ -divisor $B_S \geq 0$ on S^ν such that

$$(K_{\mathcal{F}} + S + B)|_{S^\nu} = K_{\mathcal{F}_S} + B_S.$$

By Theorem 6.2.5, $(S^\nu, \mathcal{F}_S, B_S)$ is lc, so $B_S = 0$. The theorem follows in this case. Thus we may assume that S is $\mathcal{F}$ -invariant.

We only need to check the coefficient near any closed point y on S^ν . Let x be the image of y in S . If $\mathcal{F}$ is terminal at x , then the theorem follows from Lemma 6.5.2. If $\mathcal{F}$ is not terminal at x , then the theorem follows from Lemma 6.5.1. $\square$

Theorem 6.6.2. *Theorem 6.0.1 holds when $\mathcal{F}$ is induced by a contraction $X \rightarrow Z$.*

Proof. When $\mathcal{F}$ is induced by a contraction $X \rightarrow Z$, Theorem 6.0.1(2) follows from Theorem 6.0.1(1) and Theorem 6.6.1, so we only need to prove Theorem 6.0.1(1). Since Theorem 6.0.1(1) is only about the coefficients of divisors on S^ν , which is a codimension 2 property on X , by Propositions 6.3.2 and 6.3.4, we may cut X by general elements in base-point-free linear systems and assume that $\dim X = 2$.

If $\text{rank } \mathcal{F} = 0$, then since $(X, \mathcal{F}, B, \mathbf{M})$ is lc, $B = 0$ and $\mathbf{M}$ descends to X , and the theorem is trivial. If $\text{rank } \mathcal{F} = 2$ then the theorem follows from the usual precise adjunction formula for lc g-pairs [BZ16, Page 306, Line 30]. Thus we may assume that $\text{rank } \mathcal{F} = 1$.

Let $\mathcal{F}_S$ be the restricted foliation of $\mathcal{F}$ on S^ν . If S is not $\mathcal{F}$ -invariant, then $\mathcal{F}_S$ is a foliation by points. By Lemma 6.4.11, $K_{\mathcal{F}} + S + B$ is $\mathbb{R}$ -Cartier. By [Spi20, Proposition 3.4], there exists an $\mathbb{R}$ -divisor $B_S \geq 0$ on S^ν such that

$$(K_{\mathcal{F}} + S)|_{S^\nu} = K_{\mathcal{F}_S} + B_S.$$

By Theorem 6.2.5, $(S^\nu, \mathcal{F}_S, B_S)$ is lc, so $B_S = 0$. The theorem follows in this case. Thus we may assume that S is $\mathcal{F}$ -invariant.

We only need to check the coefficient near any closed point y on S^ν . Let x be the image of y in S . If $\mathcal{F}$ is terminal at x , then the theorem follows from Lemma 6.5.2. If $\mathcal{F}$ is not terminal at x , then the theorem follows from Lemma 6.5.1. $\square$

Remark 6.6.3. Theorem 6.6.2, even without the control on the coefficients and with $\mathbf{M} = \mathbf{0}$, is already stronger than [ACSS21, Proposition 3.2] as the latter requires that X is $\mathbb{Q}$ -factorial.

The complete versions of Theorem 6.0.1 will be proven after we establish the existence of ACSS modifications in Section 8.

7. PROPERTY (*) AND ACSS GENERALIZED FOLIATED QUADRUPLES

In this section, we introduce the concepts of Property (*) and ACSS generalized foliated quadruples and study their basic properties.

7.1. Qdlt generalized pairs.

Definition 7.1.1 (Qdlt). Let $(X, B, \mathbf{M})/U$ be an lc g-pair. We say that $(X, B, \mathbf{M})$ is *qdl*t if there exists an open (possibly empty) subset $V \subset X$ satisfying the following.

- (1) $(V, B|_V)$ is $\mathbb{Q}$ -factorial toroidal. In particular, $B|_V$ is a reduced divisor.
- (2) V contains the generic point of any lc center of $(X, B, \mathbf{M})$.
- (3) The generic point of any lc center of $(X, B, \mathbf{M})$ is the generic point of an lc center of $(V, B|_V)$.

Lemma 7.1.2. *Let $(X, B, \mathbf{M})/U$ be a lc g-pair. Then the following conditions are equivalent:*

- (1) $(X, B, \mathbf{M})$ is qdlt.
- (2) For any lc center of $(X, B, \mathbf{M})$ with generic point η , near η , (X, B) is $\mathbb{Q}$ -factorial toroidal and $\mathbf{M}$ descends to X .

Proof. It is clear that (2) implies (1). Thus we only need to prove (1) implies (2).

Let η be the generic point of an lc center of $(X, B, \mathbf{M})$. Since $(X, B, \mathbf{M})$ is qdlt, there exists an open subset $V \subset X$ which satisfies Definition 7.1.1. In particular, η is an lc center of $(V, B|_V)$ and $\mathbf{M}_X|_V$ is $\mathbb{R}$ -Cartier. We let $\mathbf{M}^V := \mathbf{M}|_V$ be the restricted $\mathbf{b}$ -divisor of $\mathbf{M}$ on V , then $\mathbf{M}^V$ is nef/ V and $\mathbf{M}_V^V = \mathbf{M}_X|_V$. Suppose that $h : V' \rightarrow V$ is a resolution of V such that $\mathbf{M}^V$ descends to V' , and there exists a prime divisor E on V' such that $\text{center}_V E = \bar{\eta}$ and E is an lc place of $(V, B|_V)$. By the negativity lemma,

$$\mathbf{M}_{V'}^V = h^* \mathbf{M}_V^V - F$$

for some $F \geq 0$. Moreover, we have either $F = 0$ over η or $\text{Supp } F = \text{Supp } h^{-1}(\bar{\eta})$. Since $(X, B, \mathbf{M})$ is lc, $(V, B|_V, \mathbf{M}^V)$ is lc. Thus $F = 0$ over η . Possibly shrinking V , we may assume that $\mathbf{M}$ descends to V . The lemma follows. $\square$

Lemma 7.1.3. *Let $(X, B, \mathbf{M})$ be an lc g-pair and x a (not necessarily closed) point of X such that $\bar{x}$ is an lc center of $(X, B, \mathbf{M})$. Let $d := \dim X - \dim \bar{x}$. Then the following conditions are equivalent:*

- (1) $(X, B, \mathbf{M})$ is qdlt near x .
- (2) *There exist components $D_1, \dots, D_d$ of $\lfloor B \rfloor$, such that*
 - (a) K_X and each D_i is $\mathbb{Q}$ -Cartier near x , and
 - (b) $x \in \text{Supp } D_i$ for each i .

Proof. (1) $\Rightarrow$ (2) follows from the definition of qdlt, which in turn follows from the definition of toroidal pairs.

We prove (2) $\Rightarrow$ (1). Possibly shrinking X to a neighborhood of x , we may assume that $(X, \sum_{i=1}^d D_i)$ is a pair. Since $B \geq \sum_{i=1}^d D_i$, (X, D) is lc near x . By [dFKX17, Proposition 34], $B = \sum_{i=1}^d D_i$ near x , (X, B) is qdlt near x , and $\bar{x}$ is an lc center of (X, B) . Since $(X, B, \mathbf{M})$ is lc, $\bar{x}$ is an lc center of $(X, B, \mathbf{M})$, and $(X, B, \mathbf{M})$ is qdlt near x . $\square$

Lemma 7.1.4. *Let $(X, B, \mathbf{M})$ be a qdlt g-pair and $D \geq 0$ an $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor on X such that $D \subset \text{Supp}\{B\}$. Then there exists a positive real number δ such that $(X, B + \delta D, \mathbf{M})$ is qdlt.*

Proof. By the definition, $\text{Supp}\{B\}$ does not contain any lc center of $(X, B, \mathbf{M})$. Thus $(X, B + \epsilon D, \mathbf{M})$ is lc for some positive real number ϵ . Let $\delta := \frac{\epsilon}{2}$, then $(X, B + \delta D, \mathbf{M})$ is lc, and any lc center of $(X, B + \delta D, \mathbf{M})$ is an lc center of $(X, B, \mathbf{M})$. By the definition, $(X, B + \delta D, \mathbf{M})$ is qdlt. $\square$

Lemma 7.1.5. *Let $(X, B, \mathbf{M})/U$ be an lc g-pair and $\phi : (X, B, \mathbf{M}) \dashrightarrow (X', B', \mathbf{M})$ a sequence of steps of a $(K_X + B + \mathbf{M}_X)$ -MMP. Suppose that $(X, B, \mathbf{M})$ is qdlt. Then $(X', B', \mathbf{M})$ is qdlt.*

We remark here that ϕ may not be an MMP/ U so $(X', B', \mathbf{M})/U$ may not be a g-pair, but $(X', B', \mathbf{M})/X'$ is a g-pair.

Proof. Let S' be an lc center of $(X', B', \mathbf{M})$ with generic point $\eta_{S'}$. Let E be an lc place of $(X', B', \mathbf{M})$ such that $\text{center}_{X'} E = S'$. Since ϕ is a sequence of steps of a $(K_X + B + \mathbf{M}_X)$ -MMP,

$$0 \leq a(E, X, B, \mathbf{M}) \leq a(E, X', B', \mathbf{M}) \leq 0,$$

so E is an lc place of $(X, B, \mathbf{M})$, and ϕ^{-1} is an isomorphism near $\eta_{S'}$.

Let $S := \text{center}_X E$. Then near the generic point of S , (X, B) is $\mathbb{Q}$ -factorial toroidal and S is an lc center of (X, B) . Thus near the generic point of S' , (X', B') is $\mathbb{Q}$ -factorial toroidal and S' is an lc center of (X', B') . By Lemma 7.1.3, $(X', B', \mathbf{M})$ is qdlt. $\square$

7.2. Definition of Property (*) and ACSS generalized foliated quadruples.

Definition 7.2.1. Let $f : X \rightarrow Z$ be a projective morphism between normal quasi-projective varieties and G an $\mathbb{R}$ -divisor on X . We say that G is *super*/ Z if either Z is a point, or there exist ample Cartier divisors $H_1, \dots, H_{2\dim X+1}$ on Z such that $G \geq \sum_{i=1}^{2\dim X+1} f^* H_i$.

Definition 7.2.2 (Property (*) g-fq). Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a sub-g-fq. Let $G \geq 0$ be a reduced divisor on X and let $f : X \rightarrow Z$ be a projective morphism. We say that $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ satisfies Property (*) if the following conditions hold:

- (1) $f : (X, B + G, \mathbf{M}) \rightarrow Z$ satisfies Property (*) (See Definition 5.4.2). In particular, π is a contraction.
- (2) $\mathcal{F}$ is induced by f .
- (3) G is an $\mathcal{F}$ -invariant divisor.

If $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ satisfies Property (*), then we say that $(X, \mathcal{F}, B, \mathbf{M})$ satisfy Property (*), and say that f , Z , and G are *associated* with $(X, \mathcal{F}, B, \mathbf{M})$.

It is clear that property (*) is independent of the choice of U . We remark that the choice of f and G may not be unique. We also remark that f may not be a morphism/ U .

Definition 7.2.3 (ACSS gfq, cf. [DLM23, Definition 4.3]). Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a gfq, $G \geq 0$ a reduced divisor on X , and $f : X \rightarrow Z$ a projective morphism. We say that $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is *weak ACSS* if

- (1) $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ satisfies Property $(*)$ and $(X, \mathcal{F}, B, \mathbf{M})$ is lc, and
- (2) f is equi-dimensional.

We say that $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is *ACSS* if the following additional conditions are satisfied:

- (3) There exist an $\mathbb{R}$ -divisor $D \geq 0$ on X and a nef/ X $\mathbf{b}$ -divisor $\mathbf{N}$ such that
 - (a) $\text{Supp}\{B\} \subset \text{Supp } D$,
 - (b) $\mathbf{N} - \alpha \mathbf{M}$ is nef/ X for some $\alpha > 1$, and
 - (c) for any reduced divisor $\Sigma \geq f(G)$ such that (Z, Σ) is log smooth,

$$(X, B + D + G + f^*(\Sigma - f(G)), \mathbf{N})$$

is qdlt. In particular, $D + \mathbf{N}_X - \mathbf{M}_X$ is $\mathbb{R}$ -Cartier,

- (4) For any lc center of $(X, \mathcal{F}, B, \mathbf{M})$ with generic point η , over a neighborhood of η ,
 - (a) $\mathbf{M}$ descends to X ,
 - (b) η is the generic point of an lc center of $(X, \mathcal{F}, \lfloor B \rfloor)$, and
 - (c) $f : (X, B + G) \rightarrow (Z, f(G))$ is a toroidal morphism, in particular, (X, B) is toroidal and $B = \lfloor B \rfloor$.

If $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is ACSS, then we say that f , Z , and G are *properly associated with* $(X, \mathcal{F}, B, \mathbf{M})$. If $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is ACSS and G is super/ Z , then we say that $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is *super ACSS*.

If $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is ACSS *weak ACSS* (resp. *ACSS*, *super ACSS*), then we say that $(X, \mathcal{F}, B, \mathbf{M})/Z$ and $(X, \mathcal{F}, B, \mathbf{M})$ are *weak ACSS* (resp. *ACSS*, *super ACSS*).

Remark 7.2.4. It is possible that $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ and $(X, \mathcal{F}, B, \mathbf{M}; G')/Z$ both satisfy Property $(*)$, but $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is ACSS while $(X, \mathcal{F}, B, \mathbf{M}; G')/Z$ is not. On the other hand, by definition, if $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ and $(X, \mathcal{F}, B, \mathbf{M}; G')/Z$ both satisfy Property $(*)$, then $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is weak ACSS if and only if $(X, \mathcal{F}, B, \mathbf{M}; G')/Z$ is weak ACSS.

Remark 7.2.5. The key reason why we define the technical concept “ACSS” is because of the following two reasons, one from the classical minimal model program point of view, and the other from the foliation point of view.

From the classical minimal model program point of view, ACSS foliated triples behave more similar to qdlt pairs than Property $(*)$ foliated triples. In fact, when $\mathcal{F} = T_X$, “Property $(*)$ ” is equivalent to “lc”, while “ACSS” is equivalent to “qdlt”.

From the foliation point of view, ACSS foliated triples are very close to F-dlt foliated triples. In fact, we will show that $\mathbb{Q}$ -factorial F-dlt foliated triples are always ACSS (Theorem 17.0.1). We conjecture that the condition ACSS is equivalent to the condition F-dlt.

Conjecture 7.2.6. *Let $(X, \mathcal{F}, B, \mathbf{M})$ be a generalized foliated quadruple. Then $(X, \mathcal{F}, B, \mathbf{M})$ is F-dlt if and only if it is ACSS.*

An interesting case of Conjecture 7.2.6 is when $\mathcal{F} = T_X$ and $\mathbf{M} = \mathbf{0}$, when it says that a pair (X, B) is qdlt if and only there exists a log toroidal modification $f : Y \rightarrow X$ which only extracts divisors E such that $a(E, X, B) > -1$. We cannot find any literature even on this simplified version of the conjecture. In fact, the dlt version of this conjecture, which indicates that different definitions of dlt coincides, is not a trivial result, and is only proven by Szabó [Sza94] based on a complicated resolution lemma.

7.3. Basic properties of Property $(*)$ and ACSS generalized foliated quadruples. In this subsection, we prove several lemmas that will be very useful when applying to the minimal model program for algebraically integrable foliations.

Lemma 7.3.1. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a gfq and $f : X \rightarrow Z$ a contraction such that $(X, \mathcal{F}, B, \mathbf{M})/Z$ satisfies Property (*) (resp. is weak ACSS). Then there exists a super/ Z divisor G on X such that if $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ satisfies Property (*) (resp. is weak ACSS).*

Proof. If $(X, \mathcal{F}, B, \mathbf{M})/Z$ satisfies Property (*) (resp. is weak ACSS), then there exists a divisor $G_0 \geq 0$ on X such that $(X, \mathcal{F}, B, \mathbf{M}; G_0)/Z$ satisfies Property (*) (resp. is weak ACSS). We let $H_1, \dots, H_{2 \dim X + 1}$ be general elements of a very ample linear system on Z and let

$$G := G_0 + \sum_{i=1}^{2 \dim X + 1} f^* H_i.$$

Then $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ satisfies Property (*) (resp. is weak ACSS). $\square$

Lemma 7.3.2. *Assume that $(X, \mathcal{F}, B, \mathbf{M})/U$ and $(X, \mathcal{F}, B', \mathbf{M}')/U$ are two gfq's such that $B \geq B'$ and $\mathbf{M} - \mathbf{M}'$ is nef/ X , and all components of B are horizontal/ Z .*

Let $f : X \rightarrow Z$ be a contraction and G a divisor on X such that $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ satisfies Property () (resp. is weak ACSS, is ACSS, is super ACSS). Then $(X, \mathcal{F}, B', \mathbf{M}'; G)/Z$ satisfies Property (*) (resp. is weak ACSS, is ACSS, is super ACSS).*

Proof. The proof of this lemma is straightforward by checking the definitions. However, for the sake of clarity and to assist the reader, we offer a detailed proof below.

Step 1. Suppose that $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ satisfies Property (*). Since $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ satisfies Property (*), we have the following:

- $f : (X, B + G, \mathbf{M}) \rightarrow Z$ satisfies Property (*). Let $\Sigma_Z := f(G)$. Then we have the following:
 - (Z, Σ_Z) is log smooth.
 - Since all components of B are horizontal/ Z , $G = f^{-1}(\Sigma_Z)$. Since $B \geq B' \geq 0$, all components of B' are horizontal/ Z . Thus the vertical/ Z part of $B' + G$ is equal to G .
 - For any closed point $z \in Z$ and any reduced divisor $\Sigma \geq \Sigma_Z$ such that (Z, Σ) is log smooth near z , $(X, B + f^*(\Sigma - \Sigma_Z), \mathbf{M})$ is sub-lc over a neighborhood of z . Since $B \geq B'$ and $\mathbf{M} - \mathbf{M}'$ is nef/ U , $(X, B' + f^*(\Sigma - \Sigma_Z), \mathbf{M}')$ is sub-lc over a neighborhood of z .
- $\mathcal{F}$ is induced by f .
- G is an $\mathcal{F}$ -invariant divisor.

Therefore, $f : (X, B' + G, \mathbf{M}') \rightarrow Z$ satisfies Property (*).

Step 2. Suppose that $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is weak ACSS. Then:

- $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ satisfies Property (*) and $(X, \mathcal{F}, B, \mathbf{M})$ is lc. By **Step 1**, $(X, \mathcal{F}, B', \mathbf{M}'; G)/Z$ satisfies Property (*). Since $B \geq B'$ and $\mathbf{M} - \mathbf{M}'$ is nef/ U , $(X, \mathcal{F}, B', \mathbf{M}')$ is lc.
- f is equi-dimensional.

Thus $(X, \mathcal{F}, B', \mathbf{M}'; G)/Z$ is weak ACSS.

Step 3. Suppose that $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is ACSS. Then:

- $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is weak ACSS. By **Step 2**, $(X, \mathcal{F}, B', \mathbf{M}'; G)/Z$ is weak ACSS.
- There exists an $\mathbb{R}$ -divisor $D \geq 0$ on X and a nef/ X $\mathbf{b}$ -divisor $\mathbf{N}$ satisfying the following. Let $D' := B - B' + D$. Then:
 - $\text{Supp}\{B\} \subset \text{Supp}\{D\}$. Since $B \geq B' \geq 0$, $\text{Supp}\{B'\} \subset \text{Supp}\{D'\}$.
 - $\mathbf{N} - \alpha \mathbf{M}$ is nef/ X for any $\alpha \geq 1$. Since $\mathbf{M} - \mathbf{M}'$ is nef/ U ,

$$\mathbf{N} - \alpha \mathbf{M}' = (\mathbf{N} - \alpha \mathbf{M}) + \alpha(\mathbf{M} - \mathbf{M}')$$

is nef/ X .

- For any reduced divisor $\Sigma \geq f(G)$ such that (Z, Σ) is log smooth,
 $(X, B + D + G + f^*(\Sigma - f(G)), \mathbf{N}) = (X, B' + D' + G + f^*(\Sigma - f(G)), \mathbf{N})$
 is qdlt.
- For any lc center W of $(X, \mathcal{F}, B', \mathbf{M}')$ with generic point η_W , since $(X, \mathcal{F}, B, \mathbf{M})$ is lc, $B \geq B'$, and $\mathbf{M} - \mathbf{M}'$ is nef/ U , W is an lc center of $(X, \mathcal{F}, B, \mathbf{M})$. Moreover, over a neighborhood of η_W , $B = B'$ and $\mathbf{M} = \mathbf{M}'$. Therefore, over a neighborhood of η_W , we have the following:
 - $\mathbf{M}$ descends to X , so $\mathbf{M}'$ descends to X .
 - W is an lc center of $(X, \mathcal{F}, \lfloor B \rfloor)$. Since $B = B'$, W is an lc center of $(X, \mathcal{F}, \lfloor B' \rfloor)$.
 - $f : (X, B + G) \rightarrow (Z, f(G))$ is a toroidal morphism. Since $B = B'$, $f : (X, B' + G) \rightarrow (Z, f(G))$ is a toroidal morphism.

Thus $(X, \mathcal{F}, B', \mathbf{M}'; G)/Z$ is ACSS.

Step 4. Suppose that $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is super ACSS. Then G is super/ Z . By **Step 3**, $(X, \mathcal{F}, B', \mathbf{M}'; G)/Z$ is ACSS. Thus $(X, \mathcal{F}, B', \mathbf{M}'; G)/Z$ is super ACSS. $\square$

Lemma 7.3.3. *Let $(X, \mathcal{F}, B, \mathbf{M})$ be foliated log smooth gfq such that $\mathcal{F}$ is algebraically integrable, $f : (X, \Sigma_X, \mathbf{M}) \rightarrow (Z, \Sigma_Z)$ a contraction associated to $(X, \mathcal{F}, B, \mathbf{M})$, and G the vertical/ Z part of Σ_X . Then $(X, \mathcal{F}, B^{\mathcal{F}}, \mathbf{M}; G)/Z$ is $\mathbb{Q}$ -factorial ACSS, and $(X, \mathcal{F}, B^{\mathcal{F}}, \mathbf{M}; G')/Z$ is $\mathbb{Q}$ -factorial super ACSS for some $G' \geq G$.*

Proof. The proof of this lemma is straightforward by checking the definitions and applying [AK00, Proposition 3.2]. However, for the sake of clarity and to assist the reader, we offer a detailed proof below.

First we show that $(X, \mathcal{F}, B^{\mathcal{F}}, \mathbf{M}; G)/Z$ is $\mathbb{Q}$ -factorial ACSS. By assumption, X is $\mathbb{Q}$ -factorial. By Lemma 7.3.2, we only need to show that $(X, \mathcal{F}, \Sigma_X - G, \mathbf{M}; G)/Z$ is ACSS, and we may assume that $B = B^{\mathcal{F}} = \Sigma_X - G$.

By Proposition 5.4.5, $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ satisfies Property (*). By Lemma 6.2.2, $(X, \mathcal{F}, B, \mathbf{M})$ is lc, so $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is weak ACSS.

Let $D := 0$ and $\mathbf{N} := \mathbf{0}$. Then:

- Since $\{B\} = 0$, $\text{Supp}\{B\} \subset \text{Supp } D$.
- Since $\mathbf{M}$ descends to X , $\mathbf{N} - 2\mathbf{M}$ is nef/ X .
- For any reduced divisor $\Sigma \geq f(G)$, by [AK00, Proposition 3.2],

$$f : (X, B + D + G + f^*(\Sigma - f(G)), \mathbf{M}) \rightarrow (Z, \Sigma)$$

is toroidal.

For any lc center W of $(X, \mathcal{F}, B, \mathbf{M})$ with generic point η_W , near η_W , we have the following:

- $\mathbf{M}$ descends to X .
- Since $B = \lfloor B \rfloor$ and $\mathbf{M}$ descends to X , W is an lc center of $(X, \mathcal{F}, \lfloor B \rfloor)$.
- Since $f : (X, B + G, \mathbf{M}) \rightarrow (Z, f(G))$ is a toroidal morphism, $f : (X, B + G) \rightarrow (Z, f(G))$ is a toroidal morphism.

Therefore, $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is ACSS.

Let $H_1, \dots, H_{2 \dim X + 1}$ be ample Cartier divisors on Z . By [AK00, Proposition 3.2],

$$f : (X, \Sigma_X + \sum_{i=1}^{2 \dim X + 1} f^* H_i, \mathbf{M}) \rightarrow (Z, \Sigma_Z + \sum_{i=1}^{2 \dim X + 1} H_i)$$

is associated with $(X, \mathcal{F}, B, \mathbf{M})$. Thus

$$\left(X, \mathcal{F}, B, \mathbf{M}; G' := G + \sum_{i=1}^{2 \dim X + 1} f^* H_i \right) / Z$$

is $\mathbb{Q}$ -factorial ACSS. Since G' is super/ Z , we are done. $\square$

Lemma 7.3.4. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a sub-gfq, D an $\mathbb{R}$ -divisor on X , and $\mathbf{N}$ a $\mathbf{b}$ -divisor on X such that $D + \mathbf{N}_X$ is $\mathbb{R}$ -Cartier and $\mathbf{N}$ descends to a birational model of X . Suppose that $\mathcal{F}$ is algebraically integrable. Let*

$$t := \sup\{s \mid s \geq 0, \mathbf{M} + s\mathbf{N} \text{ is nef}/U, \text{ and } (X, \mathcal{F}, B + sD, \mathbf{M} + s\mathbf{N})/X \text{ is sub-lc}\}.$$

Then either $t = +\infty$, or

$$t = \max\{s \mid s \geq 0, \mathbf{M} + s\mathbf{N} \text{ is nef}/U, \text{ and } (X, \mathcal{F}, B + sD, \mathbf{M} + s\mathbf{N})/X \text{ is sub-lc}\}.$$

Moreover, one of the following cases hold:

- (1) $t = +\infty$.
- (2) $t < +\infty$, and $\mathbf{M} + (t + \delta)\mathbf{N}$ is not nef/ U for any $\delta > 0$.
- (3) $t < +\infty$, $\mathbf{M} + (t + \delta_0)\mathbf{N}$ is nef/ U for some $\delta_0 > 0$, and there exists a prime divisor E over X , such that

$$a(E, X, \mathcal{F}, B + tD, \mathbf{M} + t\mathbf{N}) = -\epsilon_{\mathcal{F}}(E)$$

and

$$a(E, X, \mathcal{F}, B + sD, \mathbf{M} + s\mathbf{N}) < -\epsilon_{\mathcal{F}}(E)$$

for any $s > t$.

In particular, $(X, \mathcal{F}, B + tD)$ is sub-lc and $\mathbf{M} + t\mathbf{N}$ is nef/ U if $t < +\infty$.

Proof. We may assume that $t < +\infty$. Since discrepancies of divisors are preserved under crepant pullbacks, by Definition-Theorem 5.1.2 and Lemma 6.2.4, we may assume that $\mathbf{M}$ and $\mathbf{N}$ descend to X and $(X, \mathcal{F}, \text{Supp } B \cup \text{Supp } D)$ is foliated log smooth. Then

$$t = \min\{\sup\{s \mid s \geq 0, \mathbf{M}_X + s\mathbf{N}_X \text{ is nef}/U\}, \sup\{s \mid s \geq 0, (X, \mathcal{F}, B + sD)/X \text{ is sub-lc}\}\}.$$

Since nef is a closed condition,

$$\sup\{s \mid s \geq 0, \mathbf{M}_X + s\mathbf{N}_X \text{ is nef}/U\} = \max\{s \mid s \geq 0, \mathbf{M}_X + s\mathbf{N}_X \text{ is nef}/U\} \text{ or } +\infty.$$

Thus we may assume that

$$t = \sup\{s \mid s \geq 0, (X, \mathcal{F}, B + sD)/X \text{ is sub-lc}\} < +\infty$$

and $\mathbf{M}_X + t\mathbf{N}_X$ is nef/ U . By Lemma 6.2.2,

$$\begin{aligned} t &= \sup\{s \mid 0 \leq s \leq l, a(E, X, \mathcal{F}, B + sD) \geq -\epsilon_{\mathcal{F}}(E) \text{ for any prime divisor } E \text{ on } X\} \\ &= \sup\{s \mid 0 \leq s \leq l, a(E, X, \mathcal{F}, B + sD) \geq -\epsilon_{\mathcal{F}}(E) \text{ for any prime divisor } E \subset \text{Supp } D\}. \end{aligned}$$

Since there are only finitely many components of $\text{Supp } D$ and $t < +\infty$,

$$t = \max\{s \mid 0 \leq s \leq l, a(E, X, \mathcal{F}, B + sD) \geq -\epsilon_{\mathcal{F}}(E) \text{ for any prime divisor } E \subset \text{Supp } D\}.$$

and there exists a component E of $\text{Supp } D$, such that $a(E, X, \mathcal{F}, B + tD) = -\epsilon_{\mathcal{F}}(E)$ and $\text{mult}_E D > 0$. The lemma follows. $\square$

Lemma 7.3.5. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a gfq, $f : X \rightarrow Z$ a contraction, and G a divisor on X , such that $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is ACSS. Let D be an $\mathbb{R}$ -divisor on X and $\mathbf{N}$ a $\mathbf{b}$ -divisor on X satisfying the following:*

- (1) D and $\mathbf{N}_X$ are $\mathbb{R}$ -Cartier.
- (2) $\text{Supp } D \subset \text{Supp}\{B\}$ and $\mathbf{N}$ descends to a birational model of X .
- (3) $\mathbf{M} + \mathbf{N}$ is nef/ U , and $\mathbf{M} - \delta\mathbf{N}$ is nef/ U for some $\delta > 0$.

Then there is a positive real number γ such that $(X, \mathcal{F}, B + \alpha D, \mathbf{M} + \beta\mathbf{N}; G)/Z$ is ACSS for any $\alpha, \beta \in [0, \gamma]$.

Proof. Possibly replacing δ with $\min\{1, \delta\}$ and then replacing $\mathbf{N}$ with $\delta\mathbf{N}$, we may assume that $\delta = 1$ and $\mathbf{M} - \mathbf{N}$ is nef/ U .

By assumption, $\text{Supp } D$ does not contain any lc center of $(X, \mathcal{F}, B, \mathbf{M})$, and $\mathbf{M}$ descends to X near the generic point of any lc center of $(X, \mathcal{F}, B, \mathbf{M})$. Since $\mathbf{M} - \mathbf{N}$ is nef/ X and $\mathbf{M} + \mathbf{N}$ is nef/ X , near the generic point of any lc center of $(X, \mathcal{F}, B, \mathbf{M})$, $\mathbf{N}$ is nef/ X and $-\mathbf{N}$ is nef/ X . Thus $\mathbf{N}$ descends to X near the generic point of any lc center of $(X, \mathcal{F}, B, \mathbf{M})$.

Since $\mathbf{M} + \mathbf{N}$ is nef/ U , by Lemma 7.3.4, there exists a real number $\gamma_0 \in (0, 1)$ such that $(X, \mathcal{F}, B + \gamma_0 D, \mathbf{M} + \gamma_0 \mathbf{N})$ is lc. Possibly replacing γ_0 with $\frac{1}{2}\gamma_0$, we may assume that $(X, \mathcal{F}, B + \gamma_0 D, \mathbf{M} + \gamma_0 \mathbf{N})$ and $(X, \mathcal{F}, B, \mathbf{M})$ have the same lc centers.

Since $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is ACSS, there exists an $\mathbb{R}$ -divisor $D' \geq 0$ on X and a nef/ X $\mathbf{b}$ -divisor $\mathbf{N}'$ on X , such that $\text{Supp}\{B\} \subset \text{Supp } D'$, $\mathbf{N}' - \alpha' \mathbf{M}$ is nef/ X for some $\alpha' > 1$, and for any $\Sigma \geq f(G)$ such that (Z, Σ) is log smooth,

$$(X, B + D' + G + \pi^*(\Sigma - f(G)), \mathbf{N}')$$

is qdlt. Possibly replacing α' , we may assume that $D' \geq (\alpha' - 1) \text{Supp } D'$.

In the following, we show that

$$\gamma := \min \left\{ \gamma_0, \frac{\alpha' - 1}{2} \right\}$$

satisfies our requirements. By Lemma 7.3.2, we only need to show that $(X, \mathcal{F}, B + \gamma D, \mathbf{M} + \gamma \mathbf{N}; G)/Z$ is ACSS.

- (Definition 7.2.3(3.a)) Since

$$\text{Supp } D \subset \text{Supp}\{B\} \subset \text{Supp } D',$$

we have

$$D' - \gamma D \geq 2\gamma \text{Supp } D' - \gamma \text{Supp } D = \gamma(\text{Supp } D' - \text{Supp } D) + \gamma \text{Supp } D' \geq \gamma \text{Supp } D',$$

hence

$$\text{Supp}\{B + \alpha D\} \subset \text{Supp } D' = \text{Supp}(D' - \gamma D).$$

- (Definition 7.2.3(3.b)) Let $\alpha'' := \frac{\alpha'}{1+\gamma}$. Then $\alpha'' > 1$, and

$$\mathbf{N}' - \alpha''(\mathbf{M} + \gamma \mathbf{N}) = \mathbf{N}' - \alpha' \mathbf{M} + \alpha'' \gamma(\mathbf{M} - \mathbf{N})$$

is nef/ X .

- (Definition 7.2.3(3.c)) For any reduced divisor $\Sigma \geq f(G)$ such that (Z, Σ) is log smooth, $(X, B + \gamma D + (D' - \gamma D) + f^*(\Sigma - f(G)), \mathbf{N}') = (X, B + D' + f^*(\Sigma - f(G)), \mathbf{N}')$

is qdlt. In particular, $(X, B + \gamma D + f^*(\Sigma - f(G)), \mathbf{M} + \beta \mathbf{N})$ is lc.

- (Definition 7.2.3(1-2)) Since $(X, \mathcal{F}, B + \gamma_0 D, \mathbf{M} + \gamma_0 \mathbf{N})$ is lc, $(X, \mathcal{F}, B + \gamma D, \mathbf{M} + \gamma \mathbf{N})$ is lc. Since $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is ACSS, $(Z, \Sigma_Z := f(G))$ is log smooth, $G = f^{-1}(\Sigma_Z)$, B is horizontal/ Z , $\mathcal{F}$ is induced by f , G is $\mathcal{F}$ -invariant, and f is equi-dimensional. Since $\text{Supp } D \subset \text{Supp}\{B\}$, $B + \gamma D$ is horizontal/ Z , so the horizontal/ Z part of $B + \gamma D + G$ is G .

- Definition 7.2.3(4)) Let W be an lc center of $(X, \mathcal{F}, B + \gamma D, \mathbf{M} + \gamma \mathbf{N})$. Since $(X, \mathcal{F}, B + \gamma_0 D, \mathbf{M} + \gamma_0 \mathbf{N})$ and $(X, \mathcal{F}, B, \mathbf{M})$ have the same lc centers, W is an lc center of $(X, \mathcal{F}, B, \mathbf{M})$ and an lc center of $(X, \mathcal{F}, B + \gamma D, \mathbf{M} + \gamma \mathbf{N})$. In particular, $\mathbf{N}$ descends to X near the generic point of X and $D = 0$ near the generic point of X . Since $(X, \mathcal{F}, B, \mathbf{M})$ is ACSS, near the generic point η of any lc center of $(X, \mathcal{F}, B + \gamma D, \mathbf{M} + \gamma \mathbf{N})$,
 - $\mathbf{M} + \gamma \mathbf{N}$ descends to X ,
 - η is the generic point of an lc center of $(X, \mathcal{F}, \lfloor B \rfloor) = (X, \mathcal{F}, \lfloor B + \gamma D \rfloor)$, and
 - $f : (X, B + G) \rightarrow (Z, \Sigma_Z)$ is a toroidal morphism, so $f : (X, B + \gamma D + G) \rightarrow (Z, \Sigma_Z)$ is a toroidal morphism.

□

Finally, we recall the following proposition which shows that the numerical property of the foliated log canonical divisor and the log canonical divisor are related with each other for generalized foliated quadruples satisfying Property (*).

Proposition 7.3.6 (cf. [ACSS21, Proposition 3.6]). *Let $(X, B + G, \mathbf{M})$ be a g -sub-pair and $f : X \rightarrow Z$ an equi-dimensional contraction, such that $f : (X, B + G, \mathbf{M}) \rightarrow Z$ satisfies Property (*). Assume that B is horizontal/ Z and G is vertical/ Z . Let $\mathcal{F}$ be the foliation induced by f and let $\mathbf{N}$ be the moduli part of $f : (X, B + G, \mathbf{M}) \rightarrow Z$. Then:*

- (1) $K_{\mathcal{F}} + B + \mathbf{M}_X \sim \mathbf{N}_X$.
- (2) $K_{\mathcal{F}} + B + \mathbf{M}_X \sim_Z K_X + B + G + \mathbf{M}_X$.

In particular, $K_{\mathcal{F}} + B + \mathbf{M}_X$ is $\mathbb{R}$ -Cartier.

Proof. Since $f : (X, B + G, \mathbf{M}) \rightarrow Z$ satisfies Property (*), Z is smooth. Let

$$R := \sum_{D|D \text{ is a prime divisor on } Z} (f^*D - f^{-1}(D)).$$

Since f is equi-dimensional, we have

$$K_{\mathcal{F}} = K_{X/Z} - R.$$

Let B_Z be the discriminant part of $f : (X, B + G, \mathbf{M}) \rightarrow Z$. By Lemma 5.4.3, B_Z is reduced. Since B is horizontal/ Z , $B_Z = f(G)$.

Claim 7.3.7. $f^*B_Z = R + G$.

Proof. We let D be a prime divisor on X such that D is vertical/ Z . Since f is equi-dimensional, $D_Z := f(D)$ is a divisor.

If D_Z is a component of B_Z , then D is a component of the vertical/ Z part of $B + G$. Since B is horizontal/ Z , D is a component of G . Thus $\text{mult}_D G = 1$. Therefore,

$$\begin{aligned} \text{mult}_D f^*B_Z &= \text{mult}_D f^*D_Z = \text{mult}_D f^{-1}(D_Z) + \text{mult}_D (f^*D_Z - f^{-1}(D_Z)) \\ &= \text{mult}_D G + \text{mult}_D R. \end{aligned}$$

If D_Z is not a component of B_Z , then $\text{mult}_D f^*B_Z = 0$. Since $f(G) = B_Z$, $\text{mult}_D G = 0$. Since B_Z is the discriminant part of $f : (X, B + G, \mathbf{M}) \rightarrow Z$,

$$1 = \sup\{t \mid (X, B + G + tf^*D_Z, \mathbf{M}) \text{ is sub-lc over the generic point of } D_Z\}.$$

Thus f^*D_Z is a reduced divisor, hence $\text{mult}_D R = 0$.

Since f^*B_Z and $R + G$ are both vertical/ Z , the claim follows. $\square$

Proof of Proposition 7.3.6 continued. By Claim 7.3.7, $f^*B_Z = R + G$. Thus

$$\begin{aligned} \mathbf{N}_X &\sim K_X + B + G + \mathbf{M}_X - f^*(K_Z + B_Z) = K_{X/Z} + B + G + \mathbf{M}_X - f^*B_Z \\ &= K_{\mathcal{F}} + R + B + G - f^*B_Z = K_{\mathcal{F}} + B + \mathbf{M}_X. \end{aligned}$$

(1) immediately follows. Since Z is smooth and B_Z is reduced, $K_Z + B_Z$ is Cartier. Thus

$$\mathbf{N}_X \sim K_X + B + G + \mathbf{M}_X - f^*(K_Z + B_Z) \sim_Z K_X + B + G + \mathbf{M}_X.$$

$\square$

7.4. (*)-models and ACSS models.

Definition 7.4.1. Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a gfq such that $\mathcal{F}$ is algebraically integrable. A $(*)$ -modification (resp. $\mathbb{Q}$ -factorial $(*)$ -modification, ACSS modification, super ACSS modification) of $(X, \mathcal{F}, B, \mathbf{M})$ is a birational morphism $h : X' \rightarrow X$ such that

$$(1) \quad \left(X', \mathcal{F}' := h^{-1}\mathcal{F}, B' := h_*^{-1}(B \wedge \text{Supp } B) + (\text{Supp Exc}(h))^{\mathcal{F}'}, \mathbf{M} \right)$$

is weak ACSS (resp. $\mathbb{Q}$ -factorial weak ACSS, ACSS, super ACSS),

- (2) X' is klt, and
- (3) for any h -exceptional prime divisor E ,

$$a(E, \mathcal{F}, B, \mathbf{M}) \leq -\epsilon_{\mathcal{F}}(E).$$

In particular, if $(X, \mathcal{F}, B, \mathbf{M})$ is lc, then $a(E, \mathcal{F}, B, \mathbf{M}) = -\epsilon_{\mathcal{F}}(E)$ for any h -exceptional prime divisor E .

We say that $(X', \mathcal{F}', B', \mathbf{M})$ is a $(*)$ -model (resp. $\mathbb{Q}$ -factorial $(*)$ -model, ACSS model, super ACSS model) of $(X, \mathcal{F}, B, \mathbf{M})$. Moreover, for any divisor G on X' and contraction $f : X' \rightarrow Z$ such that $(X', \mathcal{F}', B', \mathbf{M}; G)/Z$ satisfies Property $(*)$ (resp. satisfies Property $(*)$, is ACSS, is super ACSS), we say that $(X', \mathcal{F}', B', \mathbf{M}; G)/Z$ is a $(*)$ -model (resp. $\mathbb{Q}$ -factorial $(*)$ -model, ACSS model, super ACSS model) of $(X, \mathcal{F}, B, \mathbf{M})$. In addition, if

- (4) $D \subset \text{Supp } G$ for any h -exceptional $\mathcal{F}'$ -invariant divisor,

then we say that $h : X' \rightarrow X$ is a *proper* $(*)$ -modification (resp. *proper* $\mathbb{Q}$ -factorial $(*)$ -modification, *proper* ACSS modification, *great* ACSS modification) of $(X, \mathcal{F}, B, \mathbf{M})$, and say that $(X', \mathcal{F}', B', \mathbf{M})$ is a *proper* $(*)$ -model (resp. $\mathbb{Q}$ -factorial *proper* $(*)$ -model, *proper* ACSS model, *great* ACSS model) of $(X, \mathcal{F}, B, \mathbf{M})$.

Notation 7.4.2. Let $(X_0, \mathcal{F}_0, B_0, \mathbf{M})/U$ be a gfq satisfying Property $(*)$ and is associated with $X \rightarrow Z$ and G . When we say the following

$$(X_0, \mathcal{F}_0, B_0, \mathbf{M}; G_0) \xrightarrow{f_0} (X_1, \mathcal{F}_1, B_1, \mathbf{M}; G_1) \xrightarrow{f_1} \dots \xrightarrow{f_n} (X_n, \mathcal{F}_n, B_n, \mathbf{M}; G_n) \xrightarrow{f_n} \dots$$

is a (possibly infinite) sequence of steps of a $(K_{\mathcal{F}_0} + B_0 + \mathbf{M}_{X_0})$ -MMP/ U , we mean the following: for any i , $f_i : X_i \dashrightarrow X_{i+1}$ is a step of a $(K_{\mathcal{F}_i} + B_i + \mathbf{M}_{X_i})$ -MMP/ U that is not a Mori fiber space, $\mathcal{F}_{i+1} := (f_i)_*\mathcal{F}_i$, $B_{i+1} := (f_i)_*B_i$, and $G_{i+1} := (f_i)_*G_i$.

8. CONE THEOREM AND ACSS MODIFICATIONS

In this section we prove the cone theorem (Theorem 2.3.1) and the existence of ACSS modifications (Theorem 2.5.1). As an immediate corollary, we will prove the precise adjunction formula (Theorem 6.0.1) in full generality, without assuming that $\mathcal{F}$ is induced by a contraction.

8.1. Bend and break. It is important to notice that we will work under the relative setting, so the following relative bend and break theorem is crucial for our proofs.

Theorem 8.1.1 (Relative bend and break). *Let d be a positive integer, $\pi : X \rightarrow U$ a contraction from a normal quasi-projective variety to a variety such that $\dim X - \dim U = d$, $M, D_1, \dots, D_d$ $\mathbb{R}$ -divisors on X that are nef along general fibers of π , $B \geq 0$ an $\mathbb{R}$ -divisor on X , and $\mathcal{F}$ a foliation on X . Suppose that for any general fiber F of π ,*

- (1) $(D_1|_F) \cdot (D_2|_F) \cdots (D_d|_F) = 0$, and
- (2) $-(K_{\mathcal{F}} + B)|_F \cdot (D_2|_F) \cdots (D_d|_F) > 0$.

Then for any general closed point $x \in X$, there exists a rational curve C_x satisfying the following.

- (1) $x \in C_x$,
- (2) $\pi(C_x)$ is a point, and
- (3) $D_1 \cdot C_x = 0$ and

$$M \cdot C_x \leq 2d \frac{M|_F \cdot (D_2|_F) \cdots (D_d|_F)}{-K_{\mathcal{F}}|_F \cdot (D_2|_F) \cdots (D_d|_F)}.$$

Proof. Since (3) is a closed condition and M is a limit of $\mathbb{Q}$ -divisors that are nef along general fibers of π , we may assume that M is a $\mathbb{Q}$ -divisor. Possibly replacing M with a multiple, we may assume that M is a Weil divisor.

We let X^c and U^c be compactifications of X and U , such that X^c and U^c are normal projective, X is a non-empty open subset of X^c , U is a non-empty open subset of U^c , and there exists a contraction $\pi^c : X^c \rightarrow U^c$ such that $\pi^c|_X = \pi$. Let $M^c, D_1^c, \dots, D_d^c, B^c$ be the closures of

$M, D_1, \dots, D_d, B$ in X^c respectively, and let $\mathcal{F}^c$ be the natural extension of $\mathcal{F}$ in X^c [CS23b, Lemma 2.2]. Then the general fibers of π^c are general fibers of π , and $M^c, D_1^c, \dots, D_d^c$ are $\mathbb{R}$ -divisors that are nef along general fibers of π . Since we only care about properties about general fibers of π and properties near a general closed point $x \in X$, we may replace $\pi : X \rightarrow U$ with $\pi^c : X^c \rightarrow U^c$, $M, D_1, \dots, D_d, B$ with $M^c, D_1^c, \dots, D_d^c, B^c$, and $\mathcal{F}$ with $\mathcal{F}^c$, and assume that π is a projective morphism between normal projective varieties.

Let $x \in X$ be a general closed point. Then x is contained in a general fiber F of π . Let $q := \dim U$. Then there exist general hyperplane sections $H_1, \dots, H_q$ with $A_i := \pi^* H_i$, such that $F = \cap_{i=1}^q \pi^* A_i$. Let $V_k := X \cap \cap_{i=1}^k A_i$ and $W_k := U \cap \cap_{i=1}^k H_i$ for each $0 \leq k \leq q$, then

$$F = V_q \subset V_{q-1} \subset \dots \subset V_0 = X$$

and

$$z := W_q \subset W_{q-1} \subset \dots \subset W_0 = U,$$

where z is a general closed point. We may inductively define $\mathcal{F}_k$ to be the restricted foliation of $\mathcal{F}$ on V_k for each k , and let $\mathcal{F}_F := \mathcal{F}_q$. We let $M_k := M|_{V_k}$, $B_k := B|_{V_k}$, $M_F := M|_F$, and $B_F := B|_F$. Then it is clear that $M_k|_F = M|_F$, $B_k|_F = B_F$ for each k , and $B_{V_k} \geq 0$ for each k . Moreover, since $H_1, \dots, H_q$ are general hyperplane sections, M_k is a Weil divisor for each k .

Claim 8.1.2. *There exists a rational curve C_x , such that $x \in C_x$, $\pi(C_x)$ is a closed point, $D_1 \cdot C_x = 0$, and*

$$M|_F \cdot C_x \leq 2d \frac{M|_F \cdot (D_2|_F) \cdot \dots \cdot (D_d|_F)}{-K_{\mathcal{F}_k}|_F \cdot (D_2|_F) \cdot \dots \cdot (D_d|_F)}$$

for each k .

Proof. We apply induction on $q - k$. When $q - k = 0$, the existence of C_x follows from [Spi20, Corollary 2.28]. We will show that this C_x satisfies our requirement for all $q - k$ as well. In the following, we may assume that $q > k$.

We let $\pi_k : V_k \rightarrow W_k$ be the restricted contraction of π to V_k for each k . We consider W_{k+1} as a divisor on W_k and V_{k+1} as a divisor on V_k . There are two possibilities.

Case 1. V_{k+1} is $\mathcal{F}_k$ -invariant. In this case, the general fibers of π_k are tangent to $\mathcal{F}_k$, so

$$K_F = K_{\mathcal{F}_F} = K_{\mathcal{F}_k}|_F.$$

Thus by the $q - k = 0$ case,

$$M|_F \cdot C_x \leq 2d \frac{M|_F \cdot (D_2|_F) \cdot \dots \cdot (D_d|_F)}{-K_{\mathcal{F}_F} \cdot (D_2|_F) \cdot \dots \cdot (D_d|_F)} = 2d \frac{M|_F \cdot (D_2|_F) \cdot \dots \cdot (D_d|_F)}{-K_{\mathcal{F}_k}|_F \cdot (D_2|_F) \cdot \dots \cdot (D_d|_F)}.$$

Case 2. V_{k+1} is not $\mathcal{F}_k$ -invariant. In this case, by [Dru21, Proposition 3.6(1)], we have

$$(K_{\mathcal{F}_k} + V_{k+1})|_{V_{k+1}} \sim K_{\mathcal{F}_{k+1}} + D_{k+1}$$

for some $\mathbb{Q}$ -divisor $D_{k+1} \geq 0$. We remark that [Dru21, Proposition 3.6(1)] requires that $2 \leq \text{rank } \mathcal{F} \dim X - 1$, but the same lines of the proof works for the case when $\text{rank } \mathcal{F} = 1$ as well, and the $\text{rank } \mathcal{F} = \dim X$ case is the classical adjunction formula.

Since H_{k+1} is a general hyperplane section, there exists $H'_{k+1} \sim H_{k+1}$ such that H'_{k+1} does not contain z . Thus

$$V_{k+1}|_F = (H_{k+1}|_{V_k})|_F = H_{k+1}|_F \sim H'_{k+1}|_F = 0.$$

Since $H_{k+2}, \dots, H_q$ are general hyperplane sections, $D_{k+1}|_F \geq 0$. Therefore,

$$\begin{aligned} & -K_{\mathcal{F}_k}|_F \cdot (D_2|_F) \cdots (D_d|_F) \\ &= -(K_{\mathcal{F}_k} + V_{k+1})|_F \cdot (D_2|_F) \cdots (D_d|_F) \\ &= -((K_{\mathcal{F}_k} + V_{k+1})|_{V_{k+1}})|_F \cdot (D_2|_F) \cdots (D_d|_F) \\ &= -(K_{\mathcal{F}_{k+1}} + D_{k+1})|_F \cdot (D_2|_F) \cdots (D_d|_F) \\ &\leq -K_{\mathcal{F}_{k+1}}|_F \cdot (D_2|_F) \cdots (D_d|_F). \end{aligned}$$

By induction hypothesis,

$$M|_F \cdot C_x \leq 2d \frac{M|_F \cdot (D_2|_F) \cdots (D_d|_F)}{-K_{\mathcal{F}_{k+1}}|_F \cdot (D_2|_F) \cdots (D_d|_F)} = 2d \frac{M|_F \cdot (D_2|_F) \cdots (D_d|_F)}{-K_{\mathcal{F}_k}|_F \cdot (D_2|_F) \cdots (D_d|_F)}.$$

□

Proof of Lemma 8.1.1 continued. It immediately follows from Claim 8.1.2 by letting $k = 0$. □

8.2. Inductive statements to cone theorem. Similar to [ACSS21, Theorems 3.9, 3.10], the cone theorem for generalized foliated quadruples is closely related to the existence of $(*)$ -models for generalized foliated quadruples, and their proofs are done inductively. For applications to the rest of the paper as well as future works, we shall establish a much stronger version of the existence of $(*)$ -models: the existence of great ACSS models with controlled extraction of divisors. This kind of model is more technically constructed, but is also more useful in practice.

