UNIFORM RATIONAL POLYTOPES OF FOLIATED THREEFOLDS AND THE GLOBAL ACC

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ABSTRACT. In this paper, we show the existence of uniform rational lc polytopes for foliations with functional boundaries in dimension ≤ 3. As an application, we prove the global ACC for foliated threefolds with arbitrary DCC coefficients. We also provide applications on the accumulation points of lc thresholds of foliations in dimension ≤ 3.

CONTENTS

1. INTRODUCTION

We work over the field of complex numbers $\mathbb{C}$.

Foliations are important and interesting objects in geometry. Particularly in the minimal model program (MMP), they play a critical role in Miyaoka's proof of some important cases of the abundance conjecture in dimension three [Miy87] (see also [Kol+92, Chapter 9]). In recent years, progress has been made in the direction of extending the MMP to the setting of foliations, particularly in low dimension. To be specific, the foundations of the MMP for foliated surfaces (cf. [McQ08, Bru15]), foliated threefolds (cf. [CS20, Spi20, CS21, SS22]), and algebraically integrable foliations (cf. [ACSS21, CHLX23, CS23a, LMX24b] have been laid. Many classical questions from the MMP can be asked in the setting of foliations, such

Date: June 6, 2024.
2020 *Mathematics Subject Classification.* 14E30, 37F75.
Key words and phrases. Foliations. Uniform rational polytopes. Log canonical thresholds. Adjunction formulas.

as the ascending chain condition (ACC) conjecture for minimal log discrepancies and the ACC conjecture for lc thresholds [Che22, Che23b].

In a recent work [LLM23], the first two authors and Y. Luo established the rational case of the global ACC for foliated threefolds, i.e., given any lc foliated log Calabi-Yau triple $(X,\mathcal{F},B)$ of dimension 3 whose coefficients belong to a set Γ of rational numbers satisfying the descending chain condition (DCC), the coefficients of B belong to a finite set depending only on Γ. The goal of this paper is to prove the global ACC in its full generality for foliations in dimension 3.

Theorem 1.1. *Let $\Gamma\subset[0,1]$ be a DCC set. Then there exists a finite set $\Gamma_0\subset\Gamma$ satisfying the following. Let $(X,\mathcal{F},B)$ be a projective lc foliated triple of dimension ≤ 3 such that $K_{\mathcal{F}}+B\equiv 0$ and $B\in\Gamma$ (i.e. the coefficients of B belong to Γ). Then $B\in\Gamma_0$.*

Theorem 1.1 is an analogue of [HMX14, Theorem 1.5] for foliated pairs of dimension ≤ 3. Y.-A. Chen proved Theorem 1.1 in dimension 2 ([Che22, Theorem 2.5]).

Remark 1.2 ($\mathbb{Q}$-coefficients versus $\mathbb{R}$-coefficients)**.** In an earlier work [LLM23, Theorem 1.1], Theorem 1.1 was established for the case when $\Gamma\subset\mathbb{Q}$. Despite its technicality, it is important to consider pairs and foliated triple structures with real coefficients for future applications.

For example, when considering the minimal model program with scaling, we need to consider a sequence of scaling numbers λ_i. In many scenarios we need to consider the limit of the scaling numbers $\lambda:=\lim_{i\to+\infty}\lambda_i$. However, it is possible that λ is an irrational number when each λ_i is rational. Thus, we need to study pairs with real coefficients.

For example, pairs with real coefficients are necessary to develop the theory of *uniform rational polytopes*, which implies the global ACC. The theory of uniform rational polytopes allows us to reduce plenty of questions related to pairs and triples with real coefficients to the case of rational coefficients. Although seemingly technical, this theory (as well as a weaker version, the theory of rational polytopes) is known to be very useful for usual pairs in many different contexts, such as the minimal model program with scaling ([Sho92, BCHM10]), accumulation points of minimal log discrepancies ([Liu18, HLS19, HL22]), construction of special minimal model programs ([Kol21, HL23]), and the effective Iitaka fibration conjecture [CHL23]. For further details about this theory, we refer the reader to [Nak16, HLS19, HLQ21].

Due to the failure of effective birationality for foliations [SS23, Paragraph before Theorem 1.4], the proof of the rational case of the global ACC for foliated threefolds heavily relies on the index theorem of surfaces, which only works for the rational coefficient case. Therefore, there are essential difficulties in extending [LLM23, Theorem 1.1] to the general case when the coefficients of B belong to an arbitrary DCC set $\Gamma\subseteq\mathbb{R}$.

To resolve this issue, we prove the existence of uniform rational lc polytopes for foliations in dimension