Theorem 8.2.1 (Cone theorem for induction, cf. [ACSS21, Theorem 3.9]). *Let d be a positive integer. Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a gfq of dimension d such that $\mathcal{F}$ is algebraically integrable. Let $\{R_j\}_{j \in \Lambda}$ be the set of all $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -negative extremal rays/ U that are not contained in the non-lc locus of $(X, \mathcal{F}, B, \mathbf{M})$. Then*

$$\overline{NE}(X/U) = \overline{NE}(X/U)_{K_{\mathcal{F}} + B + \mathbf{M}_X \geq 0} + \overline{NE}(X/U)_{\text{Nlc}(X, \mathcal{F}, B, \mathbf{M})} + \sum_{j \in \Lambda} R_j,$$

and for any $j \in \Lambda$, R_j is exposed and is spanned by a rational curve C_j , such that C_j is tangent to $\mathcal{F}$ and

$$0 < -(K_{\mathcal{F}} + B + \mathbf{M}_X) \cdot C_j \leq 2d.$$

Theorem 8.2.2 (Existence of ACSS models, cf. [ACSS21, Theorem 3.10], [DLM23, Proposition 4.14]). *Let d be a positive integer and s a non-negative integer. Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a gfq of dimension d such that $\mathcal{F}$ is algebraically integrable, and $E_1, \dots, E_s$ lc places of $(X, \mathcal{F}, B, \mathbf{M})$, such that $(X, \mathcal{F}, B, \mathbf{M})$ is lc near the generic point of center $_X E_i$ for each i . Then $(X, \mathcal{F}, B, \mathbf{M})$ has a great ACSS model $(Y, \mathcal{F}_Y, B_Y, \mathbf{M})$, such that $E_1, \dots, E_s$ are on Y if $(X, \mathcal{F}, B, \mathbf{M})$ is lc.*

In the following, we will prove Theorems 8.2.1 and 8.2.2 by induction on d . We will often use the following useful lemma:

Lemma 8.2.3. *Let $X \rightarrow U$ be a projective morphism from a normal quasi-projective variety to a variety and R an extremal ray in $\overline{NE}(X/U)$. Let $h : Y \rightarrow X$ be a projective morphism such that R is contained in the image of the induced map $\iota : \overline{NE}(Y/U) \rightarrow \overline{NE}(X/U)$. Then there exists an extremal ray R_Y in $\overline{NE}(Y/U)$ such that $\iota(R_Y) = R$.*

Proof. Since R is contained in the image of ι , there exists a ray R' in $\overline{NE}(Y/U)$ such that $\iota(R') = R$. Then there exist extremal rays R'_i in $\overline{NE}(S/U)$ such that $R' = \sum a_i R'_i$ for some $a_i > 0$. Thus $R = \sum a_i \iota(R'_i)$. Since R is extremal/ U , for each i , either $\iota(R'_i) = R$ or $\iota(R'_i) = 0$. Since $R \neq 0$, there exists j such that $\iota(R'_j) \neq 0$. We may take $R_Y = R'_j$. □

Remark 8.2.4. We remark that our proofs of Theorems 8.2.1 and 8.2.2 generally follows from the same ideas of [ACSS21, Theorems 3.9, 3.10] but the proofs are much lengthier. This is mainly because we work in the relative setting, and include all details of the proofs. For example, we provide detailed statements when proving the exposedness of extremal rays (Propositions 8.4.3 and 8.4.4), and provide a detailed statement on why a certain minimal model program can be run (Claim 8.3.3). It is also worth to mention that we need to deal with the $\mathbb{Q}$ -factorial case first, and then deal with the non- $\mathbb{Q}$ -factorial case due to Claim 8.3.3(4).

8.3. Cone theorem to ACSS models. In this subsection, we prove Theorem 8.2.2 in dimension d provided that Theorem 8.2.1 holds in dimension $\leq d - 1$ and some $\mathbb{Q}$ -factorial properties are satisfied.

Lemma 8.3.1. *Let d be a positive integer. Assume that Theorem 8.2.1 holds in dimension $\leq d - 1$.*

Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be an lc gfq of dimension d satisfying Property $()$ associated with $f : X \rightarrow Z$. Suppose that for any $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -negative extremal ray/ U R , there exists a prime divisor E on X , such that R is contained in the image of $\overline{NE}(E/U) \rightarrow \overline{NE}(X/U)$ and $\text{mult}_E B = \epsilon_{\mathcal{F}}(E)$. Let $\{R_j\}_{j \in \Lambda}$ be the set of $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -negative extremal rays/ U . Then:*

(1)

$$\overline{NE}(X/U) = \overline{NE}(X/U)_{K_{\mathcal{F}} + B + \mathbf{M}_X \geq 0} + \sum_{j \in \Lambda} R_j.$$

(2) *Each R_j is spanned by a rational curve C_j , such that C_j is tangent to $\mathcal{F}$ and*

$$0 \leq -(K_{\mathcal{F}} + B + \mathbf{M}_X) \cdot C_j \leq 2(d - 1).$$

(3) *For any curve C'_j such that $[C'_j] \in R_j$, C'_j is contracted by f .*

(4) *Assume that f is equi-dimensional, and either X is $\mathbb{Q}$ -factorial klt or $\mathbf{M}$ is NQC/ U . Let G be any divisor associated with $(X, \mathcal{F}, B, \mathbf{M})/Z$. Then:*

(a) Λ is a countable set.

(b) *For any ample/ U $\mathbb{R}$ -divisor A on X , there exists a finite set $\Lambda_A \subset \Lambda$, such that*

$$\overline{NE}(X/U) = \overline{NE}(X/U)_{K_{\mathcal{F}} + B + A + \mathbf{M}_X \geq 0} + \sum_{j \in \Lambda_A} R_j.$$

(c) *For any $j \in \Lambda$, there exists a contraction $\phi_j : X \rightarrow X'_j$ of R_j , such that*

(i) ϕ_j is a contraction/ U as well as a contraction/ Z , and

(ii) *if ϕ_j is small, then there exists a small contraction $\phi_j^+ : X_j^+ \rightarrow X'_j$ such that the induced birational map $\psi_j : X \dashrightarrow X_j^+$ is both a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -flip/ U and a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -flip/ Z .*

(d) *For any j ,*

$$(K_{\mathcal{F}} + B + \mathbf{M}_X) \cdot R_j = (K_X + B + G + \mathbf{M}_X) \cdot R_j.$$

In particular,

(i) *each R_j is a $(K_X + B + G + \mathbf{M}_X)$ -negative extremal ray, and*

(ii) *ϕ_j is a $(K_X + B + G + \mathbf{M}_X)$ -negative extremal contraction, and if ϕ_j is small, then ψ_j is a $(K_X + B + G + \mathbf{M}_X)$ -flip.*

Proof. (1) is obvious.

Pick a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -negative extremal ray R . By our assumption, there exists a prime divisor E on X , such that R is contained in the image of $\overline{NE}(E/U) \rightarrow \overline{NE}(X/U)$ and $\text{mult}_E B = \epsilon_{\mathcal{F}}(E)$. We let S be the normalization of E , then there exists a natural surjection

$$\overline{NE}(S/U) \rightarrow \overline{NE}(E/U).$$

Thus R is contained in the image of

$$\iota : \overline{NE}(S/U) \rightarrow \overline{NE}(E/U) \rightarrow \overline{NE}(X/U).$$

By Lemma 8.2.3, there exists an extremal ray R_S in $\overline{NE}(S/U)$ such that $R = \iota(R_S)$. Let $\mathcal{F}_S$ be the restricted foliation of $\mathcal{F}$ on S which is algebraically integrable by Proposition 6.1.10, $\mathbf{M}^S := \mathbf{M}|_S$, and

$$K_{\mathcal{F}_S} + B_S + \mathbf{M}_S^S := (K_{\mathcal{F}} + B + \mathbf{M}_X)|_S.$$

Then R_S is a $(K_{\mathcal{F}} + B + \mathbf{M}_X)|_S$ -negative extremal ray. By Theorem 6.6.1, $(S, \mathcal{F}_S, B_S, \mathbf{M}^S)/U$ is an lc gfq. Since we assume Theorem 8.2.1 in dimension $\leq d-1$, R_S is spanned by a rational curve C such that C is tangent to $\mathcal{F}_S$ and

$$0 < -(K_{\mathcal{F}_S} + B_S + \mathbf{M}_S^S) \cdot C \leq 2(d-1).$$

We identify C with its image in X under the natural inclusion $S \rightarrow E \rightarrow X$. Then C spans R and

$$0 < -(K_{\mathcal{F}_S} + B_S + \mathbf{M}_S^S) \cdot C = -(K_{\mathcal{F}} + B + \mathbf{M}_X) \cdot C \leq 2(d-1).$$

Moreover, by [ACSS21, Lemma 3.3(4)], C is tangent to $\mathcal{F}$ and is contracted by f . This implies (2).

By [ACSS21, Lemma 3.3(3)], C is contained in a fiber of f , so C is contracted by f . Let C' be an irreducible curve on X such that $[C'] \in R$. If $f(C')$ is not a closed point, then there exists a general ample divisor H on Z such that H intersects $f(C')$ transversally. Thus f^*H intersects C' transversally, so $f^*H \cdot C' > 0$. Since C is contracted by f , $f^*H \cdot C = 0$. This is not possible as $C \equiv \lambda C'$ for some positive rational number λ . Therefore, $f(C')$ is a closed point, so C' is contracted by f .

For any curve C'' such that $[C''] \in R$, we let C''_i be the irreducible components of C'' . Since R is extremal, $[C''_i] \in R$ for each i , so C''_i is contracted by f for each i . Thus C'' is contracted by f , and we get (3).

We left to prove (4). We may assume that f is equi-dimensional from now on. We let G be any divisor associated with $(X, \mathcal{F}, B, \mathbf{M})/Z$. Since $(X, \mathcal{F}, B, \mathbf{M})$ is lc, all components of B are horizontal/ Z . By Proposition 7.3.6,

$$(8.1) \quad K_{\mathcal{F}} + B + \mathbf{M}_X \sim_{\mathbb{R}, Z} K_X + B + G + \mathbf{M}_X.$$

By (3), for any $j \in \Lambda$, we have

$$(K_X + B + G + \mathbf{M}_X) \cdot R_j = (K_{\mathcal{F}} + B + \mathbf{M}_X) \cdot R_j < 0,$$

so R_j is a $(K_X + B + G + \mathbf{M}_X)$ -negative extremal ray/ U . Moreover, for any ample/ U $\mathbb{R}$ -divisor A , we have

$$(K_X + B + G + A + \mathbf{M}_X) \cdot R_j = (K_{\mathcal{F}} + B + A + \mathbf{M}_X) \cdot R_j < 0.$$

By Lemma 5.4.3, $(X, B + G, \mathbf{M})$ is lc. If $\mathbf{M}$ is NQC/ U , then by [HL21a, Theorem 1.3(3)],

$$\Lambda_A := \{j \in \Lambda \mid (K_{\mathcal{F}} + B + A + \mathbf{M}_X) \cdot R_j < 0\}$$

is a finite set, hence Λ is a countable set. This implies (4.a) and (4.b). (4.c.i) follows from [Xie22, Theorem 1.5] (see also [CLX23, Theorem 1.7]), and (4.c.ii) follows from [LX23b, Theorem 1.2].

If X is $\mathbb{Q}$ -factorial klt, then by [HL22, Lemma 3.4], for any ample $\mathbb{R}$ -divisor A on X , there exists an $\mathbb{R}$ -divisor $0 \leq \Delta_A \sim_{\mathbb{R}} B + G + \frac{1}{2}A + \mathbf{M}_X$, such that (X, Δ_A) is klt. Thus

$$\Lambda_A = \left\{ j \in \Lambda \mid \left(K_{\mathcal{F}} + \Delta + \frac{1}{2}A \right) \cdot R_j < 0 \right\}$$

is a finite set by the classical cone theorem (cf. [KMM87, Theorem 4-2-1], [Fuj17, Theorem 4.5.2]), and $\Lambda = \cup_{n=1}^{+\infty} \Lambda_{\frac{1}{n}A}$ is a countable set. This implies (4.a) and (4.b). For any j , we take an ample $\mathbb{R}$ -divisor A on X , such that R_j is also a $(K_X + B + G + A + \mathbf{M}_X)$ -negative extremal

ray/ U . Then R_j is a $(K_X + \Delta_A)$ -negative extremal ray/ U , so (4.c.i) follows from the classical contraction theorem (cf. [KMM87, Theorem 3-2-1], [Fuj17, Theorem 4.5.2]) and (4.c.ii) follows from the the existence of flips [BCHM10, Corollary 1.4.1].

(4.d) follows immediately from (4.c) and (8.1) $\square$

Proposition 8.3.2. *Let d be a positive integer. Assume that Theorem 8.2.1 holds in dimension $\leq d - 1$. Let $(X, \mathcal{F}, B, \mathbf{M})/U$ a gfq of dimension d such that $\mathcal{F}$ is algebraically integrable. Let $E_1, \dots, E_s$ be lc places of $(X, \mathcal{F}, B, \mathbf{M})$ and T a reduced $\mathcal{F}$ -invariant divisor on X . Further assume that*

- either X is $\mathbb{Q}$ -factorial, or
- Theorem 8.2.1 holds for $\mathbb{Q}$ -factorial varieties in dimension d .

Then $(X, \mathcal{F}, B, \mathbf{M})$ has a great ACSS model $(Y, \mathcal{F}_Y, B_Y, \mathbf{M}; G_Y)$ such that

- (1) G_Y contains the strict transform of T on Y , and
- (2) $E_1, \dots, E_s$ are on Y if $(X, \mathcal{F}, B, \mathbf{M})$ is lc.

Proof. By Definition-Theorem 5.1.2 and Lemma 6.2.4, there exists a foliated log resolution $h : X' \rightarrow X$ of $(X, \mathcal{F}, \text{Supp } B + \text{Supp } T, \mathbf{M})$ such that $E_1, \dots, E_s$ are on X' . Then there exists a toroidal contraction $f' : (X', \Sigma_{X'}, \mathbf{M}) \rightarrow (Z, \Sigma_Z)$ such that (Z, Σ_Z) is log smooth,

$$\text{Exc}(h) \cup \text{Supp}(h_*^{-1}B) \cup \text{Supp}(h_*^{-1}T) \subset \Sigma_{X'},$$

and $\mathcal{F}' := h^{-1}\mathcal{F}$ is induced by f' . We define

$$B' := h_*^{-1}(B \wedge \text{Supp } B) + (\text{Supp } \text{Exc}(h))^{\mathcal{F}'}$$

By Lemma 7.3.3, $(X', \mathcal{F}', B', \mathbf{M}; G')$ / Z is $\mathbb{Q}$ -factorial super ACSS for some divisor G' , such that $G' \geq h_*^{-1}T$ and any $\mathcal{F}'$ -invariant h -exceptional divisor is contained in G' .

Claim 8.3.3. *Let A be an ample $\mathbb{R}$ -divisor on X . Then we may run a $(K_{\mathcal{F}'} + B' + \mathbf{M}_{X'})$ -MMP/ X*

$$(X_0, \mathcal{F}_0, B_0, \mathbf{M}; G_0) \xrightarrow{\psi_0} (X_1, \mathcal{F}_1, B_1, \mathbf{M}; G_1) \xrightarrow{\psi_1} \dots \xrightarrow{\psi_n} (X_n, \mathcal{F}_n, B_n, \mathbf{M}; G_n) \xrightarrow{\psi_n} \dots$$

where $(X_0, \mathcal{F}_0, B_0, \mathbf{M}; G_0) := (X', \mathcal{F}', B', \mathbf{M}; G')$, so that the following conditions are satisfied for each i . Let A_i be the strict transform of A on X_i .

- (1) There exists an contraction $f_i : X_i \rightarrow Z$ such that $f_{i+1} = f_i \circ \psi_i$.
- (2) There exists an contraction $h_i : X_i \rightarrow X$ such that $h_{i+1} = h_i \circ \psi_i$.
- (3) $(X_i, \mathcal{F}_i, B_i, \mathbf{M}; G_i)/Z$ is $\mathbb{Q}$ -factorial super ACSS.
- (4) If X is $\mathbb{Q}$ -factorial, then for any $(K_{\mathcal{F}_i} + B_i + \mathbf{M}_{X_i})$ -negative extremal ray/ X R , there exists a prime divisor F on X_i , such that R is contained in the image of $\overline{NE}(F/U) \rightarrow \overline{NE}(X/U)$ and $\text{mult}_F B_i = \epsilon_{\mathcal{F}_i}(F)$.
- (5) For any extremal ray/ X R on X_i such that R is either a $(K_{\mathcal{F}_i} + B_i + \mathbf{M}_{X_i})$ -negative extremal ray or a $(K_{X_i} + B_i + G_i + \mathbf{M}_{X_i})$ -negative extremal ray,
 - (a) R is an extremal ray/ Z ,
 - (b)

$$(K_{\mathcal{F}_i} + B_i + \mathbf{M}_{X_i}) \cdot R = (K_{X_i} + B_i + G_i + \mathbf{M}_{X_i}) \cdot R,$$

and

- (c) R is a $(K_{\mathcal{F}_i} + B_i + \mathbf{M}_{X_i})$ -negative extremal ray if and only if R is a $(K_{X_i} + B_i + G_i + \mathbf{M}_{X_i})$ -negative extremal ray.
- (6) ψ_i is a step of a $(K_{X_i} + B_i + G_i + \mathbf{M}_{X_i})$ -MMP/ X with scaling of A_i as well as a $(K_{\mathcal{F}_i} + B_i + \mathbf{M}_{X_i})$ -MMP/ X with scaling of A_i .
- (7) ψ_i is a step of a $(K_{\mathcal{F}_i} + B_i + \mathbf{M}_{X_i})$ -MMP/ Z as well as a step of a $(K_{X_i} + B_i + G_i + \mathbf{M}_{X_i})$ -MMP/ Z .

Moreover, there exists a positive integer m satisfying the following.

- (8) The induced birational map $X_0 \dashrightarrow X_m$ contracts any h -exceptional prime divisor F such that $a(F, \mathcal{F}, B, \mathbf{M}) > -\epsilon_{\mathcal{F}}(F)$.
- (9) If $(X, \mathcal{F}, B, \mathbf{M})$ is lc, then any divisor F contracted by $X_0 \dashrightarrow X_m$ satisfies that $a(F, \mathcal{F}, B, \mathbf{M}) > -\epsilon_{\mathcal{F}}(F)$.

Proof. Step 1. In this step we prove (1-4) for $i = 0$. (1) We have $f_0 := f$. (2) We have $h_0 := h$. (3) It follows from our construction. (4) The image of R on X is a closed point, so R is contained in an h -exceptional divisor F . By our construction, $\text{mult}_F B_0 = \epsilon_{\mathcal{F}_0}(F)$.

Step 2. In this step we prove that (1-4) for $i = n$ implies (5) for $i = n$.

First we prove (5.a). Assume that R is a $(K_{\mathcal{F}_n} + B_n + \mathbf{M}_{X_n})$ -negative extremal ray/ X . If X is $\mathbb{Q}$ -factorial, then by (4) and Lemma 8.3.1(2), R is a $(K_{\mathcal{F}_n} + B_n + \mathbf{M}_{X_n})$ -negative extremal ray/ Z . If Theorem 8.2.1 holds for $\mathbb{Q}$ -factorial varieties in dimension d , then by (3) and Theorem 8.2.1, R is a $(K_{\mathcal{F}_n} + B_n + \mathbf{M}_{X_n})$ -negative extremal ray/ Z .

Now assume that R is a $(K_{X_n} + B_n + G_n + \mathbf{M}_{X_n})$ -negative extremal ray/ X . Since G_n is super, $G_n \geq \sum_{j=1}^{2d+1} f_n^* H_j$ for some ample Cartier divisors H_j on Z . Let $L_n := G_n - \sum_{j=1}^{2d+1} f_n^* H_j$. By (3), $(X_n, B_n + G_n, \mathbf{M})$ is $\mathbb{Q}$ -factorial lc and X is klt, so $(X_n, B_n + L_n, \mathbf{M})$ is $\mathbb{Q}$ -factorial lc. By the length of extremal rays for lc g-pairs over $\mathbb{Q}$ -factorial klt varieties (cf. [HL22, Proposition 3.17]), R is spanned by a rational curve C such that

$$0 > (K_{X_n} + B_n + G_n + \mathbf{M}_{X_n}) \cdot C = (K_{X_n} + B_n + L_n + \mathbf{M}_{X_n}) \cdot C + \left(\sum_{j=1}^{2d+1} f_n^* H_j \right) \cdot C \geq -2d.$$

Therefore, $f_n^* H_j \cdot C = 0$ for each j , so R is an extremal ray/ Z . This implies (5.a).

(5.b) follows from (5.a) and Proposition 7.3.6, and (5.c) follows from (5.b). Thus (5) holds.

Step 4. In this step we prove that (1-5) for $i = n$ and (1-7) for $i \leq n - 1$ imply (6) and (7) for $i = n$, and also imply (1)(2) for $i = n + 1$.

By induction hypothesis, the induced birational map $X_0 \dashrightarrow X_n$ is a sequence of steps of a $(K_{X_0} + B_0 + G_0 + \mathbf{M}_{X_0})$ -MMP/ X with scaling of A . By Lemma 4.2.2, either this MMP already terminates at X_n and we are done, or we may run the next step of this $(K_{X_0} + B_0 + G_0 + \mathbf{M}_{X_0})$ -MMP/ X with scaling of A , which is a step of a $(K_{X_n} + B_n + G_n + \mathbf{M}_{X_n})$ -MMP/ X with scaling of A_n . (6) and (7) for $i = n$ now follow from (5) for $i = n$. (1) for $i = n + 1$ follows from (7) for $i = n$, and (2) for $i = n + 1$ follows from (6) for $i = n$.

Step 5. In this step we prove that (1-7) for $i \leq n - 1$ and (1)(2) for $i = n$ imply (3) for $i = n$.

By (3)(7) for $i = n - 1$, X_n is $\mathbb{Q}$ -factorial. By (1) for $i = n$ and (3) for $i = n - 1$, G_n is super/ Z . So we only need to show that $(X_n, \mathcal{F}_n, B_n, \mathbf{M}; G_n)/Z$ is ACSS. We check conditions (1-4) of Definition 7.2.3 for $(X_n, \mathcal{F}_n, B_n, \mathbf{M}; G_n)/Z$.

Definition 7.2.3(1) for $(X_n, \mathcal{F}_n, B_n, \mathbf{M}; G_n)/Z$: By (6) for $i = n - 1$ and Proposition 5.4.7, $(X_n, B_n + G_n, \mathbf{M})/Z$ satisfies Property (*). Since $\mathcal{F}_{n-1}$ is induced by f_{n-1} , $\mathcal{F}_n$ is induced by f_n . Since $G_{n-1} \geq 0$ is $\mathcal{F}_{n-1}$ -invariant, $G_n \geq 0$ if $\mathcal{F}_n$ -invariant. Thus $(X_n, \mathcal{F}_n, B_n, \mathbf{M}; G_n)/Z$ satisfies Property (*). By (3)(7) for $i = n - 1$, $(X_n, \mathcal{F}_n, B_n, \mathbf{M})$ is lc, so Definition 7.2.3(1) holds for $(X_n, \mathcal{F}_n, B_n, \mathbf{M}; G_n)/Z$.

Definition 7.2.3(2) for $(X_n, \mathcal{F}_n, B_n, \mathbf{M}; G_n)/Z$: It immediately follows from (3)(6) for $i = n - 1$ and Proposition 5.4.7.

Definition 7.2.3(3) for $(X_n, \mathcal{F}_n, B_n, \mathbf{M}; G_n)/Z$: By (3) for $i = n - 1$, there exist an $\mathbb{R}$ -divisor D and a $\mathbf{b}$ -divisor $\mathbf{N}$ on X_{n-1} , such that

- $\text{Supp}\{B_{n-1}\} \subset \text{Supp } D$,
- $\mathbf{N} - \alpha \mathbf{M}$ is nef/ X_{n-1} for some $\alpha > 1$, and
- For any divisor Σ on Z such that $\Sigma \geq f_{n-1}(G_{n-1})$ and (Z, Σ) is log smooth,

$$(X_{n-1}, B_{n-1} + G_{n-1} + D + f_{n-1}^*(\Sigma - f_{n-1}(G_{n-1})), \mathbf{N})$$

is qdlt,

Let $\mathbf{P} := \mathbf{N} - \mathbf{M}$. By (7) for $i = n - 1$, ψ_{n-1} is also a step of a

$$(K_{\mathcal{F}_{n-1}} + B_{n-1} + f_{n-1}^*(\Sigma - f_{n-1}(G_{n-1})) + \mathbf{M}_{X_{n-1}})\text{-MMP}/Z,$$

hence a step of a

$$(K_{\mathcal{F}_{n-1}} + B_{n-1} + \delta D + f_{n-1}^*(\Sigma - f_{n-1}(G_{n-1})) + \mathbf{M}_{X_{n-1}} + \delta \mathbf{P}_{X_{n-1}})\text{-MMP}/Z$$

for some $0 < \delta \ll 1$. By (1) for $i = n$, $f_{n-1}(G_{n-1}) = f_n(G_n)$, so

$$(X_{n-1}, B_{n-1} + G_{n-1} + \delta D + f_{n-1}^*(\Sigma - f_n(G_n)), \mathbf{M} + \delta \mathbf{P})$$

is qdlt. By Lemma 7.1.5,

$$(X_n, B_n + \delta(\psi_{n-1})_* D + G_n + f_n^*(\Sigma - f_n(G_n)), \mathbf{M} + \delta \mathbf{P})$$

is qdlt. Since $(\psi_{n-1})_* D \subset \text{Supp}\{B_n\}$ and $(\mathbf{M} + \delta \mathbf{P}) - \mathbf{M}$ is nef/ X_n , we verify Definition 7.2.3(3).

Definition 7.2.3(4) for $(X_n, \mathcal{F}_n, B_n, \mathbf{M}; G_n)/Z$: For any lc place S of $(X_n, \mathcal{F}_n, B_n, \mathbf{M})$, we have

$$-\epsilon_{\mathcal{F}}(S) = a(S, \mathcal{F}_n, B_n, \mathbf{M}) \geq a(S, \mathcal{F}_{n-1}, B_{n-1}, \mathbf{M}) \geq -\epsilon_{\mathcal{F}}(S).$$

Therefore, S is an lc place of $(X_{n-1}, \mathcal{F}_{n-1}, B_{n-1}, \mathbf{M})$, and ψ_{n-1} is an isomorphism near the generic point of $\text{center}_{X_{n-1}} S$. Since Definition 7.2.3(4) is a property near the generic point of lc places, Definition 7.2.3(4) holds for $(X_n, \mathcal{F}_n, B_n, \mathbf{M}; G_n)/Z$.

Therefore, $(X_n, \mathcal{F}_n, B_n, \mathbf{M}; G_n)/Z$ is $\mathbb{Q}$ -factorial super ACSS.

Step 6. In this step we prove (4) for $i = n$ assuming that (1-7) hold for $i = n - 1$, hence conclude the proof of (1-7). Since X is $\mathbb{Q}$ -factorial, $\text{Exc}(h_n)$ is of pure dimension, so there exists a prime h_n -exceptional divisor F such that R is contained in F . Let F' be the strict transform of F on X' , then F' is a prime h -exceptional divisor, so

$$\text{mult}_F B_n = \text{mult}_{F'} B_0 = \epsilon_{\mathcal{F}'}(E) = \epsilon_{\mathcal{F}_n}(B_n).$$

This implies (4).

By induction, (1-7) hold.

Step 7. In this step we prove (8) and (9) and conclude the proof of the claim.

If this MMP terminates, then we let m be the index such that $(X_m, \mathcal{F}_m, B_m, \mathbf{M}; G_m)$ is the last output of this MMP. In particular, $K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m}$ is nef/ X , hence it is movable/ X . If this MMP does not terminate, then we let m be the index such that ψ_i is a flip for any $i \geq m$. We let $\mu_i : X_m \dashrightarrow X_i$ be the induced birational map and let

$$\lambda_i := \inf\{t \geq 0 \mid K_{\mathcal{F}_i} + B_i + \mathbf{M}_{X_i} + tA_i \text{ is nef}/X\}$$

be the scaling numbers for each i . By (5),

$$\lambda_i = \inf\{t \geq 0 \mid K_{X_i} + B_i + G_i + \mathbf{M}_{X_i} + tA_i \text{ is nef}/X\}$$

for each i . By Lemma 4.2.2, $\lim_{i \rightarrow +\infty} \lambda_i = 0$. Therefore,

$$K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m} = \lim_{i \rightarrow +\infty} (\mu_i)_*^{-1}(K_{\mathcal{F}_i} + B_i + \mathbf{M}_{X_i} + \lambda_i A_i)$$

is a movable/ X .

Since $K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m}$ is movable/ X , for any prime divisor S on X_m any very general curve C on S over X , $(K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m}) \cdot C \geq 0$. Let $F_1, \dots, F_l$ be the h_m -exceptional prime

divisors and let $a_k := a(F_k, \mathcal{F}, B, \mathbf{M}) + \epsilon_{\mathcal{F}_m}(F_k)$ for each k , then

$$\begin{aligned} & K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m} \\ &= \phi_m^*(K_{\mathcal{F}} + B + \mathbf{M}_X) + \sum_{a_k > 0} a_k F_k - \left(\sum_{\text{mult}_D B > 1} (\text{mult}_D B - 1)(\phi_m^{-1})_* D + \sum_{a_k < 0} (-a_k) F_k \right) \\ &\sim_{\mathbb{R}, X} \sum_{a_k > 0} a_k F_k - \left(\sum_{\text{mult}_D B > 1} (\text{mult}_D B - 1)(\phi_m^{-1})_* D + \sum_{a_k < 0} (-a_k) F_k \right). \end{aligned}$$

Since each F_k is exceptional/ X , by [Bir12, Lemma 3.3], $a_k \leq 0$ for any k . This implies (8). Finally, if $(X, \mathcal{F}, B, \mathbf{M})$ is lc, then

$$K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} \sim_{\mathbb{R}, X} \sum_{F|F \subset \text{Exc}(\phi)} (\epsilon_{\mathcal{F}}(F) + a(F, \mathcal{F}, X, \mathbf{M})) F \geq 0,$$

so any divisor contracted by any $(K_{\mathcal{F}'} + B' + \mathbf{M}_{X'})$ -MMP/ X is contained in

$$\text{Supp} \sum_{F|F \subset \text{Exc}(\phi)} (\epsilon_{\mathcal{F}}(F) + a(F, \mathcal{F}, X, \mathbf{M})) F = \text{Supp} \sum_{F|F \subset \text{Exc}(\phi), a(F, \mathcal{F}, B, \mathbf{M}) > -\epsilon_{\mathcal{F}}(F)} F.$$

We get (9). The proof of the claim is concluded. $\square$

Proof of Proposition 8.3.2 continued. We let

$$(X_0, \mathcal{F}_0, B_0, \mathbf{M}; G_0) \xrightarrow{\psi_0} (X_1, \mathcal{F}_1, B_1, \mathbf{M}; G_1) \xrightarrow{\psi_1} \dots \xrightarrow{\psi_n} (X_n, \mathcal{F}_n, B_n, \mathbf{M}; G_n) \xrightarrow{\psi_n} \dots$$

and m be as in Claim 8.3.3. Then Claim 8.3.3(3)(8) guarantee that $(X_m, \mathcal{F}_m, B_m, \mathbf{M}; G_m)/Z$ is a super ACSS model of $(X, \mathcal{F}, B, \mathbf{M})$, and (9) guarantees that if $(X, \mathcal{F}, B, \mathbf{M})$ is lc then $E_1, \dots, E_s$ are on X_m . Thus $(X_m, \mathcal{F}_m, B_m, \mathbf{M}; G_m)/Z$ is a super ACSS model of $(X, \mathcal{F}, B, \mathbf{M})$ such that $E_1, \dots, E_s$ are on X_m if $(X, \mathcal{F}, B, \mathbf{M})$ is lc. Since any $\mathcal{F}'$ -invariant exceptional/ X prime divisor is contained in G' , any $\mathcal{F}_m$ -invariant exceptional/ X prime divisor is contained in G_m . Therefore, $(X_m, \mathcal{F}_m, B_m, \mathbf{M}; G_m)/Z$ is a great ACSS model of $(X, \mathcal{F}, B, \mathbf{M})$. Finally, since the strict transform of T on X' is contained in G' , the strict transform of T on X_m is contained in G_m . The proposition follows by taking

$$(Y, \mathcal{F}_Y, B_Y, \mathbf{M}; G_Y) := (X_m, \mathcal{F}_m, B_m, \mathbf{M}; G_m).$$

$\square$

Proposition 8.3.4. *Let d be a positive integer. Assume that Theorem 8.2.1 holds in dimension $\leq d - 1$. Then:*

- (1) *Theorem 8.2.2 holds for $\mathbb{Q}$ -factorial varieties in dimension d .*
- (2) *If Theorem 8.2.1 holds for $\mathbb{Q}$ -factorial varieties in dimension d , then Theorem 8.2.2 holds in dimension d .*

Proof. Notations and conditions as in Theorem 8.2.2. Further assume that either X is $\mathbb{Q}$ -factorial, or Theorem 8.2.1 holds for $\mathbb{Q}$ -factorial varieties in dimension d .

By Proposition 8.3.4, $(X, \mathcal{F}, B, \mathbf{M})$ has a great ACSS model $(Y', \mathcal{F}_{Y'}, B_{Y'}, \mathbf{M}; G_{Y'})/Z$. We let $g: Y' \rightarrow X$ be the induced birational morphism and let

$$F := \text{Supp}(K_{\mathcal{F}_{Y'}} + B_{Y'} + \mathbf{M}_{Y'} - g^*(K_{\mathcal{F}} + B + \mathbf{M}_X)).$$

Consider F as a reduced subscheme of Y' . Then for any irreducible closed subvariety $V \subset X$ such that $V \subset f(F)$, V is a non-lc center of $(X, \mathcal{F}, B, \mathbf{M})$. Therefore, the generic point of center $X E_i$ is not contained in $f(F)$ for each i , so $E_1, \dots, E_s$ are also lc places of $(Y', \mathcal{F}_{Y'}, B_{Y'}, \mathbf{M})$. Since $(Y', \mathcal{F}_{Y'}, B_{Y'}, \mathbf{M})$ is lc, by Proposition 8.3.4 again, $(Y', \mathcal{F}_{Y'}, B_{Y'}, \mathbf{M})$

has a great ACSS model $(Y, \mathcal{F}_Y, B_Y, \mathbf{M}; G_Y)$ such that $E_1, \dots, E_s$ are on Y , and G_Y contains the strict transform of $G_{Y'}$ on Y . Therefore, G_Y contains all $\mathcal{F}_Y$ -exceptional prime divisors. Since

$$g^*(K_{\mathcal{F}} + B + \mathbf{M}_X) \geq K_{\mathcal{F}_{Y'}} + B_{Y'} + \mathbf{M}_{Y'},$$

the induced birational morphism $Y \rightarrow X$ is a great ACSS modification $(X, \mathcal{F}, B, \mathbf{M})$. $\square$

8.4. ACSS models to cone theorem. In this subsection, we prove Theorem 8.2.1 in dimension d provided that Theorem 8.2.2 holds in dimension $\leq d$ and some $\mathbb{Q}$ -factorial properties are satisfied.

The following lemma is well-known to experts. For the reader's convenience, we conclude a proof here.

Lemma 8.4.1. *Let $X \rightarrow U$ be a projective morphism from a normal quasi-projective variety to a variety. Let D be an $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor on X and R a D -negative exposed ray in $\overline{NE}(X/U)$. Then there exists an ample/ U $\mathbb{R}$ -divisor A on X such that $H := D + A$ is the supporting function of R .*

Proof. Let H_R be a supporting function of R , whose existence follows from the assumption that R is exposed. Then $H_R \cdot R = 0$ and $H_R \cdot R' > 0$ for any $R' \neq R$ in $\overline{NE}(X/U)$. Let

$$C := \{D \in N^1(X/U) \mid D \cdot z \geq 0 \text{ for any } z \in \overline{NE}(X/U)_{D \geq 0}\}.$$

Then C is the dual cone of $\overline{NE}(X/U)_{D \geq 0}$ and is generated by nef/ U divisors and D . Since H_R is positive on $\overline{NE}(X/U)_{D \geq 0} \setminus \{0\}$, H_R is contained in the interior of C . Thus there exists an ample/ U $\mathbb{R}$ -divisor $\tilde{A}$ such that $H_R - \tilde{A} = L + pD$ in $N^1(X/U)$, where L is a nef/ U $\mathbb{R}$ -divisor and p is a non-negative real number. Let $A' := \tilde{A} + L$, then A' is ample/ U . We may let $H := \frac{1}{p}H_R = \frac{1}{p}\tilde{A} + D$ and $A := \frac{1}{p}A'$. $\square$

Proposition 8.4.2. *Let d be a positive integer. Assume that Theorem 8.2.1 holds in dimension $\leq d - 1$ and Theorem 8.2.2 holds for $\mathbb{Q}$ -factorial varieties in dimension d .*

Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a gfq of dimension d such that $\mathcal{F}$ is algebraically integrable. Let R be a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -negative exposed ray/ U that is not contained in $\overline{NE}(X/U)_{\text{Nlc}(X, \mathcal{F}, B, \mathbf{M})}$. Assume that

- *either X is $\mathbb{Q}$ -factorial, or*
- *Theorem 8.2.2 holds in dimension d .*

Then R is spanned by a rational curve C_j , such that C_j is tangent to $\mathcal{F}$ and

$$0 < -(K_{\mathcal{F}} + B + \mathbf{M}_X) \cdot C_j \leq 2d.$$

Proof. By Lemma 8.4.1, there exists an ample/ U $\mathbb{R}$ -divisor A on X such that

$$H_R := K_{\mathcal{F}} + B + A + \mathbf{M}_X$$

is the supporting function/ U of R . In particular, H_R is nef, $H_R \cdot R = 0$, and $H_R \cdot R' > 0$ for any $R' \in \overline{NE}(X/U) \setminus R$. In particular, $H_R \not\equiv_U 0$.

Step 1. In this step we deal with the case when H_R is not big/ U .

Let $\pi : X \rightarrow U$ be the induced projective morphism and $X \rightarrow U' \rightarrow U$ the Stein factorization of π . Since $\mathbf{M}$ is nef/ U , $\mathbf{M}$ is nef/ U' . Possibly replacing U by U' , we may assume that π is a contraction.

Let F be a general fiber of π . Then $H_F := H_R|_F$ is nef, not big, and is not numerically trivial. Let $q := \dim F$ and $A_F := A|_F$, then there exists an integer $1 \leq k \leq q - 1$ such that

$$H_F^k \cdot A_F^{q-k} > H_F^{k+1} \cdot A_F^{q-k-1} = 0.$$

Let $D_i := H_R$ for any $1 \leq i \leq k + 1$, and let $D_i := A$ for any $k + 2 \leq i \leq q$. Then

$$(D_1|_F) \cdot (D_2|_F) \cdots (D_q|_F) = H_F^{k+1} \cdot A_F^{q-k-1} = 0$$

and

$$-(K_{\mathcal{F}} + B)|_F \cdot (D_2|_F) \cdots (D_q|_F) = (A_F - H_F) \cdot H_F^k \cdot A_F^{q-k-1} = H_F^k \cdot A_F^{q-k} > 0.$$

Let $M := H_R + A = K_{\mathcal{F}} + B + 2A + \mathbf{M}_X$. Then M is ample/ U . By Theorem 8.1.1, for any general closed point $x \in X$, there exists a rational curve C_x such that $x \in C_x$, $\pi(C_x)$ is a closed point, $0 = D_1 \cdot C_x = H_R \cdot C_x$, and

$$\begin{aligned} 0 &< -(K_{\mathcal{F}} + B + \mathbf{M}_X) \cdot C_x = M \cdot C_x \\ &\leq 2d \cdot \frac{M|_F \cdot (D_2|_F) \cdots (D_q|_F)}{-K_{\mathcal{F}}|_F \cdot (D_2|_F) \cdots (D_q|_F)} = 2d \cdot \frac{-(K_{\mathcal{F}} + B + \mathbf{M}_X)|_F \cdot H_F^k \cdot A_F^{q-k-1}}{-K_{\mathcal{F}}|_F \cdot H_F^k \cdot A_F^{q-k-1}}. \end{aligned}$$

Let $\mathbf{M}^F := \mathbf{M}|_F$ and $B_F := B|_F$. Since F is a general fiber of π , $B_F \geq 0$ and $\mathbf{M}^F$ is nef. Thus $\mathbf{M}_F^F$ is pseudo-effective and $(B + \mathbf{M}_X)|_F \cdot H_F^k \cdot A_F^{q-k-1} \geq 0$. Therefore

$$0 < -(K_{\mathcal{F}} + B + \mathbf{M}_X) \cdot C_x \leq 2d.$$

Step 2. In this step we deal with the case when H_R is not big/ U . Let F be the Stein factorization of a general fiber of the π and let $q := \dim F$. Then $H_F := H_R|_F$ is nef, not big, and is not numerically trivial. Let $A_F := A|_F$, then there exists an integer $1 \leq k \leq q-1$ such that

$$H_F^k \cdot A_F^{q-k} > 0$$

and

$$H_F^{k+1} \cdot A_F^{q-k-1} = 0.$$

(Note that k is defined as the numerical dimension of H_F in some references, but since different definitions of numerical dimensions do not coincide, we do not use this notation.) Let $D_i := H_R$ for any $1 \leq i \leq k+1$, and let $D_i := A$ for any $k+2 \leq i \leq q$. Then

$$(D_1|_F) \cdot (D_2|_F) \cdots (D_q|_F) = H_F^{k+1} \cdot A_F^{q-k-1} = 0$$

and

$$-(K_{\mathcal{F}} + B)|_F \cdot (D_2|_F) \cdots (D_q|_F) = (A_F - H_F) \cdot H_F^k \cdot A_F^{q-k-1} = H_F^k \cdot A_F^{q-k} > 0.$$

We let $M := H_R + A = K_{\mathcal{F}} + B + 2A + \mathbf{M}_X$. Since H_R is nef/ U and A is ample/ U , M is ample/ U . By Theorem 8.1.1, for any general closed point $x \in X$, there exists a rational curve C_x such that $x \in C_x$, $\pi(C_x)$ is a closed point,

$$0 = D_1 \cdot C_x = H_R \cdot C_x,$$

and

$$\begin{aligned} 0 &< -(K_{\mathcal{F}} + B + \mathbf{M}_X) \cdot C_x = M \cdot C_x \\ &\leq 2d \cdot \frac{M|_F \cdot (D_2|_F) \cdots (D_q|_F)}{-K_{\mathcal{F}}|_F \cdot (D_2|_F) \cdots (D_q|_F)} = 2d \cdot \frac{-(K_{\mathcal{F}} + B + \mathbf{M}_X)|_F \cdot H_F^k \cdot A_F^{q-k-1}}{-K_{\mathcal{F}}|_F \cdot H_F^k \cdot A_F^{q-k-1}}. \end{aligned}$$

Let $\mathbf{M}^F := \mathbf{M}|_F$ and $B_F := B|_F$. Since F is a general fiber of π , $B_F \geq 0$ and $\mathbf{M}^F$ is nef. Thus $\mathbf{M}_F^F$ is pseudo-effective, so

$$(B + \mathbf{M}_X)|_F \cdot H_F^k \cdot A_F^{q-k-1} \geq 0.$$

Therefore,

$$0 < -(K_{\mathcal{F}} + B + \mathbf{M}_X) \cdot C_x \leq 2d.$$

Step 3. From now on we may assume that H_R is big/ U . In this step we construct a set Γ of tuples (W, λ) and show that it contains a minimal element. Since H_R is big/ U ,

$$H_R \sim_{\mathbb{R}, U} A' + P$$

for some ample/ U $\mathbb{R}$ -divisor A' and $\mathbb{R}$ -divisor $P \geq 0$. In particular, P is $\mathbb{R}$ -Cartier and $P \cdot R < 0$. Let S be the normalization of $\text{Supp } P$, then R is contained in the image of $\overline{NE}(S/U) \rightarrow \overline{NE}(X/U)$ induced by the natural inclusion

$$S \rightarrow \text{Supp } P \rightarrow X.$$

We let Γ be the set of all (W, λ) , such that

- (1) λ is a non-negative real number,
- (2) W is an lc center of $(X, \mathcal{F}, B + \lambda P, \mathbf{M})$ with normalization W^ν and
- (3) R is contained in the image of $\overline{NE}(W^\nu/U) \rightarrow \overline{NE}(X/U)$ induced by the natural inclusion

$$W^\nu \rightarrow W \rightarrow X.$$

By construction, there exists a component L of S such that $(L, 1) \in \Gamma$. Thus $\Gamma \neq \emptyset$.

In the rest of this step, we show that there exists $(W_0, \lambda_0) \in \Gamma$ that is minimal in the following way: for any $(W, \lambda) \in \Gamma$, one of the following cases hold.

- $\lambda_0 < \lambda$.
- $\lambda_0 = \lambda$ and $W_0 \subsetneq W$.
- $(W, \lambda) = (W_0, \lambda_0)$.

By Lemma 6.2.4, there exists a foliated log resolution $h : X' \rightarrow X$ of $(X, \mathcal{F}, \text{Supp } B \cup \text{Supp } P, \mathbf{M})$. Then there exists a toroidal morphism $f' : (X', \Sigma_X) \rightarrow (Z, \Sigma_Z)$ such that $h_*^{-1}(\text{Supp } B \cup \text{Supp } P) \cup \text{Supp } \text{Exc}(h)$ is contained in Σ_X . By Lemma 6.2.2, for any $(W, \lambda) \in \Gamma$, either W is the image of a stratum of (X', Σ) on X , or $\lambda = 0$. Therefore, the set

$$\begin{aligned} \Gamma' := \{ \lambda \mid & \text{there exists an lc center of } (X, \mathcal{F}, B + \lambda P, \mathbf{M}) \\ & \text{that is not an lc center of } (X, \mathcal{F}, B + (\lambda - \delta)P, \mathbf{M}) \text{ for any } 0 < \delta \ll 1 \} \end{aligned}$$

is a finite, so we may let

$$\lambda_0 := \min\{\lambda \mid \text{there exists } (W, \lambda) \in \Gamma\}.$$

Now by noetherian property, there exists $(W_0, \lambda_0) \in \Gamma$ such that $W_0 \subset W$ for any $(W, \lambda_0) \in \Gamma$.

Step 4. In this step we construct an $\mathbb{R}$ -divisor $\bar{B}$ on X and a $\mathbb{Q}$ -factorial ACSS model $(Y, \mathcal{F}_Y, \bar{B}_Y, \mathbf{M})$ of $(X, \mathcal{F}, B, \mathbf{M})$, so that R is the image of a $(K_{\mathcal{F}_Y} + \bar{B}_Y + \mathbf{M}_Y)$ -negative extremal ray/ U in X .

Let $\bar{B} := B + \lambda_0 P$ and let E be an lc place of $(X, \mathcal{F}, \bar{B}, \mathbf{M})$ such that $\text{center}_X E = W_0$. By our assumption, there exists an ACSS model $(Y, \mathcal{F}_Y, \bar{B}_Y, \mathbf{M}; G)$ of $(X, \mathcal{F}, \bar{B}, \mathbf{M})$ such that E is on Y with induced birational morphism $g : Y \rightarrow X$. We have

$$K_{\mathcal{F}_Y} + \bar{B}_Y + \mathbf{M}_Y + F = g^*(K_{\mathcal{F}} + \bar{B} + \mathbf{M}_X)$$

for some $F \geq 0$. Let $V := g(\text{Supp } F)$, then $V \subset \text{Nlc}(X, \mathcal{F}, \bar{B}, \mathbf{M})$ is a reduced subscheme of X .

By Lemma 8.2.3, there exists an extremal ray R_Y in Y such that $g(R_Y) = R$. Then there exist $C_{Y,i} \in NE(Y/U)$ such that $R_Y = [\lim_{i \rightarrow +\infty} C_{Y,i}]$. We let $C_i := g(C_{Y,i})$, then $R = [\lim_{i \rightarrow +\infty} C_i]$. By the projection formula,

$$\lim(K_{\mathcal{F}_Y} + \bar{B}_Y + F + \mathbf{M}_Y) \cdot C_{Y,i} = \lim(K_{\mathcal{F}} + \bar{B} + \mathbf{M}_X) \cdot C_i,$$

so

$$(K_{\mathcal{F}_Y} + \bar{B}_Y + F + \mathbf{M}_Y) \cdot R_Y = (K_{\mathcal{F}} + \bar{B} + \mathbf{M}_X) \cdot R < 0.$$

Thus R_Y is a $(K_{\mathcal{F}_Y} + \bar{B}_Y + F + \mathbf{M}_Y)$ -negative extremal ray.

If $F \cdot R_Y < 0$, then R_Y is contained in the image of $\overline{NE}(\text{Supp } F/U) \rightarrow \overline{NE}(Y/U)$. Then $R = g(R_Y)$ is contained in the image of $\overline{NE}(V/U) \rightarrow \overline{NE}(X/U)$. Thus there exists an irreducible component V_0 of V such that R is contained in the image of $\overline{NE}(V_0/U) \rightarrow \overline{NE}(X/U)$. Since R is not contained in $\overline{NE}(X/U)_{\text{Nlc}(X, \mathcal{F}, B, \mathbf{M})}$, V_0 is not an lc center of $(X, \mathcal{F}, B, \mathbf{M})$. Since $V_0 \subset V = f(F) \subset \text{Nlc}(X, \mathcal{F}, \bar{B}, \mathbf{M})$ and $\bar{B} = B + \lambda_0 P$, there exists a real number $0 < \lambda_1 < \lambda_0$ such that V_0 is an lc center of $\text{Nlc}(X, \mathcal{F}, B + \lambda_1 P, \mathbf{M})$. This contradicts the minimality of (W_0, λ_0)

as $(V_0, \lambda_1) \in \Gamma$ and $\lambda_1 < \lambda_0$. Therefore, $F \cdot R_Y \geq 0$, so R_Y is a $(K_{\mathcal{F}_Y} + \bar{B}_Y + \mathbf{M}_Y)$ -negative extremal ray.

Step 5. In this step we prove the proposition under the additional condition that X is $\mathbb{Q}$ -factorial.

Assume that X is $\mathbb{Q}$ -factorial. By [BCHM10, Lemma 3.6.2], $\text{Exc}(f)$ is a divisor, so $g^{-1}(W_0)$ is a divisor. Since R is contained in the image of $\overline{NE}(W/U) \rightarrow \overline{NE}(X/U)$ and $g(R_Y) = R$, there exists a divisor E_0 on Y such that R_Y is contained in the image of $\overline{NE}(E_0/U) \rightarrow \overline{NE}(Y/U)$. Since g is an ACSS modification of $(X, \mathcal{F}, \bar{B}, \mathbf{M})$, E_0 is an lc place of $(X, \mathcal{F}, \bar{B}, \mathbf{M})$ and an lc place of $(Y, \mathcal{F}_Y, \bar{B}_Y, \mathbf{M})$.

Let T be the normalization of E_0 , $\mathcal{F}_T := \mathcal{F}_Y|_T$ be the restricted foliation, $\mathbf{M}^T := \mathbf{M}|_T$, and

$$K_{\mathcal{F}_T} + \bar{B}_T + \mathbf{M}_T^T := (K_{\mathcal{F}_Y} + \bar{B}_Y + \mathbf{M}_Y)|_T.$$

Since R_Y is contained in the image of $\overline{NE}(E_0/U) \rightarrow \overline{NE}(Y/U)$, R_Y is contained in the image of

$$\iota : \overline{NE}(T/U) \rightarrow \overline{NE}(E_0/U) \rightarrow \overline{NE}(Y/U).$$

By Lemma 8.2.3, there exists any extremal ray $R_T \in \overline{NE}(T/U)$ such that $\iota(\tilde{R}) = R_Y$. Then R_T is a $(K_{\mathcal{F}_T} + \bar{B}_T + \mathbf{M}_T^T)$ -negative extremal ray/ U . By Theorem 6.6.1 and Theorem 8.2.1 in dimension $\leq d-1$, R_T is spanned by a rational curve C_T , such that C_T is tangent to $\mathcal{F}_T$ and

$$0 < -(K_{\mathcal{F}_T} + \bar{B}_T + \mathbf{M}_T^T) \cdot C_T \leq 2(d-1).$$

Let C_Y be the image of C_T in Y , then C_Y spans R_Y ,

$$0 < -(K_{\mathcal{F}_T} + \bar{B}_T + \mathbf{M}_T^T) \cdot C_T = -(K_{\mathcal{F}_Y} + \bar{B}_Y + \mathbf{M}_Y) \cdot C_Y \leq 2(d-1),$$

and by [ACSS21, Lemma 3.3(4)], C_Y is tangent to $\mathcal{F}_Y$. Let $C := g(C_Y)$, then C is tangent to $\mathcal{F}$. By Step 4, $F \cdot C_Y \geq 0$, so

$$2d \geq -(K_{\mathcal{F}_Y} + \bar{B}_Y + \mathbf{M}_Y) \cdot C_Y \geq -(K_{\mathcal{F}_Y} + \bar{B}_Y + F + \mathbf{M}_Y) \cdot C_Y = -(K_{\mathcal{F}} + B + \mathbf{M}_X) \cdot C > 0.$$

We are done for the case when X is $\mathbb{Q}$ -factorial.

Step 6. In this step we conclude the proof of the theorem. Since Y is $\mathbb{Q}$ -factorial and R_Y is a $(K_{\mathcal{F}_Y} + \bar{B}_Y + \mathbf{M}_Y)$ -negative extremal ray, by the $\mathbb{Q}$ -factorial case proved in Step 5, R_Y is spanned by a rational curve C_Y that is tangent to $\mathcal{F}_Y$ and

$$0 < -(K_{\mathcal{F}_Y} + \bar{B}_Y + \mathbf{M}_Y) \cdot C_Y \leq 2d.$$

Let $C := g(C_Y)$, then C is tangent to $\mathcal{F}$. Since $F \cdot C_Y \geq 0$,

$$2d \geq -(K_{\mathcal{F}_Y} + \bar{B}_Y + \mathbf{M}_Y) \cdot C_Y \geq -(K_{\mathcal{F}_Y} + \bar{B}_Y + F + \mathbf{M}_Y) \cdot C_Y = -(K_{\mathcal{F}} + B + \mathbf{M}_X) \cdot C > 0.$$

This concludes the proof of the proposition. $\square$

Proposition 8.4.3. *Let d be a positive integer. Assume that Theorem 8.2.1 holds in dimension $\leq d-1$ and Theorem 8.2.2 holds for $\mathbb{Q}$ -factorial varieties in dimension d .*

Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a gfp of dimension d such that $\mathcal{F}$ is algebraically integrable, and let A be an ample/ U $\mathbb{R}$ -divisor on X . Assume that

- *either X is $\mathbb{Q}$ -factorial, or*
- *Theorem 8.2.2 holds in dimension d .*

Then there are finitely many $(K_{\mathcal{F}} + B + A + \mathbf{M}_X)$ -negative extremal rays/ U that are not contained in $\overline{NE}(X/U)_{\text{Nlc}(X, \mathcal{F}, B, \mathbf{M})}$.

Proof. Let $d := \dim X$ and let $\omega := K_{\mathcal{F}} + B + \mathbf{M}_X$, $\rho := \rho(X/U)$, and let $A_1, \dots, A_{\rho-1}$ be ample/ U Cartier divisors on X , such that $\omega, A_1, \dots, A_{\rho-1}$ form a basis of $N_{\mathbb{R}}^1(X/U)$. Let $0 < \epsilon \ll 1$ be a rational number such that $A - \epsilon \sum_{i=1}^{\rho-1} A_i$ is ample/ U . Then we only need to show that there are finitely many $(K_{\mathcal{F}} + B + \epsilon \sum_{i=1}^{\rho-1} A_i + \mathbf{M}_X)$ -negative extremal rays/ U that are not contained in $\overline{NE}(X/U)_{\text{Nlc}(X, \mathcal{F}, B, \mathbf{M})}$. Possibly replacing A , we may assume that $A = \epsilon \sum_{i=1}^{\rho-1} A_i$.

Suppose that the proposition does not hold. Then there exist an infinite set Λ and an infinite set $\{R_j\}_{j \in \Lambda}$ of $(K_{\mathcal{F}} + B + A + \mathbf{M}_X)$ -negative extremal rays/ U that are not contained in $\overline{NE}(X/U)_{\text{Nlc}(X, \mathcal{F}, B, \mathbf{M})}$. By Definition-Lemma 3.1.8, possibly replacing Λ with a smaller infinite subset, we may assume that each R_j is a $(K_{\mathcal{F}} + B + A + \mathbf{M}_X)$ -negative exposed ray/ U . By Proposition 8.4.2, for any $j \in \Lambda$, there exists a rational curve C_j on X that is tangent to $\mathcal{F}$ and $R_j = [C_j]$, such that

$$-2d \leq \omega \cdot C_j < 0.$$

For each $j \in \Lambda$, by Lemma 8.4.1, there exists an ample/ U $\mathbb{R}$ -divisor L_j and a nef/ U $\mathbb{R}$ -divisor H_j , such that

$$H_j = L_j + (K_{\mathcal{F}} + B + A + \mathbf{M}_X) = L_j + \epsilon \sum_{i=1}^{\rho-1} A_i + \omega$$

and H_j is the supporting function of R_j . We have

$$0 = H_j \cdot C_j = L_j \cdot C_j + \epsilon \sum_{i=1}^{\rho-1} A_i \cdot C_j + \omega \cdot C_j \geq -2d + \epsilon \sum_{i=1}^{\rho-1} A_i \cdot C_j.$$

Therefore, $A_i \cdot C_j \leq \frac{2d}{\epsilon}$ for any i, j . Since $A_i \cdot C_j \in \mathbb{N}^+$, there are finitely many possibilities of $A_i \cdot C_j$. Possibly replacing Λ with an infinite subset, we may assume that $A_i \cdot C_j = A_i \cdot C_{j'}$ for any i and any $j, j' \in \Lambda$.

We may write $\omega = \sum_{i=1}^c r_i D_i$ such that $r_1, \dots, r_c$ are linearly independent over $\mathbb{Q}$ and D_i are Weil divisors. By [HLS19, Lemma 5.3], each D_i is a $\mathbb{Q}$ -Cartier divisor. Thus there exist real numbers $a_{i,k}$ and b_i , such that

$$D_i \equiv_U \sum_{k=1}^{\rho-1} a_{i,k} A_k + b_i \omega$$

for each i .

We let $\delta_1, \dots, \delta_c$ be real numbers such that $\sum_{i=1}^c b_i \delta_i > -1$ and

$$r'_i := \delta_i + r_i \in \mathbb{Q}.$$

Let $\omega' := \sum_{i=1}^c r'_i D_i$. Then

$$\omega' = \omega + \sum_{i=1}^c \delta_i D_i = \left(\sum_{k=1}^{\rho-1} \left(\sum_{i=1}^c \delta_i a_{i,k} \right) A_k \right) + \left(1 + \sum_{i=1}^c \delta_i b_i \right) \omega.$$

Since $\sum_{i=1}^c b_i \delta_i > -1$, ω' and $A_1, \dots, A_{\rho-1}$ form a basis of $N_{\mathbb{R}}^1(X/U)$. Moreover,

$$\omega' \cdot C_j = \left(\sum_{i=1}^c \sum_{k=1}^{\rho} \delta_i a_{i,k} \cdot (A_k \cdot C_j) \right) + \left(1 + \sum_{i=1}^c \delta_i b_i \right) (\omega \cdot C_j).$$

By our assumptions,

$$\alpha := \sum_{i=1}^c \sum_{k=1}^{\rho} \delta_i a_{i,k} \cdot (A_k \cdot C_j)$$

and

$$\beta := 1 + \sum_{i=1}^c \delta_i b_i > 0$$

are constants which do not depend on j , and $\omega \cdot C_j \in [-2d, 0)$. Therefore,

$$\omega' \cdot C_j \in [-2d\beta + \alpha, \alpha)$$

for any j . Since $r'_i \in \mathbb{Q}$ for any i , ω' is a $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor. Let I be the Cartier index of ω' , then

$$\omega' \cdot C_j \in [-2d\beta + \alpha, \alpha) \cap \frac{1}{I}\mathbb{Z}$$

for any j . Therefore, there are only finitely many possibilities of $\omega' \cdot C_j$. Possibly replacing Λ with an infinite subset, we may assume that $\omega' \cdot C_j = \omega' \cdot C_{j'}$ for any $j, j' \in \Lambda$. Since $\omega', A_1, \dots, A_{\rho-1}$ form a basis of $N_{\mathbb{R}}^1(X/U)$, $C_j \equiv_U C_{j'}$, which is not possible as R_j and $R_{j'}$ are different rays in $\overline{NE}(X/U)$. $\square$

Proposition 8.4.4. *Let d be a positive integer. Assume that Theorem 8.2.1 holds in dimension $\leq d-1$ and Theorem 8.2.2 holds for $\mathbb{Q}$ -factorial varieties in dimension d . Then:*

- (1) *Theorem 8.2.1 holds for $\mathbb{Q}$ -factorial varieties in dimension d .*
- (2) *If Theorem 8.2.2 holds in dimension d , then Theorem 8.2.1 holds in dimension d .*

Proof. First we show that $\overline{NE}(X/U) = V$, where

$$V := \overline{NE}(X/U)_{K_{\mathcal{F}}+B+\mathbf{M}_X \geq 0} + \overline{NE}(X/U)_{\text{Nlc}(X, \mathcal{F}, B, \mathbf{M})} + \sum_{j \in \Lambda} R_j.$$

By Definition-Lemma 3.1.8, $\overline{NE}(X/U) = \overline{V}$. Suppose that $V \neq \overline{V}$, then there exists an extremal ray R in $\overline{NE}(X/U)$ such that $R \notin V$. Since $\overline{NE}(X/U)_{K_{\mathcal{F}}+B+\mathbf{M}_X \geq 0}$ and $\overline{NE}(X/U)_{\text{Nlc}(X, \mathcal{F}, B, \mathbf{M})}$ are closed, $R \notin \overline{NE}(X/U)_{K_{\mathcal{F}}+B+\mathbf{M}_X \geq 0}$ and $R \notin \overline{NE}(X/U)_{\text{Nlc}(X, \mathcal{F}, B, \mathbf{M})}$, so R is a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -negative extremal ray R that is not contained in $\overline{NE}(X/U)_{\text{Nlc}(X, \mathcal{F}, B, \mathbf{M})}$. Thus $R = R_j$ for some j , a contradiction. Therefore, $\overline{NE}(X/U) = V$.

Next we show that any each R_j is exposed. For any fixed j , There exists an ample/ U $\mathbb{R}$ -divisor A such that R_j is a $(K_{\mathcal{F}} + B + A + \mathbf{M}_X)$ -negative extremal ray/ U . Suppose that R_j is not exposed. By Definition-Lemma 3.1.8, $R_j = \lim_{i \rightarrow +\infty} R_{j,i}$ for some exposed rays $R_{j,i} \in \overline{NE}(X/U)$. Since $(K_{\mathcal{F}} + B + A + \mathbf{M}_X) \cdot R_j < 0$, possibly passing to a subsequence, we have $(K_{\mathcal{F}} + B + A + \mathbf{M}_X) \cdot R_{j,i} < 0$ for any i . By Proposition 8.4.3, there are only finitely many $(K_{\mathcal{F}} + B + A + \mathbf{M}_X)$ -negative extremal rays that are not contained in $\overline{NE}(X/U)_{\text{Nlc}(X, \mathcal{F}, B, \mathbf{M})}$, for any $i \gg 0$, $R_{j,i}$ is contained in $\overline{NE}(X/U)_{\text{Nlc}(X, \mathcal{F}, B, \mathbf{M})}$. Since $\overline{NE}(X/U)_{\text{Nlc}(X, \mathcal{F}, B, \mathbf{M})}$ is a closed sub-cone of $\overline{NE}(X/U)$, R_j is contained in $\overline{NE}(X/U)_{\text{Nlc}(X, \mathcal{F}, B, \mathbf{M})}$, a contradiction.

By Proposition 8.4.2, for any $j \in \Lambda$, R_j is spanned by a rational curve C_j such that C_j is tangent to $\mathcal{F}$ and

$$0 < -(K_{\mathcal{F}} + B + \mathbf{M}_X) \cdot C_j \leq 2d.$$

$\square$

8.5. Proofs of Theorems 2.5.1, 6.0.1, 2.4.3, and 2.4.2.

Proofs of Theorems 8.2.1 and 8.2.2. Theorems 8.2.1 and 8.2.2 hold when $d = 1$ trivially. Therefore, we may assume that $d \geq 2$ and Theorems 8.2.1 and 8.2.2 hold in dimension $\leq d-1$.

By Proposition 8.3.4(1), Theorem 8.2.2 holds for $\mathbb{Q}$ -factorial varieties in dimension d . By Proposition 8.4.4(1), Theorem 8.2.1 holds for $\mathbb{Q}$ -factorial varieties in dimension d . By Proposition 8.3.4(2), Theorem 8.2.2 holds in dimension d . By Proposition 8.4.4(2), Theorem 8.2.1 holds in dimension d .

By induction on d , Theorems 8.2.1 and 8.2.2 hold. $\square$

Proof of Theorem 2.5.1. It is a special case of Theorem 8.2.2. $\square$

Proof of Theorem 2.4.2. By Theorem 2.5.1, there exists a $\mathbb{Q}$ -factorial ACSS model $(X', \mathcal{F}', B', \mathbf{M})$ of $(X, \mathcal{F}, B, \mathbf{M})$ with induced birational morphism $h : X' \rightarrow X$ and associated with $f : X' \rightarrow Z$. Since $(X, \mathcal{F}, B, \mathbf{M})$ is lc,

$$K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} = f^*(K_{\mathcal{F}} + B + \mathbf{M}_X).$$

Let S' be the normalization of $h_*^{-1}S$, $\mathcal{F}_{S'}$ the restricted foliation of $\mathcal{F}$ on S' , and $\mathbf{M}^S := \mathbf{M}|_{S^\nu}$. Then there exists an induced birational morphism $h_S : S' \rightarrow S^\nu$. Let

$$K_{\mathcal{F}_{S'}} + B_{S'} + \mathbf{M}_{S'}^S := (K_{\mathcal{F}'} + B' + \mathbf{M}_{X'})|_{S'},$$

then by Theorem 6.6.1, $(S', \mathcal{F}_{S'}, B_{S'}, \mathbf{M}^S)$ is lc. Since

$$K_{\mathcal{F}_{S'}} + B_{S'} + \mathbf{M}_{S'}^S = h_S^*(K_{\mathcal{F}_S} + B_S + \mathbf{M}_{S^\nu}^S),$$

$(S^\nu, \mathcal{F}_S, B_S, \mathbf{M}^S)$ is lc. $\square$

Proof of Theorem 6.0.1. By Theorem 2.5.1, there exists a $\mathbb{Q}$ -factorial ACSS model $(X', \mathcal{F}', B', \mathbf{M})$ of $(X, \mathcal{F}, B, \mathbf{M})$ with induced birational morphism $h : X' \rightarrow X$ and associated with $f : X' \rightarrow Z$. Let S' be the normalization of $h_*^{-1}S$ and $E := (\text{Supp Exc}(h))^{\mathcal{F}_Y}$. Then there exists an induced birational morphism $h_S : S' \rightarrow S^\nu$.

We let $\mathcal{F}_{S'}$ be the restricted foliation of $\mathcal{F}'$ on S' . By Theorem 6.6.2, there exist prime divisors $C'_1, \dots, C'_q, T'_1, \dots, T'_l$ on S' , positive integers $w_1, \dots, w_q$, and non-negative integers $\{w_{i,j}\}_{1 \leq i \leq q, 1 \leq j \leq m}$ and $\{v_{i,k}\}_{1 \leq i \leq q, 1 \leq k \leq n}$ satisfying the following. For any real numbers $b'_1, \dots, b'_m$ and $r'_1, \dots, r'_n$,

$$\begin{aligned} & \left(K_{\mathcal{F}'} + \epsilon_{\mathcal{F}}(S)S' + \sum_{j=1}^m b'_j B'_j + \sum_{k=1}^n r'_k \mathbf{M}_{k,X'} \right) \Big|_{S'} \\ &= K_{\mathcal{F}_{S'}} + \sum_{i=1}^q \frac{w_i - 1 + \sum_{j=1}^m w_{i,j} b'_j + \sum_{k=1}^n v_{i,k} r'_k}{w_i} C'_i + \sum_{i=1}^l T'_i + \sum_{k=1}^n r'_k \mathbf{M}_{k,S'}^S \\ &:= K_{\mathcal{F}_{S'}} + B'_{S'} + \mathbf{M}_{S'}^S. \end{aligned}$$

Let

$$K_{\mathcal{F}_S} + B'_S + \mathbf{M}_{S^\nu}^S := \left(K_{\mathcal{F}} + \sum_{j=1}^m b'_j B_j + \sum_{k=1}^n r'_k \mathbf{M}_{k,X'} \right) \Big|_{S^\nu},$$

then

$$K_{\mathcal{F}_{S'}} + B'_{S'} + \mathbf{M}_{S'}^S = (f_S)_*(K_{\mathcal{F}_S} + B'_S + \mathbf{M}_{S^\nu}^S).$$

Theorem 6.0.1(1) follows. Theorem 6.0.1(2) follows from Theorem 2.4.2. $\square$

Proof of Theorem 2.4.3. It is an immediate corollary of Theorem 6.0.1. $\square$

8.6. Proof of Theorem 2.3.1. Finally, we prove the full version of the cone theorem for generalized foliated quadruples, Theorem 2.3.1.

Lemma 8.6.1. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a gfq such that $\mathcal{F}$ is algebraically integrable. Assume that R is a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -negative extremal ray in $\overline{NE}(X/U)$ that is not contained in $\overline{NE}(X/U)_{\text{Nlc}(X, \mathcal{F}, B, \mathbf{M})}$. Then R is a rational extremal ray in $\overline{NE}(X/U)$.*

Proof. By Lemma 8.4.1, there exists an ample/ U $\mathbb{R}$ -divisor A such that $H_R := K_{\mathcal{F}} + B + A + \mathbf{M}_X$ is a supporting function of R . We let $\delta \in (0, 1)$ be a rational number such that R is a $(K_{\mathcal{F}} + B + \delta A + \mathbf{M}_X)$ -negative extremal ray/ U that is not contained in $\overline{NE}(X/U)_{\text{Nlc}(X, \mathcal{F}, B, \mathbf{M})}$. Let Λ be the set of all $(K_{\mathcal{F}} + B + \delta A + \mathbf{M}_X)$ -negative extremal ray/ U . By Proposition 8.4.3, Λ is a finite set, and we may write $\Lambda = \{R, R_1, \dots, R_l\}$ for some non-negative integer l . Then

$$V := \overline{NE}(X/U)_{K_{\mathcal{F}} + B + \delta A + \mathbf{M}_X \geq 0} + \overline{NE}(X/U)_{\text{Nlc}(X, \mathcal{F}, B, \mathbf{M})} + \sum_{i=1}^l R_i$$

is a closed sub-cone of $\overline{NE}(X/U)$ and $R \notin V$. Let C be the dual cone of V in $N^1(X/U)$, then since $H_R \cdot R' > 0$ for any $R' \in V$, H_R is contained in the interior of C . Therefore, there exists

a real number $\epsilon \in (0, 1)$ such that $H_R - \epsilon A$ is contained in the interior of C . In particular, $(H_R - \epsilon A) \cdot R' > 0$ for any $R' \in V$.

We write $H_R = \sum_{i=1}^c r_i D_i$, where $r_1, \dots, r_c$ are real numbers that are linearly independent over $\mathbb{Q}$ and D_i are Weil divisors. By [HLS19, Lemma 5.3], each D_i is a $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor. Moreover, by Theorem 8.2.1, R is spanned by a rational curve L . Since $H_R \cdot L = 0$, $D_i \cdot L = 0$ for each i .

There exist rational numbers $r'_1, \dots, r'_c$ such that $\sum_{i=1}^c (r'_i - r_i) D_i + \epsilon A$ is ample/ U . We let $H'_R := \sum_{i=1}^c r'_i D_i$. Then $H'_R \cdot R = 0$. For any extremal ray $R' \in \overline{NE}(X/U)$ such that $R' \neq R$, $R' \in V$. Thus

$$H'_R \cdot R' = H_R \cdot R' + \sum_{i=1}^c (r'_i - r_i) D_i \cdot R' = (H_R - \epsilon A) \cdot R' + \left(\sum_{i=1}^c (r'_i - r_i) D_i + \epsilon A \right) \cdot R' > 0.$$

Thus H'_R is a supporting function of R . Since H'_R is a $\mathbb{Q}$ -divisor, it is a rational supporting function of R , so R is a rational extremal ray in $\overline{NE}(X/U)$. $\square$

Proof of Theorem 2.3.1. Theorem 2.3.1(1) follows from Lemma 8.6.1 and Theorem 8.2.1. Theorem 2.3.1(2) follows from Theorem 8.2.1. Theorem 2.3.1(3) follows from Proposition 8.4.3 and that $\Lambda = \bigcup_{n=1}^{+\infty} \Lambda_{\frac{1}{n}A}$ for any ample/ U $\mathbb{R}$ -divisor A . We left to prove (4).

For any $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -negative extremal face F in $\overline{NE}(X/U)$ that is relatively ample at infinity with respect to $(X, \mathcal{F}, B, \mathbf{M})$, F is also a $(K_{\mathcal{F}} + B + \mathbf{M}_X + A)$ -negative extremal face for some ample/ U $\mathbb{R}$ -divisor A on X . Let $V := F^\perp \subset N^1(X/U)$. By (1), F is spanned by a subset of $\{R_j\}_{j \in \Lambda_A}$ and R_j is rational, so V is defined over $\mathbb{Q}$. We let

$$W_F := \overline{NE}(X/U)_{K_X+B+\mathbf{M}_X+A \geq 0} + \overline{NE}(X/U)_{\text{Nlc}(X, \mathcal{F}, B, \mathbf{M})} + \sum_{j|j \in \Lambda_A, R_j \not\subset F} R_j.$$

Then W_F is a closed cone, $\overline{NE}(X/U) = W_F + F$, and $W_F \cap F = \{0\}$. The supporting functions of F are the elements in V that are positive on $W_F \setminus \{0\}$, which is a non-empty open subset of V , and hence contains a rational element H . In particular, $F = H^\perp \cap \overline{NE}(X/U)$, hence F is rational, and we get (4). This concludes the proof of Theorem 2.3.1. $\square$

9. MINIMAL MODEL PROGRAM FOR ACSS GENERALIZED FOLIATED QUADRUPLES

With the establishment of the cone theorem, we are ready to study the minimal model program for algebraically integrable generalized foliated quadruples. Unfortunately for us, we cannot prove the contraction theorem and the cone theorem for the time being due to technical reasons. However, we are able to run some special types of the minimal model program for foliations.

9.1. Models. With the definition of ACSS singularities, we are able to define the concept of *log minimal models* and *good minimal models* for algebraically integrable foliations.

Definition 9.1.1 (Models, II). Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be an lc gfq and $(X', \mathcal{F}', B', \mathbf{M})/U$ a log birational model of $(X, \mathcal{F}, B, \mathbf{M})/U$. We say that $(X', \mathcal{F}', B', \mathbf{M})/U$ is a *log minimal model* of $(X, \mathcal{F}, B, \mathbf{M})/U$ if

- (1) $(X', \mathcal{F}', B', \mathbf{M})/U$ is a weak lc model of $(X, \mathcal{F}, B)/U$,
- (2) $(X', \mathcal{F}', B', \mathbf{M})$ is $\mathbb{Q}$ -factorial ACSS, and
- (3) for any prime divisor D on X which is exceptional over X' ,

$$a(D, \mathcal{F}, B, \mathbf{M}) < a(D, \mathcal{F}', B', \mathbf{M}).$$

We say that $(X', \mathcal{F}', B', \mathbf{M})/U$ is a *good minimal model* of $(X, \mathcal{F}, B, \mathbf{M})/U$ if $(X', \mathcal{F}', B', \mathbf{M})/U$ is a log minimal model of $(X, \mathcal{F}, B, \mathbf{M})/U$ and a semi-good minimal model of $(X, \mathcal{F}, B, \mathbf{M})/U$.

Remark 9.1.2. It is important to note that, the concept of “log minimal model” or “good minimal model” defined in Definition 9.1.1 does not coincide with the concept of “log minimal model” or “good minimal model” when $\mathcal{F} = T_X$ in the classical setting ([Bir12, Definition 2.1], [HL21a, Definition 3.2]). This is because “ACSS” is equivalent to “qdl” when $\mathcal{F} = T_X$, while the classical definition of log minimal models requires the (generalized) pair to be “dlt”. This difference will not cause trouble, mainly because the existence of log (resp. good) minimal models is equivalent to the existence of weak lc (resp. semi-good) minimal models, at least for NQC generalized pairs (cf. [TX23, Theorem 2.7]).

The following lemma is straightforward but also convenient for us to apply in some scenarios.

Lemma 9.1.3. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be an lc gfq and $(X', \mathcal{F}', B', \mathbf{M})$ a $\mathbb{Q}$ -factorial ACSS model of $(X, \mathcal{F}, B, \mathbf{M})$. Then $(X', \mathcal{F}', B', \mathbf{M})/X$ is a good minimal model of $(X, \mathcal{F}, B, \mathbf{M})/X$.*

Proof. It immediately follows from the definitions. $\square$

Lemma 9.1.4. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be an lc gfq, $\Delta \geq 0$ an $\mathbb{R}$ -divisor on X , and $\mathbf{N}$ a nef/ U $\mathbf{b}$ -divisor on X . Assume that $\mathcal{F}$ is induced by a contraction $f : X \rightarrow Z$ and*

$$K_{\mathcal{F}} + B + \mathbf{M}_X \sim_{\mathbb{R}, Z} K_X + \Delta + \mathbf{N}_X.$$

Then the followings hold.

- (1) *Any $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -negative extremal ray/ U R is a $(K_X + \Delta + \mathbf{N}_X)$ -negative extremal ray/ Z , and $(K_{\mathcal{F}} + B + \mathbf{M}_X) \cdot R = (K_X + \Delta + \mathbf{N}_X) \cdot R$.*
- (2) *Any step of a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U is a step of a $(K_X + \Delta + \mathbf{N}_X)$ -MMP/ Z . Moreover, assume that $(X, \Delta, \mathbf{N})$ is lc and either X is $\mathbb{Q}$ -factorial klt or $\mathbf{N}$ is NQC/ U , then we may run a step of a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U .*
- (3) *Assume that $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is weak ACSS for some divisor G , $\Delta = B + G$, and $\mathbf{M} = \mathbf{N}$. For any sequence of steps*

$$\phi : (X, \mathcal{F}, B, \mathbf{M}; G) \dashrightarrow (X', \mathcal{F}', B', \mathbf{M}; G')$$

of a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U , we have the following.

- (a) *$(X', \mathcal{F}', B', \mathbf{M}; G')/Z$ is weak ACSS.*
- (b) *If $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is ACSS, then $(X', \mathcal{F}', B', \mathbf{M}; G')/Z$ is ACSS.*
- (c) *If G is super/ Z , then G' is super/ Z .*
- (d) *If X is $\mathbb{Q}$ -factorial klt, then X' is $\mathbb{Q}$ -factorial klt.*
- (e) *If X is $\mathbb{Q}$ -factorial and $\mathbf{M}$ is NQC/ U , then X' is $\mathbb{Q}$ -factorial.*
- (f) *If X is $\mathbb{Q}$ -factorial and $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is (super) ACSS, then X' is $\mathbb{Q}$ -factorial and $(X', \mathcal{F}', B', \mathbf{M}; G')/Z$ is (super) ACSS.*
- (4) *Any sequence of steps of a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U is a sequence of steps of a $(K_X + \Delta + \mathbf{N}_X)$ -MMP/ Z .*

Proof. (1) By Theorem 2.3.1, any $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -negative extremal ray/ U is tangent to $\mathcal{F}$, hence is an extremal ray/ Z . We get (1).

(2) By (1), any $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -negative extremal ray/ U R is a $(K_X + \Delta + \mathbf{N}_X)$ -negative extremal ray/ U . If X is $\mathbb{Q}$ -factorial klt, then by [HL22, Lemma 3.4] and the cone theorem, contraction theorem, and the existence of flips for usual klt pairs, we get a step of a $(K_X + \Delta + \mathbf{N}_X)$ -MMP/ U associated to R , which is also a step of a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U associated to R . If $\mathbf{N}$ is NQC/ U , then by the cone theorem ([HL21a, Theorem 1.3], Theorem 2.3.1), the contraction theorem ([Xie22, Theorem 1.5], [CLX23, Theorem 1.7]), and the existence of flips ([LX23b, Theorem 1.2]), we get a step of a $(K_X + \Delta + \mathbf{N}_X)$ -MMP/ U associated to R , which is also a step of a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U associated to R . Moreover, by (1), R is a negative extremal ray/ Z , so this step of the MMP is also a step of an MMP/ Z .

(3) Without loss of generality, we may assume that ϕ is a single step of a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U . By Proposition 5.4.7, we get (3.a). (3.c) is obvious because $G' = \phi_* G$.

If $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is ACSS, then there exist an $\mathbb{R}$ -divisor $D \geq 0$ on X and a nef/ X $\mathbf{b}$ -divisor $\mathbf{M}'$, such that $\text{Supp}\{B\} \subset \text{Supp } D$, $\mathbf{M}' - \alpha\mathbf{M}$ is nef/ X for some $\alpha > 1$, and for any reduced divisor Σ on Z such that $\Sigma \geq f(G)$ and (Z, Σ) is log smooth,

$$(X, B + D + G + f^*(\Sigma - f(G)), \mathbf{M}')$$

is qdlt. Let $\mathbf{P} := \mathbf{M}' - \mathbf{M}$, then

$$(X, B + \delta D + G + f^*(\Sigma - f(G)), \mathbf{M} + \delta \mathbf{P})$$

is qdlt for any $0 \leq \delta \leq 1$, and

$$\mathbf{M} + \delta \mathbf{P} - (1 + \delta(\alpha - 1))\mathbf{M} = \delta(\mathbf{M}' - \alpha\mathbf{M})$$

is nef/ X . By (2), ϕ is a step of a $(K_X + B + G + \mathbf{M}_X)$ -MMP/ Z , hence a step of a

$$(K_X + B + \delta D + G + f^*(\Sigma - f(G)) + \mathbf{M}_X + \delta \mathbf{P}_X)\text{-MMP}/Z$$

for any $0 < \delta \ll 1$. Therefore

$$(K_{X'} + B' + \delta \phi_* D + G' + f'^*(\Sigma - f(G)), \mathbf{M}_X + \delta \mathbf{P})$$

is qdlt, where $f' : X' \rightarrow Z$ is the induced contraction. Moreover, for any lc place E of $(X', \mathcal{F}', B', \mathbf{M})$, since

$$-\epsilon_{\mathcal{F}}(E) \leq a(E, \mathcal{F}, B, \mathbf{M}) \leq a(E, \mathcal{F}', B', \mathbf{M}) \leq -\epsilon_{\mathcal{F}'}(E) = -\epsilon_{\mathcal{F}}(E),$$

E is also an lc place of $(X, \mathcal{F}, B, \mathbf{M})$ and ϕ is an isomorphism near the generic point of center $_X E$. By (3.a) $(X', \mathcal{F}', B', \mathbf{M}; G')/Z$ is ACSS. This implies (3.b).

Assume that X is $\mathbb{Q}$ -factorial. By (2), ϕ is a step of a $(K_X + B + G + \mathbf{M}_X)$ -MMP/ U , hence a step of a $(K_X + B + G + \mathbf{M}_X + A)$ -MMP/ U for some ample/ U $\mathbb{R}$ -divisor A . If X is klt, then by [HL22, Lemma 3.4], there exists a klt pair (X, Δ) such that $0 \leq \Delta \sim_{\mathbb{R}} B + G + \mathbf{M}_X + A$, so ϕ is a step of a $(K_X + \Delta)$ -MMP, and (3.d) follows from [KM98, Corollaries 3.17, 3.18]. If $\mathbf{M}$ is NQC/ U , then by [HL21a, Corollary 5.20, Theorem 6.3], X' is $\mathbb{Q}$ -factorial, and we get (3.e). Since $\mathbb{Q}$ -factorial qdlt implies that the ambient variety is klt, (3.f) follows from (3.b), (3.c) and (3.d).

(4) Since the birational transforms of $(X, \mathcal{F}, B, \mathbf{M})$ are lc under any sequence of steps of a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP, (4) follows from (2). $\square$

9.2. MMP with super divisors.

Lemma 9.2.1. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be an lc gfg, $(X, \Delta, \mathbf{N})/U$ an lc g-pair, and $f : X \rightarrow Z$ a contraction, such that $\mathcal{F}$ is induced by f , Δ is super/ Z , and*

$$K_{\mathcal{F}} + B + \mathbf{M}_X \sim_{\mathbb{R}, Z} K_X + \Delta + \mathbf{N}_X.$$

Then the followings hold.

- (1) *Any $(K_X + \Delta + \mathbf{N}_X)$ -negative extremal ray/ U R is a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -negative extremal ray/ Z and $(K_{\mathcal{F}} + B + \mathbf{M}_X) \cdot R = (K_X + \Delta + \mathbf{N}_X) \cdot R$.*
- (2) *A step of a $(K_X + \Delta + \mathbf{N}_X)$ -MMP/ U is a step of a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ Z .*
- (3) *Any sequence of steps of a $(K_X + \Delta + \mathbf{N}_X)$ -MMP/ U is a sequence of steps of a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ Z .*
- (4) *Let $D \geq 0$ be an $\mathbb{R}$ -divisor on X and $\mathbf{N}'$ a nef/ U $\mathbf{b}$ -divisor on X such that $D + \mathbf{N}'_X$ is $\mathbb{R}$ -Cartier. Then any sequence of steps of a $(K_X + \Delta + \mathbf{N}_X)$ -MMP/ U with scaling of $(D, \mathbf{N}')$ is a sequence of steps of a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U with scaling of $(D, \mathbf{N}')$, and any sequence of steps of a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U with scaling of $(D, \mathbf{N}')$ is a sequence of steps of a $(K_X + \Delta + \mathbf{N}_X)$ -MMP/ U with scaling of $(D, \mathbf{N}')$.*

Proof. (1) Let $d := \dim X$. Since Δ is super, $\Delta \geq \sum_{i=1}^{2d+1} f^* H_i$ for some ample Cartier divisors H_i on Z . Let $L := \Delta - \sum_{i=1}^{2d+1} f^* H_i$, then $(X, L, \mathbf{M})$ is lc and R is a $(K_X + L + \mathbf{N}_X)$ -negative extremal ray. By Theorem 2.2.1 (applied to $(X, T_X, L, \mathbf{N})/U$), there exists a rational curve C on X such that C spans R and

$$-2d \leq (K_X + L + \mathbf{N}_X) \cdot C < 0.$$

Therefore,

$$0 > (K_X + \Delta + \mathbf{N}_X) \cdot C = (K_X + L + \mathbf{N}_X) \cdot C + \left(\sum_{i=1}^{2d+1} f^* H_i \cdot C \right) \geq -2d + \left(\sum_{i=1}^{2d+1} f^* H_i \cdot C \right).$$

Thus $f(C)$ is a point, so R is an extremal ray/ Z . (1) follows from our assumption.

(2) immediately follows from (1). By (2), the birational transforms of $(X, \mathcal{F}, B, \mathbf{M})$ and $(X, \Delta, \mathbf{N})$ are lc after any sequences of steps of a $(K_X + \Delta + \mathbf{N}_X)$ -MMP, and (3) follows from (2). (4) follows from (1), (3) and Lemma 9.1.4(1). $\square$

Lemma 9.2.2. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be an lc gfq and $(X, \Delta, \mathbf{N})/U$ an lc g-pair such that $\mathcal{F}$ is induced by a contraction $X \rightarrow Z$ and*

$$K_{\mathcal{F}} + B + \mathbf{M}_X \sim_{\mathbb{R}, Z} K_X + \Delta + \mathbf{N}_X.$$

Then the followings hold.

- (1) $K_{\mathcal{F}} + B + \mathbf{M}_X$ is nef/ Z if and only if $K_X + \Delta + \mathbf{N}_X$ is nef/ Z .
- (2) If $K_X + \Delta + \mathbf{N}_X$ is either nef/ Z or nef/ U , then $K_{\mathcal{F}} + B + \mathbf{M}_X$ is nef/ U .
- (3) If Δ is super/ Z and $K_{\mathcal{F}} + B + \mathbf{M}_X$ is either nef/ Z or nef/ U , then $K_X + \Delta + \mathbf{N}_X$ is nef/ U .

Proof. (1) is obvious. (2) follows from Lemma 9.1.4(1). (3) follows from Lemma 9.2.1(1). $\square$

9.3. MMP with scaling and existence of Mori fiber spaces.

Notation 9.3.1. In the subsequent discussions, it is important to differentiate between “one specific MMP that adheres to certain properties” and “all MMPs that adhere to certain properties.” For instance, some argument apply to “all MMPs with scaling of an ample divisor,” whereas some only apply to “a specific MMP with scaling of an ample divisor.” Given this nuance, we will regard “MMPs” as entities, and typically represent them using symbols like $\mathcal{P}$ or similar notations.

Proposition 9.3.2. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be an lc gfq and $f : X \rightarrow Z$ a contraction, such that*

$$K_{\mathcal{F}} + B + \mathbf{M}_X \sim_{\mathbb{R}, Z} K_X + \Delta + \mathbf{N}_X$$

for some lc g-pair $(X, \Delta, \mathbf{N})/U$. Assume that either X is $\mathbb{Q}$ -factorial klt or $\mathbf{N}$ is NQC/ U . Then for any ample/ U $\mathbb{R}$ -divisor A , we can run a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U with scaling of A .

Moreover, there exists a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U with scaling of A , say $\mathcal{P}_0$, satisfying the following. Let $\mathcal{P} = \mathcal{P}_0$ if X is not $\mathbb{Q}$ -factorial, and let $\mathcal{P}$ be any $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U with scaling of A if X is $\mathbb{Q}$ -factorial. Then the followings hold.

- (1) *Suppose that there exists an lc gfq $(X, \tilde{\Delta}, \tilde{\mathbf{N}})/U$ and an ample/ U $\mathbb{R}$ -divisor H , such that either X is $\mathbb{Q}$ -factorial klt or $\tilde{\mathbf{N}}$ is NQC/ U , and $\Delta + \mathbf{N}_X \sim_{\mathbb{R}, U} \tilde{\Delta} + \tilde{\mathbf{N}}_X + H$. Then $\mathcal{P}$ terminates at a model $(X', \mathcal{F}', B', \mathbf{M})/U$ of $(X, \mathcal{F}, B, \mathbf{M})/U$, such that*
 - (a) *either there exists a $(K_{\mathcal{F}'} + B' + \mathbf{M}_{X'})$ -Mori fiber space/ U which is also a $(K_{\mathcal{F}'} + B' + \mathbf{M}_{X'})$ -Mori fiber space/ Z , or*
 - (b)

$$K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} \sim_{\mathbb{R}, Z} D$$

for some semi-ample/ U $\mathbb{R}$ -divisor D .

- (2) *Either $\mathcal{P}$ terminates, or the limit of the scaling numbers of $\mathcal{P}$ is 0.*

Proof. We first construct $\mathcal{P}_0$. Possibly replacing Δ , we may assume that Δ is super/ Z . By Lemma 4.2.2 and [TX23, Lemma 2.17], we may run a $(K_X + \Delta + \mathbf{N}_X)$ -MMP/ U with scaling of A . By Lemmas 9.2.1 and 9.2.2, this MMP is also a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U with scaling of A . This shows the existence of $\mathcal{P}_0$.

Suppose that X is not $\mathbb{Q}$ -factorial. Then $\mathbf{N}$ is NQC/ U . By [TX23, Theorem A, Theorem F, Lemma 4.3], there is a choice of $\mathcal{P}_0$ satisfying the following.

- Either $\mathcal{P}_0$ terminates, or the limit of the scaling numbers of $\mathcal{P}_0$ is 0.
- Suppose that there exists an lc gfq $(X, \tilde{\Delta}, \tilde{\mathbf{N}})/U$ and an ample/ U $\mathbb{R}$ -divisor H , such that either X is $\mathbb{Q}$ -factorial klt or $\tilde{\mathbf{N}}$ is NQC/ U , and $\Delta + \mathbf{N}_X \sim_{\mathbb{R}, U} \tilde{\Delta} + \tilde{\mathbf{N}}_X + H$. Then $\mathcal{P}$ terminates at
 - either a semi-good minimal model $(X', \Delta', \mathbf{N})/U$ of $(X, \Delta, \mathbf{N})/U$, or
 - a Mori fiber space $(X', \Delta', \mathbf{N}) \rightarrow T$ of $(X, \Delta, \mathbf{N})/U$. Moreover, by Lemmas 9.2.1 and 9.2.2, $X' \rightarrow T$ is a contraction/ Z .

This implies the proposition when X is not $\mathbb{Q}$ -factorial. In the following, we may assume that X is $\mathbb{Q}$ -factorial.

We prove (1). Possibly replacing H with a general element in $|H/U|_{\mathbb{R}}$, Δ with $\tilde{\Delta} + H$, and $\mathbf{N}$ with $\tilde{\mathbf{N}}$, we may assume that $\Delta \geq H \geq 0$. The super/ Z property of Δ is lost here, but we may replace Δ again and re-assume that Δ is super/ Z . By Lemma 9.2.1(4), any $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U with scaling of A is a $(K_X + \Delta + \mathbf{N}_X)$ -MMP/ U with scaling of A . By Lemma 4.2.1 and Proposition 4.2.3, $\mathcal{P}$ terminates. Let $(X', \mathcal{F}', B', \mathbf{M})/U$ be the output of $\mathcal{P}$ and let Δ' be the image of Δ on X' . Then Δ' is super/ Z . By Lemma 4.2.1 and Proposition 4.2.3,

- either $K_{X'} + \Delta' + \mathbf{N}_{X'}$ is semi-ample/ U and we get (1.b), or
- there exists a $(K_{X'} + \Delta' + \mathbf{N}_{X'})$ -Mori fiber space $X' \rightarrow T$ over U . By Lemma 9.2.1(1), $X' \rightarrow T$ is a $(K_{\mathcal{F}'} + \Delta' + \mathbf{N}_{X'})$ -Mori fiber space/ Z and we get (1.a).

We prove (2). Suppose that $\mathcal{P}$ does not terminate and λ is the limit of the scaling numbers of $\mathcal{P}$. Then $\mathcal{P}$ is an infinite sequence of steps of a $(K_{\mathcal{F}} + B + \frac{\lambda}{2}A + \mathbf{M}_X)$ -MMP/ U . Since $(X, \mathcal{F}, B, \mathbf{M})$ is lc, $(X, \mathcal{F}, B, \mathbf{M} + \frac{1}{2}\tilde{A})$ is lc, and

$$K_{\mathcal{F}} + B + \frac{\lambda}{2}\tilde{A}_X + \mathbf{M}_X = K_{\mathcal{F}} + B + \frac{1}{2}A + \mathbf{M}_X \sim_{\mathbb{R}, Z} K_X + \Delta + \frac{1}{2}A + \mathbf{N}_X.$$

(2) follows from (1). □

Proposition 9.3.3. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a weak ACSS gfq. Assume that*

- *either X is $\mathbb{Q}$ -factorial klt or $\mathbf{M}$ is NQC/ U , and*
- *$K_{\mathcal{F}} + B + \mathbf{M}_X$ is not pseudo-effective/ U .*

Then there exists $\mathcal{P}_0$, a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U with scaling of an ample/ U $\mathbb{R}$ -divisor A , which satisfies the following. Let $\mathcal{P} := \mathcal{P}_0$ if X is not $\mathbb{Q}$ -factorial, and let $\mathcal{P}$ be any $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U with scaling of A if X is $\mathbb{Q}$ -factorial. Then $\mathcal{P}$ terminates with a Mori fiber space of $(X, \mathcal{F}, B, \mathbf{M})/U$.

Proof. Let $(X_0, \mathcal{F}_0, B_0, \mathbf{M}) := (X, \mathcal{F}, B, \mathbf{M})$. By Proposition 9.3.2, we may suppose that $\mathcal{P}$ is an MMP with scaling of A

$$(X_0, \mathcal{F}_0, B_0, \mathbf{M}) \dashrightarrow (X_1, \mathcal{F}_1, B_1, \mathbf{M}) \dashrightarrow \dots \dashrightarrow (X_n, \mathcal{F}_n, B_n, \mathbf{M}) \dashrightarrow \dots$$

such that either this MMP terminates, or $\lim_{i \rightarrow +\infty} \lambda_i = 0$, where

$$\lambda_i := \inf\{t \geq 0 \mid K_{\mathcal{F}_i} + B_i + tA_i + \mathbf{M}_{X_i} \text{ is nef}/U\}$$

are the scaling numbers and A_i is the strict transform of A on X_i .

First we assume that $\mathcal{P}$ does not terminate. Let $0 < \epsilon \ll 1$ be a real number such that $K_{\mathcal{F}} + B + \epsilon A + \mathbf{M}_X$ is not pseudo-effective/ U . Since $\lim_{i \rightarrow +\infty} \lambda_i = 0$, there exists an integer m

such that $\lambda_m < \epsilon$. Thus $K_{\mathcal{F}_m} + B_m + \lambda_m A_m + \mathbf{M}_{X_m}$ is nef/ U but $K_{\mathcal{F}_m} + B_m + \epsilon A_m + \mathbf{M}_{X_m}$ is not pseudo-effective/ U , which is not possible. Thus $\mathcal{P}$ terminates.

Suppose that $\mathcal{P}$ terminates at $(X_m, \mathcal{F}_m, B_m, \mathbf{M})$ for some $m \geq 0$. Since $K_{\mathcal{F}} + B + \mathbf{M}_X$ is not pseudo-effective/ U , $K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m}$ is not pseudo-effective/ U . Thus $K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m}$ is not nef/ U , so there exists a $(K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m})$ -Mori fiber space/ U . The proposition follows. $\square$

Theorem 9.3.4. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be an lc gfq. Assume that $\mathcal{F}$ is algebraically integrable and $K_{\mathcal{F}} + B + \mathbf{M}_X$ is not pseudo-effective/ U . Then:*

- (1) $(X, \mathcal{F}, B, \mathbf{M})/U$ has a Mori fiber space.
- (2) Suppose that $(X, \mathcal{F}, B, \mathbf{M})$ is weak ACSS, and either X is $\mathbb{Q}$ -factorial klt or $\mathbf{M}$ is NQC/ U . Then:
 - (a) We may run a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U with scaling of an ample/ U $\mathbb{R}$ -divisor, which terminates with a Mori fiber space/ U .
 - (b) If X is $\mathbb{Q}$ -factorial, then any $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U with scaling of an ample/ U $\mathbb{R}$ -divisor terminates with a Mori fiber space/ U .

Proof. (2) follows from Proposition 9.3.3 so we only need to show (1).

By Theorem 2.5.1, $(X, \mathcal{F}, B, \mathbf{M})$ has a $\mathbb{Q}$ -factorial ACSS model $(Y, \mathcal{F}_Y, B_Y, \mathbf{M})$. Let $g : Y \rightarrow X$ be the induced birational morphism, then g only extracts divisors E such that $-\epsilon_{\mathcal{F}}(E) = a(E, \mathcal{F}, B, \mathbf{M})$, and

$$K_{\mathcal{F}_Y} + B_Y + \mathbf{M}_Y = g^*(K_{\mathcal{F}} + B + \mathbf{M}_X)$$

is not pseudo-effective/ U . By Proposition 9.3.3, we may run a $(K_{\mathcal{F}_Y} + B_Y + \mathbf{M}_Y)$ -MMP/ U which terminates with a Mori fiber space $(Y', \mathcal{F}_{Y'}, B_{Y'}, \mathbf{M}) \rightarrow T$ of $(Y, \mathcal{F}_Y, B_Y, \mathbf{M})$. Then $(Y', \mathcal{F}_{Y'}, B_{Y'}, \mathbf{M}) \rightarrow T$ is a Mori fiber space of $(X, \mathcal{F}, B, \mathbf{M})/U$. $\square$

9.4. MMP for very exceptional divisors.

Theorem 9.4.1. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a weak ACSS gfq. Let $E_1, E_2 \geq 0$ be two $\mathbb{R}$ -divisors on X such that $E_1 \wedge E_2 = 0$, E_1 is very exceptional/ U , and*

$$K_{\mathcal{F}} + B + \mathbf{M}_X \sim_{\mathbb{R}, U} (\text{resp. } \equiv_U, \sim_{\mathbb{Q}, U}) E_1 - E_2.$$

Assume that either X is $\mathbb{Q}$ -factorial klt or $\mathbf{M}$ is NQC/ U . Let A be an ample/ U $\mathbb{R}$ -divisor. Then:

- (1) We may run a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U with scaling of A .
- (2) Let $\mathcal{P}$ be the $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U constructed in (1) if X is not $\mathbb{Q}$ -factorial, and let $\mathcal{P}$ be any $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U with scaling of A if X is $\mathbb{Q}$ -factorial. Then:
 - (a) Either $\mathcal{P}$ terminates with a Mori fiber space, or $\mathcal{P}$ contracts E_1 after finitely many steps.
 - (b) Suppose that $E_2 = 0$. Then:
 - (i) $\mathcal{P}$ terminates with a weak lc model $(X', \mathcal{F}', B', \mathbf{M})/U$ of $(X, \mathcal{F}, B, \mathbf{M})/U$. In particular, $K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} \sim_{\mathbb{R}, U}$ (resp. $\equiv_U, \sim_{\mathbb{Q}, U}$) 0 .
 - (ii) The divisors contracted by the induced birational map $X \dashrightarrow X'$ are exactly $\text{Supp } E_1$.
 - (iii) If $(X, \mathcal{F}, B, \mathbf{M})$ is $\mathbb{Q}$ -factorial ACSS, then $(X', \mathcal{F}', B', \mathbf{M})/U$ is a good minimal model of $(X, \mathcal{F}, B, \mathbf{M})/U$.

Proof. (1) is a direct corollary of Proposition 9.3.2. Moreover, by Proposition 9.3.2, we may suppose that $\mathcal{P}$ is an MMP/ U with scaling of A

$$(X_0, \mathcal{F}_0, B_0, \mathbf{M}) \dashrightarrow (X_1, \mathcal{F}_1, B_1, \mathbf{M}) \dashrightarrow \dots \dashrightarrow (X_n, \mathcal{F}_n, B_n, \mathbf{M}) \dashrightarrow \dots$$

such that either this MMP terminates, or $\lim_{i \rightarrow +\infty} \lambda_i = 0$, where

$$\lambda_i := \inf\{t \geq 0 \mid K_{\mathcal{F}_i} + B_i + tA_i + \mathbf{M}_{X_i} \text{ is nef}/U\}$$

are the scaling numbers and A_i is the strict transform of A on X_i .

(2.a) We let m be the integer satisfying the following: if $\mathcal{P}$ terminates, then $(X_m, \mathcal{F}_m, B_m, \mathbf{M})$ is the output of $\mathcal{P}$. If we already get a $(K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m})$ -Mori fiber space/ U then we are done, so we may assume that $K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m}$ is nef/ U . In particular, $K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m}$ is movable/ U .

Otherwise, we let m be a positive integer such that f_i is small for any $i \geq m$. Let $\psi_i : X_m \dashrightarrow X_i$ be the induced birational maps. Since $K_{\mathcal{F}_i} + B_i + \lambda_i A_i + \mathbf{M}_{X_i}$ is nef/ U for any i ,

$$K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m} = \lim_{i \rightarrow +\infty} (\psi_i^{-1})_*(K_{\mathcal{F}_i} + B_i + \lambda_i A_i + \mathbf{M}_{X_i})$$

is movable/ U .

Since $K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m}$ is movable/ U , for any prime divisor S on X_m and any very general curve C on S over U , $(K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m}) \cdot C \geq 0$. Let $E_{1,m}$ and $E_{2,m}$ be the images of E_1 and E_2 on X_m respectively. Then $E_{1,m}$ is very exceptional/ U and

$$K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m} \sim_{\mathbb{R},U} (\text{resp. } \equiv_U, \sim_{\mathbb{Q},U}) E_{1,m} - E_{2,m}.$$

By [Bir12, Lemma 3.3], $E_{1,m} = 0$. This implies (2.a).

(2.b) Now we assume that $E_2 = 0$. Then $K_{\mathcal{F}} + B + \mathbf{M}_X \equiv_U E_1 \geq 0$, so $\mathcal{P}$ does not terminate with a Mori fiber space. By (2.a), $\mathcal{P}$ contracts E_1 and achieves a log birational model $(X', \mathcal{F}', B', \mathbf{M})/U$ of $(X, \mathcal{F}, B, \mathbf{M})$ after finitely many steps. Since the image of E_1 on X' is 0,

$$K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m} \sim_{\mathbb{R},U} (\text{resp. } \equiv_U, \sim_{\mathbb{Q},U}) 0.$$

In particular, $\mathcal{P}$ terminates at X' . Since the induced birational map $X \dashrightarrow X'$ does not extract any divisor, $(X', \mathcal{F}', B', \mathbf{M})/U$ is a weak lc model of $(X, \mathcal{F}, B, \mathbf{M})/U$, which implies (2.b.i). Since $\mathcal{P}$ is also an E_1 -MMP/ U , we get (2.b.ii). (2.b.iii) follows from Lemma 9.1.4(3.f). $\square$

10. ACC FOR LC THRESHOLDS AND THE GLOBAL ACC

10.1. The global ACC.

Lemma 10.1.1. *Let X be a normal projective variety and $\mathbf{M}$ a nef $\mathbf{b}$ -divisor on X . If $\mathbf{M}_X \equiv 0$, then $\mathbf{M} \equiv 0$.*

Proof. Let $f : Y \rightarrow X$ be a birational morphism such that $\mathbf{M}$ descends to Y . By the negativity lemma, $\mathbf{M}_Y = f^*\mathbf{M}_X - E \equiv -E$ for some $E \geq 0$. Since $\mathbf{M}_Y$ is nef, $\mathbf{M}_Y$ is pseudo-effective, so $-E$ is pseudo-effective. Thus $E = 0$ and $\mathbf{M}_Y \equiv 0$, so $\mathbf{M} \equiv 0$. $\square$

Proof of Theorem 2.4.5. By Theorem 2.5.1, possibly replacing Γ with $\Gamma \cup \{1\}$ and replacing $(X, \mathcal{F}, B, \mathbf{M})$ with an ACSS model, we may assume that there exists a contraction $f : X \rightarrow Z$ such that $(X, \mathcal{F}, B, \mathbf{M})/Z$ is $\mathbb{Q}$ -factorial ACSS. Let F be a general fiber of f , $B_F := B|_F$, $\mathbf{M}^F := \mathbf{M}|_F$, and $\mathbf{M}_j^F := \mathbf{M}_j|_F$ for each j . Since $K_F = K_X|_F = K_{\mathcal{F}}|_F$,

$$(F, B_F, \mathbf{M}^F = \sum \gamma_j \mathbf{M}_j^F)$$

is an lc g-pair of dimension r such that $K_F + B_F + \mathbf{M}_F^F \equiv 0$. Moreover, $B_F \in \Gamma$. By [BZ16, Theorem 1.6], there exists a finite set $\Gamma_1 \subset \Gamma$ depending only on r and Γ such that $B_F \in \Gamma_1$. Since $(X, \mathcal{F}, B, \mathbf{M})$ is lc, B is horizontal/ Z . Thus $B \in \Gamma_1$.

Possibly rewrite $\mathbf{M}$, we may assume that $\mathbf{M}_j \not\equiv 0$ and $\gamma_j > 0$ for any j . By Lemma 10.1.1, $\mathbf{M}_{j,X} \not\equiv 0$ for each j . For any j , we let $\delta_j \in (0, \gamma_j)$ be a real number and run a

$$(K_{\mathcal{F}} + B + \mathbf{M}_X - \delta_j \mathbf{M}_{j,X})\text{-MMP}/Z.$$

Since $\mathbf{M}_{j,X} \not\equiv 0$, $K_{\mathcal{F}} + B + \mathbf{M}_X - \delta_j \mathbf{M}_{j,X} \equiv -\delta_j \mathbf{M}_{j,X}$ is not pseudo-effective/ Z . By Theorem 9.3.4, this MMP terminates with a Mori fiber space $\pi_j : (X_j, \mathcal{F}_j, B_j, \mathbf{M} - \delta_j \mathbf{M}_j) \rightarrow T_j$ of $(X, \mathcal{F}, B, \mathbf{M} - \delta_j \mathbf{M}_j)$.

Since $K_{\mathcal{F}} + B + \mathbf{M}_X \equiv 0$, $(X, \mathcal{F}, B, \mathbf{M})$ and $(X_j, \mathcal{F}_j, B_j, \mathbf{M})$ are crepant, so $(X_j, \mathcal{F}_j, B_j, \mathbf{M})$ is lc and $K_{\mathcal{F}_j} + B_j + \mathbf{M}_{X_j} \equiv 0$. Since $K_{\mathcal{F}_j} + B_j + \mathbf{M}_{X_j} - \delta_j \mathbf{M}_{j,X_j}$ is anti-ample/ T_j , $\mathbf{M}_{j,X_j}$

is ample/ T_j . Let F_j be a general fiber of π_j , $r_j := \dim F_j$, $B_{F_j} := B_j|_{F_j}$, $\mathbf{M}^j := \mathbf{M}|_{F_j}$, and $\mathbf{M}_i^j := \mathbf{M}_i|_{F_j}$. Then $r_j \leq r$. Since $K_{F_j} = K_{X_j}|_{F_j} = K_{\mathcal{F}_j}|_{F_j}$,

$$\left(F_j, B_{F_j}, \mathbf{M}^j = \sum_i \gamma_i \mathbf{M}_i^j \right)$$

is an lc g-pair of dimension r_j , and $B_{F_j} \in \Gamma$. Moreover, since $\mathbf{M}_{j,X_j}$ is ample/ T_j , $\mathbf{M}_{j,X_j}^j$ is ample. Thus $\mathbf{M}_j^j \neq \mathbf{0}$. By [BZ16, Theorem 1.6], there exists a finite set $\Gamma_2 \subset \Gamma$ depending only on Γ such that $\gamma_j \in \Gamma_2$. Since j can be any index, we may take $\Gamma_0 := \Gamma_1 \cup \Gamma_2$. $\square$

10.2. ACC for lc thresholds.

Lemma 10.2.1. *Let $(X, \mathcal{F}, B, \mathbf{M})/X$ be an lc gfq, D an $\mathbb{R}$ -divisor on X , and $\mathbf{N}$ a $\mathbf{b}$ -divisor on X satisfying the following.*

- (i) $\mathcal{F}$ is algebraically integrable.
- (ii) $\text{Supp } B = \text{Supp}(B + D)$.
- (iii) $\mathbf{M} + \mathbf{N}$ and $\mathbf{M} - \delta \mathbf{N}$ are nef/ X for some $\delta \in (0, 1)$.
- (iv) $(X, \mathcal{F}, B + D, \mathbf{M} + \mathbf{N})/X$ is an lc gfq. In particular, $D + \mathbf{N}_X$ is $\mathbb{R}$ -Cartier.
- (v) $(X, \mathcal{F}, B + (1 + \epsilon)D, \mathbf{M} + (1 + \epsilon)\mathbf{N})$ is not lc for any positive real number ϵ .
- (vi) For any prime divisor P on X with $a(P, \mathcal{F}, B + D, \mathbf{M} + \mathbf{N}) = -\epsilon_{\mathcal{F}}(D)$, $\text{mult}_P D = 0$.

Then for any real number $t \in (0, 1)$, there exist two projective birational morphisms $h : X' \rightarrow X$ and $g : Y' \rightarrow X'$ satisfying the following.

- (1) h is an ACSS modification of $(X, \mathcal{F}, B + tD, \mathbf{M} + t\mathbf{N})$.
- (2) For any prime h -exceptional divisor P , $a(P, \mathcal{F}, B, \mathbf{M}) = -\epsilon_{\mathcal{F}}(P)$. In particular,

$$a(D, \mathcal{F}, B + sD, \mathbf{M} + s\mathbf{N}) = -\epsilon_{\mathcal{F}}(D)$$

for any real number s .

- (3) g extracts a unique prime divisor E . In particular, $-E$ is ample over X' .
- (4) $a(E, \mathcal{F}, B + D, \mathbf{M} + \mathbf{N}) = -\epsilon_{\mathcal{F}}(E)$ and $a(E, \mathcal{F}, B, \mathbf{M}) > -\epsilon_{\mathcal{F}}(E)$. In particular,

$$a(E, \mathcal{F}, B + sD, \mathbf{M} + s\mathbf{N}) > -\epsilon_{\mathcal{F}}(E)$$

for any real number $s < 1$.

- (5) Let $B_{Y'}, D_{Y'}$ be the strict transforms of B, D on Y' respectively, $\mathcal{F}_{Y'} := (h \circ g)^{-1} \mathcal{F}$, and $F_{Y'} := (\text{Supp Exc}(h \circ g))^{\mathcal{F}_{Y'}}$. Then

$$(Y', \mathcal{F}_{Y'}, B_{Y'} + tD_{Y'} + F_{Y'}, \mathbf{M} + t\mathbf{N})$$

is $\mathbb{Q}$ -factorial ACSS.

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & Y' \\ f \downarrow & & \downarrow g \\ X & \xleftarrow{\quad h \quad} & X' \end{array}$$

Proof. By condition (v), there exists a prime divisor P_0 over X such that

$$a(P_0, \mathcal{F}, B + D, \mathbf{M} + \mathbf{N}) = -\epsilon_{\mathcal{F}}(P_0)$$

and

$$a(P_0, \mathcal{F}, B + \alpha D, \mathbf{M} + \alpha \mathbf{N}) < -\epsilon_{\mathcal{F}}(P_0)$$

for any $\alpha > 1$. In particular,

$$a(P_0, \mathcal{F}, B + tD, \mathbf{M} + t\mathbf{N}) > -\epsilon_{\mathcal{F}}(P_0).$$

By condition (vi), P_0 is exceptional/ X . By Theorem 8.2.2, there exists a proper ACSS model $(Y, \mathcal{F}_Y, \tilde{B}_Y, \mathbf{M} + \mathbf{N}; G_Y)/Z$ of $(X, \mathcal{F}, B + D, \mathbf{M} + \mathbf{N})$ such that P_0 is on Y . Let $f : Y \rightarrow X$ be

the induced birational morphism, B_Y, D_Y the strict transforms of B, D on Y respectively, and $F_Y := (\text{Supp Exc}(f))^{\mathcal{F}_Y}$. Then $\tilde{B}_Y = B_Y + D_Y + F_Y$.

By conditions (ii) and (iii) and Lemma 7.3.2,

$$(Y, \mathcal{F}_Y, B_Y + tD_Y + F_Y, \mathbf{M} + t\mathbf{N}; G_Y)/Z$$

is ACSS. Let $E_1, \dots, E_n$ be the prime f -exceptional divisors, then

$$K_{\mathcal{F}_Y} + B_Y + tD_Y + F_Y + \mathbf{M}_Y + t\mathbf{N}_Y \sim_{\mathbb{R}, X} \sum_{i=1}^n (\epsilon_{\mathcal{F}}(E_i) + a(E_i, \mathcal{F}, B + tD, \mathbf{M} + t\mathbf{N})) E_i \geq 0.$$

By Theorem 9.4.1, we may run a $(K_{\mathcal{F}_Y} + B_Y + tD_Y + \mathbf{M}_Y + t\mathbf{N}_Y + F_Y)$ -MMP/ X which terminates with a good minimal model $(X', \mathcal{F}', B' + tD' + F', \mathbf{M} + t\mathbf{N})/X$ of $(Y, \mathcal{F}_Y, B_Y + tD_Y + F_Y, \mathbf{M} + t\mathbf{N})$, such that

$$K_{\mathcal{F}'} + B' + tD' + F' + \mathbf{M}_{X'} + t\mathbf{N}_{X'} \sim_{\mathbb{R}, X} 0,$$

where B', D', F' are the strict transforms of B_Y, D_Y, F_Y on X' respectively. Let G' be the image of G_Y on X' . By Lemma 9.1.4, $(X', \mathcal{F}', B' + tD' + F', \mathbf{M} + t\mathbf{N}; G')/Z$ is $\mathbb{Q}$ -factorial ACSS. In particular, the induced morphism $h : X' \rightarrow X$ is an ACSS modification of $(X, \mathcal{F}, B + tD, \mathbf{M} + t\mathbf{N})$.

By construction, the divisors contracted by the induced birational map $Y \dashrightarrow X'$ are all divisors E_i satisfying the inequality

$$a(E_i, \mathcal{F}, B_Y + tD_Y, \mathbf{M} + t\mathbf{N}) > -\epsilon_{\mathcal{F}}(E_i).$$

Thus $Y \dashrightarrow X'$ contracts P_0 . Therefore, $Y \dashrightarrow X'$ contains a divisorial contraction, so it is not the identity morphism. We let $g : Y' \dashrightarrow X'$ be the last step of this MMP. Since X' is $\mathbb{Q}$ -factorial and

$$K_{\mathcal{F}'} + B' + tD' + F' + \mathbf{M}_{X'} + t\mathbf{N}_{X'} \sim_{\mathbb{R}, X} 0,$$

g is a divisorial contraction of a prime divisor E .

We show that h, g , and t satisfy our requirements. (1) and (5) immediately follow from our construction. (3) follows from our construction and the negativity lemma. For any prime divisor Q on X' that is exceptional over X ,

$$\begin{aligned} a(Q, \mathcal{F}, B + D, \mathbf{M} + \mathbf{N}) &= -\epsilon_{\mathcal{F}}(Q) && (D \text{ is also on } Y \text{ and is exceptional}/X) \\ &= a(Q, \mathcal{F}, B + tD, \mathbf{M} + t\mathbf{N}) && (D \text{ is on } X' \text{ and is exceptional}/X), \end{aligned}$$

so $a(Q, \mathcal{F}, B + sD, \mathbf{M} + s\mathbf{N}) = -\epsilon_{\mathcal{F}}(Q)$ for any real number s . This implies (2).

Since g is a divisorial contraction of the prime divisor E ,

$$\begin{aligned} a(E, \mathcal{F}, B + tD, \mathbf{M} + t\mathbf{N}) &= a(E, \mathcal{F}', B' + tD' + F', \mathbf{M} + t\mathbf{N}) \\ &> a(E, \mathcal{F}_Y, B_Y + tD_Y + F_Y, \mathbf{M} + t\mathbf{N}) = -\epsilon_{\mathcal{F}}(E), \end{aligned}$$

so $a(E, \mathcal{F}, B, \mathbf{M}) > -\epsilon_{\mathcal{F}}(E)$. This implies (4) and completes the proof. $\square$

Proof of Theorem 2.4.4. Suppose that the theorem does not hold. Then there exists a sequence of NQC lc gfq $(X_i, \mathcal{F}_i, B_i, \mathbf{M}_i)$, $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisors D_i on X_i , and $\mathbf{b}$ -divisors $\mathbf{N}_i$ on X_i , such that $\text{rank } \mathcal{F}_i = r$, $B_i, D_i \in \Gamma$, $\mathbf{M}_i, \mathbf{N}_i$ are Γ -linear combination of $\mathbf{b}$ -nef/ X $\mathbf{b}$ -divisors, and

$$t_i := \text{lct}(X_i, \mathcal{F}_i, B_i, \mathbf{M}_i; D_i, \mathbf{N}_i)$$

is strictly increasing. By Lemma 10.2.1, possibly replacing Γ with $\Gamma \cup \{1\}$, we may assume that

- (i) $(X_i, \mathcal{F}_i, B_i + t'_i D_i, \mathbf{M}_i + t'_i \mathbf{N}_i)$ is $\mathbb{Q}$ -factorial ACSS for some $0 < t'_i < t_i$,
- (ii) there exists a divisorial contraction $f_i : Y_i \rightarrow X_i$ of a prime divisor E_i , such that $-E_i$ is ample/ X_i ,

$$a(E_i, \mathcal{F}_i, B_i + t_i D_i, \mathbf{M}_i + t_i \mathbf{N}_i) = -\epsilon_{\mathcal{F}_i}(E_i)$$

and

$$a(E_i, \mathcal{F}_i, B_i + sD_i, \mathbf{M}_i + s\mathbf{N}_i) \neq -\epsilon_{\mathcal{F}_i}(E_i)$$

for any $s \neq t_i$, and

(iii) let B_{Y_i}, D_{Y_i} be the strict transforms of B_i, D_i on Y_i respectively, $\mathcal{F}_{Y_i} := f_i^{-1}\mathcal{F}_i$, and $F_i := (\text{Supp Exc}(f_i))^{\mathcal{F}_i}$. Then

$$(Y_i, \mathcal{F}_{Y_i}, B_{Y_i} + t'_i D_{Y_i} + F_i, \mathbf{M}_i + t'_i \mathbf{N}_i)$$

is $\mathbb{Q}$ -factorial ACSS.

We let E_i^ν be the normalization of E_i , $\mathcal{F}_{E_i}$ the restricted foliation of $\mathcal{F}_{Y_i}$ on E_i , $\mathbf{M}_i^E := \mathbf{M}_i|_{E_i}$, and $\mathbf{N}_i^E := \mathbf{N}_i|_{E_i}$. For any real number t , we let $\mathbf{M}^E(t)_i := \mathbf{M}_i^E + t\mathbf{N}_i^E$, and

$$K_{\mathcal{F}_{E_i}} + B_{E_i}(t) + \mathbf{M}^E(t)_{i, E_i^\nu} := (K_{\mathcal{F}_{Y_i}} + B_{Y_i} + tD_{Y_i} + F_i + \mathbf{M}_{i, Y_i} + t\mathbf{N}_{i, Y_i})|_{E_i^\nu}.$$

Let V_i be the center of E_i on X_i . Then there exists an induced birational morphism $\phi_i : E_i^\nu \rightarrow V_i$ such that

$$K_{\mathcal{F}_{E_i}} + B_{E_i}(t_i) + \mathbf{M}^E(t_i)_{i, E_i^\nu} \sim_{\mathbb{R}, V_i} 0.$$

Since $-E_i$ is ample/ X_i ,

$$K_{\mathcal{F}_{E_i}} + B_{E_i}(t'_i) + \mathbf{M}^E(t'_i)_{i, E_i^\nu}$$

is anti-ample/ V_i .

By Proposition 6.1.10, $\mathcal{F}_{E_i}$ is algebraically integrable. By Theorem 6.0.1,

$$(E_i^\nu, \mathcal{F}_{E_i}, B_{E_i}(t_i), \mathbf{M}^E(t_i)_i)/V_i$$

is lc, and

$$(E_i^\nu, \mathcal{F}_{E_i}, B_{E_i}(t), \mathbf{M}^E(t)_i)/V_i$$

is lc for any $0 \leq t \leq t_i$. By Theorem 2.5.1, we may let $(W_i, \mathcal{F}_{W_i}, B_{W_i}(t_i), \mathbf{M}^E(t_i)_i; G_i)/Z_i$ be an ACSS model of

$$(E_i^\nu, \mathcal{F}_{E_i}, B_{E_i}(t_i), \mathbf{M}^E(t_i)_i)$$

with induced birational morphism $g_i : W_i \rightarrow E_i^\nu$, and let

$$B_{W_i}(t) := (g_i^{-1})_* B_{E_i}(t) + (\text{Supp Exc}(g_i))^{\mathcal{F}_{W_i}}$$

for each i . Since

$$K_{\mathcal{F}_{E_i}} + B_{E_i}(t'_i) + \mathbf{M}^E(t'_i)_{i, E_i^\nu}$$

is anti-ample/ V_i , $K_{\mathcal{F}_{W_i}} + B_{W_i}(t'_i) + \mathbf{M}^E(t'_i)_{i, W_i}$ is not pseudo-effective/ V_i . Moreover, since $t_i > t'_i$, $B_{E_i}(t_i) \geq B_{E_i}(t'_i)$, so $(W_i, \mathcal{F}_{W_i}, B_{W_i}(t'_i), \mathbf{M}^E(t'_i)_i)$ is lc. Thus we may run a

$$(K_{\mathcal{F}_{W_i}} + B_{W_i}(t'_i) + \mathbf{M}^E(t'_i)_{i, W_i})\text{-MMP}/V_i$$

with scaling of an ample/ V_i divisor. By Theorem 9.3.4, this MMP terminates with a Mori fiber space $\psi_i : (\bar{W}_i, \mathcal{F}_{\bar{W}_i}, B_{\bar{W}_i}(t'_i), \mathbf{M}^E(t'_i)_i) \rightarrow T_i$ of $(W_i, \mathcal{F}_{W_i}, B_{W_i}(t'_i), \mathbf{M}^E(t'_i)_i)/V_i$.

$$\begin{array}{ccc} W_i & \dashrightarrow & \bar{W}_i \\ g_i \downarrow & & \downarrow \psi_i \\ E_i^\nu & & T_i \\ & \searrow \phi_i & \swarrow \\ & V_i & \end{array}$$

By Lemma 9.1.4, ψ_i is also a Mori fiber space/ Z_i . Since

$$K_{\mathcal{F}_{W_i}} + B_{W_i}(t_i) + \mathbf{M}^E(t_i)_{i, W_i} \sim_{\mathbb{R}, V_i} 0,$$

$(\bar{W}_i, \mathcal{F}_{\bar{W}_i}, B_{\bar{W}_i}(t_i), \mathbf{M}^E(t_i)_i)$ and $(W_i, \mathcal{F}_{W_i}, B_{W_i}(t_i), \mathbf{M}^E(t_i)_i)$ are crepant, where $B_{\bar{W}_i}(t)$ is the image of $B_{W_i}(t)$ on $\bar{W}_i$ for any t . Then

$$K_{\mathcal{F}_{\bar{W}_i}} + B_{\bar{W}_i}(t_i) + \mathbf{M}^E(t_i)_{i, \bar{W}_i} \sim_{\mathbb{R}, V_i} 0,$$

so

$$K_{\mathcal{F}_{\bar{W}_i}} + B_{\bar{W}_i}(t_i) + \mathbf{M}^E(t_i)_{i, \bar{W}_i} \sim_{\mathbb{R}, T_i} 0.$$

Let L_i be a general fiber of ψ_i , $B_{L_i}(t) := B_{\bar{W}_i}(t)|_{L_i}$ for any t , and $\mathbf{M}^L(t)_i := \mathbf{M}^E(t)_i|_{L_i}$ for any t . Then $K_{\mathcal{F}_{\bar{W}_i}}|_{L_i} = K_{L_i}$, $(L_i, B_{L_i}(t_i), \mathbf{M}^L(t_i)_i)$ is lc,

$$K_{L_i} + B_{L_i}(t_i) + \mathbf{M}^L(t_i)_{i, L_i} \equiv 0,$$

and

$$K_{L_i} + B_{L_i}(t'_i) + \mathbf{M}^L(t'_i)_{i, L_i}$$

is anti-ample. Moreover, since ψ_i is a Mori fiber space/ Z_i , by Proposition 6.1.10,

$$\dim L_i \leq \text{rank } \mathcal{F}_{\bar{W}_i} = \text{rank } \mathcal{F}_{E_i} \leq \text{rank } \mathcal{F}_i = r.$$

We get a contradiction to [BZ16, Theorem 1.6] by considering the coefficients of $B_{L_i}(t_i)$ and $\mathbf{M}^L(t_i)_{i, L_i}$, which can be precisely computed by Theorem 6.0.1. Theorem 2.4.4 is proven. $\square$

10.3. Uniform rational polytopes.

Definition 10.3.1. Let X be a normal variety, D_i $\mathbb{R}$ -divisors on X , $\mathbf{M}_i$ $\mathbf{b}$ -divisors on X , and $d_i(t) : \mathbb{R} \rightarrow \mathbb{R}$ $\mathbb{R}$ -affine functions. Then we call the formal finite sum $\sum d_i(t)D_i$ an $\mathbb{R}$ -affine functional divisor, and call the formal finite sum $\sum d_i(t)\mathbf{M}_i$ an $\mathbb{R}$ -affine $\mathbf{b}$ -divisor.

Definition 10.3.2. Let c be a non-negative real number, and $\Gamma \subset [0, +\infty)$ a set of real numbers. Let X be a normal variety.

For any $\mathbb{R}$ -affine functional divisor $\Delta(t)$ on X , we write $\Delta(t) \in \mathcal{D}_c(\Gamma)$ if we may write $\Delta(t) = \sum_i d_i(t)D_i$, where D_i are distinct prime divisors, and the following condition is satisfied: For any i , either $d_i(t) = 1$, or

$$d_i(t) = \frac{m - 1 + \gamma + kt}{m},$$

where $m \in \mathbb{N}^+$, $\gamma \in \Gamma_+$, $k \in \mathbb{Z}$, and $f + kt = \sum_j (f_j + k_j t)$, where $f_j \in \Gamma \cup \{0\}$, $k_j \in \mathbb{Z}$, and $f_j + k_j c \geq 0$ for any j .

For any $\mathbb{R}$ -affine functional $\mathbf{b}$ -divisor $\mathbf{M}(t)$ on X and any projective morphism $X \rightarrow Z$, we write $\mathbf{M}(t) \in \mathcal{D}_c(\Gamma/Z)$ if we can write $\mathbf{M}(t) = \sum_i \mu_i(t)\mathbf{M}_i$, where $\mathbf{M}_i$ are nef/ Z $\mathbf{b}$ -Cartier $\mathbf{b}$ -divisors, and the following condition is satisfied: For any i , either $\mu_i(t) = 1$, or

$$\mu_i(t) = v + nt = \sum_j (v_j + n_j t),$$

where $v_j \in \Gamma$, $n_j \in \mathbb{Z}$, and $v_j + n_j c \geq 0$ for any j . Moreover, if $Z = \{pt\}$, then we may omit Z and write $\mathbf{M}(t) \in \mathcal{D}_c(\Gamma)$.

Definition 10.3.3. Let d be a positive integer and $\Gamma \subset [0, +\infty)$ a set of real numbers. We define $\mathcal{B}_d(\Gamma), \mathcal{B}'_d(\Gamma) \subset [0, +\infty)$ as follows: $c \in \mathcal{B}_d(\Gamma)$ (resp. $\mathcal{B}'_d(\Gamma)$) if and only if there exist a normal projective variety X (resp. a $\mathbb{Q}$ -factorial normal projective variety X), an $\mathbb{R}$ -affine functional divisor $\Delta(t)$ on X , and an $\mathbb{R}$ -affine functional $\mathbf{b}$ -divisor $\mathbf{M}(t)$ satisfying the following.

- (1) $\dim X \leq d$,
- (2) $\Delta(t) \in \mathcal{D}_c(\Gamma)$, $\mathbf{M}(t) \in \mathcal{D}_c(\Gamma)$,
- (3) $(X, \Delta(c), \mathbf{M}(c))$ is lc,
- (4) $K_X + \Delta(c) + \mathbf{M}(c)_X \equiv 0$, and
- (5) $K_X + \Delta(c') + \mathbf{M}(c')_X \not\equiv 0$ for any $c' \neq c$.

Definition 10.3.4. Let r be a positive integer and $\Gamma \subset [0, +\infty)$ a set of real numbers. We define $\mathcal{C}_r(\Gamma), \mathcal{C}'_r(\Gamma) \subset [0, +\infty)$ as follows: $c \in \mathcal{C}_r(\Gamma)$ (resp. $c \in \mathcal{C}'_r(\Gamma)$) if and only if there exist a normal projective variety X (resp. a $\mathbb{Q}$ -factorial normal projective variety X), an algebraically integrable foliation $\mathcal{F}$ on X , an $\mathbb{R}$ -affine functional divisor $\Delta(t)$ on X , and an $\mathbb{R}$ -affine functional $\mathbf{b}$ -divisor $\mathbf{M}(t)$ on X satisfying the following.

- (1) $\text{rank } \mathcal{F} \leq r$,
- (2) $\Delta(t) \in \mathcal{D}_c(\Gamma)$, $\mathbf{M}(t) \in \mathcal{D}_c(\Gamma)$,
- (3) $(X, \mathcal{F}, \Delta(c), \mathbf{M}(c))$ is lc,
- (4) $K_{\mathcal{F}} + \Delta(c) + \mathbf{M}(c)_X \equiv 0$, and
- (5) $K_{\mathcal{F}} + \Delta(c') + \mathbf{M}(c')_X \not\equiv 0$ for any $c' \neq c$.

Proposition 10.3.5. *Let r be a positive integer and $\Gamma \subset [0, +\infty)$ a set of real numbers. Then $\mathcal{B}_r(\Gamma) = \mathcal{C}_r(\Gamma) = \mathcal{B}'_r(\Gamma) = \mathcal{C}'_r(\Gamma)$.*

Proof. By considering the foliation $\mathcal{F} = T_X$, we have $\mathcal{B}_r(\Gamma) \subset \mathcal{C}_r(\Gamma)$. By the existence of dlt modifications, $\mathcal{B}_r(\Gamma) = \mathcal{B}'_r(\Gamma)$. By Theorem 2.5.1, $\mathcal{C}_r(\Gamma) = \mathcal{C}'_r(\Gamma)$. We only need show that $\mathcal{C}_r(\Gamma) \subset \mathcal{B}_r(\Gamma)$.

Pick $c \in \mathcal{C}_r(\Gamma)$. Then there exists a $\mathbb{Q}$ -factorial normal projective variety X , an algebraically integrable foliation $\mathcal{F}$ on X , an $\mathbb{R}$ -affine functional divisor $\Delta(t)$ on X , and an $\mathbb{R}$ -affine functional $\mathbf{b}$ -divisor $\mathbf{M}(t)$ on X , such that

- (1) $\text{rank } \mathcal{F} \leq r$,
- (2) $\Delta(t) \in \mathcal{D}_c(\Gamma)$, $\mathbf{M}(t) \in \mathcal{D}_c(\Gamma)$,
- (3) $(X, \mathcal{F}, \Delta(c), \mathbf{M}(c))$ is lc,
- (4) $K_{\mathcal{F}} + \Delta(c) + \mathbf{M}(c)_X \equiv 0$, and
- (5) $K_{\mathcal{F}} + \Delta(t) + \mathbf{M}(t)_X \not\equiv 0$ for any $t \neq c$.

By Theorem 2.5.1, we may let $f : X' \rightarrow X$ be an ACSS modification of $(X, \mathcal{F}, \Delta(c), \mathbf{M}(c))$, $\mathcal{F}' := f^{-1}\mathcal{F}$, $E := (\text{Supp Exc}(f))^{\mathcal{F}'}$, and $\Delta'(t) := f_*^{-1}\Delta(t) + E$ for any real number t . Then $\text{rank } \mathcal{F}' \leq r$, $\Delta'(t) \in \mathcal{D}_c(\Gamma)$, $(X', \mathcal{F}', \Delta'(c), \mathbf{M}(c))$ is lc, and $K_{\mathcal{F}'} + \Delta'(c) + \mathbf{M}(c)_{X'} \equiv 0$. Moreover, for any $t \neq c$, since

$$0 \not\equiv K_{\mathcal{F}} + \Delta(t) + \mathbf{M}(t)_X = f_*(K_{\mathcal{F}'} + \Delta'(t) + \mathbf{M}(t)_{X'}),$$

$K_{\mathcal{F}'} + \Delta'(t) + \mathbf{M}(t)_{X'} \not\equiv 0$. Therefore, we may replace $(X, \mathcal{F}, \Delta(t), \mathbf{M}(t))$ with $(X', \mathcal{F}', \Delta'(t), \mathbf{M}(t))$, and assume that $(X, \mathcal{F}, \Delta(c), \mathbf{M}(c))$ is $\mathbb{Q}$ -factorial ACSS. Thus there exists a contraction $f : X \rightarrow Z$ and a reduced divisor G such that $(X, \mathcal{F}, \Delta(c), \mathbf{M}(c); G)/Z$ is ACSS.

Suppose that for any $0 < \delta \ll 1$, $(X, \mathcal{F}, \Delta(c + \delta), \mathbf{M}(c + \delta); G)/Z$ or $(X, \mathcal{F}, \Delta(c - \delta), \mathbf{M}(c - \delta); G)/Z$ is not ACSS. By Lemmas 7.3.2 and 7.3.5,

- either there exists a component D of $\Delta(c)$, such that $\text{mult}_D \Delta(c) = 1$ and $\text{mult}_D \Delta(t) \neq 1$ for any $t \neq c$, or
- $\mathbf{M}(t) = \sum \mu_i(t) \mathbf{M}_i$, where each $\mathbf{M}_i$ is $\mathbf{b}$ -nef, and $\mu_i(t) = v_i + n_i t = \sum_j (v_{i,j} + n_{i,j} t)$ for any $v_{i,j} \in \Gamma$, $n_{i,j} \in \mathbb{Z}$, $v_i + n_i c \geq 0$ for any i , and $v_i + n_i c = 0$ for some i .

By [Nak16, Lemma 3.7], $c \in \mathcal{B}_1(\Gamma) \subset \mathcal{B}_r(\Gamma)$. Therefore, we may assume that $(X, \mathcal{F}, \Delta(c + \delta), \mathbf{M}(c + \delta); G)/Z$ and $(X, \mathcal{F}, \Delta(c - \delta), \mathbf{M}(c - \delta); G)/Z$ are ACSS for any $0 < \delta \ll 1$.

Fix $0 < \delta \ll 1$. Since $K_{\mathcal{F}} + \Delta(t) + \mathbf{M}(t)_X \not\equiv 0$ for any $t \neq c$ and $K_{\mathcal{F}} + \Delta(c) + \mathbf{M}(c)_X \equiv 0$, either $K_{\mathcal{F}} + \Delta(c + \delta) + \mathbf{M}(c + \delta)_X$ or $K_{\mathcal{F}} + \Delta(c - \delta) + \mathbf{M}(c - \delta)_X$ is not pseudo-effective. By Theorem 9.3.4, we may run a $(K_{\mathcal{F}} + \Delta(c + \delta) + \mathbf{M}(c + \delta)_X)$ -MMP (resp. $(K_{\mathcal{F}} + \Delta(c - \delta) + \mathbf{M}(c - \delta)_X)$ -MMP) with scaling of an ample divisor if $K_{\mathcal{F}} + \Delta(c + \delta) + \mathbf{M}(c + \delta)_X$ (resp. $K_{\mathcal{F}} + \Delta(c - \delta) + \mathbf{M}(c - \delta)_X$) is not pseudo-effective, which terminates with a Mori fiber space $\phi : (X'', \mathcal{F}'', \Delta''(c + \delta), \mathbf{M}(c + \delta)) \rightarrow T$ (reps. $\phi : (X'', \mathcal{F}'', \Delta''(c - \delta), \mathbf{M}(c - \delta)) \rightarrow T$) of $(X, \mathcal{F}, \Delta(c + \delta), \mathbf{M}(c + \delta))$ (resp. $(X, \mathcal{F}, \Delta(c - \delta), \mathbf{M}(c - \delta))$), where $\Delta''(t)$ is the image of $\Delta(t)$ on X'' for any t . By Lemma 9.1.4(4), this MMP is also an MMP/ Z and ϕ is a contraction/ Z .

Since $K_{\mathcal{F}} + \Delta(c) + \mathbf{M}(c)_X \equiv 0$, $(X'', \mathcal{F}'', \Delta''(c), \mathbf{M}(c))$ and $(X, \mathcal{F}, \Delta(c), \mathbf{M}(c))$ are crepant, so $K_{\mathcal{F}''} + \Delta''(c) + \mathbf{M}(c)_{X''} \equiv 0$ and $(X'', \mathcal{F}'', \Delta''(c), \mathbf{M}(c))$ is lc.

Let F be a general fiber of ϕ . By Theorem 2.3.1, F is tangent to $\mathcal{F}''$, so $K_{\mathcal{F}''}|_F = K_{X''}|_F = K_F$. Let $\Delta_F(t) := \Delta''(t)|_F$ and $\mathbf{M}^F(t) := \mathbf{M}(t)|_F$. Then

- $\dim F \leq \dim X - \dim Z = \text{rank } \mathcal{F} \leq r$,
- $\Delta_F(t) \in \mathcal{D}_c(\Gamma)$ and $\mathbf{M}^F(t) \in \mathcal{D}_c(\Gamma)$,

- $(F, \Delta_F(c), \mathbf{M}^F(c))$ is lc,
- $K_F + \Delta_F(c) + \mathbf{M}^F(c)_F \equiv 0$, and
- $K_F + \Delta_F(c + \delta) + \mathbf{M}^F(c + \delta)_F$ or $K_F + \Delta_F(c - \delta) + \mathbf{M}^F(c - \delta)_F$ is anti-ample.

Thus $c \in \mathcal{B}_r(\Gamma)$. $\square$

Theorem 10.3.6. *Let d, c, m, n be positive integers, $r_1, \dots, r_c$ real numbers such that $1, r_1, \dots, r_c$ are linearly independent over $\mathbb{Q}$, $\mathbf{r} := (r_1, \dots, r_c)$, and $s_1, \dots, s_m, \mu_1, \dots, \mu_n : \mathbb{R}^{c+1} \rightarrow \mathbb{R}$ $\mathbb{Q}$ -linear functions. Then there exists a positive real number δ depending only on $d, \mathbf{r}$ and $s_1, \dots, s_m, \mu_1, \dots, \mu_n$ satisfying the following. Assume that*

(1)

$$\left(X, \mathcal{F}, B = \sum_{i=1}^m s_i(1, r_1, \dots, r_{c-1}, t) B_i, \mathbf{M} = \sum_{i=1}^n \mu_i(1, r_1, \dots, r_{c-1}, t) \mathbf{M}_i \right) \Big/ X$$

is an lc gfq such that $\mathcal{F}$ is algebraically integrable and $\text{rank } \mathcal{F} \leq d$,

(2) $B_i \geq 0$ are distinct Weil divisors (possibly 0) and $s_i(1, \mathbf{r}) \geq 0$ for each i ,

(3) $\mathbf{M}_i$ are nef/ X $\mathbf{b}$ -Cartier $\mathbf{b}$ -divisors and $\mu_i(1, \mathbf{r}) \geq 0$ for each i , and

(4) $B(t) := \sum_{i=1}^m s_i(1, r_1, \dots, r_{c-1}, t) B_i$ and $\mathbf{M}(t) := \sum_{i=1}^n \mu_i(1, r_1, \dots, r_{c-1}, t) \mathbf{M}_i$ for any $t \in \mathbb{R}$.

Then $(X, \mathcal{F}, B(t), \mathbf{M}(t))$ is lc for any $t \in (r_c - \delta, r_c + \delta)$.

Proof. We let $s_i(t) := s_i(1, r_1, \dots, r_{c-1}, t)$ and $\mu_i(t) := \mu_i(1, r_1, \dots, r_{c-1}, t)$ for any $t \in \mathbb{R}$. If $s_i(r_c) = 0$, then $s_i(t) = 0$ for any i , so we may assume that $s_i(r_c) \neq 0$ for any i . Let $(X', \mathcal{F}', B'(r_c), \mathbf{M}(r_c))$ be an ACSS model of $(X, \mathcal{F}, B(r_c), \mathbf{M}(r_c))$, $f : X' \rightarrow X$ the induced birational morphism, $E := (\text{Supp}(\text{Exc}(f)))^{\mathcal{F}'}$, and $B'(t) := f_*^{-1} B(t) + E$ for any t . Then

$$K_{\mathcal{F}'} + B'(r_c) + \mathbf{M}(r_c)_{X'} = f^*(K_{\mathcal{F}} + B(r_c) + \mathbf{M}(r_c)_X).$$

Since $1, r_1, \dots, r_c$ are linearly independent over $\mathbb{Q}$, $B'(t) := f_*^{-1} B(t) + E$ for any t , and

$$K_{\mathcal{F}'} + B'(t) + \mathbf{M}(t)_{X'} = f^*(K_{\mathcal{F}} + B(t) + \mathbf{M}(t)_X)$$

for any $t \in \mathbb{R}$. Thus possibly replacing $(X, \mathcal{F}, B(t), \mathbf{M}(t))$ with $(X', \mathcal{F}', B'(t), \mathbf{M}(t))$, we may assume that $(X, \mathcal{F}, B(r_c), \mathbf{M}(r_c))$ is $\mathbb{Q}$ -factorial ACSS.

Let

$$t_1 := \inf\{t \geq r_c \mid (X, \mathcal{F}, B(r_c), \mathbf{M}(r_c)) \text{ is lc}\}$$

and

$$t_2 := \sup\{t \leq r_c \mid (X, \mathcal{F}, B(r_c), \mathbf{M}(r_c)) \text{ is lc}\}.$$

If $|t_1 - t_0| \leq |t_2 - t_0|$ then we let $t_0 := t_1$. Otherwise, we let $t_0 := t_2$. We only need to show that there exists a positive real number ϵ depending only on $d, \mathbf{r}$, and $s_1, \dots, s_m, \mu_1, \dots, \mu_n$, such that $|t_0 - r_c| \geq \epsilon$.

Since $1, r_1, \dots, r_c$ are linearly independent over $\mathbb{Q}$, there exists a positive real number δ_1 depending only on $\mathbf{r}$ and $s_1, \dots, s_m, \mu_1, \dots, \mu_n$, such that $s_i(t) > 0$ and $\mu_i(t) > 0$ for any $t \in (r_c - \delta_1, r_c + \delta_1)$. We may assume that $|t_0 - r_c| < \delta_1$. In particular, for any $0 < \delta \ll 1$, $B(t_0 + \delta(t_0 - r_c)) \geq 0$, and $\mu_i(t_0 + \delta(t_0 - r_c)) > 0$. Thus $(X, \mathcal{F}, B(t_0), \mathbf{M}(t_0))$ has an lc center V_0 such that $\dim V_0 \leq \dim X - 2$, and V_0 is not an lc center of $(X, \mathcal{F}, B(r_c), \mathbf{M}(r_c))$.

By Lemma 10.2.1, possibly replacing $(X, \mathcal{F}, B(t), \mathbf{M}(t))$, we may assume that there exist a divisorial contraction $g : Y \rightarrow X$ of a prime divisor $\tilde{E}$ and a real number s satisfying the following: let $B_Y(t)$ be the strict transform of $B(t)$ on Y for any t and $\mathcal{F}_Y := g^{-1}\mathcal{F}$, then

- (i) $s \in (r_c, t_0)$ if $r_c > t_0$, and $s \in (t_0, r_c)$ if $t_0 < r_c$,
- (ii) $(X, \mathcal{F}, B(s), \mathbf{M}(s))$ is $\mathbb{Q}$ -factorial ACSS, $(X, \mathcal{F}, B(r_c), \mathbf{M}(r_c))$ is lc, and $(X, \mathcal{F}, B(t_0), \mathbf{M}(t_0))$ is lc,
- (iii) $-\tilde{E}$ is ample over X ,
- (iv) $(Y, \mathcal{F}_Y, B_Y(s) + \epsilon_{\mathcal{F}}(\tilde{E}), \mathbf{M}(s))$ is $\mathbb{Q}$ -factorial ACSS, and

(v) $a(E, \mathcal{F}, B(t_0), \mathbf{M}(t_0)) = -\epsilon_{\mathcal{F}}(\tilde{E})$ and $a(E, \mathcal{F}, B(r_c), \mathbf{M}(r_c)) > -\epsilon_{\mathcal{F}}(\tilde{E})$. In particular, $(Y, \mathcal{F}_Y, B_Y(t_0) + \epsilon_{\mathcal{F}}(\tilde{E}), \mathbf{M}(t_0))$ is lc and $a(E, \mathcal{F}, B(s), \mathbf{M}(s)) > -\epsilon_{\mathcal{F}}(\tilde{E})$.

We let E be the normalization of $\tilde{E}$, $\mathcal{F}_E$ the restricted foliation of $\mathcal{F}_Y$ on E , $V := \text{center}_X \tilde{E}$, $\mathbf{M}^E(t) := \mathbf{M}(t)|_E$, and

$$K_{\mathcal{F}_E} + B_E(t) + \mathbf{M}^E(t)_E := (K_{\mathcal{F}_Y} + B_Y(t) + \epsilon_{\mathcal{F}}(\tilde{E}) + \mathbf{M}(t)_Y)|_E$$

for any real number t . By Theorem 6.0.1, $B_E(t)$ is an $\mathbb{R}$ -affine functional divisor, $\mathbf{M}^E(t)$ is an $\mathbb{R}$ -affine functional $\mathbf{b}$ -divisor, and

$$(E, \mathcal{F}_E, B_E(t_0), \mathbf{M}^E(t_0)), (E, \mathcal{F}_E, B_E(s), \mathbf{M}^E(s))$$

are lc gfqs. By Proposition 6.1.10, $\mathcal{F}_E$ is algebraically integrable and $\text{rank } \mathcal{F} \leq d$.

Let $\phi : E \rightarrow V$ be the induced projective surjective morphism. Since $-\tilde{E}$ is ample/ X ,

$$K_{\mathcal{F}_E} + B_E(t_0) + \mathbf{M}^E(t_0)_E \sim_{\mathbb{R}, V} 0$$

and

$$K_{\mathcal{F}_E} + B_E(s) + \mathbf{M}^E(s)_E$$

is anti-ample/ V . Thus

$$K_{\mathcal{F}_E} + B_E(t) + \mathbf{M}^E(t)_E$$

is anti-ample/ V for any $t \in (t_0, s)$ if $t_0 < r_c$, and for any $t \in (s, t_0)$ if $t_0 > r_c$.

By Theorem 2.5.1, we may let

$$(W, \mathcal{F}_W, B_W(t_0), \mathbf{M}^E(t_0); G)/Z$$

be an ACSS model of $(E, \mathcal{F}_E, B_E(t_0), \mathbf{M}^E(t_0))$ with induced birational morphism $g : W \rightarrow E$. Let $F_W := (\text{Supp Exc}(g))^{\mathcal{F}_W}$ and let $B_E(t) := g_*^{-1} B_E(t_0) + F_W$ for any $t \in \mathbb{R}$. Since $s \in (r_c - \delta_1, r_c + \delta_1)$, $s_i(s) > 0$ and $\mu_i(s) > 0$. By Theorem 6.0.1 and Lemma 7.3.5, there exists a real number u , such that $u \in (t_0, s)$ if $t_0 < r_c$, $u \in (s, t_0)$ if $t_0 > r_c$, and $(W, \mathcal{F}_W, B_W(u), \mathbf{M}^E(u); G)/Z$ is ACSS. Since

$$K_{\mathcal{F}_E} + B_E(u) + \mathbf{M}^E(u)_E$$

is anti-ample/ V , $K_{\mathcal{F}_W} + B_W(u) + \mathbf{M}^E(u)_W$ is not pseudo-effective/ V . Thus we may run a

$$(K_{\mathcal{F}_W} + B_W(u) + \mathbf{M}^E(u)_W)\text{-MMP}/V$$

with scaling of an ample/ V divisor. By Theorem 9.3.4, this MMP terminates with a Mori fiber space/ V $\psi : (\bar{W}, \mathcal{F}_{\bar{W}}, B_{\bar{W}}(u), \mathbf{M}^E(u)) \rightarrow T$ of $(W, \mathcal{F}_W, B_W(u), \mathbf{M}^E(u))/V$. By Lemma 9.1.4, ψ is also a Mori fiber space/ Z .

$$\begin{array}{ccc} W & \dashrightarrow & \bar{W} \\ g \downarrow & & \downarrow \psi \\ E & & T \\ & \searrow \phi & \swarrow \\ & & V \end{array}$$

Let $B_{\bar{W}}(t)$ be the image of $B_W(t)$ on $\bar{W}$ for any t . Since

$$K_{\mathcal{F}_E} + B_E(t_0) + \mathbf{M}^E(t_0)_E \sim_{\mathbb{R}, V} 0,$$

we have

$$K_{\mathcal{F}_W} + B_W(t_0) + \mathbf{M}^E(t_0)_W \sim_{\mathbb{R}, V} 0,$$

so $(\bar{W}, \mathcal{F}_{\bar{W}}, B_{\bar{W}}(u), \mathbf{M}^E(u))$ and $(W, \mathcal{F}_W, B_W(u), \mathbf{M}^E(u))$ are crepant, and

$$K_{\mathcal{F}_{\bar{W}}} + B_{\bar{W}}(t_0) + \mathbf{M}^E(t_0)_{\bar{W}} \sim_{\mathbb{R}, V} 0.$$

Let L be a general fiber of ψ , $B_L(t) := B_{\bar{W}}(t)|_L$ for any t , and $\mathbf{M}^L(t) := \mathbf{M}^E(t)|_L$ for any t . Since ψ is Mori fiber space/ Z , the general fibers of ϕ are tangent to $\mathcal{F}_{\bar{W}}$. Thus $K_{\mathcal{F}_{\bar{W}}}|_L = K_L$, $(L, B_L(t_0), \mathbf{M}^L(t_0))$ is lc,

$$K_L + B_L(t_0) + \mathbf{M}^L(t_0)_L \equiv 0,$$

and

$$K_L + B_L(u) + \mathbf{M}^L(u)_L$$

is anti-ample. Moreover, since ψ is a Mori fiber space/ Z ,

$$\dim L \leq \text{rank } \mathcal{F}_{\bar{W}} = \text{rank } \mathcal{F}_E \leq d.$$

By [Che20, Theorem 3.6] and considering the coefficients of $B_L(t_0)$ and $\mathbf{M}^L(u)$, which can be precisely computed by Theorem 6.0.1, there exists a positive real number ϵ depending only on $d, \mathbf{r}, s_1, \dots, s_m, \mu_1, \dots, \mu_n$, such that $|t_0 - r_c| \geq \epsilon$. This concludes the proof of the theorem. $\square$

Theorem 10.3.7. *Let d, c, m, n be positive integers, $r_1, \dots, r_c$ real numbers such that $1, r_1, \dots, r_c$ are linearly independent over $\mathbb{Q}$, $\mathbf{r} := (r_1, \dots, r_c)$, and $s_1, \dots, s_m, \mu_1, \dots, \mu_n : \mathbb{R}^{c+1} \rightarrow \mathbb{R}$ $\mathbb{Q}$ -linear functions. Then there exists an open subset $U \ni \mathbf{r}$ depending only on $d, \mathbf{r}$ and $s_1, \dots, s_m, \mu_1, \dots, \mu_n$ satisfying the following. Assume that*

(1)

$$\left(X, \mathcal{F}, B(\mathbf{r}) := \sum_{i=1}^m s_i(1, \mathbf{r}) B_i, \mathbf{M}(\mathbf{r}) := \sum_{i=1}^n \mu_i(1, \mathbf{r}) \mathbf{M}_i \right) / X$$

is an lc gfq such that $\mathcal{F}$ is algebraically integrable and $\text{rank } \mathcal{F} \leq d$,

(2) $B_i \geq 0$ are distinct Weil divisors (possibly 0) and $s_i(1, \mathbf{r}) \geq 0$,

(3) $\mathbf{M}_i$ are nef/ X $\mathbf{b}$ -Cartier $\mathbf{b}$ -divisors and $\mu_i(1, \mathbf{r}) \geq 0$, and

(4) $B(\mathbf{v}) := \sum_{i=1}^m s_i(1, \mathbf{v}) B_i$ and $\mathbf{M}(\mathbf{v}) := \sum_{i=1}^n \mu_i(1, \mathbf{v}) \mathbf{M}_i$ for any $t \in \mathbb{R}$.

Then $(X, \mathcal{F}, B(\mathbf{v}), \mathbf{M}(\mathbf{v}))$ is lc for any $\mathbf{v} \in U$.

Proof. We apply induction on c . When $c = 1$, Theorem 10.3.7 directly follows from Theorem 10.3.6. When $c \geq 2$, by Theorem 10.3.6, there exists a positive integer δ depending only on $r_1, \dots, r_c, s_1, \dots, s_m$, such that for any $t \in (r_c - \delta, r_c + \delta)$,

$$\left(X, \mathcal{F}, \sum_{i=1}^m s_i(1, r_1, \dots, r_{c-1}, t) B_i, \sum_{i=1}^n \mu_i(1, r_1, \dots, r_{c-1}, t) \mathbf{M}_i \right)$$

is lc. We pick rational numbers $r_{c,1} \in (r_c - \delta, r_c)$ and $r_{c,2} \in (r_c, r_c + \delta)$ depending only on $r_1, \dots, r_c, s_1, \dots, s_m$. By induction on c , there exists an open subset $U_0 \in (r_1, \dots, r_{c-1})$ of $\mathbb{R}^{c-1}$, such that for any $\mathbf{v} \in U_0$,

$$\left(X, \mathcal{F}, \sum_{i=1}^m s_i(1, \mathbf{v}, r_{c,1}) B_i, \sum_{i=1}^n \mu_i(1, \mathbf{v}, r_{c,1}) \mathbf{M}_i \right)$$

and

$$\left(X, \mathcal{F}, \sum_{i=1}^m s_i(1, \mathbf{v}, r_{c,2}) B_i, \sum_{i=1}^n \mu_i(1, \mathbf{v}, r_{c,2}) \mathbf{M}_i \right)$$

are lc. We may pick $U := U_0 \times (r_{c,1}, r_{c,2})$. $\square$

Part III. Canonical bundle formula and MMP for generalized pairs

11. CANONICAL BUNDLE FORMULA FOR LC-TRIVIAL FIBRATIONS

11.1. Stability of generalized foliated quadruples.

Proposition 11.1.1 (cf. [ACSS21, Proposition 3.7]). *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a sub-gfq satisfying Property $(*)$ associated with $f : X \rightarrow Z$ and G . Assume that f is equi-dimensional and B is horizontal/ Z . Then $f : (X, B + G, \mathbf{M}) \rightarrow Z$ is BP semi-stab if and only if $(X, \mathcal{F}, B, \mathbf{M})$ is sub-lc.*

Proof. Since $(X, \mathcal{F}, B + \mathbf{M})/U$ satisfies Property $(*)$, $f : (X, B + G, \mathbf{M}) \rightarrow Z$ satisfies Property $(*)$. By Lemma 5.4.3(1), $(X, B + G, \mathbf{M})$ is sub-lc. Let $f' : (X', \Sigma_{X'}, \mathbf{M}) \rightarrow (Z', \Sigma_{Z'})$ be any equi-dimensional model of $(X, B + G, \mathbf{M})$ associated with $h : X' \rightarrow X$ and $h_Z : Z' \rightarrow Z$. By Proposition 5.4.6, there exists an $\mathbb{R}$ -divisor $\bar{B}$ on X' satisfying the followings.

- $\text{Supp } \bar{B} \subset \text{Supp } \Sigma_{X'}$.
- $K_{X'} + \bar{B} + \mathbf{M}_{X'} = h^*(K_X + B + G + \mathbf{M}_X) + F$ for some $F \geq 0$ that is vertical/ Z' .
- $(X', \bar{B}, \mathbf{M})$ and $(X, B + G, \mathbf{M})$ are crepant over the generic point of Z . In particular, $(X', \bar{B}, \mathbf{M})$ and $(X, B, \mathbf{M})$ are crepant over the generic point of Z .
- $f' : (X', \bar{B}, \mathbf{M}) \rightarrow Z$ satisfies Property $(*)$. By Lemma 5.4.3(1), $(X', \bar{B}, \mathbf{M})$ is sub-lc.

Let $\mathcal{F}' := h^{-1}\mathcal{F}$, G' the vertical/ Z' part of $\bar{B}$, and B' the horizontal/ Z' part of $\bar{B}$. Then $(X', \mathcal{F}', B', \mathbf{M}; G')/Z'$ satisfies Property $(*)$. Since F is vertical/ Z' and $\mathcal{F}'$ is induced by f' , F is $\mathcal{F}'$ -invariant.

We let B_Z and $\mathbf{N}$ be the discriminant part and moduli part of $f : (X, B + G, \mathbf{M}) \rightarrow Z$ respectively, and let $\bar{B}_{Z'}$ and $\mathbf{N}'$ be the discriminant part and moduli part of $f' : (X', \bar{B}, \mathbf{M}) \rightarrow Z$ respectively. Since $(X, \mathcal{F}, B, \mathbf{M})/Z$ satisfies Property $(*)$, Z is smooth, so $K_Z + B_Z$ is $\mathbb{R}$ -Cartier, and we may define

$$K_{Z'} + B_{Z'} := h_Z^*(K_Z + B_Z).$$

Since $f' : (X', \bar{B}, \mathbf{M}) \rightarrow Z'$ satisfies Property $(*)$, $\bar{B}_{Z'} = f'(G')$. Since $f : (X, B + G, \mathbf{M}) \rightarrow Z$ satisfies Property $(*)$, $B_Z = f(G)$.

By Proposition 7.3.6, $K_{\mathcal{F}} + B + \mathbf{M}_X \sim \mathbf{N}_X$ and $K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} \sim \mathbf{N}'_{X'}$. In particular, $\mathbf{N}_X$ is $\mathbb{R}$ -Cartier. Let $A := \mathbf{N}'_{X'} - h^*\mathbf{N}_{X'}$. Then

$$\begin{aligned} A &= K_{X'} + \bar{B} + \mathbf{M}_{X'} - f'^*(K_{Z'} + \bar{B}_{Z'}) - h^*(K_X + B + G + \mathbf{M}_X - f^*(K_Z + B_Z)) \\ &= F - f'^*(K_{Z'} + \bar{B}_{Z'}) + f'(K_{Z'} + B_{Z'}) = F - f'^*(\bar{B}_{Z'} - B_{Z'}). \end{aligned}$$

In particular, A vertical/ Z' .

Claim 11.1.2. $(X, \mathcal{F}, B, \mathbf{M})$ is sub-lc if and only if $A \geq 0$.

Proof. Let

$$A' := K_{\mathcal{F}'} + h_*^{-1}B + \mathbf{M}_{X'} + (\text{Supp Exc}(h))^{\mathcal{F}'} - h^*(K_{\mathcal{F}} + B + \mathbf{M}_X).$$

By Lemma 6.2.4, h is a foliated log resolution of $(X, \mathcal{F}, B, \mathbf{M})$, so $(X, \mathcal{F}, B, \mathbf{M})$ is sub-lc if and only if $A' \geq 0$. Since

$$A \sim K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} - h^*(K_{\mathcal{F}} + B + \mathbf{M}_X),$$

for suitable choices of $K_{\mathcal{F}}$ and $K_{\mathcal{F}'}$, we have

$$A' - A = h_*^{-1}B - B' + (\text{Supp Exc}(h))^{\mathcal{F}'}.$$

For any horizontal/ Z prime divisor P on X' , if P is not exceptional/ X , then

$$\text{mult}_P A' = \text{mult}_P(h_*^{-1}B - \bar{B}) = 0$$

as G and F are vertical/ Z . If P is exceptional/ X , then

$$\text{mult}_P A' = 1 + \text{mult}_P(h_*^{-1}B - \bar{B}) \geq 1 - \text{mult}_P \bar{B}.$$

Since $f' : (X', \bar{B}, \mathbf{M}) \rightarrow Z'$ satisfies Property $(*)$, $\text{mult}_P \bar{B} \leq 1$, so $\text{mult}_P A' \geq 0$.

For any vertical/ Z' prime divisor P on X' , since B is horizontal/ Z and B' is horizontal/ Z' , $\text{mult}_P A' = \text{mult}_P A$. The claim follows. $\square$

Let $B'_{Z'}$ be the discriminant part of $f' : (X', \bar{B} - F, \mathbf{M}) \rightarrow Z'$. Then for any prime divisor D on Z' ,

$$\text{mult}_D \bar{B}_{Z'} = 1 - \sup\{t \mid (X', \bar{B} + tf'^*D, \mathbf{M}) \text{ is sub-lc over the generic point of } D\}$$

and

$$\text{mult}_D B'_{Z'} = 1 - \sup\{t \mid (X', \bar{B} - F + tf'^*D, \mathbf{M}) \text{ is sub-lc over the generic point of } D\}.$$

Since $\text{Supp } \bar{B} \subset \text{Supp } \Sigma_{X'}$, by the definition of equi-dimensional model, $\text{Supp}(\bar{B} - F) \subset \text{Supp } \Sigma_{X'}$. Therefore, if $D \subset \text{Supp } \Sigma_{Z'}$, then

$$\text{mult}_D B'_{Z'} = \text{mult}_D \bar{B}_{Z'} = \text{mult}_{f^*D} F = 0.$$

Otherwise,

$$\text{mult}_D(\bar{B}_{Z'} - B'_{Z'}) = \sup\{t \geq 0 \mid F - tf'^*D \geq 0\}.$$

Therefore,

- $F - f'^*(\bar{B}_{Z'} - B'_{Z'}) \geq 0$, and
- $F - f'^*(\bar{B}_{Z'} - B'_{Z'}) - \delta f'^*D \geq 0$ for any prime divisor D on Z' and any $\delta > 0$.

Since

$$A = f'^*(B_{Z'} - B'_{Z'}) + (F - f'^*(\bar{B}_{Z'} - B'_{Z'})),$$

we have that $A \geq 0$ if and only if $B_{Z'} - B'_{Z'} \geq 0$. The proposition follows from Claim 11.1.2. $\square$

Proposition 11.1.3. *Let $(X, B, \mathbf{M})/U$ be an lc g-pair and $f : X \rightarrow Z$ a contraction such that*

- $f : (X, B, \mathbf{M}) \rightarrow Z$ satisfies Property $(*)$,
- $(X, B, \mathbf{M})$ is BP semi-stable/ Z ,
- f is equi-dimensional, and
- $K_X + B + \mathbf{M}_X$ is nef/ Z .

Let $\mathbf{N}$ be the moduli part of $f : X \rightarrow Z$. Then:

- (1) $\mathbf{N}_X$ is nef/ U .
- (2) If $(X, B, \mathbf{M})$ is BP stable/ Z , then $\mathbf{N}$ descends to X . In particular $\mathbf{N}$ is nef/ U .

Proof. Let $\mathcal{F}$ be the foliation induced by $\mathcal{F}$ and B^h the horizontal/ Z part of B . By Proposition 11.1.1, $(X, \mathcal{F}, B^h, \mathbf{M}; G)/Z$ is weak ACSS. Since $K_X + B + \mathbf{M}_X$ is nef/ U , by Lemma 9.2.2, $K_{\mathcal{F}} + B + \mathbf{M}_X$ is nef/ U . By Proposition 7.3.6,

$$\mathbf{N}_X \sim K_{\mathcal{F}} + B^h + \mathbf{M}_X,$$

so $\mathbf{N}_X$ is nef/ U . This implies (1). If $(X, B, \mathbf{M})$ is BP stable/ Z , then $\mathbf{N}$ descends to X , and (2) follows from (1). $\square$

Lemma 11.1.4. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be an lc gfq such that $\mathcal{F}$ is induced by a contraction $f : X \rightarrow Z$. Let D_Z be a divisor over Z . Then there exists an ACSS model $(X', \mathcal{F}', B', \mathbf{M})/Z'$ of $(X, \mathcal{F}, B, \mathbf{M})$ with induced morphisms $f' : X' \rightarrow Z'$ and $g : X' \rightarrow X$, and a birational morphism $h_Z : Z' \rightarrow Z$, such that $h_Z \circ f' = f \circ g$ and D_Z is on Z' .*

Proof. By Definition-Theorem 5.1.2, there exists an equi-dimensional model $(Y, \Sigma_Y, \mathbf{M}) \rightarrow Z$ of $f : (X, B, \mathbf{M}) \rightarrow Z$ associated with $h : Y \rightarrow X$ and $h_Z : Z' \rightarrow Z$, such that D_Z is on Z . Let $\mathcal{F}_Y := h^{-1}\mathcal{F}$ and $B_Y := h_*^{-1}B + (\text{Supp Exc}(h))^{\mathcal{F}_Y}$, then $(Y, \mathcal{F}_Y, B_Y, \mathbf{M})$ is foliated log smooth, and

$$K_{\mathcal{F}_Y} + B_Y + \mathbf{M}_Y \sim_{\mathbb{R}, X} \sum_{E \in \text{Exc}(h)} (\epsilon_{\mathcal{F}}(E) - a(E, \mathcal{F}, B, \mathbf{M}))E \geq 0.$$

By Theorem 9.4.1, we may run a $(K_{\mathcal{F}_Y} + B_Y + \mathbf{M}_Y)$ -MMP/ X which terminates with a good minimal model $(X', \mathcal{F}', B', \mathbf{M})/X$ of $(Y, \mathcal{F}_Y, B_Y, \mathbf{M})/X$. By Lemma 9.1.3, $(X', \mathcal{F}', B', \mathbf{M})$ is an ACSS model of $(X, \mathcal{F}, B, \mathbf{M})$. By Lemma 9.1.4, this MMP is also a $(K_{\mathcal{F}_Y} + B_Y + \mathbf{M}_Y)$ -MMP/ Z' . Then $(X', \mathcal{F}', B', \mathbf{M})/Z'$ satisfies our requirements. $\square$

Theorem 11.1.5. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be an lc gfq, $f : X \rightarrow Z$ a contraction, and G a reduced divisor on X such that $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is weak ACSS. Then $(X, B + G, \mathbf{M})$ is BP stable/ Z .*

Proof. For any prime divisor D_Z over Z , by Lemma 11.1.4, there exist two birational morphisms $h_Z : Z' \rightarrow Z$ and $h : X' \rightarrow X$, and an ACSS model $(X', \mathcal{F}', B', \mathbf{M})/Z'$ of $(X, \mathcal{F}, B, \mathbf{M})$ with induced morphism $f' : X' \rightarrow Z'$, such that $f \circ h = h_Z \circ f'$ and D_Z is on Z' .

We let G' be a divisor on X' such that $(X', \mathcal{F}', B', \mathbf{M}; G')/Z'$ is ACSS. Let B_Z and $\mathbf{N}$ be the discriminant and moduli part of $f : (X, B + G, \mathbf{M}) \rightarrow Z$ respectively, $K_{Z'} + B_{Z'} := h_Z^*(K_Z + B_Z)$, and let $B'_{Z'}$ and $\mathbf{N}'$ be the discriminant part and moduli part of $f' : (X', B' + G', \mathbf{M}) \rightarrow Z'$ respectively.

By Proposition 7.3.6, we have

$$\mathbf{N}'_{X'} \sim K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} = h^*(K_{\mathcal{F}} + B + \mathbf{M}_X) \sim h^*\mathbf{N}_X.$$

Thus for suitable choices of $\mathbf{N}'$ and $\mathbf{N}$, we may assume that $\mathbf{N}'_{X'} = h^*\mathbf{N}_X$. Let

$$K_{X'} + \tilde{B}' + \mathbf{M}_{X'} := h^*(K_X + B + G + \mathbf{M}_X).$$

Let $\tilde{B}_{Z'}$ and $\tilde{\mathbf{N}}$ be the discriminant and moduli parts of $f' : (X', \tilde{B}, \mathbf{M}) \rightarrow Z'$ respectively. Since

$$\begin{aligned} & \mathbf{N}'_{X'} - h^*\mathbf{N}_X \\ &= ((K_{X'} + B' + G' + \mathbf{M}_{X'}) - f'^*(K_{Z'} + B'_{Z'})) - h^*(K_X + B + G + \mathbf{M}_X - f^*(K_Z + B_Z)) \\ &= B' + G' - \tilde{B}' - f'^*(B'_{Z'} - B_{Z'}), \end{aligned}$$

we have that

$$B' + G' - \tilde{B}' - f'^*(B'_{Z'} - B_{Z'}) = 0.$$

Moreover, by Proposition 11.1.1, $(X, B + G, \mathbf{M})$ is BP semi-stable/ Z . Thus

$$\text{mult}_{D_Z} \tilde{B}_{Z'} \leq \text{mult}_{D_Z} B_{Z'}.$$

For any prime divisor D on X with $f'(D) = D_Z$, since B' is horizontal/ Z' , $\text{mult}_D B' = 0$. Thus

$$\text{mult}_D G' = \text{mult}_D (\tilde{B}' + f'^*(B'_{Z'} - B_{Z'})).$$

There are two cases.

Case 1. D_Z is not a component of $B'_{Z'}$. In this case, $\text{mult}_D G' = 0$, and

$$\text{mult}_D \tilde{B}' = \text{mult}_D f'^* B_{Z'} = \text{mult}_{D_Z} B_{Z'} \cdot \text{mult}_D f'^* D_Z.$$

Thus $\text{mult}_D (\tilde{B}' - \text{mult}_{D_Z} B_{Z'} f'^* D_Z) = 0$, and

$$\text{mult}_D (\tilde{B}' + (1 - \text{mult}_{D_Z} B_{Z'}) f'^* D_Z) = \text{mult}_D f'^* D_Z \geq 1.$$

Therefore,

$$\begin{aligned} 1 - \text{mult}_{D_Z} \tilde{B}_{Z'} &= \sup \left\{ t \mid (X', \tilde{B}' + t f'^* D_Z, \mathbf{M}) \text{ is lc over the generic point of } D_Z \right\} \\ &\leq 1 - \text{mult}_{D_Z} B_{Z'}. \end{aligned}$$

Thus

$$\text{mult}_{D_Z} B_{Z'} = \text{mult}_{D_Z} \tilde{B}_{Z'}.$$

Case 2. D_Z is a component of $B'_{Z'}$. In this case, $\text{mult}_D G' = 1$ and $\text{mult}_D B'_{Z'} = 1$. Therefore,

$$1 = \text{mult}_D (\tilde{B}' + (\text{mult}_D B'_{Z'} - \text{mult}_D B_{Z'}) f'^* D_Z) = \text{mult}_D (\tilde{B}' + (1 - \text{mult}_D B_{Z'}) f'^* D_Z).$$

Thus

$$\begin{aligned} 1 - \text{mult}_{D_Z} \tilde{B}_{Z'} &= \sup \{ t \mid (X', \tilde{B}' + t f'^* D_Z, \mathbf{M}) \text{ is lc over the generic point of } D_Z \} \\ &\leq 1 - \text{mult}_{D_Z} B_{Z'}, \end{aligned}$$

and hence

$$\text{mult}_{D_Z} B_{Z'} = \text{mult}_{D_Z} \tilde{B}_{Z'}.$$

In either case, we have $\text{mult}_{D_Z} B_{Z'} = \text{mult}_{D_Z} \tilde{B}_{Z'}$. Since D_Z can be any prime divisor over Z , $f : (X, B + G, \mathbf{M}) \rightarrow Z$ is BP stable. $\square$

Remark 11.1.6. When $\mathbf{M} = \mathbf{0}$ and X is projective, Theorem 11.1.5 becomes [ACSS21, Theorem 4.3] without the condition that $K_X + B$ is f -nef. This seems to be an interesting discovery and may be useful for future applications.

11.2. Numerical dimension zero generalized foliated quadruples.

Proposition 11.2.1. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be an lc gfq such that*

- $(X, \mathcal{F}, B, \mathbf{M})$ is weak ACSS,
- $\kappa_\sigma(X/U, K_{\mathcal{F}} + B + \mathbf{M}_X) = 0$, and
- either X is $\mathbb{Q}$ -factorial klt or $\mathbf{M}$ is NQC/U .

Then for any ample/ U $\mathbb{R}$ -divisor A , there exists a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U with scaling of A , say $\mathcal{P}_0$, satisfying the following. Let $\mathcal{P} = \mathcal{P}_0$ if X is not $\mathbb{Q}$ -factorial, and let $\mathcal{P}$ be any $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ U with scaling of A if X is $\mathbb{Q}$ -factorial. After a sequence of steps in $\mathcal{P}$, we get a log birational model $(X', \mathcal{F}', B', \mathbf{M})/U$ of $(X, \mathcal{F}, B, \mathbf{M})/U$ satisfying the following.

- (1) *For any very general fiber F' of $X' \rightarrow U$, $(K_{\mathcal{F}'} + B' + \mathbf{M}_{X'})|_{F'} \equiv 0$, and if $\kappa_t(X/U, K_{\mathcal{F}} + B + \mathbf{M}_X) = 0$, then $(K_{\mathcal{F}'} + B' + \mathbf{M}_{X'})|_{F'} \sim_{\mathbb{R}} 0$.*
- (2) *Suppose that the associated morphism $\pi : X \rightarrow U$ is an equi-dimensional contraction and U is $\mathbb{Q}$ -factorial.*
 - (a) *Assume that $K_{\mathcal{F}} + B + \mathbf{M}_X \equiv_U$ (resp. $\sim_{\mathbb{R}, U}$) $E^h + E^v$ for some $\mathbb{R}$ -divisors E^h and E^v such that $E^h \geq 0$ and E^v is vertical/ U . Then*

$$K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} \equiv_U \text{ (resp. } \sim_{\mathbb{R}, U} \text{) } 0.$$

In particular, $(X', \mathcal{F}', B', \mathbf{M})/U$ is a weak lc model of $(X, \mathcal{F}, B, \mathbf{M})/U$.

- (b) *If $\kappa_t(X/U, K_{\mathcal{F}} + B + \mathbf{M}_X) = 0$, then:*
 - (i) $K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} \sim_{\mathbb{R}, U} 0$.
 - (ii) $(X', \mathcal{F}', B', \mathbf{M})/U$ is a weak lc model of $(X, \mathcal{F}, B, \mathbf{M})/U$.
 - (iii) If $(X, \mathcal{F}, B, \mathbf{M})/U$ is ACSS, then $(X', \mathcal{F}', B', \mathbf{M})/U$ is a good minimal model of $(X, \mathcal{F}, B, \mathbf{M})/U$.

Proof. Let $(X_0, \mathcal{F}_0, B_0, \mathbf{M}) := (X, \mathcal{F}, B, \mathbf{M})$. We denote $\mathcal{P}$ by

$$(X_0, \mathcal{F}_0, B_0, \mathbf{M}) \dashrightarrow (X_1, \mathcal{F}_1, B_1, \mathbf{M}) \dashrightarrow \dots \dashrightarrow (X_n, \mathcal{F}_n, B_n, \mathbf{M}) \dashrightarrow \dots$$

Let A_i be the strict transform of A on X_i , $\pi_i : X_i \rightarrow U$ the induced contraction for each i , and

$$\lambda_i := \inf\{t \geq 0 \mid K_{\mathcal{F}_i} + B_i + \mathbf{M}_{X_i} \text{ is nef}/U\}$$

the scaling numbers. By Proposition 9.3.2, we may choose $\mathcal{P}_0$ so that either $\mathcal{P}$ terminates, or $\lim_{i \rightarrow +\infty} \lambda_i = 0$.

If $\mathcal{P}$ terminates, then we let m be the index so that $(X_m, \mathcal{F}_m, B_m, \mathbf{M})/U$ is output of $\mathcal{P}$. If $\mathcal{P}$ does not terminate, then we let m be a positive integer such that f_i is a flip for any $i \geq m$. We let $\phi_i : X_m \dashrightarrow X_i$ be the induced birational map for any $i \geq m$. Since $\mathcal{P}$ contains countable many steps, there are at most countably many closed point $z \in Z$, such that for some $i \geq m$, $\pi_i^{-1}(z)$ is contained in either the flipping locus of f_i or the flipped locus of f_{i-1} . Therefore, for a very general point $z \in Z$ and any $i \geq m$, $\pi_i^{-1}(z)$ is neither contained in the flipping locus of f_i nor the flipped locus of f_{i-1} for any $i \geq m$. We let F_m be a very general fiber of π_m , $z_0 := \pi_m(F_m)$, and let F_i be the fiber of π_i over z_0 for each i . Then the induced birational map

$$\phi_{F,i} := \phi_i|_{F_m} : F_m \dashrightarrow F_i$$

is small for any $i \geq m$. Let $\mathbf{M}^F := \mathbf{M}|_{F_m}$, $B_{F_i} := B_i|_{F_i}$, and $A_{F_i} := A_i|_{F_i}$ for each $i \geq m$. Since F_i is a very general fiber of π_i , $K_{F_i} = K_{\mathcal{F}_i}|_{F_i}$ for any $i \geq m$.

We will show that $(X', \mathcal{F}', B', \mathbf{M})/U := (X_m, \mathcal{F}_m, B_m, \mathbf{M})/U$ satisfies our requirements.

Claim 11.2.2. $K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m}$ is movable/ U and $(K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m})|_{F_m}$ is movable.

Proof. If $K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m}$ is nef/ U then the claim is obvious, so we may assume that $K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m}$ is not nef/ U . In particular, $\mathcal{P}$ does not terminate. Since $K_{\mathcal{F}_i} + B_i + \mathbf{M}_{X_i} + tA_i$ is nef/ U for any $i \geq m$,

$$K_{X_m} + B_m + \mathbf{M}_{X_m} = \lim_{i \rightarrow +\infty} (\phi_i^{-1})_*(K_{X_i} + B_i + \mathbf{M}_{X_i} + tA_i)$$

is movable/ U , and

$$K_{F_i} + B_{F_i} + \mathbf{M}_{F_i}^F + tA_{F_i} = (K_{\mathcal{F}_i} + B_i + \mathbf{M}_{X_i} + tA_i)|_{F_i}$$

is nef for each $i \geq m$. Thus

$$K_{F_m} + B_{F_m} + \mathbf{M}_{F_m}^F = \lim_{i \rightarrow +\infty} (\phi_{F,i}^{-1})_*(K_{F_i} + B_{F_i} + \mathbf{M}_{F_i}^F + tA_{F_i})$$

is movable, and the claim follows. $\square$

Proof of Proposition 11.2.1 continued. Since $\kappa_\sigma(X/U, K_{\mathcal{F}} + B + \mathbf{M}_X) = 0$, $\kappa_\sigma(X_m/U, K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m}) = 0$ and $\kappa_\sigma(K_{F_m} + B_{F_m} + \mathbf{M}_{F_m}^F) = 0$. By Claim 11.2.2 and Lemma 4.2.4, $K_{F_m} + B_{F_m} + \mathbf{M}_{F_m}^F \equiv 0$. Moreover, if $\kappa_\ell(X/U, K_{\mathcal{F}} + B + \mathbf{M}_X) = 0$, then $\kappa_\ell(X_m/U, K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m}) = 0$, so $\kappa_\ell(K_{F_m} + B_{F_m} + \mathbf{M}_{F_m}^F) = 0$, and hence $K_{F_m} + B_{F_m} + \mathbf{M}_{F_m}^F \sim_{\mathbb{R}} 0$. This implies (1).

We prove (2). From now on, we may assume that π is equi-dimensional and U is $\mathbb{Q}$ -factorial. Suppose that $K_{\mathcal{F}} + B + \mathbf{M}_X \equiv_U$ (resp. $\sim_{\mathbb{R},U}$) $E^h + E^v$ where $E^h \geq 0$ and E^v is vertical/ U . Since U is $\mathbb{Q}$ -factorial, for any prime divisor D on X , D is $\mathbb{Q}$ -Cartier, and we may define

$$t_D := \sup\{s \mid E^v - s\pi^*D \geq 0 \text{ over the generic point of } D\}.$$

Let

$$\tilde{E}^v := E^v - \sum_{D|D \text{ is a prime divisor on } U} t_D D.$$

Since π is equi-dimensional, $\tilde{E}^v \geq 0$ and $\tilde{E}^v$ is very exceptional/ U . Possibly replacing E^v with $\tilde{E}^v$, we may assume that $0 \leq E^v$ is very exceptional/ U . Let E_m^h and E_m^v be the strict transforms of E^h and E^v on $X' = X_m$ respectively. Then $E_m^h|_{F_m} \equiv 0$, so $E_m^h|_{F_m} = 0$ and

$$E_m^v \equiv_U \text{ (resp. } \sim_{\mathbb{R},U} \text{) } K_{\mathcal{F}_m} + B_m + \mathbf{M}_{X_m}$$

is movable/ U . Therefore, for any prime divisor S on X_m and very general curves C on S over U , $E_m^v \cdot C \geq 0$. By [Bir12, Lemma 3.3], $E_m^v = 0$. This implies (2.a).

If $\kappa_\ell(X/U, K_{\mathcal{F}} + B + \mathbf{M}_X) = 0$, then $K_{\mathcal{F}} + B + \mathbf{M}_X \sim_{\mathbb{R},U} E \geq 0$ for some $\mathbb{R}$ -divisor E on X (cf. [HH20, Definition 2.6]). Then (2.b) immediately follows from (2.a) and Lemma 9.1.4. $\square$

The following proposition is a direct consequence of Proposition 11.2.1.

Proposition 11.2.3. *Let $(X, \mathcal{F}, B, \mathbf{M})$ be a projective lc gfq such that*

- $(X, \mathcal{F}, B, \mathbf{M})$ is weak ACSS,
- $\kappa_\sigma(K_{\mathcal{F}} + B + \mathbf{M}_X) = 0$, and
- either X is $\mathbb{Q}$ -factorial klt or $\mathbf{M}$ is NQC.

Then for any ample $\mathbb{R}$ -divisor A , there exists a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP with scaling of A , say $\mathcal{P}_0$, satisfying the following. Let $\mathcal{P} = \mathcal{P}_0$ if X is not $\mathbb{Q}$ -factorial, and let $\mathcal{P}$ be any $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP with scaling of an ample $\mathbb{R}$ -divisor if X is $\mathbb{Q}$ -factorial. Then:

- (1) $\mathcal{P}$ terminates with a weak lc model $(X', \mathcal{F}', B', \mathbf{M})$ of $(X, \mathcal{F}, B, \mathbf{M})$ such that

$$K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} \equiv 0.$$

- (2) Suppose that $\kappa_\ell(K_{\mathcal{F}} + B + \mathbf{M}_X) = 0$. Then:
- (a) $K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} \sim_{\mathbb{R}, U} 0$.
 - (b) If $(X, \mathcal{F}, B, \mathbf{M})$ is $\mathbb{Q}$ -factorial ACSS, then $(X', \mathcal{F}', B', \mathbf{M})$ is a good minimal model of $(X, \mathcal{F}, B, \mathbf{M})$.

Proof. It immediately follows from Proposition 11.2.1 by taking $U = \{pt\}$. $\square$

11.3. Refined definition of lc-trivial fibrations.

Definition 11.3.1 (Lc-trivial fibration). Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a sub-gfq and $f : X \rightarrow Z$ a contraction/ U , such that the general fibers of f are tangent to $\mathcal{F}$. We say that $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$ is an *lc-trivial fibration* if

- (1) $(X, \mathcal{F}, B, \mathbf{M})$ is sub-lc over the generic point of Z ,
- (2) $K_{\mathcal{F}} + B + \mathbf{M}_X \sim_{\mathbb{R}, Z} 0$, and
- (3) there exists a birational morphism $h : Y \rightarrow X$ with $\mathcal{F}_Y := h_*^{-1}\mathcal{F}$ and $K_{\mathcal{F}_Y} + B_Y + \mathbf{M}_Y = h^*(K_{\mathcal{F}} + B + \mathbf{M}_X)$, such that $-B_Y^{\leq 0}$ is $\mathbb{R}$ -Cartier and

$$\kappa_\sigma(Y/Z, -B_Y^{\leq 0}) = 0.$$

It is clear the lc-trivial fibration does not depend on the choice of U .

Remark 11.3.2. It is essential to note that our definition of lc-trivial fibration differs from the classical one, even when $\mathbf{M} = \mathbf{0}$ and $\mathcal{F} = T_X$. We have valid reasons for this deviation. For the sake of simplicity, in the rest part of this remark, we will assume that $\mathcal{F} = T_X$.

In the classical definition, condition (3) is substituted by

$$(3') \text{ rank } f_* \mathcal{O}_X([\mathbf{A}^*(X, B, \mathbf{M})]) = 1.$$

This condition (3') appears in the initial version of the canonical bundle formula [Kaw98, Condition (3) of Theorem 2]. It has also been adopted in subsequent versions, for instance, [Amb05, Theorem 0.2] for sub-plt sub-pairs and [Kol07, Theorem 8.3.7] (also see [FG14, Theorem 3.6]) for lc-trivial fibrations of sub-lc sub-pairs.

However, for generalized sub-pairs, does not we cannot prove a complete version of the canonical bundle formula under condition (3'). Specifically, for lc-trivial fibrations of NQC generalized pairs defined using condition (3') instead of (3), one must incorporate one of the subsequent conditions to derive the canonical bundle formula:

- (4.1) $B \geq 0$ over the generic point of Z (rational coefficient case [FS23, Theorem 2.20]; real coefficient case [JLX22, Theorem 2.23]).
- (4.2) $\mathbf{M}$ is $\mathbf{b}$ -semi-ample/ Z (rational coefficient case [Fil19, Chapter 6, Theorem 7]; real coefficient case [JLX22, Theorem 2.23]).

While the canonical bundle formula for NQC generalized pairs under conditions (4.1) or (4.2) usually suffices for studying generalized pairs, troubles arise when examining the canonical bundle formula for generalized foliated quadruples. This is because we need to construct equi-dimensional models during the construction of the canonical bundle formula for generalized foliated quadruples, as outlined in [LLM23, Definition-Theorem 6.12]. This approach could yield a sub-lc g-sub-pair with negative coefficients, typically not complying with (4.1) or (4.2). Consequently, defining the canonical bundle formula for generalized foliated quadruples becomes challenging. Condition (3) was introduced to address this issue.

Indeed, the most important cases of the canonical bundle formula arise when $B \geq 0$ over the generic point of Z . But as we often need to consider the coefficients of the discriminant part across any high model of the base Z in order to study the corresponding singularities, base changes are inevitable. Therefore, we need to take consideration of crepant pullbacks of $(X, B, \mathbf{M})$. Furthermore, running minimal model programs over the base to introduce new structures means that crepant transformations over the generic point of Z also become inevitable. This will inevitably introduce more sub-pairs or g-sub-pairs, necessitating a broader category of

structures for which the canonical bundle formula needs to be applied. Specifically, we aim to identify a category $\mathcal{D}$ of structures

$$f : (X, B, \mathbf{M}) \rightarrow Z,$$

which satisfies the following two conditions.

- (i) For any g-sub-pair $(X, B, \mathbf{M})/U$ and contraction $f : X \rightarrow Z$ such that $K_X + B + \mathbf{M}_X \sim_{\mathbb{R}, Z} 0$ and $(X, B, \mathbf{M})$ is lc over the generic point of Z , $f : (X, B, \mathbf{M}) \rightarrow Z$ belongs to $\mathcal{D}$.
- (ii) For any g-sub-pairs $(X, B, \mathbf{M})/U$ and $(X', B', \mathbf{M}')/U$ and birationally equivalent contractions $f : X \rightarrow Z$, $f' : X' \rightarrow Z'$ such that $K_X + B + \mathbf{M}_X \sim_{\mathbb{R}, Z} 0$, $K_{X'} + B' + \mathbf{M}_{X'} \sim_{\mathbb{R}, Z'} 0$, and $(X, B, \mathbf{M})$ and $(X', B', \mathbf{M}')$ are crepant over the generic point of Z , $f : (X, B, \mathbf{M}) \rightarrow Z$ belongs to $\mathcal{D}$ if and only if $f' : (X', B', \mathbf{M}') \rightarrow Z$ belongs to $\mathcal{D}$.

Condition (3') is one natural condition to add in order to form the category $\mathcal{D}$. For generalized pairs, however, the $\mathcal{D}$ category shaped by incorporating condition (3') becomes overly expansive to consistently prove the canonical bundle formula. By comparing our condition (3) with (3'), it becomes evident that (3') can be loosely interpreted as

$$\kappa(Y/Z, -B_Y^{\leq 0}) = 0$$

(cf. [Kol07, Definitions 8.4.1, 8.4.2]). This essentially indicates some kind of existence of good minimal models should hold for generalized pairs with Kodaira dimension 0. But such an assertion is absurd by numerous examples (e.g., [Sho00, 1.1 Example]). In fact, even for usual pairs, since we don't know the existence of good minimal models for pairs with Kodaira dimension of 0, the theory of mixed Hodge structures is inevitably used to prove the canonical bundle formula in almost all literature, with the exception of [ACSS21]. We also note that [ACSS21] does not extensively address lc-trivial fibrations.

Given these considerations, we redirect our focus to a new category $\mathcal{D}$ of g-sub-pairs which adhere to (1) and (2) but do not depend on condition (3'). It turns out that condition (3) is a natural alternative choice for us to form the category $\mathcal{D}$. This enables us to bypass the abundance conjecture or the mixed Hodge structure by replacing (3) with (3'). This will eventually lead us to prove the canonical bundle formula for generalized pairs and generalized foliated quadruples in full generality.

The following lemmas are analogues of Lemma 11.3.3, 11.3.4, and 11.3.5 for foliations, and their proofs are similar.

Lemma 11.3.3. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a sub-gfq. Assume that $-B^{\leq 0}$ is $\mathbb{R}$ -Cartier and $\kappa_\sigma(X/U, -B^{\leq 0}) = 0$. Then for any birational morphism $g : W \rightarrow X$, such that*

- (1) $K_{g^{-1}\mathcal{F}} + B_W + \mathbf{M}_W := g^*(K_{\mathcal{F}} + B + \mathbf{M}_X)$ satisfies that $-B_W^{\leq 0}$ is $\mathbb{R}$ -Cartier, and
- (2) there exists an $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor $0 \leq F \subset \text{Supp Exc}(g)$,

we have that

$$\kappa_\sigma(W/U, -B_W^{\leq 0}) = 0.$$

Proof. Let $D := -B^{\leq 0}$, $D_W := -B_W^{\leq 0}$, and $m \gg 0$ an integer. Then we have

$$D_W = g_*^{-1}D + E$$

for some $E \geq 0$ that is exceptional/ X . Thus

$$0 = \kappa_\sigma(X/U, D) = \kappa_\sigma(W/U, g^*D + mF) \geq \kappa_\sigma(W/U, g_*^{-1}D + E) = \kappa_\sigma(W/U, D_W) \geq 0.$$

So $\kappa_\sigma(W/U, D_W) = 0$. □

Lemma 11.3.4. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ and $(X', \mathcal{F}', B', \mathbf{M}')/U$ be two sub-gfqs. Let $f : X \rightarrow Z$ and $f' : X' \rightarrow Z'$ be two birationally equivalent contractions/ U , such that $(X, \mathcal{F}, B, \mathbf{M})$ and $(X', \mathcal{F}', B', \mathbf{M}')$ are crepant over the generic point of Z . Assume that $K_{\mathcal{F}} + B + \mathbf{M}_X \sim_{\mathbb{R}, Z} 0$ and $K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} \sim_{\mathbb{R}, Z'} 0$.*

Then $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$ is an lc-trivial fibration if and only if $f' : (X', \mathcal{F}', B', \mathbf{M}') \rightarrow Z'$ is an lc-trivial fibration.

Proof. By symmetry, we only need to prove the only if part, and we may assume that $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$ is an lc-trivial fibration.

Let $p : W \rightarrow X$ and $q : W \rightarrow X'$ be a resolution of indeterminacy of the induced birational map $\phi : X \dashrightarrow X'$ such that $\mathbf{M}$ descends to W , $\mathcal{F}_W := p^{-1}\mathcal{F} = q^{-1}\mathcal{F}'$, $K_{\mathcal{F}_W} + B_W + \mathbf{M}_W := p^*(K_{\mathcal{F}} + B + \mathbf{M}_X)$, and $K_{\mathcal{F}_W} + B'_W + \mathbf{M}'_W := q^*(K_{\mathcal{F}'} + B' + \mathbf{M}'_{X'})$. Moreover, by Lemma 11.3.3, possibly replacing W with a higher resolution, we may assume that W is smooth and $\kappa_{\sigma}(W/Z, -B_W^{\leq 0}) = 0$.

Since $(X, \mathcal{F}, B, \mathbf{M})$ and $(X', \mathcal{F}', B', \mathbf{M}')$ are crepant over the generic point of Z , over the generic point of Z , we have that $B_W = B'_W$, $\mathcal{F} = \mathcal{F}'$, and $\mathbf{M}_W = \mathbf{M}'_W$. Thus $\kappa_{\sigma}(W/Z, -B_W^{\leq 0}) = 0$. Moreover, since $(X, \mathcal{F}, B, \mathbf{M})$ is sub-lc over the generic point of Z , $(W, \mathcal{F}, B_W, \mathbf{M})$ is sub-lc over the generic point of Z , so $(W, \mathcal{F}', B'_W, \mathbf{M}')$ is sub-lc over the generic point of Z , and so $(W, \mathcal{F}', B'_W, \mathbf{M}')$ is sub-lc over the generic point of Z' , so $(X', \mathcal{F}', B', \mathbf{M}')$ is sub-lc over the generic point of Z' . The lemma follows. $\square$

Lemma 11.3.5. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a sub-gfq and $f : X \rightarrow Z$ a contraction/ U , such that $(X, \mathcal{F}, B, \mathbf{M})$ is lc over the generic point of Z and $K_{\mathcal{F}} + B + \mathbf{M}_X \sim_{\mathbb{R}, Z} 0$. Then $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$ is an lc-trivial fibration.*

Proof. Over the generic point of Z , $B^{\leq 0} = 0$, so $\kappa_{\sigma}(X/Z, B^{\leq 0}) = 0$. The lemma follows from the definition. $\square$

11.4. Canonical bundle formula for generalized pairs.

Definition 11.4.1. Let $(X, B, \mathbf{M})/U$ be a g-sub-pair and $f : X \rightarrow Z$ a contraction/ U such that $f : (X, B, \mathbf{M}) \rightarrow Z$ is an lc-trivial fibration. Then there exists an $\mathbb{R}$ -divisor L on Z such that $K_X + B + \mathbf{M}_X \sim_{\mathbb{R}} f^*L$. There exists a unique $\mathbf{b}$ -divisor $\mathbf{M}^Z$ on Z satisfying the following.

Let $f' : X' \rightarrow Z'$ be any contraction that is birationally equivalent to f such that the induced birational maps $h : X' \dashrightarrow X$ and $h_Z : Z' \dashrightarrow Z$ are morphisms. We let

$$K_{X'} + B' + \mathbf{M}_{X'} := h^*(K_X + B + \mathbf{M}_X)$$

and let $B_{Z'}$ be the discriminant part of $f' : (X', B', \mathbf{M}) \rightarrow Z'$. Then

$$\mathbf{M}_{Z'}^Z = h_Z^*L - K_{Z'} - B_{Z'}.$$

We call $\mathbf{M}^Z$ the *base moduli part* of $f : (X, B, \mathbf{M}) \rightarrow Z$. If there is no confusion, we may also call $\mathbf{M}^Z$ as the *moduli part* of $f : (X, B, \mathbf{M}) \rightarrow Z$. It is clear that for any fixed choice of L , $\mathbf{M}^Z$ is unique. In particular, $\mathbf{M}^Z$ is unique up to $\mathbb{R}$ -linear equivalence.

Lemma 11.4.2. *Let $(X, B, \mathbf{M})/U$ be a g-sub-pair and $f : X \rightarrow Z$ a contraction/ U such that $f : (X, B, \mathbf{M}) \rightarrow Z$ is an lc-trivial fibration. Suppose that $n(K_X + B + \mathbf{M}_X) \sim 0$ over the generic point of Z for some positive integer n . Then there exists a choice $\mathbf{M}^Z$ of the base moduli part of $f : (X, B, \mathbf{M}) \rightarrow Z$ such that*

$$n(K_X + B + \mathbf{M}_X) \sim n f^*(K_Z + B_Z + \mathbf{M}_Z^Z),$$

where B_Z be the discriminant part of $f : (X, B, \mathbf{M}) \rightarrow Z$.

Proof. By assumption, there exists a rational function $\psi \in K(X)$ such that $n(K_X + B + \mathbf{M}_X) + (\psi)$ is vertical/ Z . By [CHL23, Lemma 2.5], there exists an $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor L on Z such that

$$n(K_X + B + \mathbf{M}_X) + (\psi) = n f^*L.$$

The lemma follows from our construction of $\mathbf{M}^Z$ as in Definition 11.4.1. $\square$

Lemma 11.4.3. *Let $(X, B, \mathbf{M})/U$ and $(X', B', \mathbf{M})/U$ be two g -sub-pairs. Let $f : (X, B, \mathbf{M}) \rightarrow Z$ and $f' : (X', B', \mathbf{M}) \rightarrow Z'$ be two lc-trivial fibrations/ U such that f and f' are birationally equivalent, and $(X, B, \mathbf{M})$ and $(X', B', \mathbf{M})$ are crepant over the generic point of Z . Let $\mathbf{M}^Z$ be the base moduli part of $f : (X, B, \mathbf{M}) \rightarrow Z$ and let $\mathbf{M}^{Z'}$ be the base moduli part of Z' . Then $\mathbf{M}^Z \sim_{\mathbb{R}} \mathbf{M}^{Z'}$.*

Proof. Possibly passing to a common base and resolve indeterminacy of the induced birational map $X \dashrightarrow X'$, we may assume that $f = f'$, $X = X'$, and $Z = Z'$. Now $K_X + B + \mathbf{M}_X = K_X + B' + \mathbf{M}_X$ over the generic point of Z , so $B - B'$ is vertical/ Z . Since $K_X + B + \mathbf{M}_X \sim_{\mathbb{R}, Z} 0$ and $K_X + B' + \mathbf{M}_X \sim_{\mathbb{R}, Z} 0$, $B - B' \sim_{\mathbb{R}, Z} 0$, so $B - B' = f^*P$ for some $\mathbb{R}$ -divisor P on Z .

Let B_Z and B'_Z be the discriminant parts of $f : (X, B, \mathbf{M}) \rightarrow Z$ and $f : (X, B', \mathbf{M}) \rightarrow Z$ respectively. By the definition of the discriminant part, $B_Z = B'_Z + P$. Since

$$K_Z + B'_Z + P + \mathbf{M}_Z^{Z'} \sim_{\mathbb{R}} K_Z + B_Z + \mathbf{M}_Z^Z,$$

$\mathbf{M}_Z^{Z'} \sim_{\mathbb{R}} \mathbf{M}_Z^Z$. Since we may pass to an arbitrarily high base change, we have $\mathbf{M}^Z \sim_{\mathbb{R}} \mathbf{M}^{Z'}$. $\square$

Theorem 11.4.4. *Let $(X, B, \mathbf{M})/U$ be a g -sub-pair and $f : X \rightarrow Z$ a contraction/ U such that $f : (X, B, \mathbf{M}) \rightarrow Z$ is an lc-trivial fibration. Let B_Z and $\mathbf{M}^Z$ be the discriminant part and a base moduli part of $f : (X, B, \mathbf{M}) \rightarrow Z$ respectively. Then $\mathbf{M}^Z$ is nef/ U . Moreover:*

- (1) $(Z, B_Z, \mathbf{M}^Z)/U$ is a g -sub-pair.
- (2) If the vertical/ Z part of B is ≥ 0 , then $(Z, B_Z, \mathbf{M}^Z)/U$ is a g -pair.
- (3) If $(X, B, \mathbf{M})$ is sub-lc (resp. lc, sub-klt, klt), then $(Z, B_Z, \mathbf{M}^Z)$ is sub-lc (resp. lc, sub-klt, klt).
- (4) Any lc center of $(Z, B_Z, \mathbf{M}^Z)$ is the image of an lc center of $(X, B, \mathbf{M})$.
- (5) The image of any lc center of $(X, B, \mathbf{M})$ on Z is an lc center of $(Z, B_Z, \mathbf{M}^Z)$.
- (6) If $\mathbf{M}$ is NQC/ U , then $\mathbf{M}^Z$ is NQC/ U .

Proof. By Lemmas 11.3.3 and 11.3.4, possibly replacing f , we may assume that $-B^{\leq 0}$ is $\mathbb{R}$ -Cartier and $\kappa_{\sigma}(X/Z, -B^{\leq 0}) = 0$. By Definition-Theorem 5.1.2, $f : (X, B, \mathbf{M}) \rightarrow Z$ has an equi-dimensional model $f' : (X', \Sigma_{X'}, \mathbf{M}) \rightarrow Z'$ with associated morphisms $h : X' \rightarrow X$ and $h_Z : Z' \rightarrow Z$. Let

$$K_{X'} + \tilde{B}' + \mathbf{M}_{X'} := h^*(K_X + B + \mathbf{M}_X),$$

$\tilde{B}^h$ the horizontal/ Z' part of $\tilde{B}'$, and $B' := (\tilde{B}^h)^{\geq 0}$. Let G' be the vertical/ Z' part of $\Sigma_{X'}$, $\tilde{B}^v$ the vertical/ Z' part of $\tilde{B}'$, $E^h := -(\tilde{B}^h)^{\leq 0}$, and $E^v := G' - \tilde{B}^v$. Then $E^h \geq 0$ and E^v is vertical/ Z' . By Lemma 11.3.3, $\kappa_{\sigma}(X'/Z, E^h) = 0$. We have

$$\begin{aligned} K_{X'} + B' + G' + \mathbf{M}_{X'} &= h^*(K_X + B + \mathbf{M}_X) + B' + G' - \tilde{B}' \\ &\sim_{\mathbb{R}, Z'} (B' - \tilde{B}^h) + G' - \tilde{B}^v = -(\tilde{B}^h)^{\leq 0} + (G' - \tilde{B}^v) = E^h + E^v. \end{aligned}$$

Since $(X, B, \mathbf{M})$ is sub-lc over the generic point of Z , $\Sigma_{X'} \geq B' \geq 0$. Let $\mathcal{F}'$ be the foliation induced by $f' : X' \rightarrow Z'$. By Lemma 6.2.4, $(X', \mathcal{F}', B', \mathbf{M}; G')/Z'$ is ACSS. By Proposition 7.3.6,

$$K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} \sim_{\mathbb{R}, Z'} K_{X'} + B' + G' + \mathbf{M}_{X'} \sim_{\mathbb{R}, Z'} E^h + E^v.$$

Thus

$$\kappa_{\sigma}(X'/Z', K_{\mathcal{F}'} + B' + \mathbf{M}_{X'}) = \kappa_{\sigma}(X'/Z', E^h) = 0.$$

By Proposition 11.2.1, we may run a $(K_{\mathcal{F}'} + B' + \mathbf{M}_{X'})$ -MMP/ Z' which terminates with a good minimal model $(X'', \mathcal{F}'', B'', \mathbf{M})/Z'$ of $(X', \mathcal{F}', B', \mathbf{M})/Z'$. Let G'' be the image of G' on X'' . By Lemma 9.1.4, $(X'', \mathcal{F}'', B'', \mathbf{M}; G'')/Z'$ is ACSS.

Since $X' \rightarrow U$ factors through Z' , $X' \dashrightarrow X''$ is a sequences of steps of a $(K_{\mathcal{F}'} + B' + \mathbf{M}_{X'})$ -MMP/ U . By Lemma 9.2.2, $K_{\mathcal{F}''} + B'' + \mathbf{M}_{X''}$ is nef/ U . By Theorem 11.1.5, $(X'', B'' + G'', \mathbf{M})$

is BP stable/ Z' . Let $f'' : X'' \rightarrow Z'$ be the induced contraction and let $\mathbf{N}$ be the moduli part of $f'' : (X'', B'' + G'', \mathbf{M}) \rightarrow Z'$. By Proposition 11.1.3, $\mathbf{N}$ is nef/ U and $\mathbf{N}$ descends to X . By Proposition 7.3.6,

$$K_{X''} + B'' + G'' + \mathbf{M}_{X''} \sim_{\mathbb{R}, Z'} 0.$$

Let $\mathbf{M}'$ be the base moduli part of $f'' : (X'', B'' + G'', \mathbf{M}) \rightarrow Z'$, then by the definition of base moduli part, $\mathbf{M}'$ descends to Z' and $f''^* \mathbf{M}'_{X'} = \mathbf{N}_{X''}$ is nef, so $\mathbf{M}'_{X'}$ is nef, hence $\mathbf{M}'$ is nef.

Let $\tilde{B}''^h$ be the image of $\tilde{B}''^h$ on X'' . Since $K_{X'} + \tilde{B}' + \mathbf{M}_{X'} \sim_{\mathbb{R}, Z'} 0$, $K_{X'} + \tilde{B}''^h + \mathbf{M}_{X'} \sim_{\mathbb{R}} 0$ over the generic point of Z' . Thus $K_{X''} + \tilde{B}''^h + \mathbf{M}_{X''} \sim_{\mathbb{R}} 0$ over the generic point of Z' . Since $B'' \geq \tilde{B}''^h$ and $K_{X''} + B'' + \mathbf{M}_{X''} \sim_{\mathbb{R}, Z'} 0$, $B'' = \tilde{B}''^h$ over the generic point of Z' . Since $(X', \tilde{B}''^h, \mathbf{M})$ and $(X'', \tilde{B}''^h, \mathbf{M})$ are crepant over the generic point of Z' , $(X', \tilde{B}', \mathbf{M})$ and $(X'', B'' + G'', \mathbf{M})$ are crepant over the generic point of Z' . Thus $(X, B, \mathbf{M})$ and $(X'', B'' + G'', \mathbf{M})$ are crepant over the generic point of Z . By Lemma 11.4.3, $\mathbf{M}^Z = \mathbf{M}'$. The main part of the theorem follows. (1) immediately follows.

(2-4) immediately from the definition of the discriminant part. (5) follows from the definition of the discriminant part and Lemma 7.3.4. By [JLX22, Theorem 2.23], if $\mathbf{M}$ is NQC/ U , then $\mathbf{M}'$ is NQC/ U , hence $\mathbf{M}^Z$ is NQC/ U . (6) follows. $\square$

11.5. Canonical bundle formula for generalized foliated quadruples.

Definition-Lemma 11.5.1. Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a sub-gfq and $f : X \rightarrow Z$ a contraction/ U , such that the general fibers of f are tangent to $\mathcal{F}$ and $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$ is an lc-trivial fibration. We define two $\mathbf{b}$ -divisors $\mathbf{B}$ and $\mathbf{M}^Z$ on Z in the following way.

By Lemma 6.1.8, there exists a foliation $\mathcal{F}_Z$ on Z such that $\mathcal{F} = f^{-1} \mathcal{F}_Z$. Let $f' : (X', \Sigma_{X'}, \mathbf{M}) \rightarrow (Z', \Sigma_{Z'})$ be any equi-dimensional model of $f : (X, B, \mathbf{M}) \rightarrow Z$ with associated morphisms $h : X' \rightarrow X$ and $h_Z : Z' \rightarrow Z$. Let $\mathcal{F}_{Z'} := h_Z^{-1} \mathcal{F}_Z$ and $\mathcal{F}' := h^* \mathcal{F}$, then $\mathcal{F}' = f'^{-1} \mathcal{F}_{Z'}$. We define

$$R' := \sum (f'^* D - f^{-1}(D)),$$

where D runs over all $\mathcal{F}_{Z'}$ -invariant prime divisors on Z . By [Dru17, 2.9], we have

$$K_{\mathcal{F}'/\mathcal{F}_{Z'}} = K_{X'/Z'} - R'.$$

Let $K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} := h^*(K_{\mathcal{F}} + B + \mathbf{M}_X)$. Then $K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} \sim_{\mathbb{R}, Z'} 0$, so

$$K_{X'} + B' - R' + \mathbf{M}_{X'} \sim_{\mathbb{R}, Z'} 0.$$

Since $R' = 0$ and $K_{X'} = K_{\mathcal{F}'}$ over the generic point of Z' , $f' : (X', B' - R', \mathbf{M}) \rightarrow Z'$ is an lc-trivial fibration. By Theorem 11.4.4, there exist two $\mathbf{b}$ -divisors $\mathbf{B}$ and $\mathbf{M}^Z$ on Z , such that $\mathbf{B}$ is uniquely determined and $\mathbf{M}^Z$ is uniquely determined up to $\mathbb{R}$ -linear equivalence, and the following conditions are satisfied:

- (i) $K_{X'} + B' - R' + \mathbf{M}_{X'} \sim_{\mathbb{R}} f'^*(K_{Z'} + \mathbf{B}_{Z'} + \mathbf{M}_{Z'}^Z)$.
- (ii) $\mathbf{M}^Z$ is nef/ U .
- (iii) For any birational morphism $g_Z : Z'' \rightarrow Z'$ and $g : X'' \rightarrow X'$ such that the induced map $f'' : X'' \dashrightarrow Z''$ is a morphism, we let

$$K_{X''} + \tilde{B}'' + \mathbf{M}_{X''} := g^*(K_{X'} + B' - R' + \mathbf{M}_{X'}),$$

then $\mathbf{B}_{Z''}$ is the discriminant part of $f'' : (X'', \tilde{B}'', \mathbf{M}) \rightarrow Z''$.

We call $\mathbf{B}$ as the *discriminant $\mathbf{b}$ -divisor* of $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$ and call $\mathbf{M}^Z$ as the *base moduli part* of $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$. We also call $\mathbf{B}_Z$ the *discriminant part* of $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$. Then:

- (1) $\mathbf{B}$ and $\mathbf{M}^Z$ are well-defined, i.e. $\mathbf{B}$ and the $\mathbb{R}$ -linear equivalence class of $\mathbf{M}^Z$ are independent of the choices of the equi-dimensional model of $f : (X, B, \mathbf{M}) \rightarrow Z$.
- (2) $(Z, \mathcal{F}_Z, B_Z := \mathbf{B}_Z, \mathbf{M}^Z)/U$ is a sub-gfq. We say that $(Z, \mathcal{F}_Z, B_Z, \mathbf{M}^Z)/U$ is a sub-gfq induced by a canonical bundle formula/ U of $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$.

(3) If $\mathbf{M}$ is NQC/U , then $\mathbf{M}^Z$ is NQC/U .

Proof. By [LLM23, Definition-Lemma 6.11], $\mathbf{B}$ is independent of the choices of the equi-dimensional model of $f : (X, B, \mathbf{M}) \rightarrow Z$.

Since $K_{\mathcal{F}} + B + \mathbf{M}_X \sim_{\mathbb{R}, Z} 0$, there exists an $\mathbb{R}$ -divisor L on Z which is uniquely determined up to $\mathbb{R}$ -linear equivalence, such that

$$K_{\mathcal{F}} + B + \mathbf{M}_X \sim_{\mathbb{R}} f^* L.$$

By condition (i), we have

$$K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} \sim_{\mathbb{R}} f'^* (K_{\mathcal{F}_{Z'}} + \mathbf{B}_{Z'} + \mathbf{M}_{Z'}^Z).$$

Therefore, for any birational morphism $g_Z : Z'' \rightarrow Z'$ with $\mathcal{F}_{Z''} := g_Z^{-1} \mathcal{F}_{Z'}$, we have

$$\mathbf{M}_{Z''}^Z \sim_{\mathbb{R}} (h_Z \circ g_Z)^* L - K_{\mathcal{F}_{Z''}} - \mathbf{B}_{Z''}.$$

Thus $\mathbf{M}_{Z''}^Z$ is uniquely determined up to the choices of L in its $\mathbb{R}$ -linear equivalence class, so $\mathbf{M}^Z$ is uniquely determined up to $\mathbb{R}$ -linear equivalence. This implies (1).

We have

$$L = (h_Z)_* h_Z^* L \sim_{\mathbb{R}} (h_Z)_* (K_{\mathcal{F}_{Z'}} + \mathbf{B}_{Z'} + \mathbf{M}_{Z'}^Z) = K_{\mathcal{F}_Z} + B_Z + \mathbf{M}_Z^Z,$$

so $K_{\mathcal{F}_Z} + B_Z + \mathbf{M}_Z^Z$ is $\mathbb{R}$ -Cartier. By condition (ii), $(Z, \mathcal{F}_Z, B_Z := \mathbf{B}_Z, \mathbf{M}^Z)/U$ is a sub-gfq. This implies (2).

(3) follows from Theorem 11.4.4(6). $\square$

Lemma 11.5.2. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a sub-gfq and $f : X \rightarrow Z$ a contraction/ U such that $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$ is an lc-trivial fibration. Let B_Z be the discriminant part of $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$ and $\mathcal{F}_Z$ a foliation on Z such that $\mathcal{F} = f^{-1} \mathcal{F}_Z$. Let n be a positive integer such that $n(K_{\mathcal{F}} + B + \mathbf{M}_X) \sim 0$ over the generic point of Z . Then there is a choice $\mathbf{M}^Z$ of the base moduli part of $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$, such that*

$$n(K_{\mathcal{F}} + B + \mathbf{M}_X) \sim n f^* (K_{\mathcal{F}_Z} + B_Z + \mathbf{M}_Z^Z).$$

Proof. Let $f' : (X', \Sigma_{X'}, \mathbf{M}) \rightarrow (Z', \Sigma_{Z'})$ be a sufficiently high equi-dimensional model of $f : (X, B, \mathbf{M}) \rightarrow Z$ with associated morphisms $h : X' \rightarrow X$ and $h_Z : Z' \rightarrow Z$. Let $\mathcal{F}_{Z'} := h_Z^{-1} \mathcal{F}_Z$ and let

$$R' := \sum_{D|D \text{ is an } \mathcal{F}_{Z'}\text{-invariant prime divisor}} (f'^* D - f'^{-1}(D)).$$

Then $f' : (X', B' - R', \mathbf{M}) \rightarrow Z'$ is an lc-trivial fibration. Since R' is vertical/ Z' , $n(K_{X'} + B' - R' + \mathbf{M}_{X'}) \sim 0$ over the generic point of Z . The lemma follows from Lemma 11.4.2. $\square$

Lemma 11.5.3. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ and $(X', \mathcal{F}', B', \mathbf{M})/U$ be two sub-gfqs. Let $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$ and $f' : (X', \mathcal{F}', B', \mathbf{M}) \rightarrow Z'$ be two lc-trivial fibrations/ U such that f and f' are birationally equivalent, and $(X, \mathcal{F}, B, \mathbf{M})$ and $(X', \mathcal{F}', B', \mathbf{M})$ are crepant over the generic point of Z . Let $\mathbf{M}^Z$ be the base moduli part of $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$ and let $\mathbf{M}^{Z'}$ be the base moduli part of $f' : (X', \mathcal{F}', B', \mathbf{M}) \rightarrow Z'$. Then $\mathbf{M}^Z \sim_{\mathbb{R}} \mathbf{M}^{Z'}$.*

Proof. Possibly passing to a common base and resolve indeterminacy of the induced birational map $X \dashrightarrow X'$, we may assume that $f = f'$, $X = X'$, $Z = Z'$, and $\mathcal{F} = \mathcal{F}'$ over the generic point of Z , $f : (X, \Sigma) \rightarrow (Z, \Sigma_Z)$ is equi-dimensional toroidal for some $\Sigma \supset \text{Supp } B \cup \text{Supp } B'$, and (Z, Σ_Z) is log smooth. Let $\mathcal{F}_Z$ and $\mathcal{F}'_Z$ be two foliations on Z such that $\mathcal{F} = f^{-1} \mathcal{F}_Z$ and $\mathcal{F}' = f'^{-1} \mathcal{F}'_Z$,

$$R := \sum_{D|D \text{ is an } \mathcal{F}_Z\text{-invariant prime divisor}} (f^* D - f^{-1}(D)),$$

and

$$R' := \sum_{D|D \text{ is an } \mathcal{F}'_Z\text{-invariant prime divisor}} (f^*D - f^{-1}(D)).$$

Then $\mathbf{M}^Z$ and $\mathbf{M}^{Z'}$ are the moduli parts of $f : (X, B - R, \mathbf{M}) \rightarrow Z$ and $f' : (X, B' - R', \mathbf{M}) \rightarrow Z$ respectively. Since $(X, B - R, \mathbf{M})$ and $(X', B' - R', \mathbf{M})$ are crepant over the generic point of Z , by Lemma 11.4.3, $\mathbf{M}^Z \sim_{\mathbb{R}} \mathbf{M}^{Z'}$. $\square$

Lemma 11.5.4. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a sub-gfq and $f : X \rightarrow Z$ a contraction/ U such that $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$ is an lc-trivial fibration with discriminant $\mathbf{b}$ -divisor $\mathbf{B}$. Let $\mathcal{F}_Z$ be a foliation on Z such that $\mathcal{F} = f^{-1}\mathcal{F}_Z$. Then for any prime divisor D on Z ,*

$$\text{mult}_D \mathbf{B}_Z = \epsilon_{\mathcal{F}_Z}(D) - \sup\{t \geq 0 \mid (X, \mathcal{F}, B + tf^*D, \mathbf{M}) \text{ is sub-lc over the generic point of } D\}.$$

Moreover, there exists an lc center of

$$(X, \mathcal{F}, B + (\epsilon_{\mathcal{F}_Z}(D) - \text{mult}_D \mathbf{B}_Z)f^*D, \mathbf{M})$$

over the generic point of D .

Proof. Let $B_Z := \mathbf{B}_Z$. By Definition-Lemma 11.5.1, possibly replacing $f : X \rightarrow Z$ with an equi-dimensional model of $f : (X, B, \mathbf{M}) \rightarrow Z$, we may assume that X is $\mathbb{Q}$ -factorial klt with at most toric quotient singularities, f is equi-dimensional, $\mathbf{M}$ descends to X , and there exists a toroidal morphism $f : (X, \Sigma_X, \mathbf{M}) \rightarrow (Z, \Sigma_Z)$ such that $\text{Supp } B \subset \Sigma_X$. We define

$$R := \sum_{D|D \text{ is an } \mathcal{F}_Z\text{-invariant prime divisor}} (f^*D - f^{-1}(D)).$$

For any prime divisor D on Z , we define

$$b_D := 1 - \sup\{t \geq 0 \mid (X, B - R + tf^*D, \mathbf{M}) \text{ is lc over the generic point of } D\}$$

and

$$t_D := \epsilon_{\mathcal{F}_Z}(D) - \sup\{t \geq 0 \mid (X, \mathcal{F}, B + tf^*D, \mathbf{M}) \text{ is lc over the generic point of } D\}.$$

By definition, $\text{mult}_D B_Z = b_D$ for any prime divisor D on Z . There are three cases.

Case 1. D is not $\mathcal{F}_Z$ -invariant. In this case, $R = 0$ and $K_{\mathcal{F}} = K_X$ over the generic point of D , so

$$\begin{aligned} & \sup\{t \mid (X, B - R + tf^*D, \mathbf{M}) \text{ is sub-lc over the generic point of } D\} \\ &= \sup\{t \mid (X, \mathcal{F}, B + tf^*D, \mathbf{M}) \text{ is sub-lc over the generic point of } D\}. \end{aligned}$$

Thus $b_D = t_D$. Moreover, any lc center of $(X, B - R + (1 - b_D)f^*D, \mathbf{M})$ over the generic point of D is an lc center of $(X, \mathcal{F}, B + (1 - b_D)f^*D, \mathbf{M})$ over the generic point of D . Since $(X, B - R + (1 - b_D)f^*D, \mathbf{M})$ is a g-sub-pair over the generic point of D , by Lemma 7.3.4, there exists an lc center of $(X, B - R + (1 - b_D)f^*D, \mathbf{M})$ over the generic point of D . Thus there exists an lc center of $(X, \mathcal{F}, B + (1 - b_D)f^*D, \mathbf{M})$ over the generic point of D .

Case 2. D is $\mathcal{F}_Z$ -invariant and $D \not\subset \Sigma_Z$. Let B^h be the horizontal/ Z part of B , then $B = B^h$ over the generic point of D . Since $(X, \mathcal{F}, B, \mathbf{M})$ is sub-lc over the generic point of Z , $\Sigma_X \geq B^h$. By [LLM23, Lemma 6.6], $(X, B^h + f^{-1}(D), \mathbf{M})$ is sub-lc over the generic point of D . Since

$$(X, B - R + f^*D, \mathbf{M}) = (X, B^h + f^{-1}(D), \mathbf{M})$$

over the generic point of D , $b_D = 0$. Thus $\text{mult}_D B_Z = 0$. Since D is $\mathcal{F}_Z$ -invariant, any component of $f^{-1}(D)$ is $\mathcal{F}$ -invariant. Since $B = B^h$ over the generic point of D , any component of $f^{-1}(D)$ is an lc center of $(X, \mathcal{F}, B, \mathbf{M})$. In particular, $b_D = 0 = t_D$.

Case 3. D is $\mathcal{F}_Z$ -invariant and $D \subset \Sigma_Z$. Then

$$-b_D = \sup\{t \mid (X, B + f^{-1}(D) + tf^*D, \mathbf{M}) \text{ is sub-lc over the generic point of } D\}.$$

Since $f : (X, \Sigma_X, \mathbf{M}) \rightarrow (Z, \Sigma_Z)$ is toroidal, there exists a component S of f^*D , such that

- $\text{mult}_S(B + f^{-1}(D) - b_D f^*D) = 1$, and
- $0 \geq \lfloor B + f^{-1}(D) - t f^*D \rfloor$ over the generic point of D for any $t < -b_D$.

Therefore, $\text{mult}_S(B - b_D f^*D) = 0$, and $0 \geq B - b_D f^*D$ over the generic point of D . Thus

$$-b_D \geq \sup\{t \geq 0 \mid (X, \mathcal{F}, B + t f^*D, \mathbf{M}) \text{ is sub-lc over the generic point of } D\} = -t_D.$$

Suppose that $-b_D > -t_D$. Let $s \in (-t_D, -b_D)$ be a real number, then

$$(X, B + f^{-1}(D) + s f^*D, \mathbf{M})$$

is sub-lc over the generic point of D , and $(X, \mathcal{F}, B + s f^*D, \mathbf{M})$ is not sub-lc over the generic point of D . Then there exists a prime divisor D_X over X , such that the image of D_X on Z is D , and $a(D_X, \mathcal{F}, B + s f^*D, \mathbf{M}) < -\epsilon_{\mathcal{F}}(D_X)$. By Definition-Theorem 5.1.2, there exists an equi-dimensional model $f' : (X', \Sigma_{X'}, \mathbf{M}) \rightarrow (Z', \Sigma_{Z'})$ of $f : (X, \text{Supp } B + \text{Supp } f^*D, \mathbf{M}) \rightarrow Z$ associated with $h : X' \rightarrow X$ and $h_Z : Z' \rightarrow Z$, such that D_X is on X' . Let $\mathcal{F}' := h^{-1}\mathcal{F}$, $\mathcal{F}_{Z'} := h_Z^{-1}\mathcal{F}_Z$, $K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} := h^*(K_{\mathcal{F}} + B + \mathbf{M}_X)$, $D' := (h_Z^{-1})_*D$, and

$$R' := \sum_{L|L \text{ is an } \mathcal{F}_{Z'}\text{-invariant prime divisor}} (f'^*L - f'^{-1}(L)).$$

Then D_X is a component of $f'^{-1}(D')$. Since D' is $\mathcal{F}_{Z'}$ -invariant and $\mathcal{F}' = f'^{-1}\mathcal{F}_{Z'}$, D_X is $\mathcal{F}'$ -invariant. Since $a(D_X, \mathcal{F}, B + s f^*D, \mathbf{M}) < -\epsilon_{\mathcal{F}}(D_X)$, $\text{mult}_{D_X}(B' + s f'^*D') > 0$. By Definition-Lemma 11.5.1(1),

$$\begin{aligned} -b_D &= \sup\{t \geq 0 \mid (X', B' - R' + t f'^*D', \mathbf{M}) \text{ is lc over the generic point of } D'\} - 1 \\ &= \sup\{t \geq 0 \mid (X', B' + f'^{-1}(D) + t f'^*D', \mathbf{M}) \text{ is lc over the generic point of } D'\} \\ &< s < -b_D, \end{aligned}$$

a contradiction. Thus $b_D = t_D$. Since $\text{mult}_S(B - t_D f^*D) = 0$, S is an lc center of $(X, B - b_D f^*D, \mathbf{M})$ over the generic point of D . The lemma follows in this case. $\square$

Proposition 11.5.5. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a sub-gfq and $f : X \rightarrow Z$ a contraction/ U such that $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$ is an lc-trivial fibration. Let $\mathbf{B}$ be the discriminant $\mathbf{b}$ -divisor of $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$, $B_Z := \mathbf{B}_Z$, and $\mathbf{M}^Z$ the base moduli part of $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$. Let $\mathcal{F}_Z$ be a foliation on Z such that $\mathcal{F} = f^{-1}\mathcal{F}_Z$. Then:*

- (1) *If the vertical/ Z part of B is ≥ 0 , then $B_Z \geq 0$.*
- (2) *If $(X, \mathcal{F}, B, \mathbf{M})$ is sub-lc (resp. lc), then $(Z, \mathcal{F}_Z, B_Z, \mathbf{M}^Z)$ is sub-lc (resp. lc).*
- (3) *Any lc center of $(Z, \mathcal{F}_Z, B_Z, \mathbf{M}^Z)$ is the image of an lc center of $(X, \mathcal{F}, B, \mathbf{M})$.*
- (4) *The image of any lc center of $(X, \mathcal{F}, B, \mathbf{M})$ on Z is an lc center of $(Z, \mathcal{F}_Z, B_Z, \mathbf{M}^Z)$.*

Proof. The proposition immediately follows from Lemma 11.5.4. $\square$

Finally, we state the following proposition that can be useful for inductive purposes.

Proposition 11.5.6. *Let $(X, \mathcal{F}, B, \mathbf{M})$ be a sub-gfq and $X \xrightarrow{f} Y \xrightarrow{g} Z$ two contractions/ U . Let $h := g \circ f$. Suppose that $h : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$ is an lc-trivial fibration. Let $(Z, \mathcal{F}_Z, B_Z, \mathbf{M}^Z)$ be the sub-gfq induced by $h : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$. Then:*

- (1) *$f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Y$ is an lc-trivial fibration.*
- (2) *Let $(Y, \mathcal{F}_Y, B_Y, \mathbf{M}^Y)$ be a sub-gfq induced by $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Y$. Then:*
 - (a) *$g : (Y, \mathcal{F}_Y, B_Y, \mathbf{M}^Y) \rightarrow Z$ is an lc-trivial fibration.*
 - (b) *The discriminant part of $g : (Y, \mathcal{F}_Y, B_Y, \mathbf{M}^Y) \rightarrow Z$ is B_Z .*
 - (c) *$(Z, \mathcal{F}_Z, B_Z, \mathbf{M}^Z)$ is a sub-gfq induced by $g : (Y, \mathcal{F}_Y, B_Y, \mathbf{M}^Y) \rightarrow Z$.*

Proof. Possibly replacing X and Y with high models, we may assume that X and Y are smooth, and $\kappa_\sigma(X/Z, -B^{\leq 0}) = 0$.

(1) Since $(X, \mathcal{F}, B, \mathbf{M})$ is sub-lc over the generic point of Z , $(X, \mathcal{F}, B, \mathbf{M})$ is sub-lc over the generic point of Y . Since $K_{\mathcal{F}} + B + \mathbf{M}_X \sim_{\mathbb{R}, Z} 0$, $K_{\mathcal{F}} + B + \mathbf{M}_X \sim_{\mathbb{R}, Y} 0$. Since $\kappa_\sigma(X/Z, -B^{\leq 0}) = 0$, $\kappa_\sigma(X/Y, -B^{\leq 0}) = 0$. This implies (1).

(2.a) Since $(X, \mathcal{F}, B, \mathbf{M})$ is sub-lc over the generic point of Z , by Theorem 11.4.4, $(Y, \mathcal{F}_Y, B_Y, \mathbf{M}^Y)$ is sub-lc over the generic point of Z . Since

$$f^*(K_{\mathcal{F}_Y} + B_Y + \mathbf{M}_Y) \sim_{\mathbb{R}} K_{\mathcal{F}} + B + \mathbf{M}_X \sim_{\mathbb{R}, Z} 0,$$

$K_{\mathcal{F}_Y} + B_Y + \mathbf{M}_Y \sim_{\mathbb{R}, Z} 0$. By Lemma 11.5.4, for any component D of $B_Y^{\leq 0}$ and any irreducible component D_X of $f^{-1}(D)$ over the generic point of D , D_X is a component of $B^{\leq 0}$. Therefore, over the generic point of Z , there exists a positive real number δ such that

$$-B^{\leq 0} \geq \epsilon f^*(-B_Y^{\leq 0}).$$

Thus

$$0 \leq \kappa_\sigma(X/Z, f^*(-B_Y^{\leq 0})) = \kappa_\sigma(X/Z, \epsilon f^*(-B_Y^{\leq 0})) \leq \kappa_\sigma(X/Z, -B^{\leq 0}) = 0,$$

so

$$\kappa_\sigma(Y/Z, -B_Y^{\leq 0}) = \kappa_\sigma(X/Z, f^*(-B_Y^{\leq 0})) = 0.$$

Therefore, $g : (Y, \mathcal{F}_Y, B_Y, \mathbf{M}^Y) \rightarrow Z$ is an lc-trivial fibration.

(2.b) Let B'_Z be the discriminant part of $g : (Y, \mathcal{F}_Y, B_Y, \mathbf{M}^Y) \rightarrow Z$. For any prime divisor D over Z , let $s_D := \epsilon_{\mathcal{F}_Z}(D) - \text{mult}_D B_Z$ and $s'_D := \epsilon_{\mathcal{F}_Z}(D) - \text{mult}_D B'_Z$.

By Lemma 11.5.4, for any positive real number t and any prime divisor D on Z ,

$$(Y, \mathcal{F}_Y, B_Y + tg^*D, \mathbf{M})$$

is the sub-gfq induced by $f : (X, \mathcal{F}, B + th^*D, \mathbf{M}) \rightarrow Y$ over the generic point of D . By Proposition 11.5.5(3)(4),

$$\begin{aligned} s'_D &= \sup\{t \geq 0 \mid (Y, \mathcal{F}_Y, B_Y + tg^*D, \mathbf{M}^Y) \text{ is sub-lc over the generic point of } D\} \\ &= \sup\{t \geq 0 \mid (X, \mathcal{F}, B + th^*D, \mathbf{M}) \text{ is sub-lc over the generic point of } D\} = s_D. \end{aligned}$$

Thus $B_Z = B'_Z$.

(2.c) By applying (2.b) to all high models of Z , we get (2.c). $\square$

12. CANONICAL BUNDLE FORMULA FOR LC-TRIVIAL MORPHISMS AND SUBADJUNCTION FORMULA

12.1. Canonical bundle formula for lc-trivial morphisms.

Definition-Lemma 12.1.1 ([Dru21, Proposition 3.4]; cf. [Spi20, Proposition 3.7]). Let $f : X' \rightarrow X$ be a surjective finite morphism between normal varieties and $\mathcal{F}$ a foliation on X . Assume that $K_{\mathcal{F}}$ is $\mathbb{Q}$ -Cartier and $\mathcal{F}' := f^{-1}\mathcal{F}$. For any prime divisor on X , we let r_D be the ramification index of f along D . We call

$$R := \sum_{D \mid D \text{ is a non-}\mathcal{F}\text{-invariant prime divisor}} (r_D - 1)D$$

the *ramification divisor* of f with respect to $\mathcal{F}$. Then we have

$$K_{\mathcal{F}'} = f^*K_{\mathcal{F}} + R.$$

Definition-Lemma 12.1.2. Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a sub-gfq and $f : X \rightarrow Z$ a finite morphism/ U . Suppose that there exists a foliation $\mathcal{F}_Z$ on Z such that $\mathcal{F} = f^{-1}\mathcal{F}_Z$, and suppose that $K_{\mathcal{F}} + B + \mathbf{M}_X \sim_{\mathbb{R}, Z} 0$.

We define two $\mathbf{b}$ -divisors, $\mathbf{B}$ on $\mathbf{M}^Z$ on Z , in the following way. Let $h_Z : Z' \rightarrow Z$ be any birational morphism, X' the main component of $Z' \times_Z X$, $f' : X' \rightarrow Z'$ and $h : X' \rightarrow X$ the induced morphisms, $\mathcal{F}' := h^{-1}\mathcal{F}$, and $\mathcal{F}_{Z'} := h_Z^{-1}\mathcal{F}_Z$. We let

$$K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} := h^*(K_{\mathcal{F}} + B + \mathbf{M}_X).$$

Let Z'^0 be the largest open subset of Z' which does not contain $\text{Sing}(\mathcal{F}_{Z'}) \cup \text{Sing}(Z')$ and let $X'^0 := f'^{-1}(Z'^0)$. By Definition-Lemma 12.1.1,

$$K_{\mathcal{F}'|_{X'^0}} = (f'|_{X'^0})^* K_{\mathcal{F}_{Z'}|_{Z'^0}} + R'^0$$

where R'^0 is the ramification divisor of $f'|_{X'^0}$ with respect to $\mathcal{F}_{Z'}|_{Z'^0}$. We let R' be the closure of R'^0 in X'^0 . We let $\mathbf{B}$ and $\mathbf{M}^Z$ be the $\mathbf{b}$ -divisors such that $\mathbf{B}_{Z'} = \frac{1}{\deg f} f'_*(R' + B')$ and $\mathbf{M}_{Z'}^Z = \frac{1}{\deg f} f'_* \mathbf{M}_{X'}$ for any choices of Z' . Then:

- (1) $\mathbf{B}$ and $\mathbf{M}$ are well-defined and uniquely determined.
- (2) For any choice of Z' ,

$$K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} \sim_{\mathbb{R}} f'^*(K_{\mathcal{F}_{Z'}} + B_{Z'} + \mathbf{M}_{Z'}^Z).$$

- (3) $\mathbf{M}^Z$ is nef/ U .
- (4) If $B \geq 0$, then $\mathbf{B}_Z \geq 0$.
- (5) If $(X, \mathcal{F}, B, \mathbf{M})$ is (sub-)lc, then $(Z, \mathcal{F}_Z, \mathbf{B}_Z, \mathbf{M}^Z)$ is (sub-)lc, and for any lc center T of $(Z, \mathcal{F}_Z, B_Z, \mathbf{M}^Z)$, any component of $f^{-1}(T)$ is an lc center of $(X, \mathcal{F}, B, \mathbf{M})$.
- (6) If $\mathbf{M}$ is NQC/ U , then $\mathbf{M}^Z$ is NQC/ U .

We call $\mathbf{B}$ the *discriminant $\mathbf{b}$ -divisor* of $f : (X, B, \mathbf{M}) \rightarrow Z$, and call $B_Z := \mathbf{B}_Z$ the *discriminant part* of $f : (X, B, \mathbf{M}) \rightarrow Z$. We call $\mathbf{M}^Z$ the *base moduli part* of $f : (X, B, \mathbf{M}) \rightarrow Z$. We say that $(Z, \mathcal{F}_Z, B_Z, \mathbf{M}^Z)/U$ is the sub-gfq induced by $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$.

Proof. (1) We only need to show that for any birational morphism $g_Z : Z'' \rightarrow Z'$, $(g_Z)_* \mathbf{B}_{Z''} = \mathbf{B}_{Z'}$ and $(g_Z)_* \mathbf{M}_{Z''}^Z = \mathbf{M}_{Z'}^Z$. We let X'' be the main component of $X' \times_{Z'} Z''$ and $g : X'' \rightarrow X'$, $f'' : X'' \rightarrow Z''$ the induced morphisms. Let $\mathcal{F}'' := g^{-1}\mathcal{F}'$, $\mathcal{F}_{Z''} := g_Z^{-1}\mathcal{F}_{Z'}$, Z''^0 be the largest open subset of Z'' which does not contain $\text{Sing}(\mathcal{F}_{Z''}) \cup \text{Sing}(Z'')$, $X''^0 := f''^{-1}(Z''^0)$, R''^0 the ramification divisor of $f''|_{X''^0}$ with respect to $\mathcal{F}_{Z''}|_{Z''^0}$, and R'' the closure of R''^0 in X'' . Then

$$\mathbf{B}_{Z'} = \frac{1}{\deg f} f'_*(B' + R') = \frac{1}{\deg f} f'_* g_*(B'' + R'') = \frac{1}{\deg f} (g_Z)_* f''_*(B'' + R'') = (g_Z)_* \mathbf{B}_{Z''}$$

and

$$\mathbf{M}_{Z'}^Z = \frac{1}{\deg f} f'_* \mathbf{M}_{X'} = \frac{1}{\deg f} f'_* g_* \mathbf{M}_{X''} = \frac{1}{\deg f} (g_Z)_* f''_* \mathbf{M}_{X''} = (g_Z)_* \mathbf{M}_{Z''}^Z.$$

(2) By (1), we only need to prove (2) for any sufficiently high model Z' of Z . In particular, we may assume that Z' is $\mathbb{Q}$ -factorial. Then $f'^*(\frac{1}{\deg f} f'_* R') = R'$, $f'^*(\frac{1}{\deg f} f'_* B') = B'$, and $f'^*(\frac{1}{\deg f} f'_* \mathbf{M}_{X'}) = \mathbf{M}_{X'}$, so (2) immediately follows.

(3)(6) By [HL21b, Lemma 4.2], there exists a birational morphism $h_Z : Z'' \rightarrow Z$ satisfying the following. Let X'' be the main component of $Z'' \times_Z X$, then $\mathbf{M}$ descends to X'' . By definition, $\mathbf{M}^Z$ descends to Z'' . Since $\mathbf{M}_{X''}$ is nef, $\mathbf{M}_{Z''}^Z$ is nef. Thus $\mathbf{M}^Z$ is nef. This implies (3). Moreover, if $\mathbf{M}$ is NQC/ U , then $\mathbf{M}_{X''}$ is NQC/ U , so $\mathbf{M}_{Z''}^Z$ is NQC/ U , hence $\mathbf{M}^Z$ is NQC/ U . This implies (6).

(4) It is obvious from the definition of $\mathbf{B}$.

(5) By (4), we only need to prove the sub-lc case. Suppose that $(X, \mathcal{F}, B, \mathbf{M})$ is sub-lc, then $(X', \mathcal{F}', B', \mathbf{M})$ is sub-lc. Let D be a prime divisor on Z' . Let $E_1, \dots, E_m$ be all components of $f'^{-1}(D)$ and let r_i be the ramification index of r_i along E_i .

If D is $\mathcal{F}_{Z'}$ -invariant, then each E_i is $\mathcal{F}$ -invariant $E_i \not\subset \text{Supp } R'$. Since $(X', \mathcal{F}', B', \mathbf{M})$ is sub-lc, $\text{mult}_{E_i} B' \leq 0$ for any i . Thus

$$\text{mult}_D \mathbf{B}_{Z'} = \text{mult}_D \frac{1}{\deg f} f'_*(B' + R') = \sum_{i=1}^m \frac{1}{\deg f} (\text{mult}_{E_i} B') \leq 0 = \epsilon_{\mathcal{F}_{Z'}}(D).$$

Moreover, if D is an lc place of $(Z, \mathcal{F}_Z, B_Z, \mathbf{M}^Z)$, then $\text{mult}_D \mathbf{B}_{Z'} = 0$, so $\text{mult}_{E_i} B' = 0$ for each i . Therefore, each E_i is an lc place of $(X', \mathcal{F}', B', \mathbf{M})$, hence an lc place of $(X, \mathcal{F}, B, \mathbf{M})$.

If D is $\mathcal{F}_Z$ -invariant, then each E_i is not $\mathcal{F}$ -invariant, and $\sum_{i=1}^m r_i \leq \deg f$. Since $(X', \mathcal{F}', B', \mathbf{M})$ is sub-lc, $\text{mult}_{E_i} B' \leq 1$ for any i . Thus

$$\text{mult}_D \mathbf{B}_{Z'} = \text{mult}_D \frac{1}{\deg f} f'_*(B' + R') = \sum_{i=1}^m \frac{1}{\deg f} (r_i - 1 + \text{mult}_{E_i} B') \leq \frac{\sum_{i=1}^m r_i}{\deg f} \leq 1 = \epsilon_{\mathcal{F}_{Z'}}(D).$$

Moreover, if D is an lc place of $(Z, \mathcal{F}_Z, B_Z, \mathbf{M}^Z)$, then $\text{mult}_D \mathbf{B}_{Z'} = 1$, so $\text{mult}_{E_i} B' = 1$ for each i . Therefore, each E_i is an lc place of $(X', \mathcal{F}', B', \mathbf{M})$, hence an lc place of $(X, \mathcal{F}, B, \mathbf{M})$.

Since $h_Z : Z' \rightarrow Z$ can be any birational morphism, we get (5). $\square$

Definition 12.1.3 (lc-trivial morphism). Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a sub-gfq and $f : X \rightarrow Z$ a projective surjective morphism over U . Let $X \xrightarrow{\tau} \tilde{Z} \xrightarrow{\gamma} Z$ be the Stein factorization of f . We say that $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$ is an *lc-trivial morphism*, if

- (1) $K_{\mathcal{F}} + B + \mathbf{M}_X \sim_{\mathbb{R}, Z} 0$,
- (2) $\tau : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow \tilde{Z}$ is an lc-trivial fibration, and
- (3) there exists a foliation $\mathcal{F}_Z$ on Z such that $\mathcal{F} = f^{-1}\mathcal{F}_Z$.

Definition-Theorem 12.1.4 (Canonical bundle formula for lc-trivial morphisms). Let

$$(X, \mathcal{F}, B, \mathbf{M})/U$$

be a sub-gfq and $f : X \rightarrow Z$ an lc-trivial morphism/ U , and let $\mathcal{F}_Z$ be a foliation on Z such that $\mathcal{F} = f^{-1}\mathcal{F}_Z$. Then there is a sub-gfq $(Z, \mathcal{F}_Z, B_Z, \mathbf{M}^Z)/U$, such that B_Z is uniquely determined and $\mathbf{M}^Z$ is determined up to $\mathbb{R}$ -linear equivalence, defined in the following way.

Let $X \xrightarrow{\tau} \tilde{Z} \xrightarrow{\gamma} Z$ be the Stein factorization of f . By Definition-Lemma 11.5.1, there exists a sub-gfq

$$(\tilde{Z}, \mathcal{F}_{\tilde{Z}}, B_{\tilde{Z}}, \tilde{\mathbf{M}}^Z)/U$$

induced by $\tau : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow \tilde{Z}$, such that $B_{\tilde{Z}}$ is uniquely determined, and $\tilde{\mathbf{M}}^Z$ is uniquely determined up to $\mathbb{R}$ -linear equivalence. Moreover, we have $\mathcal{F}_{\tilde{Z}} = \gamma^{-1}\mathcal{F}_Z$ and

$$K_{\mathcal{F}_{\tilde{Z}}} + B_{\tilde{Z}} + \tilde{\mathbf{M}}_{\tilde{Z}}^Z \sim_{\mathbb{R}, Z} 0.$$

By Definition-Lemma 12.1.2, there exists a sub-gfq

$$(Z, \mathcal{F}_Z, B_Z, \mathbf{M}^Z)/U$$

induced by $\gamma : (\tilde{Z}, \mathcal{F}_{\tilde{Z}}, B_{\tilde{Z}}, \tilde{\mathbf{M}}^Z) \rightarrow Z$, such that B_Z is uniquely determined, and $\mathbf{M}^Z$ is uniquely determined up to $\mathbb{R}$ -linear equivalence. We say that B_Z is the *discriminant part* of $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$, $\mathbf{M}^Z$ the *base moduli part* of $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$, and say that $(Z, \mathcal{F}_Z, B_Z, \mathbf{M}^Z)$ is a sub-gfq induced by $f : (X, \mathcal{F}, B, \mathbf{M}) \rightarrow Z$.

Moreover, we have the following:

- (1) If the vertical/ Z part of B is ≥ 0 , then $B_Z \geq 0$.
- (2) If $(X, \mathcal{F}, B, \mathbf{M})$ is (sub-)lc, then $(Z, \mathcal{F}_Z, B_Z, \mathbf{M}^Z)$ is (sub-)lc.
- (3) B_Z is uniquely determined, and $\mathbf{M}^Z$ is uniquely determined up to $\mathbb{R}$ -linear equivalence.
- (4) Suppose that $(X, \mathcal{F}, B, \mathbf{M})$ is sub-lc. Then for any lc center T of $(Z, \mathcal{F}_Z, B_Z, \mathbf{M}^Z)$, T is the image of an lc center of $(X, \mathcal{F}, B, \mathbf{M})$ on Z .

Proof. (1) It follows from Definition-Lemma 12.1.2(4) and Proposition 11.5.5(1).

(2) It follows from Definition-Lemma 12.1.2(5) and Proposition 11.5.5(2).

(3) It follows from Definition-Lemma 11.5.1(1) and Definition-Lemma 12.1.2(1).

(4) It follows from Proposition 11.5.5(3) and Definition-Lemma 12.1.2(5). $\square$

12.2. Subadjunction formula for g-pairs. In this section, we shall introduce and discussion the subadjunction formula for lc g-pairs. Since the canonical bundle formula for lc-trivial fibrations for gfqs requires that the general fibers are tangent to the foliation, the subadjunction formula for foliations is more subtle and we will omit it in this paper.

Definition-Theorem 12.2.1 (Subadjunction formula via log resolutions). Let $(X, B, \mathbf{M})/U$ be a g-sub-pair and V an lc center of $(X, B, \mathbf{M})$ with normalization $\nu : W \rightarrow V$, such that $B \geq 0$ near the generic point of V . Then there exists a naturally defined g-sub-pair $(W, B_W, \mathbf{M}^W)/U$ defined in the following way.

Let S be an lc place of $(X, B, \mathbf{M})$ so that $\text{center}_X S = V$. Let $h : Y \rightarrow X$ be a log resolution of $(X, \text{Supp } B)$ such that $\mathbf{M}$ descends to Y and S is on Y . We let

$$K_Y + B_Y + \mathbf{M}_Y := h^*(K_X + B + \mathbf{M}_X)$$

and let $(S, B_S, \mathbf{M}^S)/U$ be the g-sub-pair induced by the adjunction

$$K_S + B_S + \mathbf{M}_S^S := (K_Y + B_Y + \mathbf{M}_Y)|_S.$$

Then there exists an induced projective surjective morphism $h_S : S \rightarrow W$ such that $\nu \circ f_S = h|_S$. By construction, we have

$$K_S + B_S + \mathbf{M}_S^S \sim_{\mathbb{R}, W} 0.$$

Since $B \geq 0$ near the generic point of V , $B_W \geq 0$ near the generic point of S . Therefore, $h_S : (S, B_S, \mathbf{M}^S) \rightarrow W$ is an lc-trivial morphism. By Definition-Theorem 12.1.4, there exists a g-sub-pair $(W, B_W, \mathbf{M}^W)/U$ induced by $h_S : (S, B_S, \mathbf{M}^S) \rightarrow W$. Moreover, we have the following:

- (1) For any fixed choice and S , B_W is uniquely determined, and $\mathbf{M}^W$ is uniquely determined up to $\mathbb{R}$ -linear equivalence. In particular, B_W and the $\mathbb{R}$ -linear equivalence class of $\mathbf{M}^W$ are independent of the choice of h .
- (2) $K_W + B_W + \mathbf{M}_W \sim_{\mathbb{R}} (K_X + B + \mathbf{M}_X)|_W$.
- (3) If $(X, B, \mathbf{M})$ is sub-lc near V , then $(W, B_W, \mathbf{M}^W)$ is sub-lc.
- (4) Suppose that $(X, B, \mathbf{M})$ is sub-lc near V . Then for any lc center T of $(W, B_W, \mathbf{M}^W)$, $\nu(T)$ is an lc center of $(X, B, \mathbf{M})$.

We say that $(W, B_W, \mathbf{M}^W)/U$ is a g-sub-pair induced by subadjunction

$$K_W + B_W + \mathbf{M}_W^W := (K_X + B + \mathbf{M}_X)|_W$$

and say that $(W, B_W, \mathbf{M}^W)$ is associated with S .

Proof. The construction is clear so we only need to prove (1-4).

(1) We let $h' : Y' \rightarrow X$ be a log resolution of $(X, \text{Supp } B)$ such that $\mathbf{M}$ descends to Y' and S on Y , so that the induced birational map $g : Y' \rightarrow Y$ is a morphism. Let $S' := g_*^{-1}S$,

$$K_{Y'} + B_{Y'} + \mathbf{M}_{Y'} := h'^*(K_X + B + \mathbf{M}_X),$$

and let $(S', B_{S'}, \mathbf{M}^S)/U$ be the g-sub-pair induced by the adjunction

$$K_{S'} + B_{S'} + \mathbf{M}_{S'}^S := (K_{Y'} + B_{Y'} + \mathbf{M}_{Y'})|_{S'}.$$

Then $g|_{S'} : S' \rightarrow S$ is a morphism, and we have

$$\begin{aligned} K_{S'} + B_{S'} + \mathbf{M}_{S'}^S &= (K_{Y'} + B_{Y'} + \mathbf{M}_{Y'})|_{S'} = g^*(K_Y + B_Y + \mathbf{M}_Y)|_{S'} \\ &= g|_{S'}^*((K_Y + B_Y + \mathbf{M}_Y)|_S) = g|_{S'}^*(K_S + B + S + \mathbf{M}_S^S). \end{aligned}$$

By our construction, the g-sub-pair induced by $h_S \circ g|_{S'} : (S', B_{S'}, \mathbf{M}^S) \rightarrow W$ is equal to the g-sub-pair induced by $h_S : (S, B_S, \mathbf{M}^S) \rightarrow W$ modulo $\mathbb{R}$ -linear equivalence of the base moduli part. Since h' can be any high log resolution of $(X, \text{Supp } B)$, (1) follows.

(2) It immediately follows from the definition.

(3) Since $(X, B, \mathbf{M})$ is sub-lc near V , $(W, B_W, \mathbf{M})$ is sub-lc near S . Thus $(S, B_S, \mathbf{M}^S)$ is sub-lc. By Definition-Lemma 12.1.4(1), we get (3).

(4) By Definition-Theorem 12.1.4, T is the image of an lc center T_S of $(S, B_S, \mathbf{M}^S)$ on W . Since $(S, B_S, \mathbf{M}^S)$ is log smooth, T_S is also an lc center of $(Y, B_Y, \mathbf{M})$. Thus $h(T_S)$ is an lc center of $(X, B, \mathbf{M})$. By construction, $\nu(T) = h(T_S)$. $\square$

Proposition 12.2.2 (Subadjunction formula via dlt models). *Let $(X, B, \mathbf{M})/U$ be an g-sub-pair and V an lc center of $(X, B, \mathbf{M})$ with normalization $\nu : W \rightarrow V$, such that $(X, B, \mathbf{M})$ is lc near W . Let S be an lc place of $(X, B, \mathbf{M})$ such that $\text{center}_X S = V$. Let $(W, B_W, \mathbf{M}^W)/U$ be a g-sub-pair induced by subadjunction*

$$K_W + B_W + \mathbf{M}_W^W := (K_X + B + \mathbf{M}_X)|_W$$

and is associated with S .

Suppose that $f : Y \rightarrow X$ is a dlt modification of $(X, B, \mathbf{M})$ near W such that S is on Y . Let

$$K_Y + B_Y + \mathbf{M}_Y := f^*(K_X + B + \mathbf{M}_X),$$

$(S, B_S, \mathbf{M}^S)/U$ the g-sub-pair induced by the adjunction

$$K_S + B_S + \mathbf{M}_S^S := (K_Y + B_Y + \mathbf{M}_Y)|_S,$$

and $f_S : S \rightarrow W$ the induced projective surjective morphism such that $\nu \circ f_S = f|_S$. Then:

(1) $(W, B_W, \mathbf{M}^W)$ is the g-pair induced by $f_S : (S, B_S, \mathbf{M}^S) \rightarrow W$.

(2) $(W, B_W, \mathbf{M}^W)$ is lc.

Proof. Let $g : Y' \rightarrow Y$ be a log resolution of $(Y, \text{Supp } B_Y)$ such that $\mathbf{M}$ descends to Y' ,

$$K_{Y'} + B_{Y'} + \mathbf{M}_{Y'} := g^*(K_Y + B_Y + \mathbf{M}_Y),$$

$S' := g_*^{-1}S$, and let $(S', B_{S'}, \mathbf{M}^S)/U$ be the g-sub-pair induced by the adjunction

$$K_{S'} + B_{S'} + \mathbf{M}_{S'}^S := (K_{Y'} + B_{Y'} + \mathbf{M}_{Y'})|_{S'}.$$

Then $g|_{S'} : S' \rightarrow S$ is a morphism, and we have

$$\begin{aligned} K_{S'} + B_{S'} + \mathbf{M}_{S'}^S &= (K_{Y'} + B_{Y'} + \mathbf{M}_{Y'})|_{S'} = g^*(K_Y + B_Y + \mathbf{M}_Y)|_S \\ &= g|_{S'}^*((K_Y + B_Y + \mathbf{M}_Y)|_S) = g|_{S'}^*(K_S + B_S + \mathbf{M}_S^S). \end{aligned}$$

By our construction, $(W, B_W, \mathbf{M}^W)/U$ is the g-sub-pair induced by $f_S \circ g|_{S'} : (S', B_{S'}, \mathbf{M}^S) \rightarrow W$, which is equal to the g-sub-pair induced by $f_S : (S, B_S, \mathbf{M}^S) \rightarrow W$ modulo $\mathbb{R}$ -linear equivalence of the base moduli part. By Definition-Theorem 12.1.4(2), $(W, B_W, \mathbf{M}^W)$ is lc. $\square$

Definition-Theorem 12.2.3. Let $(X, B, \mathbf{M})/U$ be a dlt g-pair and $f : (X, B, \mathbf{M}) \rightarrow Y$ a dlt crepant log structure/ U (Definition 4.3.4). Let $Z \subset Y$ be an lc center of $f : (X, B, \mathbf{M}) \rightarrow Y$ with normalization $\nu : Z^n \rightarrow Z$. Let $\mathcal{S}$ be the set of all lc centers of $(X, B, \mathbf{M})$ which dominate Z and let $S \in \mathcal{S}$ be an element that is minimal under inclusion. Let $(S, B_S, \mathbf{M}^S)$ be the g-pair induced by adjunction

$$K_S + B_S + \mathbf{M}_S^S := (K_X + B + \mathbf{M}_X)|_S,$$

$f_S : S \rightarrow Z^n$ the induced morphism such that $\nu \circ f_S = f|_S$, and let $f_S^n : S \xrightarrow{\tau} V \xrightarrow{\gamma} Z^n$ be the Stein factorization of $f|_S : S \rightarrow Z$. Then:

- (1) (Crepant log structure) $(S, B_S, \mathbf{M}^S)$ is dlt, $K_S + B_S + \mathbf{M}_S^S \sim_{\mathbb{R}, Z} 0$, and $(S, B_S, \mathbf{M}^S)$ is klt over the generic point of Z . In particular, $f|_S : (S, B_S, \mathbf{M}^S) \rightarrow Z$ is a dlt crepant log structure and an lc-trivial morphism.

We let

$$(V, B_V, \mathbf{M}^V)/U$$

be the g-pair induced by the lc-trivial fibration $\tau : (S, B_S, \mathbf{M}^S) \rightarrow V$. Then:

- (2) (Uniqueness of sources) The crepant birational equivalence class of $(S, B_S, \mathbf{M}^S)$ does not depend on the choice of S . We call the crepant birational equivalence class of $(S, B_S, \mathbf{M}^S)$ as the *source* of Z with respect to $f : (X, B, \mathbf{M}) \rightarrow Y$, and is denoted by $\text{Src}(Z, X, B, \mathbf{M})$.
- (3) (Uniqueness of springs) $(V, B_V, \mathbf{M}^V)$ modulo the $\mathbb{R}$ -linear equivalence class of $\mathbf{M}^V$ is unique up to isomorphism. We call $(V, B_V, \mathbf{M}^V)$ as the *spring* of Z with respect to $f : (X, B, \mathbf{M}) \rightarrow Y$, and is denoted by $\text{Spr}(Z, X, B, \mathbf{M})$.
- (4) (Adjunction) Let $W \subset X$ be an lc center such that $Z \subset Y_W := f(W)$, and let $(W, B_W, \mathbf{M}^W)/U$ be the lc g-pair induced by repeatedly applying adjunction

$$K_W + B_W + \mathbf{M}_W^W := (K_X + B + \mathbf{M}_X)|_W.$$

Let $\nu_Y : Y_W^n \rightarrow Y_W$ be the normalization of Y_W , $f_W : W \rightarrow Y_W^n$ the induced morphism such that $\nu_Y \circ f_W = f|_W$, and let

$$W \xrightarrow{\tau_W} V_W \xrightarrow{\gamma_W} Y_W$$

be the Stein factorization of f_W . Let $Z_W \subset V_W$ be an irreducible subvariety such that $(\nu_Y \circ \gamma_W)(Z_W) = Z$, and $(V_W, B_{V_W}, \mathbf{M}^{V_W})/U$ a g-pair induced by the lc-trivial fibration $\tau_W : (W, B_W, \mathbf{M}^W) \rightarrow V_W$. Then:

- (a) Z_W is an lc center of $(V_W, B_{V_W}, \mathbf{M}^{V_W})$.
- (b) $\text{Src}(Z, X, B, \mathbf{M}) = \text{Src}(Z_W, W, B_W, \mathbf{M}^W)$.
- (c) $\text{Spr}(Z, X, B, \mathbf{M}) = \text{Spr}(Z_W, W, B_W, \mathbf{M}^W)$.

Proof. (1) By [HL22, Lemma 2.9], $(S, B_S, \mathbf{M}^S)$ is dlt. Since $K_X + B + \mathbf{M}_X \sim_{\mathbb{R}, Z} 0$, $K_S + B_S + \mathbf{M}_S^S \sim_{\mathbb{R}, Z} 0$. By Lemma 4.3.2 and since S is minimal in $\mathcal{S}$, $(S, B_S, \mathbf{M}^S)$ is klt over the generic point of Z . (1) follows.

(2) By Theorem 4.3.8, different choices of S are $\mathbb{P}^1$ -linked to each other, hence they are crepant equivalent to each other by Definition 4.3.7(3).

(3) It follows from (2) and Definition 11.4.1.

(4) By Lemma 4.3.11(3) and Theorem 11.4.4, Z_W is an lc center of $(V_W, B_{V_W}, \mathbf{M}^{V_W})$ and an lc center of $\tau_W : (W, B_W, \mathbf{M}^W) \rightarrow V_W$. This implies (4.a).

Let S' be a minimal lc center of $(W, B_W, \mathbf{M}^W)$ which dominates Z_W , then S' is also an lc center of $(X, B, \mathbf{M})$ which dominates Z_W . In particular, S' dominates Z . If S' is not minimal in $\mathcal{S}$, then there exists $S'' \subsetneq S'$ such that S'' dominate Z , so $\tau_W(S'') \subset Z_W$ and $\tau_W(S'')$ dominates Z . This is not possible as Z_W is irreducible and γ_W is finite. Therefore, S' is minimal in $\mathcal{S}$. This implies (4.b). (4.c) follows from (4.b) and (3). $\square$

Definition-Lemma 12.2.4 (Subadjunction formula via minimal lc centers). Let $(X, B, \mathbf{M})/U$ be an g-sub-pair and V an lc center of $(X, B, \mathbf{M})$ with normalization $\nu : W \rightarrow V$, such that $(X, B, \mathbf{M})$ is lc near W .

Suppose that $f : Y \rightarrow X$ is a dlt modification of $(X, B, \mathbf{M})$ near W and let

$$K_Y + B_Y + \mathbf{M}_Y := f^*(K_X + B + \mathbf{M}_X).$$

Let $\mathcal{S}$ be the set of all lc center of $(Y, B_Y, \mathbf{M})$ whose image on X is V , and let S be a minimal element of $\mathcal{S}$ up to inclusion. Let $(S, B_S, \mathbf{M}^S)/U$ be the g-pair induced by repeating applying adjunction

$$K_S + B_S + \mathbf{M}_S^S := (K_Y + B_Y + \mathbf{M}_Y)|_S,$$

and let $f_S : S \rightarrow W$ be the induced projective surjective morphism such that $\nu \circ f_S = f|_S$.

We let $(W, B_W, \mathbf{M}^W)/U$ be a g-pair induced by a canonical bundle formula of $f_S : (S, B_S, \mathbf{M}^S) \rightarrow W$. Then:

- (1) There exists an lc place S' of $(X, B, \mathbf{M})$ such that $\text{center}_X S' = V$, $(W, B_W, \mathbf{M}^W)$ is a g-pair induced by subadjunction

$$K_W + B_W + \mathbf{M}_W^W := (K_X + B + \mathbf{M}_X)|_W,$$

and $(W, B_W, \mathbf{M}^W)$ is associated with S' .

- (2) $K_W + B_W + \mathbf{M}_W \sim_{\mathbb{R}} (K_X + B + \mathbf{M}_X)|_W$.
 (3) $(W, B_W, \mathbf{M}^W)$ is lc.
 (4) For any lc center T of $(W, B_W, \mathbf{M}^W)$, $\nu(T)$ is an lc center of $(X, B, \mathbf{M})$.
 (5) W does not depend on the choice of S (but may depend on the choice of f).

We say that $(W, B_W, \mathbf{M}^W)/U$ is *associated to* f .

Proof. (1) We let $g : Y' \rightarrow Y$ be the blow-up of the generic point of S and let S' be the reduced exceptional divisor. Let

$$K_{Y'} + B_{Y'} + \mathbf{M}_{Y'} = g^*(K_Y + B_Y + \mathbf{M}_Y).$$

Then $(Y', B_{Y'}, \mathbf{M})$ is dlt over a neighborhood of W . Let $(S', B_{S'}, \mathbf{M}^{S'})/U$ be the g-pair induced by adjunction

$$K_{S'} + B_{S'} + \mathbf{M}_{S'}^{S'} := (K_{Y'} + B_{Y'} + \mathbf{M}_{Y'})|_{S'}.$$

Since (Y, B_Y) is log smooth near the generic point of S and $\mathbf{M}$ descends to Y near the generic point of S , $g|_{S'} : S' \rightarrow S$ is a contraction, and $(S, B_S, \mathbf{M}^S)$ is induced by $g|_{S'}(S', B_{S'}, \mathbf{M}^{S'}) \rightarrow S$.

Thus the Stein factorization of the induced morphism $S' \rightarrow W$ factors through S . By Proposition 11.5.6, we get (1).

- (2) It follows from (1) and Definition-Theorem 12.2.1(2).
 (3) It follows from (1) and Proposition 12.2.2(2).
 (4) It follows from (1) and Definition-Theorem 12.2.1(4).
 (5) It follows from Definition-Theorem 12.2.3. $\square$

13. STRATIFICATION OF GENERALIZED PAIRS AND DU BOIS PROPERTY

The goal of this section is to study the stratification properties of lc generalized pairs and prove Theorem 2.2.9.

13.1. Stratification. In this subsection we recall some basic definitions of stratifications.

Definition 13.1.1 ([Kol13, Definition 9.15]). Let X be a scheme. A *stratification* of X is a decomposition of X into a finite disjoint union of reduced locally closed subschemes. We will consider stratifications where the strata are of pure dimensions and are indexed by their dimensions. We write $X = \cup_i S_i X$ where $S_i X \subset X$ is the i -th dimensional stratum. Such a stratified scheme is denoted by (X, S_*) . We also assume that $\cup_{i \leq j} S_i X$ is closed for every j . The *boundary* of (X, S_*) is the closed subscheme

$$B(X, S_*) := \cup_{i < \dim X} S_i X = X \setminus S_{\dim X} X,$$

and is denoted by $B(X)$ if the stratification S_* is clear.

Let (X, S_*) and (Y, S_*) be stratified schemes. We say that $f : X \rightarrow Y$ is a *stratified morphism* if $f(S_i X) \subset S_i Y$ for every i . Since $S_i X$ are disjoint with each other, $f : X \rightarrow Y$ is a stratified morphism if and only if $S_i X = f^{-1}(S_i Y)$.

Let (Y, S_*) be a stratified scheme and $f : X \rightarrow Y$ a quasi-finite morphism such that $f^{-1}(S_i Y)$ has pure dimension i for every i . Then $S_i X := f^{-1}(S_i Y)$ defines a stratification of X . We denote it by $(X, f^{-1}S_*)$, and we say that $f : X \rightarrow (Y, S_*)$ is *stratifiable*.

Definition 13.1.2 ([Kol13, Definition 9.16]). Let (X, S_*) be stratified variety. A relation $(\sigma_1, \sigma_2) : R \rightrightarrows (X, S_*)$ is *stratified* if each σ_i is stratifiable and $\sigma_1^{-1}S_* = \sigma_2^{-1}S_*$. Equivalently, there exists a stratification $(R, \sigma^{-1}S_i)$, such that $r \in \sigma^{-1}S_i R$ if and only if $\sigma_1(r) \in S_i X$ and if and only if $\sigma_2(r) \in S_i X$.

Definition 13.1.3 ([Kol13, Definition 9.18]). Let (X, S_*) be a stratified scheme such that X is an excellent scheme. The normality conditions (N), (SN), (HN), and (HSN) are defined in the following ways.

- (N) We say that (X, S_*) has *normal strata*, or that it satisfies (N), if each $S_i X$ is normal.
- (SN) We say that (X, S_*) has *semi-normal boundary*, or that it satisfies (SN), if X and $B(X, S_*)$ are both semi-normal.
- (HN) We say that (X, S_*) has *hereditarily normal strata*, or that it satisfies (HN), if
 - (a) the normalization $\pi : (X^n, \pi^{-1} S_*) \rightarrow (X, S_*)$ is stratifiable,
 - (b) $(X^n, \pi^{-1} S_*)$ satisfies (N), and
 - (c) $B(X^n, \pi^{-1} S_*)$ satisfies (HN).
- (HSN) We say that (X, S_*) has *hereditarily semi-normal boundary*, or that it satisfies (HSN), if
 - (a) the normalization $\pi : (X^n, \pi^{-1} S_*) \rightarrow (X, S_*)$ is stratifiable,
 - (b) $(X, \pi^{-1} S_*)$ satisfies (SN), and
 - (c) $B(X^n, \pi^{-1} S_*)$ satisfies (HSN).

Next we give a special stratification that is induced by the lc crepant log structure.

Definition 13.1.4 (Lc stratification for generalized pairs). Let $f : (X, \Delta, \mathbf{M}) \rightarrow Z$ be an lc crepant log structure. Let $S_i^*(Z, X, \Delta, \mathbf{M}) \subset Z$ be the union of all $\leq i$ -dimensional lc centers of $f : (X, \Delta, \mathbf{M}) \rightarrow Z$, and

$$S_i(Z, X, \Delta, \mathbf{M}) := S_i^*(Z, X, \Delta, \mathbf{M}) \setminus S_{i-1}^*(Z, X, \Delta, \mathbf{M}).$$

If the lc crepant log structure $f : (X, \Delta, \mathbf{M}) \rightarrow Z$ is clear from the context, we will use $S_i(Z)$ for abbreviation. It is clear that each $S_i(Z)$ is a locally closed subspace of Z of pure dimension i , and Z is the disjoint union of all $S_i(Z)$.

The stratification of Z induced by $S_i(Z)$ is called the *lc stratification* of Z induced by $f : (X, \Delta, \mathbf{M}) \rightarrow Z$. Since this is the only stratification we are going to use in the rest of this paper, we usually will not emphasize the lc crepant structure $f : (X, \Delta, \mathbf{M}) \rightarrow Z$, and we will denote the corresponding stratified scheme by (Z, S_*) . The *boundary* of (Z, S_*) is the closed subspace

$$B(Z, S_*) := Z \setminus S_{\dim Z}(Z) = \cup_{i < \dim Z} S_i(Z).$$

Definition 13.1.5. We say that a semi-normal stratified space (Y, S_*) is *of lc origin* if $S_i(Y)$ is unibranch for any i , and there are lc crepant log structures $f_j : (X_j, \Delta_j, \mathbf{M}^j) \rightarrow Z_j$ with lc stratifications (Z_j, S_*^j) and a finite surjective stratified morphism $\pi : \coprod_j (Z_j, S_*^j) \rightarrow (Y, S_*)$.

13.2. Semi-normality of lc centers and lc origin. In this subsection we show that lc centers of lc generalized pairs are semi-normal.

Theorem 13.2.1. *Let $f : (X, \Delta, \mathbf{M}) \rightarrow Z$ be an lc crepant log structure. Let $W \subset Z$ be the union of all lc centers of $f : (X, \Delta, \mathbf{M}) \rightarrow Z$ except Z , and $B(W) \subset W$ the union of all non-maximal (with respect to inclusion) lc centers that are contained in W . Then*

- (1) W is semi-normal, and
- (2) $W \setminus B(W)$ is normal.

Proof. Let $(Z, \Delta_Z, \mathbf{N})/U$ be an lc g-pair induced by the canonical bundle formula/ U of $f : (X, \Delta, \mathbf{M}) \rightarrow Z$. By Theorem 11.4.4, the lc centers of $(Z, \Delta_Z, \mathbf{N})$ are exactly the lc centers of $f : (X, \Delta, \mathbf{M}) \rightarrow Z$. Possibly replacing $(X, \Delta, \mathbf{M})$ with a dlt model of $(Z, \Delta_Z, \mathbf{N})$, we may assume that f is birational and $(X, \Delta, \mathbf{M})$ is $\mathbb{Q}$ -factorial dlt. We have $W = f([\Delta])$. Let $\Delta' := \{\Delta\}$. We consider the exact sequence

$$0 \rightarrow \mathcal{O}_X(-[\Delta]) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{[\Delta]}$$

and its push-forward

$$\mathcal{O}_Z = f_* \mathcal{O}_X \rightarrow f_* \mathcal{O}_{[\Delta]} \xrightarrow{\delta} R^1 f_* \mathcal{O}_X(-[\Delta]).$$

By [HL22, Lemma 3.4], we can find an $\mathbb{R}$ -divisor $\Delta'' \geq 0$ such that

$$-\lfloor \Delta \rfloor \sim_{\mathbb{R}, Z} K_X + \Delta' + \mathbf{M}_X \sim_{\mathbb{R}, Z} K_X + \Delta''$$

and (X, Δ'') is klt. Since $-\lfloor \Delta \rfloor$ is a Weil divisor, by [HLS19, Lemma 5.3, Theorem 5.6], possibly perturbing Δ'' , we may assume that Δ'' is a $\mathbb{Q}$ -divisor and

$$-\lfloor \Delta \rfloor \sim_{\mathbb{Q}, Z} K_X + \Delta''.$$

By [Kol13, Corollary 10.40], $R^i f_* \mathcal{O}_X(-\lfloor \Delta \rfloor)$ is torsion free for every i . On the other hand, $f_* \mathcal{O}_{\lfloor \Delta \rfloor}$ is supported on W , hence it is a torsion sheaf. Thus the connecting map δ is zero, hence $\mathcal{O}_Z \twoheadrightarrow f_* \mathcal{O}_{\lfloor \Delta \rfloor}$ is surjective. Since this map factors through $\mathcal{O}_W$, we conclude that $\mathcal{O}_W \twoheadrightarrow f_* \mathcal{O}_{\lfloor \Delta \rfloor}$ is also surjective, hence an isomorphism.

Note that $\lfloor \Delta \rfloor$ has only nodes at codimension 1 points and it is S_2 by [Kol13, Corollary 2.88]. By [Kol13, Lemma 10.14], $\lfloor \Delta \rfloor$ is semi-normal. By [Kol13, Lemma 10.15], W is semi-normal. This is (1).

To prove (2), let $V \subset \lfloor \Delta \rfloor$ be an irreducible component of its non-normal locus. Then V is an lc center of (X, Δ) , hence an lc center of $(X, \Delta, \mathbf{M})$. Thus $f(V)$ is an lc center of $f : (X, \Delta, \mathbf{M}) \rightarrow Z$. Hence either $f(V)$ is an irreducible component of W , or $f(V) \subset B(W)$. Thus [Kol13, Complement 10.15.1] implies that $W \setminus B(W)$ is normal. $\square$

Corollary 13.2.2. *Let $(X, \Delta, \mathbf{M})$ be an lc g-pair. Then $\text{Nklt}(X, \Delta, \mathbf{M})$ is semi-normal.*

Proof. It follows from Theorem 13.2.1 when f is the identity morphism. $\square$

Lemma 13.2.3. (cf. [Kol13, Lemma 5.26]) *Let $f : (X, \Delta, \mathbf{M}) \rightarrow Z$ be a lc crepant log structure and (Z, S_*) the induced lc stratification. Then*

- (1) $S_i(Z)$ is unibranch for every i , and
- (2) $B(Z, S_*)$ is semi-normal.

Proof. (1) follows from Lemma 4.3.10(2) and (2) follows from Theorem 13.2.1. $\square$

Lemma 13.2.4. (cf. [Kol13, Proposition 4.42]) *Let $f : (X, \Delta, \mathbf{M}) \rightarrow Z$ be a dlt crepant log structure, (Z, S_*) its induced lc stratification, and $Y \subset X$ an lc center of $(X, \Delta, \mathbf{M})$. Let $(Y, \Delta, \mathbf{M}^Y)/Z$ be the dlt g-pair induced by adjunction to higher-codimensional lc center Y , i.e.*

$$K_Y + \Delta_Y + \mathbf{M}_Y^Y := (K_X + \Delta + \mathbf{M}_X)|_Y.$$

We consider the Stein factorization of $f|_Y$

$$(Y, \Delta_Y, \mathbf{M}^Y) \xrightarrow{f_Y} W \xrightarrow{\pi} Z.$$

Then:

- (1) $f_Y : (Y, \Delta_Y, \mathbf{M}^Y) \rightarrow W$ is a dlt crepant log structure which induces an lc stratification (W, S_*) .
- (2) $S_i(W) = \pi^{-1}(S_i(Z))$ for every i .

Proof. It follows from Lemma 4.3.10. $\square$

Theorem 13.2.5. *Let $f : (X, \Delta, \mathbf{M}) \rightarrow Z$ be an lc crepant log structure and (Z, S_*) the induced lc stratification. Then (Z, S_*) satisfies (HN) and (HSN).*

Proof. By Lemma 13.2.3 and [Kol13, Definitions 9.18, 9.19], (Z, S_*) satisfies (HU) and (HSN). By [Kol13, Theorem 9.21], (Z, S_*) satisfies (HN). $\square$

Lemma 13.2.6. (cf. [Kol13, 5.29]) *Every lc stratification is of lc origin. More precisely, let $f : (X, \Delta, \mathbf{M}) \rightarrow W$ be an lc crepant log structure and $Y \subset W$ any union of lc centers. Then (Y, S_*) is of lc origin, where $S_i(Y) = Y \cap S_i(W)$ for each i .*

Proof. By Theorem 13.2.5 and [Kol13, Theorem 9.26], we know that Y is semi-normal and $S_i(Y)$ is unibranch for each i . Then we can apply Lemma 13.2.4 to each lc center of $f : (X, \Delta, \mathbf{M})$ contained in Y to conclude that (Y, S_*) is of lc origin. $\square$

13.3. Du Bois property. In this subsection, we show that lc generalized pairs have Du Bois singularities. This subsection is parallel to [LX23b, Section 6].

We recall the following definition in [Kov11] (cf. [Kol13, Definition 6.10]).

Definition 13.3.1. A *DB pair* (X, Σ) consists of a reduced scheme X of finite type and a closed reduced subscheme Σ in X such that the natural morphism

$$\mathcal{I}_{\Sigma \subset X} \rightarrow \underline{\Omega}_{X, \Sigma}^0$$

is a quasi-isomorphism. We will also say (X, Σ) is *DB* in this case.

The definition of DB pairs is subtle but what really matters here is the following lemma:

Lemma 13.3.2 ([Kol13, Proposition 6.15]). *Let (X, Σ) be a DB pair. Then X has Du Bois singularities if and only if Σ has Du Bois singularities.*

The following theorems are analogues of [Kol13, Theorems 6.31, 6.33] for g-pairs and the proofs are similar. For the reader's convenience, we provide full proofs here.

Theorem 13.3.3. *Let $(X, B, \mathbf{M})/U$ be an lc g-pair and $f : (X, B, \mathbf{M}) \rightarrow Z$ an lc-trivial fibration. Let $W \subset Z$ be the union of lc centers of $f : (X, B, \mathbf{M}) \rightarrow Z$ except Z . Then (Z, W) is a DB pair.*

Proof. Let $(Z, B_Z, \mathbf{M}^Z)/U$ be a g-pair induced by $f : (X, B, \mathbf{M}) \rightarrow Z$. By Theorem 11.4.4, the lc centers of $(Z, B_Z, \mathbf{M}^Z)$ are exactly the lc centers of $f : (X, B, \mathbf{M}) \rightarrow Z$. Thus we can assume that f is the identity morphism, $(X, B, \mathbf{M}) = (Z, B_Z, \mathbf{M}^Z)$, and $W = \text{Nklt}(X, B, \mathbf{M})$.

Let $g : Y \rightarrow X$ be a log resolution of $(X, \text{Supp } B)$ such that $\mathbf{M}$ descends to Y and $F := g^{-1}(W)$ is an snc divisor. Let

$$K_Y + B_Y + \mathbf{M}_Y := g^*(K_X + B + \mathbf{M}_X)$$

and $D := B_Y^{-1}$. Since $\mathbf{M}_Y$ is nef/ X and big/ X , there exists $0 \leq B'_Y \sim_{\mathbb{R}, X} \mathbf{M}_Y$ such that $(Y, B_Y - D + B'_Y)$ is sub-klt. Possibly replacing Y with a higher resolution, we may assume that $(Y, \text{Supp } B_Y \cup \text{Supp } D \cup \text{Supp } B'_Y)$ is log smooth. Let

$$\bar{B}_Y := (B_Y - D + B'_Y)^{\geq 0} + \{(B_Y - D + B'_Y)^{\leq 0}\}$$

and

$$E := \lfloor (B_Y - D + B'_Y)^{\leq 0} \rfloor,$$

then $\lfloor \bar{B}_Y \rfloor = 0$ and E is a g-exceptional Weil divisor. In particular, $(Y, \bar{B}_Y)$ is klt.

Since $E - D \geq -F$, we have natural maps:

$$g_*\mathcal{O}_Y(-F) \rightarrow Rg_*\mathcal{O}_Y(-F) \rightarrow Rg_*\mathcal{O}_Y(E - D).$$

Since $E - D \sim_{\mathbb{R}, X} K_Y + \bar{B}_Y$ and $E - D$ is a Weil divisor, by [HLS19, Lemma 5.3, Theorem 5.6], $E - D \sim_{\mathbb{Q}, X} K_Y + \bar{B}'_Y$ for some klt $\mathbb{Q}$ -pair $(Y, \bar{B}'_Y)$. by [Kol13, Theorem 10.41],

$$Rg_*\mathcal{O}_Y(E - D) \simeq_{qis} \sum_i R^i g_*\mathcal{O}_Y(E - D)[i].$$

Thus we get a morphism

$$g_*\mathcal{O}_Y(-F) \rightarrow Rg_*\mathcal{O}_Y(-F) \rightarrow Rg_*\mathcal{O}_Y(E - D) \rightarrow g_*\mathcal{O}_Y(E - D).$$

Note that

$$g_*\mathcal{O}_Y(E - D) = g_*\mathcal{O}_Y(E - D) \cap g_*\mathcal{O}_Y(E) = g_*\mathcal{O}_Y(E - D) \cap g_*\mathcal{O}_Y = g_*\mathcal{O}_Y(-D).$$

Since D is reduced and $g(D) = W$, we have $g_*\mathcal{O}_Y(-D) = \mathcal{I}_W$, the ideal sheaf of W in $Z = X$. Moreover, $g_*\mathcal{O}_Y(-F) = \mathcal{I}_W$ since F is also reduced. Therefore, we get an isomorphism $\mathcal{I}_W = g_*\mathcal{O}_Y(-F) \rightarrow g_*\mathcal{O}_Y(E - D)$, which implies that

$$\rho : \mathcal{I}_W \simeq g_*\mathcal{I}_F \rightarrow Rg_*\mathcal{I}_F$$

has a left inverse. Since Y is smooth and F is an snc divisor, we see that (Y, F) is a DB pair, thus by [Kov12, Theorem 3.3] (cf. [Kol13, Theorem 6.27]), (Z, W) is also a DB pair. $\square$

Definition 13.3.4. We say a commutative diagram of schemes

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{j} & Y \\ q \downarrow & & \downarrow p \\ \mathcal{D} & \xrightarrow{i} & X \end{array}$$

is a *universal push-out diagram* if for any scheme T , the induced diagram

$$\begin{array}{ccc} \mathrm{Hom}(X, T) & \xrightarrow{\circ i} & \mathrm{Hom}(\mathcal{D}, T) \\ \circ p \downarrow & & \downarrow \circ q \\ \mathrm{Hom}(Y, T) & \xrightarrow{\circ j} & \mathrm{Hom}(\mathcal{C}, T) \end{array}$$

is a universal pull-back diagram of sets.

Theorem 13.3.5. Let (X, S_*) be a stratified scheme of lc origin (Definition 13.1.5). Then X is Du Bois.

Proof. We use induction on the dimension. When $\dim X = 1$ the theorem is trivial.

Let $\pi : (X^n, S_*^n) \rightarrow (X, S_*)$ be the normalization. Let $B(X) \subset X$ and $B(X^n) \subset X^n$ denote the corresponding boundaries. By [Kol13, 9.15.1], we have a universal push-out diagram

$$\begin{array}{ccc} B(X^n) & \hookrightarrow & X^n \\ \downarrow & & \downarrow \pi \\ B(X) & \hookrightarrow & X \end{array}$$

Notice that $B(X)$ and $B(X^n)$ are of lc origin by Lemma 13.2.6, hence Du Bois by induction.

Since π is finite, it follows that $R\pi_* \mathcal{I}_{B(X^n) \subset X^n} = \pi_* \mathcal{I}_{B(X^n) \subseteq X^n}$. Furthermore, $\pi_* \mathcal{I}_{B(X^n) \subseteq X^n} = \mathcal{I}_{B(X) \subseteq X}$ by [Kol13, Theorem 9.30]. By [Kov12, Theorem 3.3] and Lemma 13.3.2, we only need to show that X^n is Du Bois. By assumption, for each irreducible component $X_i^n \subset X^n$, there exists an lc crepant log structure $f_i : (Y_i, \Delta_i, \mathbf{M}) \rightarrow Z_i$ and a finite surjection $Z_i \rightarrow X_i^n$. By [Kov99, Corollary 2.5], we only need to show that Z_i is Du Bois for each i . Let $B(Z_i) \subset Z_i$ be the boundary of the lc stratification of Z_i . Then $B(Z_i)$ is of lc origin by Lemma 13.2.6, hence Du Bois by induction. By Theorem 13.3.3, $(Z_i, B(Z_i))$ is a DB pair, hence Z_i is Du Bois and we are done. $\square$

Proof of Theorem 2.2.9. Let W be any union of the glc centers, then by Lemma 13.2.6 the induced stratified space (W, S_*) is of lc origin. Theorem 2.2.9 follows from Theorem 13.3.5. $\square$

14. VANISHING AND CONTRACTION THEOREMS FOR LC GENERALIZED PAIRS

The goal of this section is to prove the vanishing theorems and contraction theorems for lc generalized pairs. This section is parallel to [CLX23], except that the canonical bundle formula and the subadjunction formulas are replaced by the ones established in Sections 11 and 12.

14.1. Adjacent lc centers and universal push-out diagram.

Definition 14.1.1 (Union of lc centers). Let $(X, B, \mathbf{M})$ be an lc g-pair. A *union of lc centers* of $(X, B, \mathbf{M})$ is a reduced scheme $Y = \cup Y_i$, where each Y_i is an lc center of $(X, B, \mathbf{M})$. We denote by $S(X, B, \mathbf{M})$ the set of all unions of lc centers of $(X, B, \mathbf{M})$. We remark that

- (1) $\emptyset$ is also considered as a union of lc centers, and

- (2) a union of lc center may be represented in different ways. For example, if Y_1 and Y_2 are two lc centers such that $Y_1 \subsetneq Y_2$, then $Y_1 \cup Y_2$ and Y_2 are the same union of lc centers.

Definition 14.1.2 (Adjacent unions of lc centers). Let $(X, B, \mathbf{M})$ be an lc g-pair. For any two unions of lc centers $Y, Y' \in S(X, B, \mathbf{M})$, we say that Y and Y' are *adjacent* in $S(X, B, \mathbf{M})$ if

- (1) $Y \subsetneq Y'$ or $Y' \subsetneq Y$, and
- (2) there does not exist any $Y'' \in S(X, B, \mathbf{M})$ such that $Y \subsetneq Y'' \subsetneq Y'$ or $Y' \subsetneq Y'' \subsetneq Y$.

An lc center V is called *minimal* in $S(X, B, \mathbf{M})$ if V and $\emptyset$ are adjacent in $S(X, B, \mathbf{M})$.

Lemma 14.1.3. *Let $(X, B, \mathbf{M})$ be an lc g-pair. Let Y and Y' be two unions of lc centers, such that $Y' \subsetneq Y$, and Y and Y' are adjacent in $S(X, B, \mathbf{M})$. Let $\pi : Y^n \rightarrow Y$ be the normalization of Y and let $Y'' := \pi^{-1}(Y')$ with the reduced scheme structure. Denote the induced morphism $Y'' \rightarrow Y'$ by π'' . Then there exist a universal push-out diagram*

$$\begin{array}{ccc} Y'' & \xrightarrow{j} & Y^n \\ \pi'' \downarrow & & \downarrow \pi \\ Y' & \xrightarrow{i} & Y \end{array}$$

and a short exact sequence

$$0 \rightarrow \mathcal{O}_Y \xrightarrow{\pi^* \oplus i^*} \pi_* \mathcal{O}_{Y^n} \oplus \mathcal{O}_{Y'} \xrightarrow{j^* - \pi''^*} \pi''_* \mathcal{O}_{Y''} \rightarrow 0,$$

where i, j are the natural closed immersions.

Proof. By Theorem 13.2.1 and [Kol13, Theorem 9.26], Y is semi-normal. Let L be an lc center contained in Y but not contained in Y' . Since Y' and Y are adjacent in $S(X, B, \mathbf{M})$, we have

$$Y \setminus Y' = L \setminus (L \cap Y'),$$

and $L \cap Y'$ is the union of all lc centers of $(X, B, \mathbf{M})$ that are contained in L but not equal to L . By Theorem 13.2.1, $Y \setminus Y'$ is normal. The lemma follows from [CLX23, Lemma 2.6]. $\square$

14.2. Vanishing theorems. The following lemma is very similar to [Xie22, Lemma 2.4].

Lemma 14.2.1. *Let $(X, B, \mathbf{M})/U$ be an lc g-pair, and L a nef $\mathbb{R}$ -divisor such that $L - (K_X + B + \mathbf{M}_X)$ is nef/ U and big/ U . Then there exists an $\mathbb{R}$ -divisor $\Delta \geq 0$ such that $L - (K_X + \Delta)$ is ample over U and $\text{Nlc}(X, \Delta) = \text{Nklt}(X, B, \mathbf{M})$.*

Proof. Let $f : Y \rightarrow X$ be a log resolution of $(X, \text{Supp } B)$ such that $\mathbf{M}$ descends on Y , and let

$$K_Y + B_Y + \mathbf{M}_Y := f^*(K_X + B + \mathbf{M}_X).$$

Since $L - (K_X + B + \mathbf{M}_X)$ is nef/ U and big/ U , $f^*L - (K_Y + B_Y + \mathbf{M}_Y)$ is nef/ U and big/ U . Then there exists an $\mathbb{R}$ -divisor $E \geq 0$ on Y , such that for any positive integer n , there exists an ample/ U $\mathbb{R}$ -divisor A_n on Y such that

$$f^*L - (K_Y + B_Y + \mathbf{M}_Y) \sim_{\mathbb{R}, U} A_n + \frac{1}{n}E.$$

We let m be a positive integer such that

$$\text{Nklt}(Y, B_Y, \mathbf{M}) = \lfloor B_Y \rfloor = \text{Nklt}(Y, B_Y + \frac{1}{m}E, \mathbf{M}).$$

Let $0 < \delta \ll 1$ a real number such that $A_m - \delta \lfloor B_Y \rfloor$ is ample/ U . Pick a general ample $\mathbb{R}$ -divisor $A_Y \in |(A_m + \mathbf{M}_Y - \delta \lfloor B_Y \rfloor)/U|_{\mathbb{R}}$, set

$$B'_Y := B_Y + A_Y + \delta \lfloor B_Y \rfloor + \frac{1}{m}E,$$

and $B' := f_* B'_Y$. Then $0 \leq B' \sim_{\mathbb{R}, U} B + \mathbf{M}_X$ and

$$\text{Nlc}(X, B') = \text{Nklt}(X, B') = \text{Nklt}(X, B, \mathbf{M}).$$

Since $L - (K_X + B + \mathbf{M}_X)$ is big/ U and nef/ U , there exist an $\mathbb{R}$ -divisor $F \geq 0$ on X and an ample/ U $\mathbb{R}$ -divisor H on X such that

$$L - (K_X + B + \mathbf{M}_X) \sim_{\mathbb{R}, U} H + F.$$

Let $l \gg 0$ be an integer, then

$$L - \left(K_X + B' + \frac{1}{l}F\right) \sim_{\mathbb{R}, U} L - \left(K_X + B + \mathbf{M}_X + \frac{1}{l}F\right) \sim_{\mathbb{R}, U} \frac{1}{l}H + \frac{l-1}{l}(L - (K_X + B + \mathbf{M}_X))$$

is ample/ U , and

$$\mathrm{Nlc}\left(X, B' + \frac{1}{l}F\right) \subset \mathrm{Nklt}(X, B') = \mathrm{Nlc}(X, B') \subset \mathrm{Nlc}\left(X, B' + \frac{1}{l}F\right).$$

Hence

$$\mathrm{Nlc}\left(X, B' + \frac{1}{n}F\right) = \mathrm{Nlc}(X, B') = \mathrm{Nklt}(X, B, \mathbf{M}).$$

Thus $\Delta := B' + \frac{1}{l}F$ has the required property. $\square$

Lemma 14.2.2. *Let $f : X \rightarrow U$ be a projective morphism, $h : Y \rightarrow X$ a finite morphism between normal schemes, and $g := f \circ h$. Let $W \subset X$ and $V \subset Y$ be two reduced subschemes such that $h^{-1}(W) = V$ with defining ideal sheaves $\mathcal{I}_W$ and $\mathcal{I}_V$. Let L be a line bundle on X such that $R^i g_*(h^*L \otimes \mathcal{I}_V) = 0$ for some positive integer i . Then $R^i f_*(L \otimes \mathcal{I}_W) = 0$.*

Proof. Notice that $\mathcal{I}_W$ is a direct summand of $h_*\mathcal{I}_V$ (via the splitting $\mathcal{O}_X \rightarrow h_*\mathcal{O}_Y \rightarrow \mathcal{O}_X$), so it suffices to prove that $R^i f_*(L \otimes h_*\mathcal{I}_V) = 0$. Since $R^i h_*(G) = 0$ for any coherent sheaf G and $i > 0$, we have

$$R^i f_*(L \otimes h_*\mathcal{I}_V) = R^i f_*(h_*(h^*L \otimes \mathcal{I}_V)) = R^i g_*(h^*L \otimes \mathcal{I}_V) = 0.$$

$\square$

Lemma 14.2.3. *Let $f : X \rightarrow U$ be a projective morphism, L an $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor on X , and D a Cartier divisor on X . Let $h : Y \rightarrow X$ be a finite morphism, $(Y, B_Y, \mathbf{M}^Y)/U$ an lc g -pair such that*

$$K_Y + B_Y + \mathbf{M}_Y^Y \sim_{\mathbb{R}, U} h^*L,$$

*and $V := \mathrm{Nklt}(Y, B_Y, \mathbf{M}^Y)$ with the reduced scheme structure. Set $W := h(V)$ with the reduced scheme structure. Let $\mathcal{I}_W, \mathcal{I}_V$ be the defining ideal sheaves of W and V respectively, $g := f \circ h$, and $D_Y := h^*D$. Suppose that $D_Y - (K_Y + B_Y + \mathbf{M}_Y^Y)$ is nef/ U and log big/ U with respect to $(Y, B_Y, \mathbf{M}^Y)$. Then:*

- (1) $R^i g_*(\mathcal{I}_V \otimes \mathcal{O}_Y(D_Y)) = 0$ for any $i > 0$.
- (2) $g_*\mathcal{O}_Y(D_Y) \rightarrow g_*\mathcal{O}_V(D_Y)$ is surjective.
- (3) Suppose that $V = h^{-1}(W)$. Then $R^i f_*(\mathcal{I}_W \otimes \mathcal{O}_X(D)) = 0$ for any $i > 0$.
- (4) Suppose that $V = h^{-1}(W)$. Then $f_*\mathcal{O}_X(D) \rightarrow f_*\mathcal{O}_W(D)$ is surjective.

Proof. By Lemma 14.2.1, there exists a pair (Y, Δ_Y) such that $D_Y - (K_Y + \Delta_Y)$ is ample/ U and $V = \mathrm{Nlc}(Y, \Delta_Y)$. (1) follows from [Fuj11, Theorem 8.1]. (2) follows from (1) and the long exact sequence

$$0 \rightarrow g_*(\mathcal{I}_V \otimes \mathcal{O}_Y(D_Y)) \rightarrow g_*\mathcal{O}_Y(D_Y) \rightarrow g_*\mathcal{O}_V(D_Y) \rightarrow R^1 g_*(\mathcal{I}_V \otimes \mathcal{O}_Y(D_Y)) \rightarrow \dots$$

(3) follows from (1) and Lemma 14.2.2. (4) follows from (3) and the long exact sequence

$$0 \rightarrow f_*(\mathcal{I}_W \otimes \mathcal{O}_X(D)) \rightarrow f_*\mathcal{O}_X(D) \rightarrow f_*\mathcal{O}_W(D) \rightarrow R^1 f_*(\mathcal{I}_W \otimes \mathcal{O}_X(D)) \rightarrow \dots$$

$\square$

Lemma 14.2.4. *Let $(X, B, \mathbf{M})/U$ be an lc g-pair associated with morphism $f : X \rightarrow U$, and D a Cartier divisor on X such that $D - (K_X + B + \mathbf{M}_X)$ is nef/ U and log big/ U with respect to $(X, B, \mathbf{M})$. Let Y and Y' be two unions of lc centers, such that $Y' \subsetneq Y$, and Y and Y' are adjacent in $S(X, B, \mathbf{M})$. Let $\pi : Y^n \rightarrow Y$ be the normalization of Y , $Y'' := \pi^{-1}(Y')$ with the reduced scheme structure, and $\pi'' := \pi|_{Y''}$.*

$$\begin{array}{ccc} Y'' & \xrightarrow{j} & Y^n \\ \pi'' \downarrow & & \downarrow \pi \\ Y' & \xrightarrow{i} & Y \end{array}$$

Then the induced map

$$f_*\pi_*\mathcal{O}_{Y^n}(D|_{Y^n}) \rightarrow f_*\pi''_*\mathcal{O}_{Y''}(D|_{Y''})$$

is surjective.

Proof. We only need to show that

$$f_*\pi_*\mathcal{O}_{Y_0}(D|_{Y_0^n}) \rightarrow f_*\pi''_*\mathcal{O}_{Y_0''}(D|_{Y_0^n \cap Y''})$$

is surjective for any connected component Y_0^n of Y^n . Let Y_0^n be a connected component of Y^n , $Y_0'' := Y'' \cap Y_0^n$, $Y_0 := \pi(Y_0^n)$, and $Y'_0 := \pi''(Y_0'')$. Since Y and Y' are adjacent in $S(X, B, \mathbf{M})$, either $Y_0 = Y'_0$, or Y'_0 and Y_0 are adjacent in $S(X, B, \mathbf{M})$ and $Y'_0 \subsetneq Y_0$. Possibly replacing Y with Y_0 and Y' with Y'_0 , we may assume that Y is an lc center of $(X, B, \mathbf{M})$. Since Y and Y' are adjacent in $S(X, B, \mathbf{M})$, Y' is the union of all lc centers of $(X, B, \mathbf{M})$ that are contained in Y .

We let $(W, B_W, \mathbf{M})$ be a dlt model of $(X, B, \mathbf{M})$ with induced birational morphism $h : W \rightarrow X$. Let S be an lc center of $(W, B_W, \mathbf{M})$ which is minimal in all lc centers which dominate Y , $(S, B_S, \mathbf{M}^S)/U$ the dlt g-pair induced by adjunction

$$K_S + B_S + \mathbf{M}_S^S := (K_W + B_W + \mathbf{M}_W)|_W,$$

and $h_S : S \rightarrow Y^n$ the induced morphism such that $\pi \circ h_S = h|_S$. Let

$$S \xrightarrow{\tau} Z \xrightarrow{\gamma} Y^n$$

be the Stein factorization of h_S , and let $(Z, B_Z, \mathbf{M}^Z)/U$ be the lc g-pair induced by $\tau : (S, B_S, \mathbf{M}^S) \rightarrow Z$. Let $(Y^n, B_{Y^n}, \mathbf{M}^{Y^n})/U$ be the lc g-pair induced by $\gamma : (Z, B_Z, \mathbf{M}^Z) \rightarrow Y^n$, and let $Y'_Z := \gamma^{-1}(Y'')$.

By Lemma 4.3.11(3) and Theorem 11.4.4(5), for any lc center V of $(X, B, \mathbf{M})$ such that $V \subset Y^n$, any irreducible component of $\gamma^{-1}(V)$ is an lc center of $(Z, B_Z, \mathbf{M}^Z)$. In particular, any irreducible component of Y'_Z is an lc center of $(Z, B_Z, \mathbf{M}^Z)$, so $\text{Nklt}(Z, B_Z, \mathbf{M}^Z) \subset Y'_Z$. By Theorem 11.4.4(4), $(\pi \circ \gamma)(\text{Nklt}(Z, B_Z, \mathbf{M}^Z))$ is a union of lc centers of $(X, B, \mathbf{M})$ that are contained in Y , so $\text{Nklt}(Z, B_Z, \mathbf{M}^Z) \subset Y'_Z$. Thus

$$\text{Nklt}(Z, B_Z, \mathbf{M}^Z) = Y'_Z = \gamma^{-1}(Y'').$$

Let $D_{Y^n} := D|_{Y^n}$ and $D_Z := \gamma^*D_{Y^n}$. Since $D - (K_X + B + \mathbf{M}_X)$ is nef/ U , $D_{Y^n} - (K_{Y^n} + B_{Y^n} + \mathbf{M}_{Y^n}^{Y^n})$ is nef/ U , so $D_Z - (K_Z + B_Z + \mathbf{M}_Z^Z)$ is nef/ U . For any lc center V_Z of $(Z, B_Z, \mathbf{M}^Z)$ with normalization V_Z^n , $(\pi \circ \gamma)(V_Z)$ is an lc center of $(X, B, \mathbf{M})$, so $(D - (K_X + B + \mathbf{M}_X))|_{\pi(V_Z)^n}$ is big/ U , where $\pi(V_Z)^n$ is the normalization of $\pi(W)$. Since π is finite, $(D_Z - (K_Z + B_Z + \mathbf{M}_Z^Z))|_{V_Z^n}$ is big/ U . Therefore, $D_Z - (K_Z + B_Z + \mathbf{M}_Z^Z)$ is log big/ U .

The lemma follows from Lemma 14.2.3(4). $\square$

Theorem 14.2.5. *Let $(X, B, \mathbf{M})/U$ be an lc g-pair associated with projective morphism $f : X \rightarrow U$, D a Cartier divisor on X such that $D - (K_X + B + \mathbf{M}_X)$ is nef/ U and log big/ U with respect to $(X, B, \mathbf{M})$, and Y a union of lc centers of $(X, B, \mathbf{M})$ such that $Y \neq X$. Then:*

- (1) $R^i f_* \mathcal{O}_Y(D) = 0$ for any positive integer i .

- (2) $R^i f_* \mathcal{O}_X(D) = 0$ for any positive integer i .
- (3) The map $f_* \mathcal{O}_X(D) \rightarrow f_* \mathcal{O}_Y(D)$ is surjective.
- (4) $R^i f_*(\mathcal{I}_Y \otimes \mathcal{O}_X(D)) = 0$ for any positive integer i , where $\mathcal{I}_Y$ is the defining ideal sheaf of Y on X .

Proof. We apply induction on $\dim X$. When $\dim X = 1$ the theorem is obvious.

For any union of lc centers Z of $(X, B, \mathbf{M})$, we define $m(Z)$ to be the number of lc centers of $(X, B, \mathbf{M})$ that are contained in Z . We let $W := \text{Nklt}(X, B, \mathbf{M})$, associated with the reduced scheme structure.

Step 1. In this step we prove (1) when Y is minimal in $S(X, B, \mathbf{M})$ the set of all unions of lc centers of $(X, B + \mathbf{M})/U$.

By Theorem 13.2.5, Y is normal. If $\dim Y = 0$ then we are done. Otherwise, by Definition-Theorem 12.2.1 and Proposition 12.2.2, there exists a klt g-pair $(Y, B_Y, \mathbf{M}^Y)/U$ such that $K_Y + B_Y + \mathbf{M}_Y^Y \sim_{\mathbb{R}, U} (K_X + B + \mathbf{M}_X)|_Y$. Hence $D|_Y - (K_Y + B_Y + \mathbf{M}_Y^Y)$ is nef/ U and big/ U . By Lemma 14.2.1, there exists a klt pair (Y, Δ_Y) such that $D|_Y - (K_Y + \Delta_Y)$ is ample/ U . (1) follows from the usual Kawamata-Viehweg vanishing theorem (cf. [KMM87, Theorem 1-2-7]).

Step 2. In this step we prove (1).

We apply induction on $m(Y)$. When $m(Y) = 1$, Y is minimal in $S(X, B, \mathbf{M})$ and we are done by **Step 1**. Thus we may assume that $m(Y) > 1$. In particular, $\dim Y \geq 1$. We let $Y' \in S(X, B, \mathbf{M})$ be a union of lc centers such that $Y' \subsetneq Y$ and Y', Y are adjacent in $S(X, B, \mathbf{M})$. Let $\pi : Y^n \rightarrow Y$ be the normalization of Y , $Y'' := \pi^{-1}(Y')$ with the reduced scheme structure, and $\pi'' := \pi|_{Y''}$. By Lemma 14.1.3, there exists a universal push-out diagram

$$\begin{array}{ccc} Y'' & \xrightarrow{j} & Y^n \\ \pi'' \downarrow & & \downarrow \pi \\ Y' & \xrightarrow{i} & Y \end{array}$$

and a short exact sequence

$$(14.1) \quad 0 \rightarrow \mathcal{O}_Y \xrightarrow{\pi^* \oplus i^*} \pi_* \mathcal{O}_{Y^n} \oplus \mathcal{O}_{Y'} \xrightarrow{j^* - \pi''^*} \pi''_* \mathcal{O}_{Y''} \rightarrow 0.$$

where i, j are the natural closed immersions. Since $m(Y') < m(Y)$, by induction on $m(Y)$, we have

$$(14.2) \quad R^i f_* \mathcal{O}_{Y'}(D) = 0$$

for any positive integer i .

By Definition-Theorem 12.2.1 and Proposition 12.2.2, there exists an lc g-pair $(Y^n, B_{Y^n}, \mathbf{M}^{Y^n})/U$ such that $K_{Y^n} + B_{Y^n} + \mathbf{M}_{Y^n}^{Y^n} \sim_{\mathbb{R}} (K_X + B + \mathbf{M}_X)|_{Y^n}$, and the image of any lc center of $(Y^n, B_{Y^n}, \mathbf{M}^{Y^n})$ in X is an lc center of $(X, B, \mathbf{M})$. Since $\dim Y^n < \dim X$ and π is a finite morphism, by induction on $\dim X$, we have

$$(14.3) \quad R^i(f \circ \pi)_* \mathcal{O}_{Y^n}(D|_{Y^n}) = R^i f_* (\pi_* (\mathcal{O}_{Y^n}(D|_{Y^n}))) = 0.$$

Claim 14.2.6. For any positive integer i ,

$$(14.4) \quad R^i(f \circ \pi'')_* \mathcal{O}_{Y''}(D|_{Y''}) = R^i f_* (\pi''_* \mathcal{O}_{Y''}(D|_{Y''})) = 0.$$

Proof. We only need to show that,

$$R^i(f \circ \pi'')_* \mathcal{O}_{Y'' \cap Y_0^n}(D|_{Y'' \cap Y_0^n}) = R^i f_* (\pi''_* \mathcal{O}_{Y'' \cap Y_0^n}(D|_{Y'' \cap Y_0^n})) = 0$$

for any irreducible component Y_0^n of Y^n . Let Y_0^n be a connected component of Y^n , $Y_0'' := Y'' \cap Y_0^n$, $Y_0 := \pi(Y_0^n)$, and $Y_0' := \pi''(Y_0'')$. Since Y and Y' are adjacent in $S(X, B, \mathbf{M})$, either $Y_0 = Y_0'$, or Y_0' and Y_0 are adjacent in $S(X, B, \mathbf{M})$ and $Y_0' \subsetneq Y_0$. Possibly replacing Y with

Y_0 and Y' with Y'_0 , we may assume that Y is an lc center of $(X, B, \mathbf{M})$. Since Y and Y' are adjacent in $S(X, B, \mathbf{M})$, Y' is the union of all lc centers of $(X, B, \mathbf{M})$ that are contained in Y .

We let $(X', B', \mathbf{M})$ be a dlt model of $(X, B, \mathbf{M})$ with induced birational morphism $h : X' \rightarrow X$. Let S be an lc center of $(X', B', \mathbf{M})$ which is minimal in all lc centers which dominate Y , $(S, B_S, \mathbf{M}^S)/U$ the dlt g-pair induced by adjunction

$$K_S + B_S + \mathbf{M}_S^S := (K_{X'} + B' + \mathbf{M}_{X'})|_S,$$

and $h_S : S \rightarrow Y^n$ the induced morphism such that $\pi \circ h_S = h|_S$. Let

$$S \xrightarrow{\tau} Z \xrightarrow{\gamma} Y^n$$

be the Stein factorization of h_S , and let $(Z, B_Z, \mathbf{M}^Z)/U$ be the lc g-pair induced by $\tau : (S, B_S, \mathbf{M}^S) \rightarrow Z$. Let $(Y^n, B_{Y^n}, \mathbf{M}^{Y^n})/U$ be the lc g-pair induced by $\gamma : (Z, B_Z, \mathbf{M}^Z) \rightarrow Y^n$, and let $Y'_Z := \gamma^{-1}(Y^n)$.

By Lemma 4.3.11(3) and Theorem 11.4.4(5), for any lc center V of $(X, B, \mathbf{M})$ such that $V \subset Y^n$, any irreducible component of $\gamma^{-1}(V)$ is an lc center of $(Z, B_Z, \mathbf{M}^Z)$. In particular, any irreducible component of Y'_Z is an lc center of $(Z, B_Z, \mathbf{M}^Z)$, so $\text{Nklt}(Z, B_Z, \mathbf{M}^Z) \subset Y'_Z$. By Theorem 11.4.4(4), $(\pi \circ \gamma)(\text{Nklt}(Z, B_Z, \mathbf{M}^Z))$ is a union of lc centers of $(X, B, \mathbf{M})$ that are contained in Y , so $\text{Nklt}(Z, B_Z, \mathbf{M}^Z) \subset Y'_Z$. Thus

$$\text{Nklt}(Z, B_Z, \mathbf{M}^Z) = Y'_Z = \gamma^{-1}(Y^n).$$

Since $\dim Z < \dim X$, by induction on $\dim X$,

$$R^i(f \circ \pi'' \circ \gamma)_* \mathcal{O}_{Y'_Z}(D|_{Y'_Z}) = 0.$$

By Lemma 14.2.2, the claim follows. $\square$

Proof of Theorem 14.2.5 continued. By the short exact sequence (14.1), we have a short exact sequence

$$0 \rightarrow \mathcal{O}_Y(D) \xrightarrow{\pi^* \oplus i^*} \pi_* \mathcal{O}_{Y^n}(D|_{Y^n}) \oplus \mathcal{O}_{Y'}(D) \xrightarrow{j^* - \pi''^*} \pi''_* \mathcal{O}_{Y''}(D|_{Y''}) \rightarrow 0,$$

which induces a long exact sequence

$$\begin{aligned} 0 \rightarrow f_* \mathcal{O}_Y(D) \rightarrow f_* \pi_* \mathcal{O}_{Y^n}(D|_{Y^n}) \oplus f_* \mathcal{O}_{Y'}(D) &\xrightarrow{j^* - \pi''^*} f_* \pi''_* \mathcal{O}_{Y''}(D|_{Y''}) \rightarrow \cdots \\ \cdots \rightarrow R^i f_* \mathcal{O}_Y(D) \rightarrow R^i f_* (\pi_* (\mathcal{O}_{Y^n}(D|_{Y^n})) \oplus R^i f_* \mathcal{O}_{Y'}(D) &\rightarrow R^i f_* (\pi''_* \mathcal{O}_{Y''}(D|_{Y''})) \rightarrow \cdots \end{aligned}$$

Hence, it follows from (14.2), (14.3), (14.4) and Lemma 14.2.4 that $R^i f_* \mathcal{O}_Y(D) = 0$ for any positive integer i .

Step 3. In this step we prove (2) and prove (3)(4) when $Y = W = \text{Nklt}(X, B, \mathbf{M})$.

We have the long exact sequence

$$\begin{aligned} 0 \rightarrow f_*(\mathcal{I}_W \otimes \mathcal{O}_X(D)) \rightarrow f_* \mathcal{O}_X(D) \rightarrow f_* \mathcal{O}_W(D) \rightarrow \cdots \\ \cdots \rightarrow R^i f_*(\mathcal{I}_W \otimes \mathcal{O}_X(D)) \rightarrow R^i f_* \mathcal{O}_X(D) \rightarrow R^i f_* \mathcal{O}_W(D) \rightarrow \cdots \end{aligned}$$

By (1), $R^i f_* \mathcal{O}_W(D) = 0$ for any positive integer i . By Lemma 14.2.3(1), $R^i(\mathcal{I}_W \otimes f_* \mathcal{O}_X(D)) = 0$ for any positive integer i . This implies (2), and also implies (3)(4) when $Y = W$.

Step 4. We prove (3)(4) in this step, hence conclude the proof of the theorem.

We apply induction on $m(W) - m(Y)$. When $m(W) - m(Y) = 0$, $Y = W$ and we are done by **Step 3**. Thus we may assume that $m(W) - m(Y) > 0$. Then there exists a union of lc centers $\tilde{Y}$ such that $Y \subsetneq \tilde{Y} \subset W$, and Y and $\tilde{Y}$ are adjacent in $S(X, B, \mathbf{M})$.

Let $\tilde{\pi} : \tilde{Y}^n \rightarrow \tilde{Y}$ be the normalization of $\tilde{Y}$, and let $\hat{Y} := \tilde{\pi}^{-1}(Y)$ with the reduced scheme structure. Let $\tilde{i} : Y \hookrightarrow \hat{Y}$ and $\tilde{j} : \hat{Y} \hookrightarrow \tilde{Y}^n$ be the natural inclusions, and let $\hat{\pi} := \tilde{\pi}|_{\hat{Y}}$. By Lemma 14.1.3, there exists a universal push-out diagram

$$\begin{array}{ccc}
\widehat{Y} & \xrightarrow{\tilde{j}} & \tilde{Y}^n \\
\downarrow \widehat{\pi} & & \downarrow \tilde{\pi} \\
Y & \xrightarrow{\tilde{i}} & \tilde{Y}
\end{array}$$

and a short exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{Y}} \xrightarrow{\tilde{\pi}^* \oplus \tilde{i}^*} \tilde{\pi}_* \mathcal{O}_{\tilde{Y}^n} \oplus \mathcal{O}_Y \xrightarrow{\tilde{j}^* - \widehat{\pi}^*} \widehat{\pi}_* \mathcal{O}_{\widehat{Y}} \rightarrow 0.$$

which induces a short exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{Y}}(D) \xrightarrow{\tilde{\pi}^* \oplus \tilde{i}^*} \tilde{\pi}_* \mathcal{O}_{\tilde{Y}^n}(D|_{\tilde{Y}^n}) \oplus \mathcal{O}_Y(D) \xrightarrow{\tilde{j}^* - \widehat{\pi}^*} \widehat{\pi}_* \mathcal{O}_{\widehat{Y}}(D|_{\widehat{Y}}) \rightarrow 0.$$

So we have the left exact sequence

$$(14.5) \quad 0 \rightarrow f_* \mathcal{O}_{\tilde{Y}}(D) \xrightarrow{\tilde{\pi}^* \oplus \tilde{i}^*} f_* \tilde{\pi}_* \mathcal{O}_{\tilde{Y}^n}(D|_{\tilde{Y}^n}) \oplus f_* \mathcal{O}_Y(D) \xrightarrow{\tilde{j}^* - \widehat{\pi}^*} f_* \widehat{\pi}_* \mathcal{O}_{\widehat{Y}}(D|_{\widehat{Y}}).$$

By Lemma 14.2.4,

$$\tilde{j}^* : f_* \tilde{\pi}_* \mathcal{O}_{\tilde{Y}^n}(D|_{\tilde{Y}^n}) \rightarrow f_* \widehat{\pi}_* \mathcal{O}_{\widehat{Y}}(D|_{\widehat{Y}})$$

is surjective. Thus by an easy map tracing of (14.5) we have that

$$\tilde{i}^* : f_* \mathcal{O}_{\tilde{Y}}(D) \rightarrow f_* \mathcal{O}_Y(D)$$

is also surjective. Since $m(W) - m(\tilde{Y}) < m(W) - m(Y)$, by induction on $m(W) - m(Y)$,

$$f_* \mathcal{O}_X(D) \rightarrow f_* \mathcal{O}_{\tilde{Y}}(D)$$

is surjective. This implies (3).

We have the long exact sequence

$$\begin{aligned}
0 \rightarrow f_*(\mathcal{I}_Y \otimes \mathcal{O}_X(D)) &\rightarrow f_* \mathcal{O}_X(D) \rightarrow f_* \mathcal{O}_Y(D) \rightarrow \dots \\
\dots \rightarrow R^i f_*(\mathcal{I}_Y \otimes \mathcal{O}_X(D)) &\rightarrow R^i f_* \mathcal{O}_X(D) \rightarrow R^i f_* \mathcal{O}_Y(D) \rightarrow \dots,
\end{aligned}$$

so (4) follows immediately from (1)(2)(3). $\square$

14.3. Base-point-freeness theorem and contraction theorem.

Lemma 14.3.1. *Let $(X, B, \mathbf{M})/U$ be an lc g-pair and D a nef/ U Cartier divisor on X such that $aD - (K_X + B + \mathbf{M}_X)$ is ample/ U for some positive real number a . Let Y be a minimal lc center of $(X, B, \mathbf{M})$ if $(X, B, \mathbf{M})$ is not klt, and let $Y := X$ if $(X, B, \mathbf{M})$ is klt. Let $D_Y := D|_Y$. Then for any integer $m \gg 0$,*

- (1) $\mathcal{O}_Y(mD_Y)$ is globally generated over U ,
- (2) $|mD/U| \neq \emptyset$, and
- (3) Y is not contained in $\text{Bs}|mD/U|$.

Proof. When $(X, B, \mathbf{M})$ is klt, by [HL22, Lemma 3.4], there exists a klt pair (X, Δ) such that $D - (K_X + \Delta)$ is ample/ U . By the usual base-point-freeness theorem (cf. [KMM87, Theorem 3-1-1]), the lemma follows.

When $(X, B, \mathbf{M})$ is not klt, by Theorem 13.2.5, Y is normal. By Theorem 14.2.5(3), the map $f_* \mathcal{O}_X(mD) \rightarrow f_* \mathcal{O}_Y(mD_Y)$ is surjective for any positive integer $m \geq a$. Thus (2)(3) follow from (1) and we only need to prove (1). If $\dim Y = 0$ then there is nothing left to prove. If $\dim Y > 0$, then by Definition-Lemma 12.2.4, there exists a klt g-pair $(Y, B_Y, \mathbf{M}^Y)/U$ such that $K_Y + B_Y + \mathbf{M}_Y^Y \sim_{\mathbb{R}, U} (K_X + B + \mathbf{M}_X)|_Y$. Thus $D_Y - (K_Y + B_Y + \mathbf{M}_Y^Y)$ is nef/ U and log big/ U with respect to $(Y, B_Y, \mathbf{M}^Y)$. By [HL22, Lemma 3.4], there exists a klt pair (Y, Δ_Y) such that $D_Y - (K_Y + \Delta_Y)$ is ample/ U . By the usual base-point-freeness theorem (cf. [KMM87, Theorem 3-1-1]), the lemma follows. $\square$

Proof of Theorem 2.2.6. By Lemma 14.3.1, we may let m_0 be the minimal positive integer such that $|mD| \neq \emptyset$ for any integer $m \geq m_0$.

Claim 14.3.2. *Let $\{p_i\}_{i=1}^{+\infty}$ be a strictly increasing sequence of positive integers. There exist a non-negative integer M and integers $i_1 < i_2 < \dots < i_{M+1}$ satisfying the following. Let $s_k := \prod_{i=1}^k p_{i_i}$ for any $1 \leq k \leq M+1$, then*

- (1) $|s_1 D/U| \neq \emptyset$,
- (2) $\text{Bs}|s_k D/U| \supsetneq \text{Bs}|s_{k+1} D/U|$ for any $1 \leq k \leq M$, and
- (3) $\text{Bs}|s_{M+1} D/U| = \emptyset$.

Proof. We may take i_1 to be any integer such that $p_{i_1} \geq m_0$, then (1) holds.

Suppose that we have already found $i_1, \dots, i_k$ for some positive integer k . Let $d := \dim X$, let $H_1, \dots, H_{d+1}$ be $d+1$ general elements in $|s_k D/U|$, and let $H := H_1 + \dots + H_{d+1}$. Then $(X, B + H, \mathbf{M})$ is lc outside $\text{Bs}|s_k D/U|$. If $\text{Bs}|s_k D/U| = \emptyset$, then we may let $M := k-1$ and we are done. Thus we may assume that $\text{Bs}|s_k D/U| \neq \emptyset$.

Since every H_j contains $\text{Bs}|s_k D/U|$, by [Kol⁺92, Theorem 18.22], $(X, B + H, \mathbf{M})$ is not lc near $\text{Bs}|s_k D/U|$. Let

$$c := \sup\{t \mid t \geq 0, (X, B + tH, \mathbf{M}) \text{ is lc}\},$$

then $c \in [0, 1)$, and there exists at least one lc center of $(X, B + cH, \mathbf{M})$ which is contained in $\text{Bs}|s_k D/U|$. Let $\mathcal{S}$ be the set of all lc centers of $(X, B + cH, \mathbf{M})$ that are contained in $\text{Bs}|s_k D/U|$, and let Y be a minimal lc center in $\mathcal{S}$. Since

$$(a + s_k(d+1))D - (K_X + B + cH + \mathbf{M}_X) \sim_{\mathbb{R}} s_k(d+1)(1-c)D + (aD - (K_X + B + \mathbf{M}_X))$$

is ample/ U , by Lemma 14.3.1, there exists a positive integer n , such that for any integer $m \geq n$, $|ms_k D/U| \neq \emptyset$ and $\text{Bs}|ms_k D/U|$ does not contain Y . In particular, $\text{Bs}|ms_k D/U| \subsetneq \text{Bs}|s_k D/U|$. We may let i_{k+1} be any integer such that $i_{k+1} > i_k$ and $p_{i_{k+1}} \geq n$. This construction implies (2). (3) follows from (2) and the Noetherian property. $\square$

Proof of Theorem 2.2.6 continued. We let p and q be two different prime numbers. By Claim 14.3.2, there exist two non-negative integers M, N such that $\mathcal{O}_X(p^M D)$ and $\mathcal{O}_X(q^N D)$ are globally generated/ U . Since p^M and q^N are coprime, for any integer $m \gg 0$, we may write $m = bp^M + cq^N$ for some non-negative integers b, c , hence

$$\text{Bs}|mD/U| \subset \text{Bs}|p^M D/U| \cup \text{Bs}|q^N D/U| = \emptyset.$$

Therefore, $\mathcal{O}_X(mD)$ is globally generated over U for any integer $m \gg 0$. $\square$

Theorem 14.3.3 (Contraction theorem for lc generalized pairs, cf. [Xie22, Theorem 1.5]). *Let $(X, B, \mathbf{M})/U$ be an lc generalized pair and F a $(K_X + B + \mathbf{M}_X)$ -negative extremal face/ U . Then there exists a contraction/ U $\text{cont}_F : X \rightarrow Z$ of F satisfying the following.*

- (1) *For any integral curve C on X such that the image of C in U is a closed point, $\text{cont}_F(C)$ is a point if and only if $[C] \in F$.*
- (2) *$\mathcal{O}_Y = (\text{cont}_F)_* \mathcal{O}_X$. In other words, cont_F is a contraction.*
- (3) *For any Cartier divisor D on Y such that $D \cdot C = 0$ for any curve C contracted by cont_F , there exists a Cartier divisor D_Y on Y such that $D = \text{cont}_F^* D_Y$.*

Proof of Theorem 14.3.3. (1)(2) By Theorem 2.3.1, F is a finitely dimensional rational $(K_X + B + \mathbf{M}_X)$ -negative extremal face/ U . Thus there exists a nef Cartier divisor L on X that is the supporting function of F . Then $L - (K_X + B + \mathbf{M}_X)$ is ample. By Theorem 2.2.6, mL is base-point-free/ U , hence defines a contraction/ U . Denote this contraction by cont_F . Then cont_F satisfies (1) and (2).

(3) Since $D - (K_X + B + \mathbf{M}_X)$ is ample/ Z , by Theorem 2.2.6, $\mathcal{O}_X(mD)$ is globally generated over Z for any integer $m \gg 0$. Since $D \cdot C$ for any curve C contracted by cont_F , cont_F is defined by $|mD|$ for any integer $m \gg 0$. Thus $mD = f^* D_{Y,m}$ and $(m+1)D = f^* D_{Y,m+1}$ for any integer $m \gg 0$. We may let $D_Y := D_{Y,m+1} - D_{Y,m}$. $\square$

Proof of Theorem 2.2.7. We write $D = \sum_{i=1}^c r_i D_i$ where $r_1, \dots, r_c$ are linearly independent over $\mathbb{Q}$ and each D_i is a $\mathbb{Q}$ -divisor. We define $D(\mathbf{v}) := \sum_{i=1}^c v_i D_i$ for any $\mathbf{v} = (v_1, \dots, v_c) \in \mathbb{R}^c$, and let $\mathbf{r} := (r_1, \dots, r_c)$. By [HLS19, Lemma 5.3], each D_i is $\mathbb{Q}$ -Cartier, so $D(\mathbf{v})$ is $\mathbb{Q}$ -Cartier for any $\mathbf{v} \in \mathbb{R}^c$.

Let $L := D - (K_X + B + \mathbf{M}_X)$. Since ample/ U is an open condition, there exists an open set $V \ni \mathbf{r}$ in $\mathbb{R}^c$, such that $\frac{1}{2}L + D(\mathbf{v}) - D$ is ample/ U for any $\mathbf{v} \in V$.

By Theorem 2.3.1, there exist finitely many $(K_X + B + \mathbf{M}_X + \frac{1}{2}L)$ -negative extremal rays/ U $R_1, \dots, R_l$, and each $R_j = \mathbb{R}_+[C_j]$ for some rational curve C_j such that

$$-2 \dim X \leq (K_X + B + \mathbf{M}_X + \frac{1}{2}L) \cdot C_j < 0.$$

Since D is nef, $D \cdot C_j \geq 0$ for each j . Thus possibly shrinking V , we may assume that for any $\mathbf{v} \in V$, we have that $D(\mathbf{v}) \cdot C_j > 0$ for any j such that $D \cdot C_j > 0$. Since $r_1, \dots, r_c$ are linearly independent over $\mathbb{Q}$, for any j such that $D \cdot C_j = 0$, we have $D(\mathbf{v}) \cdot C_j = 0$ for any $\mathbf{v} \in \mathbb{R}^c$. Therefore, $D(\mathbf{v}) \cdot C_j \geq 0$ for any j and any $\mathbf{v} \in V$.

For any extremal ray R in $\overline{NE}(X/U)$ and any $\mathbf{v} \in V$, if $R = R_j$ for some j , then $D(\mathbf{v}) \cdot R_j \geq 0$. If $R \neq R_j$ for any j , then

$$D(\mathbf{v}) \cdot R_j = (K_X + B + \mathbf{M}_X + \frac{1}{2}L) \cdot R + (\frac{1}{2}L + D(\mathbf{v}) - D) \cdot R > 0.$$

Therefore, $D(\mathbf{v})$ is nef/ U for any $\mathbf{v} \in V$. Moreover,

$$D(\mathbf{v}) - (K_X + B + \mathbf{M}_X) = \frac{1}{2}L + \frac{1}{2}L + D(\mathbf{v}) - D$$

is ample/ U .

We let $\mathbf{v}_1, \dots, \mathbf{v}_{c+1} \in V \cap \mathbb{Q}^c$ be rational points such that $\mathbf{r}$ is in the interior of the convex hull of $\mathbf{v}_1, \dots, \mathbf{v}_{c+1}$. Then there exists positive real numbers $a_1, \dots, a_{c+1}$ such that $\sum_{i=1}^{c+1} a_i = 1$ and $\sum_{i=1}^{c+1} a_i \mathbf{v}_i = \mathbf{r}$. Since $D(\mathbf{v}_i)$ is a nef/ U $\mathbb{Q}$ -divisor and $D(\mathbf{v}_i) - (K_X + B + \mathbf{M}_X)$ is ample/ U , by Theorem 2.2.6, $D(\mathbf{v}_i)$ is semi-ample/ U for any i . Therefore, $D = \sum a_i D(\mathbf{v}_i)$ is semi-ample/ U . $\square$

15. EXISTENCE OF FLIPS FOR GENERALIZED PAIRS

The goal of this section is to show the existence of flips for $\mathbb{Q}$ -factorial lc generalized pairs. We remark that [HL21a, Theorem 1.2] proves the case when the generalized pairs are NQC. Our proof does not rely on the results in [HL21a].

The following lemma is crucial for the proof of the existence of flips.

Lemma 15.0.1. *Let $(X, B, \mathbf{M})/U$ be an lc g -pair such that the induced morphism $\pi : X \rightarrow U$ is birational. Assume that there exists a non-empty open subset $U^0 \subset U$, such that*

- (1) *all lc centers of $(X, B, \mathbf{M})$ intersect $X^0 := X \times_U U^0$, and*
- (2) *$\mathbf{M}^0 := \mathbf{M} \times_U U^0$ descends to X^0 , and $\mathbf{M}_{X^0}^0 \sim_{\mathbb{R}, U^0} 0$.*

Then there exists an $\mathbb{R}$ -divisor $0 \leq G \sim_{\mathbb{R}, U} \mathbf{M}_X$ such that $(X, B + G)$ is lc and $\text{Nklt}(X, B + G) = \text{Nklt}(X, B, \mathbf{M})$.

Proof. Let $f : \tilde{X} \rightarrow X$ be a log resolution of $(X, \text{Supp } B)$ such that $\mathbf{M}$ descends to $\tilde{X}$, we may write

$$K_{\tilde{X}} + \tilde{B} + \mathbf{M}_{\tilde{X}} = f^*(K_X + B + \mathbf{M}_X)$$

for some $\mathbb{R}$ -divisor $\tilde{B}$. Since π is birational, $\mathbf{M}_{\tilde{X}}$ is big/ U and nef/ U . Thus there exists a $\mathbb{Q}$ -divisor $E \geq 0$, such that for any positive integer k , there exists an ample/ U $\mathbb{R}$ -divisor A_k on $\tilde{X}$ such that

$$\mathbf{M}_{\tilde{X}} = A_k + \frac{1}{k}E.$$

Moreover, for any $k \gg 1$, $\text{Nklt}(X, \tilde{B} + \frac{1}{k}E)$ is contained in the strata of $[\tilde{B}]$. By [HLS19, Lemma 5.3], there exist $\mathbb{Q}$ -divisors $\tilde{M}_i$ and positive real numbers a_i , such that

- $\sum a_i = 1$,
- $\mathbf{M}_{\tilde{X}} = \sum a_i \tilde{M}_i$,
- $\tilde{M}_i - \mathbf{M}_{\tilde{X}} + A_k$ is ample/ U for each i , and
- $\tilde{M}_i|_{\tilde{X}^0} \sim_{\mathbb{Q}, U^0} 0$, where $\tilde{X}^0 := \tilde{X} \times_U U_0$.

Let m be a positive integer such that $m\tilde{M}_i$ is a Weil divisor and $m\tilde{M}_i|_{\tilde{X}^0} \sim_{U^0} 0$ for each i . Then there exists a very ample divisor $H \geq 0$ on U , such that $\phi_* \mathcal{O}_{\tilde{X}}(m\tilde{M}_i) \otimes H$ is globally generated for each i , where $\phi := \pi \circ f$. In particular, $\mathcal{O}_{\tilde{X}}(m\tilde{M}_i) \otimes \phi^* H$ is globally generated over U^0 . Thus for any general element $D_i \in |m\tilde{M}_i + \phi^* H|$, any non-klt center of $(X, \tilde{B} + \frac{1}{m} \sum a_i D_i)$ is contained in $\tilde{X} \setminus \tilde{X}^0$.

Since

$$m\tilde{M}_i + \phi^* H = m(\tilde{M}_i - \mathbf{M}_{\tilde{X}} + A_k) + \frac{m}{k}E + \phi^* H,$$

possibly replacing m by a multiple, we may assume that $\frac{m}{k}E$ is Cartier. Thus $m(\tilde{M}_i - \mathbf{M}_{\tilde{X}} + A_k)$ is ample/ U and Cartier, and for any general element $\tilde{G}_i \in |m\tilde{M}_i + \phi^* H|$, any non-klt center of $(X, \tilde{B} + \frac{1}{m} \sum a_i \tilde{G}_i)$ is a stratum of $[\tilde{B}]$ which intersects $\tilde{X}^0$.

Therefore, for each i , we may choose $\tilde{G}_i \in |m\tilde{M}_i + \phi^* H|$, such that

$$\left(X, \tilde{B} + \frac{1}{m} \sum a_i \tilde{G}_i \right)$$

is sub-lc. Let $\tilde{G} = \frac{1}{m} \sum a_i \tilde{G}_i$, and $G := f_* \tilde{G}$. Then $0 \leq G \sim_{\mathbb{R}, U} \mathbf{M}_X$, $(X, B + G)$ is lc, and $\text{Nklt}(X, B + G) = \text{Nklt}(X, B, \mathbf{M})$. $\square$

Definition 15.0.2 (Flipping contraction). Let $X \rightarrow U$ be a projective morphism such that X is normal quasi-projective and D an $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor on X . A D -*flipping contraction* over U is a contraction $f : X \rightarrow Z$ over U satisfying the following:

- (1) X is $\mathbb{Q}$ -factorial,
- (2) f is a small birational morphism, and
- (3) f is the contraction of a D -negative extremal ray R in $\overline{NE}(X/U)$. In particular, $\rho(X/Z) = 1$.

Definition 15.0.3 (Flip). Let X be a normal quasi-projective variety, D an $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor on X , and $f : X \rightarrow Z$ a D -flipping contraction. A D -*flip* is a birational contraction $f^+ : X^+ \rightarrow Z$ satisfying the following.

- (1) X^+ is a normal quasi-projective variety,
- (2) f^+ is small, and
- (3) D^+ is $\mathbb{R}$ -Cartier and ample/ Z , where D^+ is the strict transform of D on X^+ .

We call f^+ the *flip* of f .

Lemma 15.0.4. Let X be a normal quasi-projective variety and D and D' two $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisors on X . Let $f : X \rightarrow Z$ a D -flipping contraction such that $D \equiv_Z rD'$ for some real number $r > 0$. Then:

- (1) f is a D' -flipping contraction.
- (2) Suppose that $f^+ : X^+ \rightarrow Z$ is a D' -flip. Assume that either $D \sim_{\mathbb{R}, Z} rD'$, or $D = K_X + B + \mathbf{M}_X$ for some lc g -pair $(X, B, \mathbf{M})/Z$. Then f^+ is also a D -flip.

Proof. Since $D \sim_{\mathbb{R}, U} rD'$, the unique D -negative extremal ray in $\overline{NE}(X/Z)$ is also a D' -negative extremal ray, and we get (1).

By Theorem 14.3.3(3) we may assume that $D \sim_{\mathbb{R}, Z} rD'$. Let D^+ and D'^+ be the strict transform of D and D' on X^+ respectively. We have $D - rD' \sim_{\mathbb{R}} f^* L$ for some $\mathbb{R}$ -Cartier

$\mathbb{R}$ -divisor L on Z . Since $X \dashrightarrow X^+$ is small, $D^+ - rD'^+ \sim_{\mathbb{R}} (f^+)^*L$. Since D^+ is $\mathbb{R}$ -Cartier and ample/ Z , D'^+ is $\mathbb{R}$ -Cartier and ample/ Z . This implies (2). $\square$

Lemma 15.0.5. *Let $(X, B, \mathbf{M})/U$ be an lc g-pair and $f : X \rightarrow Z$ a $(K_X + B + \mathbf{M}_X)$ -flipping contraction/ U . Suppose that the flip $f^+ : X^+ \rightarrow Z$ of f exists. Then:*

- (1) f^+ is unique.
- (2) For any $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor D on X , the strict transform of D on X^+ is $\mathbb{R}$ -Cartier.
- (3) If X is $\mathbb{Q}$ -factorial, then X^+ is $\mathbb{Q}$ -factorial and $\rho(X) = \rho(X^+)$.

Proof. Let H be an anti-ample/ Z divisor on X . Since $\rho(X/Z) = 1$ and $K_X + B + \mathbf{M}_X$ is anti-ample/ Z , there exist a positive real number r and a real number s such that $H - r(K_X + B + \mathbf{M}_X) \equiv_Z 0$ and $D - s(K_X + B + \mathbf{M}_X) \equiv_Z 0$. By Theorem 14.3.3(3), $H - r(K_X + B + \mathbf{M}_X) \sim_{\mathbb{R}, Z} 0$ and $D - s(K_X + B + \mathbf{M}_X) \sim_{\mathbb{R}, Z} 0$.

- (1) By Lemma 15.0.4, f^+ is an H -flip. Thus

$$X^+ = \text{Proj} \left(\bigoplus_{m=0}^{+\infty} f_* \mathcal{O}_X(mH) \right)$$

is unique.

(2) We have $D - s(K_X + B + \mathbf{M}_X) \sim_{\mathbb{R}} f^*L$ for some $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor L on Z . Let D^+ and B^+ be the strict transform of D and B on X^+ respectively. Since $X \dashrightarrow X^+$ is small, $D^+ - s(K_{X^+} + B^+ + \mathbf{M}_{X^+}) \sim_{\mathbb{R}} (f^+)^*L$. Since $K_{X^+} + B^+ + \mathbf{M}_{X^+}$ is $\mathbb{R}$ -Cartier, D^+ is $\mathbb{R}$ -Cartier.

(3) Since $X \dashrightarrow X^+$ is an isomorphism in codimension 1, there is a natural isomorphism between the groups of Weil divisors on X and X^+ . By (2), if X is $\mathbb{Q}$ -factorial, then X^+ is $\mathbb{Q}$ -factorial. Since X and X^+ are both $\mathbb{Q}$ -factorial, $\rho(X) = \rho(X^+)$. $\square$

Theorem 15.0.6 (Existence of flips). *Let $(X, B, \mathbf{M})/U$ be an lc g-pair and $f : X \rightarrow Z$ a $(K_X + B + \mathbf{M}_X)$ -flipping contraction/ U . Assume that $\mathbf{M}_X$ is $\mathbb{R}$ -Cartier. Then the $(K_X + B + \mathbf{M}_X)$ -flip $f^+ : X^+ \rightarrow Z$ of f exists. Moreover,*

- (1) $\rho(X^+/Z) = 1$,
- (2) $\mathbf{M}_{X^+}$ is $\mathbb{R}$ -Cartier, and
- (3) If X is $\mathbb{Q}$ -factorial, then X^+ is $\mathbb{Q}$ -factorial and $\rho(X) = \rho(X^+)$.

Proof of Theorem 15.0.6. Let $h : \tilde{X} \rightarrow X$ be a birational morphism such that $\mathbf{M}$ descends to $\tilde{X}$. Since $\mathbf{M}_X$ is $\mathbb{R}$ -Cartier and $\mathbf{M}_{\tilde{X}}$ is nef/ X , we have

$$\mathbf{M}_{\tilde{X}} + E = h^* \mathbf{M}_X$$

for some $E \geq 0$ that is exceptional over X . Let C be any flipping curve contracted by f .

There are two cases:

Case 1. $\mathbf{M}_X \cdot C \geq 0$. Then $(K_X + B) \cdot C < 0$, and f is also a $(K_X + B)$ -flipping contraction. By [Bir12, Corollary 1.2], [HX13, Corollary 1.8], there exists a $(K_X + B)$ -flip $f^+ : X^+ \rightarrow Z$, such that $\rho(X^+/Z) = 1$. (1) follows. By Lemma 15.0.4, $f^+ : X^+ \rightarrow Z$ is a $(K_X + B + \mathbf{M}_X)$ -flip. (2-3) follow from Lemma 15.0.5.

Case 2. $\mathbf{M}_X \cdot C < 0$. In this case, $C \subset h(E)$. Let $Z^0 := Z \setminus \{f(h(\text{Supp } E))\}$, $X^0 := X \times_Z Z^0$, $B^0 := B \times_Z Z^0$, and $\mathbf{M}^0 := \mathbf{M} \times_Z Z^0$. Since $h(\text{Supp } E)$ does not contain any lc center of $(X, B, (1 - \epsilon)\mathbf{M})$, for any $\epsilon \in (0, 1)$,

- all lc centers of $(X, B, (1 - \epsilon)\mathbf{M})$ intersect X^0 ,
- $\mathbf{M}^0$ descends to X^0 and $\mathbf{M}_{X^0}^0 \sim_{\mathbb{R}, Z^0} 0$.

Let $\epsilon_0 \in (0, 1)$ be a real number such that f is also a $(K_X + B + (1 - \epsilon_0)\mathbf{M}_X)$ -flipping contraction. By Lemma 15.0.1, there exists an $\mathbb{R}$ -divisor $G \sim_{\mathbb{R}, Z} (1 - \epsilon_0)\mathbf{M}_X$ such that $(X, B + G)$ is lc. By Lemma 15.0.4, f is a $(K_X + B + G)$ -flip. By [Bir12, Corollary 1.2], [HX13, Corollary 1.8], the flip $f^+ : X^+ \rightarrow Z$ of f exists and $\rho(X^+/Z) = 1$. (1) follows. By Lemma 15.0.4, f^+ is a $(K_X + B + \mathbf{M}_X)$ -flip. (2-3) follow from Lemma 15.0.5. $\square$

Part IV. Good minimal model and the proofs of the main theorems

16. EXISTENCE OF GOOD MINIMAL MODELS AND $\mathbf{b}$ -SEMI-AMPLENESS

16.1. Good minimal models for polarized foliations. We note that the subsequent lemma does not necessarily require $\mathcal{F}$ to be algebraically integrable, allowing its application to other scenarios.

Lemma 16.1.1. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be an lc gfq and A an ample/ U $\mathbb{R}$ -divisor on X . Let*

$$\phi : (X, \mathcal{F}, B + A, \mathbf{M}) \dashrightarrow (X', \mathcal{F}', B' + A', \mathbf{M})$$

be a sequence of steps of a $(K_{\mathcal{F}} + B + A + \mathbf{M}_X)$ -MMP/ U , where B' and A' are the images of B and A on X' respectively. Then there exist a nef/ U $\mathbf{b}$ -divisor $\mathbf{N}$ and an ample/ U $\mathbb{R}$ -divisor $\tilde{A}'$ on X' , such that

- (1) $(X', \mathcal{F}', B', \mathbf{N})/U$ is lc,
- (2) $\mathbf{N}_{X'} + \tilde{A}' \sim_{\mathbb{R}, U} \mathbf{M}_{X'} + A'$,
- (3) $\mathbf{N} - \mathbf{M}$ is nef/ U , and
- (4) for any contraction $f : X \rightarrow Z$ and divisor G on X such that $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ satisfies Property (*) (resp. is weak ACSS, is ACSS), $(X', \mathcal{F}', B', \mathbf{N}; G' := \phi_*G)/Z$ satisfies Property (*) (resp. is weak ACSS, is ACSS).

Proof. We may assume that ϕ is a single step of a MMP/ U , and we have the following diagram/ U

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ & \searrow g & \swarrow h \\ & T & \end{array}$$

such that either $\phi = g$ is a divisorial contraction, or ϕ is a flip, g is the flipping contraction, and h is the flipped contraction. Then there exists an ample/ T divisor H on X such that $K_{\mathcal{F}} + B + A + H + \mathbf{M}_X \sim_{\mathbb{R}, T} 0$. Let $H' := \phi_*H$, then $-H'$ is ample/ T . Since A is ample/ U , there exists an ample/ U $\mathbb{R}$ -divisor C on T such that $A - g^*C$ is ample/ U .

Let $0 < \epsilon \ll 1$ be a real number. Then $\tilde{A}' := h^*C - \epsilon H'$ and $L := A - g^*C + \epsilon H$ are ample/ U , and ϕ is a step of a $(K_{\mathcal{F}} + B + A + \mathbf{M}_X + \epsilon H)$ -MMP/ T . Since

$$K_{\mathcal{F}} + B + A + \mathbf{M}_X \sim_{\mathbb{R}, T} K_{\mathcal{F}} + B + L + \mathbf{M}_X - \epsilon H,$$

ϕ is a step of a $(K_{\mathcal{F}} + B + L + \mathbf{M}_X)$ -MMP/ T , hence a step of a $(K_{\mathcal{F}} + B + L + \mathbf{M}_X)$ -MMP/ U . Let $L' := \phi_*L$, then

$$\phi_*L + \tilde{A}' = A'.$$

We let $\mathbf{N} := \mathbf{M} + \tilde{L}$. By our construction, $\mathbf{N}$ and $\tilde{A}'$ satisfy (2) and (3). Since $\mathbf{N} - \mathbf{M}$ descends to X and $(X, \mathcal{F}, B, \mathbf{M})$ is lc, $(X, \mathcal{F}, B, \mathbf{N})$ is lc. Since ϕ is a step of a $(K_{\mathcal{F}} + B + \mathbf{N}_X)$ -MMP/ U , $(X', \mathcal{F}', B', \mathbf{N})$ is lc, which implies (1). If $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ satisfies Property (*) (resp. is weak ACSS, is ACSS), then $(X, \mathcal{F}, B, \mathbf{N}; G)/Z$ satisfies Property (*) (resp. is weak ACSS, is ACSS). (4) follows from Lemma 9.1.4. $\square$

Lemma 16.1.2. *Let $(X, B, \mathbf{M})/U$ be an lc g-pair and $f : X \rightarrow Z$ a contraction such that B is super/ Z . Assume that $\phi : X \rightarrow T$ is a contraction/ U such that $K_X + B + \mathbf{M}_X \sim_{\mathbb{R}, T} 0$ and ϕ is also a contraction/ Z . Let B_T be the discriminant part of $f : (X, B, \mathbf{M}) \rightarrow T$. Then B_T is super/ Z .*

Proof. Let $d := \dim X$. Since B is super/ Z , there exist ample Cartier divisors $H_1, \dots, H_{2d+1}$ on Z such that $B \geq \sum_{i=1}^{2d+1} f^*H_i$. In particular, $B - \sum_{i=1}^{2d+1} f^*H_i \geq 0$, and

$$K_X + B - \sum_{i=1}^{2d+1} f^*H_i + \mathbf{M}_X \sim_{\mathbb{R}, T} 0.$$

Let B'_T be the discriminant part of $\phi : (X, B - \sum_{i=1}^{2d+1} f^* H_i, \mathbf{M}) \rightarrow T$ and let $\psi : T \rightarrow Z$ be the induced contraction. Then $B_T = B'_T + \sum_{i=1}^{2d+1} \psi^* H_i$, so B_T is super/ Z . $\square$

Theorem 16.1.3. *Let d and m be two positive integers, $(X, \mathcal{F}, B, \mathbf{M})/U$ an lc gfq of dimension d , and $A \geq 0$ an ample/ U $\mathbb{R}$ -divisor on X , such that*

- $\mathcal{F}$ is induced by a contraction $f : X \rightarrow Z$,
- $K_{\mathcal{F}} + B + A + \mathbf{M}_X$ is nef/ U , and
- $K_{\mathcal{F}} + B + \mathbf{M}_X \sim_{\mathbb{R}, Z} K_X + \Delta + \mathbf{N}_X$ for some lc g-pair $(X, \Delta, \mathbf{N})/U$.

Then the followings hold.

- (1) $K_{\mathcal{F}} + B + A + \mathbf{M}_X$ is semi-ample/ U .
- (2) The contraction/ U defined by $K_{\mathcal{F}} + B + A + \mathbf{M}_X$ is a contraction/ Z .
- (3) Suppose that

$$m(K_{\mathcal{F}} + B + \mathbf{M}_X) \sim_Z m(K_X + \Delta + \mathbf{N}_X)$$

and $m(K_{\mathcal{F}} + B + A + \mathbf{M}_X)$ is Cartier. Then

$$\mathcal{O}_X(nm(K_{\mathcal{F}} + B + A + \mathbf{M}_X))$$

is globally generated/ U for any integer $n \gg 0$.

Proof. Let $\pi : X \rightarrow U$ be the induced morphism and let H' be a sufficiently ample Cartier divisor on U . Possibly replacing A with $A + \pi^* H'$, we may assume that A is ample. Let $H_1, \dots, H_{2 \dim X + 1}$ be ample Cartier divisor on Z . Possibly replacing Δ with $\Delta + \sum_{i=1}^{2 \dim X + 1} f^* H_i$, we may assume that Δ is super/ Z . By Lemma 9.2.2, $K_X + \Delta + A + \mathbf{N}_X$ is nef/ U . By Theorem 2.2.7, $K_X + \Delta + A + \mathbf{N}_X$ is semi-ample/ U , so $K_X + \Delta + A + \mathbf{N}_X$ defines a contraction/ U $\phi : X \rightarrow T$. Since Δ is super/ Z , by the the length of extremal rays (Theorem 2.3.1(2)), any $(K_X + \Delta + A + \mathbf{N}_X)$ -trivial extremal ray in $\overline{NE}(X/U)$ is an extremal ray/ Z , so ϕ is a contraction/ Z . Therefore, $K_{\mathcal{F}} + B + A + \mathbf{M}_X \sim_{\mathbb{R}, T} 0$. Let $\mathcal{F}_T$ be the foliation induced by the induced contraction $\psi : T \rightarrow Z$, then $\mathcal{F} = \phi^{-1} \mathcal{F}_T$.

We let H_T be a general ample/ U $\mathbb{R}$ -divisor on T such that $H := A - \phi^* H_T$ is ample/ U . By Theorem 2.3.2, there exist an lc gfq $(T, \mathcal{F}_T, B_T, \mathbf{M}_T^T)/U$ induced by a canonical bundle formula of $\phi : (X, \mathcal{F}, B, \overline{H} + \mathbf{M}) \rightarrow T$, and an lc g-pair $(T, \Delta_T, \mathbf{N}_T^T)/U$ induced by a canonical bundle formula of $\phi : (X, \Delta, \overline{H} + \mathbf{N}) \rightarrow T$. By Lemma 16.1.2, Δ_T is super/ Z .

We have

$$K_{\mathcal{F}_T} + B_T + \mathbf{M}_T^T \sim_{\mathbb{R}, Z} K_T + \Delta_T + \mathbf{N}_T^T.$$

Since ϕ is the morphism/ U defined by $K_X + \Delta + A + \mathbf{N}_X$, $K_T + \Delta_T + H_T + \mathbf{N}_T^T$ is ample/ U . Thus $K_T + \Delta_T + (1 - \delta)H_T + \mathbf{N}_T^T$ is ample/ U for any $0 < \delta \ll 1$. By Theorem 2.3.1, $(K_{\mathcal{F}_T} + B_T + (1 - \delta)H_T + \mathbf{M}_T^T)$ is nef/ U . Thus $K_{\mathcal{F}_T} + B_T + H_T + \mathbf{M}_T^T$ is ample/ U , so $K_{\mathcal{F}} + B + A + \mathbf{M}_X$ is semi-ample/ U , and ϕ is the contraction/ U defined by $K_{\mathcal{F}} + B + A + \mathbf{M}_X$. This implies (1)(2).

We prove (3). Since ϕ is a contraction defined by $K_X + \Delta + A + \mathbf{N}_X$, $m(K_{\mathcal{F}} + B + A + \mathbf{M}_X)$ is Cartier, and $K_{\mathcal{F}} + B + A + \mathbf{M}_X \sim_{\mathbb{Q}, T} 0$, by Theorem 14.3.3(3), there exists a Cartier divisor L on T such that

$$m(K_{\mathcal{F}} + B + A + \mathbf{M}_X) = \phi^* L.$$

Since $L \sim_{\mathbb{R}} K_{\mathcal{F}_T} + B_T + H_T + \mathbf{M}_T^T$, L is ample/ U . Thus nL is very ample/ U for any integer $n \gg 0$, so

$$\mathcal{O}_X(nm(K_{\mathcal{F}} + B + A + \mathbf{M}_X)) = \mathcal{O}_X(\phi^*(nL))$$

is globally generated/ U for any integer $n \gg 0$. $\square$

Theorem 16.1.4. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be an lc gfq, and A, H two ample/ U $\mathbb{R}$ -divisors on X . Assume that*

- $\mathcal{F}$ is induced by a contraction $X \rightarrow Z$,
- $K_{\mathcal{F}} + B + A + \mathbf{M}_X$ is pseudo-effective/ U ,
- $K_{\mathcal{F}} + B + \mathbf{M}_X \sim_{\mathbb{R}, Z} K_X + \Delta + \mathbf{N}_X$ for some lc g-pair $(X, \Delta, \mathbf{N})/U$, and

- either X is $\mathbb{Q}$ -factorial klt or $\mathbf{M}$ is NQC/U .

Then there exists a $(K_{\mathcal{F}} + B + A + \mathbf{M}_X)$ -MMP/ U with scaling of H , say $\mathcal{P}_0$, satisfying the following. Let $\mathcal{P} = \mathcal{P}_0$ if X is not $\mathbb{Q}$ -factorial, and let $\mathcal{P}$ be any $(K_{\mathcal{F}} + B + A + \mathbf{M}_X)$ -MMP/ U with scaling of an ample/ U $\mathbb{R}$ -divisor if X is $\mathbb{Q}$ -factorial. Then

- (1) $\mathcal{P}$ terminates at a model X' such that $K_{\mathcal{F}'} + B' + A' + \mathbf{M}_X$ is semi-ample/ U , where B', A' are the images of B, A on X' respectively, and $\mathcal{F}'$ is the pushforward of $\mathcal{F}$ on X' , and
- (2) the contraction/ U defined by $K_{\mathcal{F}'} + B' + A' + \mathbf{M}_{X'}$ is a contraction/ Z .

Proof. Let $\mathbf{M}' := \mathbf{M} + \bar{A}$. Then $\mathcal{P}$ is a $(K_{\mathcal{F}} + B + \mathbf{M}'_X)$ -MMP/ U and $(X, \mathcal{F}, B, \mathbf{M}')$ is lc. By Proposition 9.3.2, $\mathcal{P}$ terminates with a weak lc model $(X', \mathcal{F}', B', \mathbf{M}')/U$ of $(X, \mathcal{F}, B, \mathbf{M}')/U$. By Lemma 16.1.1, there exists a nef/ U $\mathbf{b}$ -divisor $\mathbf{M}''$ and an ample/ U $\mathbb{R}$ -divisor A'' such that $\mathbf{M}''_{X'} + A'' \sim_{\mathbb{R}, U} \mathbf{M}_{X'} + A'$ and $(X', \mathcal{F}', B', \mathbf{M}'')/Z$ is lc. By Lemma 9.1.4, $\mathcal{P}$ is also a $(K_X + \Delta + A + \mathbf{N}_X)$ -MMP/ U . Since $(X, \Delta, \mathbf{N} + \bar{A})$ is lc, $(X', \Delta', \mathbf{N} + \bar{A})$ is lc, where Δ' is the image of Δ on X' . Moreover,

$$K_{\mathcal{F}'} + B' + A'' + \mathbf{M}''_{X'} \sim_{\mathbb{R}, Z} K_{X'} + \Delta' + \mathbf{N}_{X'} + \bar{A}_{X'}.$$

The theorem follows from Theorem 16.1.3. $\square$

16.2. A special case of Prokhorov-Shokurov's effective $\mathbf{b}$ -semi-ampleness conjecture.

Theorem 16.2.1. *Let $(X, B, \mathbf{M})/U$ be an lc g -pair and $f : (X, B, \mathbf{M}) \rightarrow Z$ a contraction satisfying Property (*). Let $\mathbf{N}$ be the moduli part of $f : (X, B, \mathbf{M}) \rightarrow Z$. Assume that*

- (1) f is equi-dimensional,
- (2) $K_X + B + \mathbf{M}_X$ is nef/ Z ,
- (3) $(X, B, \mathbf{M})$ is BP semi-stable/ Z , and
- (4) there exists an ample/ U $\mathbb{R}$ -divisor H such that either $B^h \geq H$ or $\mathbf{M} - \bar{H}$ is nef/ U , where B^h the horizontal/ Z part of B .

Then $\mathbf{N}$ descends to X and $\mathbf{N}_X$ is semi-ample/ U .

Proof. Let $\mathcal{F}$ be the foliation induced by f . By Proposition 11.1.1, $(X, \mathcal{F}, B^h, \mathbf{M})$ is lc and $(X, \mathcal{F}, B^h, \mathbf{M}; B - B^h)/Z$ is weak ACSS. By Theorem 11.1.5, $(X, B, \mathbf{M})$ is BP stable/ Z . By Proposition 11.1.3, $\mathbf{N}$ descends to X and is nef/ U . By Proposition 7.3.6, $K_{\mathcal{F}} + B^h + \mathbf{M}_X = \mathbf{N}_X$ is nef/ U . By Theorem 16.1.3, $\mathbf{N}_X = K_{\mathcal{F}} + B^h + \mathbf{M}_X$ is semi-ample/ U . $\square$

When we have an lc-trivial fibration, we can prove stronger $\mathbf{b}$ -semi-ampleness.

Theorem 16.2.2. *Let d and m be two positive integers. Then there exists a positive integer I depending only on d and m satisfying the following.*

Assume that $(X, B, \mathbf{M})/U$ is an lc g -pair and $f : (X, B, \mathbf{M}) \rightarrow Z$ is a contraction/ U satisfying Property (). Let $\mathbf{N}$ be the moduli part of $f : (X, B, \mathbf{M}) \rightarrow Z$. Assume that*

- (1) f is equi-dimensional,
- (2) X is of Fano type over Z ,
- (3) $K_X + B + \mathbf{M}_X \sim_{\mathbb{Q}, Z} 0$,
- (4) $(X, B, \mathbf{M})$ is BP semi-stable/ Z ,
- (5) mB is a Weil divisor and $m\mathbf{M}$ is $\mathbf{b}$ -base-point-free/ U , and
- (6) there exists an ample/ U $\mathbb{R}$ -divisor H such that either $B^h \geq H$ or $\mathbf{M} - \bar{H}$ is nef/ U , where B^h the horizontal/ Z part of B .

Then $\mathbf{N}$ descends to X , $I\mathbf{N}_X$ is Cartier, and $\mathcal{O}_X(nI\mathbf{N}_X)$ is globally generated/ U for any integer $n \gg 0$.

Proof. Let B_Z be the discriminant part of $f : (X, B, \mathbf{M}) \rightarrow Z$. By [Has22, Lemma 4.2] (cf. [Bir19, Proposition 6.3]), there exist a positive integer q depending only on d and m and a choice $\mathbf{M}^Z$ of the moduli part of $f : (X, B, \mathbf{M}) \rightarrow Z$, such that

$$q(K_X + B + \mathbf{M}_X) \sim qf^*(K_Z + B_Z + \mathbf{M}_Z^Z)$$

and $q\mathbf{M}^Z$ is nef/ U . Since $f : (X, B, \mathbf{M}) \rightarrow Z$ satisfies Property (*), Z is smooth and B_Z is reduced. In particular, $q(K_X + B + \mathbf{M}_X)$ is Cartier.

By Theorem 16.2.1, $\mathbf{N}$ descends to X and $\mathbf{N}_X$ is semi-ample/ U . By Proposition 7.3.6,

$$\mathbf{N}_X \sim K_{\mathcal{F}} + B^h + \mathbf{M}_X \sim_Z K_X + B + \mathbf{M}_X$$

is semi-ample/ U , where $\mathcal{F}$ is the foliation induced by f . Since Z is smooth and $q(K_X + B + \mathbf{M}_X)$ is Cartier, $q\mathbf{N}_X$ and $q(K_{\mathcal{F}} + B^h + \mathbf{M}_X)$ are Cartier, and we may let $I := q$. By Theorem 16.1.3(3), $\mathcal{O}_X(nI(K_{\mathcal{F}} + B^h + \mathbf{M}_X)) = \mathcal{O}_X(nI\mathbf{N}_X)$ is globally generated/ U for any integer $n \gg 0$. $\square$

Theorem 16.2.3. *Let $(X, B, \mathbf{M})/U$ be an lc g-pair, $G \geq 0$ an $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor on X , and $f : X \rightarrow Z$ an equi-dimensional contraction/ U . Assume that*

- G is vertical/ Z ,
- $f : (X, B + G, \mathbf{M}) \rightarrow Z$ satisfies Property (*),
- $(X, B + G, \mathbf{M})$ is BP semi-stable/ Z ,
- $K_X + B + \mathbf{M}_X \sim_{\mathbb{R}, Z} 0$,
- there exists an ample/ U $\mathbb{R}$ -divisor H such that either $B^h \geq H$ or $\mathbf{M} - \bar{H}$ is nef/ U , where B^h the horizontal/ Z part of B , and
- either X is $\mathbb{Q}$ -factorial klt or $\mathbf{M}$ is NQC/ U .

Let $\mathbf{N}$ be the moduli part of $f : (X, B, \mathbf{M}) \rightarrow Z$. Then:

- (1) $\mathbf{N}$ descends to X and $\mathbf{N}_X$ is semi-ample/ U .
- (2) Suppose that there exists a positive integer m such that mB^h is a Weil divisor, $m\mathbf{M}$ is $\mathbf{b}$ -base-point-free/ U , and X is of Fano type over Z . Then there exists a positive integer I depending only on $\dim X$ and m , such that $I\mathbf{N}_X$ is Cartier and $\mathcal{O}_X(nI\mathbf{N}_X)$ is globally generated/ U for any integer $n \gg 0$.

Proof. For any prime divisor D on Z , we let

$$t_D := \sup\{t \geq 0 \mid G - tf^*D \geq 0\}$$

and let

$$G_0 := G - \sum_{D \mid D \text{ is a prime divisor on } Z} t_D \pi^* D.$$

Then $G_0 \geq 0$ and G_0 is very exceptional/ Z .

Let $\mathcal{F}$ be the foliation induced by f . By Proposition 11.1.1, $(X, \mathcal{F}, B^h, \mathbf{M})$ is lc, so $(X, \mathcal{F}, B^h, \mathbf{M}; G + B - B^h)/Z$ is weak ACSS. By Proposition 7.3.6,

$$K_{\mathcal{F}} + B^h + \mathbf{M}_X \sim_{\mathbb{R}, Z} K_X + B + G + \mathbf{M}_X \sim_{\mathbb{R}, Z} G \sim_{\mathbb{R}, Z} G_0.$$

By Theorem 9.4.1, we may run a $(K_{\mathcal{F}} + B^h + \mathbf{M}_X)$ -MMP/ Z which terminates with a weak lc model $(X', \mathcal{F}', (B^h)', \mathbf{M})/Z$ of $(X, \mathcal{F}, B^h, \mathbf{M})/Z$, such that $K_{\mathcal{F}'} + (B^h)' + \mathbf{M}_{X'} \sim_{\mathbb{R}, Z} 0$.

Let B' and G' be the images of B and G on X' respectively, $f' : X' \rightarrow Z$ the induced contraction, and let $\mathbf{N}'$ be the moduli part of $f' : (X', B' + G', \mathbf{M}) \rightarrow Z$. By Lemma 9.1.4, $(X', B' + G', \mathbf{M})/Z$ satisfies Property (*). By Proposition 11.1.1, $(X', B' + G', \mathbf{M})/Z$ is BP semi-stable. By Proposition 7.3.6,

$$K_{X'} + B' + G' + \mathbf{M}_{X'} \sim_{\mathbb{R}, Z} K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} \sim_{\mathbb{R}, Z} 0.$$

By Lemma 16.1.1, there exists an ample/ U $\mathbb{R}$ -divisor $H' \geq 0$ on X' such that either $(B^h)' \geq H' \geq 0$ or $\mathbf{M} - \bar{H}'$ is nef/ U . By Theorem 16.2.1, $\mathbf{N}'$ descends to X' and $\mathbf{N}'_{X'}$ is semi-ample/ U .

Moreover, under the condition of (2), there exists a positive integer I depending only on d and m such that $IN'_{X'}$ is Cartier and $\mathcal{O}_X(nIN'_{X'})$ is globally generated/ U for any integer $n \gg 0$.

Let $\mathbf{M}^Z$ and $\mathbf{M}'^Z$ be the base moduli part of $f : (X, B, \mathbf{M}) \rightarrow Z$ and $f' : (X', B' + G', \mathbf{M}) \rightarrow Z$ respectively. Since $\mathbf{N}'$ descends to X' and $\mathbf{N}'_{X'}$ is semi-ample/ U , $\mathbf{M}^Z$ descends to Z and $\mathbf{M}'^Z_Z$ is semi-ample/ U . Since the induced birational map ϕ is a G' -MMP and G' is vertical/ Z , ϕ is an isomorphism over the generic point of Z . Thus $f : (X, B, \mathbf{M}) \rightarrow Z$ and $f' : (X', B' + G', \mathbf{M}) \rightarrow Z$ are crepant over the generic point of Z . By Lemma 11.4.3, $\mathbf{M}^Z = \mathbf{M}'^Z$. Thus $\mathbf{N} = \mathbf{N}'$, and the theorem follows. $\square$

17. PROOFS OF THE MAIN THEOREMS

In this section, we prove all theorems that are listed in Sections 1 and 2. We recall the theorems that are already proven in the previous parts of the paper.

- (1) Theorems 2.2.6 and 2.2.7 were proven in Subsection 14.3.
- (2) Theorem 2.2.9 was proven in Subsection 13.3.
- (3) Theorem 2.3.1 was proven in Subsection 8.6.
- (4) Theorems 2.4.2, 2.4.3, 2.5.1 were proven in Subsection 8.5.
- (5) Theorem 2.4.5 was proven in Subsection 10.1.
- (6) Theorem 2.4.4 was proven in Subsection 10.2.

Theorem 17.0.1 (cf. [CS23a, Conjecture 4.2(1)]). *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a $\mathbb{Q}$ -factorial F -dlt gfq. Then $(X, \mathcal{F}, B, \mathbf{M})$ is ACSS.*

Proof. Let $h : Y \rightarrow X$ be a foliated log resolution of $(X, \mathcal{F}, B, \mathbf{M})$ such that $a(D, \mathcal{F}, B, \mathbf{M}) > -\epsilon_{\mathcal{F}}(D)$ for any prime h -exceptional divisor D . Let $\mathcal{F}_Y := h^{-1}\mathcal{F}$ and $B_Y := h_*^{-1}B + (\text{Supp Exc}(h))^{\mathcal{F}_Y}$, then $(Y, \mathcal{F}_Y, B_Y, \mathbf{M})$ is $\mathbb{Q}$ -factorial ACSS and $K_{\mathcal{F}_Y} + B_Y + \mathbf{M}_Y \sim_{\mathbb{R}, X} E \geq 0$ for some h -exceptional prime divisor E such that $\text{Supp } E = \text{Supp Exc}(h)$. By Theorem 9.4.1, we may run a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ X with scaling of an ample/ U divisor A which terminates with a good minimal model $(X', \mathcal{F}', B', \mathbf{M})/X$ of $(X, \mathcal{F}, B, \mathbf{M})/X$, such that E is contracted by this MMP. Thus the induced birational morphism $X' \rightarrow X$ is small. Since X is $\mathbb{Q}$ -factorial, $X' \rightarrow X$ is the identity morphism. The theorem follows. $\square$

Proof of Theorem 2.1.1. It follows from Theorem 17.0.1 and Proposition 9.3.2. $\square$

Proof of Theorem 2.1.2. It follows from Theorems 17.0.1, 16.1.4, and Proposition 9.3.3. $\square$

Proof of Theorem 2.1.3. We may assume that $K_{\mathcal{F}} + B + A$ is pseudo-effective/ U . The theorem follows from Theorems 17.0.1 and 16.1.4. $\square$

Proof of Theorem 2.1.4. It follows from Theorems 17.0.1 and 16.1.3. $\square$

Proof of Theorem 2.1.5. It is a special case of Theorem 9.3.4. $\square$

Proof of Theorem 2.1.6. First we prove (3). By Theorem 17.0.1 and Proposition 11.2.3, we may run a $(K_{\mathcal{F}} + B)$ -MMP with scaling of an ample $\mathbb{R}$ -divisor, and any such MMP terminates with a log minimal model $(X', \mathcal{F}', B')$ of $(X, \mathcal{F}, B)$ such that $K_{\mathcal{F}'} + B' \equiv 0$. By [DLM23, Theorem 1.4], $K_{\mathcal{F}'} + B' \sim_{\mathbb{R}} 0$.

(2) follows from (3) and Theorem 2.5.1. (1) follows from (2). $\square$

Proof of Theorem 2.1.7. Let $(Y, \mathcal{F}_Y, \bar{B}_Y; G)/Z$ be a proper ACSS model of $(X, \mathcal{F}, B)$ with induced birational morphism $g : Y \rightarrow X$, whose existence is guaranteed by Theorem 8.2.2. Let $K_{\mathcal{F}_Y} + B_Y := g^*(K_{\mathcal{F}} + B)$ and $K_Y + B'_Y := g^*(K_X + B)$. Since (X, B) is lc, the coefficient of any component of B'_Y is ≤ 1 . In particular, any coefficient of B is ≤ 1 . We let

$$\bar{B}_Y := g_*^{-1}B + (\text{Supp Exc}(g))^{\mathcal{F}_Y}$$

and $E := B_Y - \bar{B}_Y$. Then $E \geq 0$ and E is exceptional/ X .

Suppose that $K_{\mathcal{F}_Y} + \bar{B}_Y$ is not pseudo-effective. We let

$$F := \sum_{D|D \text{ is a } g\text{-exceptional prime divisor}} D.$$

Then $G \geq F$. Since $(Y, \bar{B}_Y + G)$ is lc and $G \geq F$, $(Y, \bar{B}_Y + F)$ is lc. By [ACSS21, Theorem 5.3], $K_Y + \bar{B}_Y + F$ is not pseudo-effective. For any prime f -exceptional divisor D such that D is not $\mathcal{F}_Y$ -invariant, we have $\text{mult}_D \bar{B}_Y = 1$. Therefore,

$$\bar{B}_Y + F = g_*^{-1}B + \text{Supp Exc}(g).$$

Since the coefficient of any component of B'_Y is ≤ 1 , we have

$$E' := g_*^{-1}B + \text{Supp Exc}(g) - B'_Y \geq 0$$

and E' is exceptional/ X . Therefore,

$$\begin{aligned} -\infty &= \kappa_\sigma(K_Y + \bar{B}_Y + F) = \kappa_\sigma(K_Y + g_*^{-1}B + \text{Supp Exc}(g)) \\ &= \kappa_\sigma(g^*(K_X + B) + E') = \kappa_\sigma(K_X + B) \geq 0, \end{aligned}$$

a contradiction. Thus $K_{\mathcal{F}_Y} + \bar{B}_Y$ is not pseudo-effective. Since

$$0 \leq \kappa_\sigma(K_{\mathcal{F}_Y} + \bar{B}_Y) \leq \kappa_\sigma(K_{\mathcal{F}_Y} + B_Y) = \kappa_\sigma(K_{\mathcal{F}} + B) = 0,$$

we have $\kappa_\sigma(K_{\mathcal{F}_Y} + \bar{B}_Y) = 0$. By Theorem 2.1.6(1), $\kappa_\ell(K_{\mathcal{F}_Y} + \bar{B}_Y) = 0$, so

$$0 = \kappa_\ell(K_{\mathcal{F}_Y} + \bar{B}_Y) \leq \kappa_\ell(K_{\mathcal{F}_Y} + B_Y) = \kappa_\ell(K_{\mathcal{F}} + B) \leq \kappa_\sigma(K_{\mathcal{F}} + B) = 0.$$

Thus $\kappa_\ell(K_{\mathcal{F}} + B) = 0$ and we are done. $\square$

Proof of Theorem 2.1.9. It immediately follows from Theorem 17.0.1. $\square$

Theorem 17.0.2. *Let $(X, \mathcal{F}, B, \mathbf{M})$ be a $\mathbb{Q}$ -factorial lc generalized foliated quadruple such that $\mathcal{F}$ is algebraically integrable. Assume that there exists a foliated log resolution $h : Y \rightarrow X$ such that $a(D, \mathcal{F}, B, \mathbf{M}) > -1$ for any h -exceptional prime divisor D . Then $\mathcal{F}$ is induced by an almost holomorphic map.*

Proof. Let $\mathcal{F}_Y := h^{-1}\mathcal{F}$ and let $B_Y := h_*^{-1}B + (\text{Supp Exc}(h))^{\mathcal{F}_Y}$, then $(Y, \mathcal{F}_Y, B_Y, \mathbf{M})$ is $\mathbb{Q}$ -factorial ACSS and $K_{\mathcal{F}_Y} + B_Y + \mathbf{M}_Y \sim_{\mathbb{R}, X} E \geq 0$ for some h -exceptional prime divisor E . Let F be the sum of all non- $\mathcal{F}_Y$ -invariant prime h -exceptional divisors. By assumption, $F \subset \text{Supp } E$.

By Theorem 9.4.1, we may run a $(K_{\mathcal{F}} + B + \mathbf{M}_X)$ -MMP/ X with scaling of an ample/ U divisor A which terminates with a good minimal model $(X', \mathcal{F}', B', \mathbf{M})/X$ of $(X, \mathcal{F}, B, \mathbf{M})/X$, such that E is contracted by this MMP. Then F is contracted by this MMP, and $(X', \mathcal{F}', B', \mathbf{M})/Z$ is ACSS for some contraction $f' : X' \rightarrow Z$. Since X is $\mathbb{Q}$ -factorial, The induced morphism $g : X' \rightarrow X$ only extracts $\mathcal{F}'$ -invariant divisors, so g is an isomorphism over the generic point of Z . In particular,

$$f := f' \circ g^{-1} : X \dashrightarrow Z$$

is an almost holomorphic map which induces $\mathcal{F}$. $\square$

Proof of Theorem 2.1.10. It is a special case of Theorem 17.0.2. $\square$

Proof of Theorem 2.2.1. It follows from Theorems 2.3.1 and 14.3.3. $\square$

Proof of Theorem 2.2.2. It is a special case of Theorem 15.0.6. $\square$

Proof of Theorem 2.2.3. It follows from Theorems 2.2.1 and 2.2.2. $\square$

Proof of Theorem 2.2.4. It immediately follows from Theorem 14.2.5(2) by letting $U = \{pt\}$. $\square$

Proof of Theorem 2.2.5. It immediately follows from Theorem 14.2.5(2). $\square$

Proof of Theorem 2.2.8. It is a special case of Definition-Lemma 12.2.4. $\square$

Proof of Theorem 2.3.2. (1-4) follow from Definition-Theorem 12.1.4. (5) follows from Definition-Theorem 12.1.4 and Proposition 11.5.5. (6) follows from Proposition 11.5.5. (7) follows from Lemma 11.5.4. (8) follows from Lemma 11.5.3 and Definition-Lemma 12.1.2. (9) follows from Definition-Lemma 11.5.1(3) and Definition-Lemma 12.1.2(6). $\square$

Proof of Theorem 2.4.1. It is a special case of Theorem 2.4.2. $\square$

Proof of Corollary 2.4.6. If $K_{\mathcal{F}}$ is pseudo-effective, then $K_{\mathcal{F}} \equiv B \equiv \mathbf{M}_X \equiv 0$. By Lemma 10.1.1, $\mathbf{M} \equiv \mathbf{0}$ so there is nothing left to prove. So we may assume that $K_{\mathcal{F}}$ is not pseudo-effective. Since $\text{rank } \mathcal{F} = 1$, by Theorem 6.1.11, $\mathcal{F}$ is algebraically integrable. The corollary follows from Theorem 2.4.5. $\square$

Proof of Theorem 2.4.7. It immediately follows from Theorem 10.3.7. $\square$

Proof of Theorem 2.5.2. It is a special case of Theorem 9.4.1. $\square$

Proof of Theorem 2.5.3. It follows from Theorem 16.2.1. $\square$

Proof of Theorem A. (1) follows from Theorem 17.0.1, Proposition 7.3.6, Theorem 2.3.1, and the contraction theorem and the existence of flips for lc pairs. (2) is a special case of Theorem 2.1.4. (3) is a special case of Theorem 2.1.2. (4) follows from Theorems 17.0.1 and 16.1.3. $\square$

Proof of Theorem B. (1) follows from Theorem 2.2.1 and 2.2.2 and (2) follows from Theorem 2.2.7. $\square$

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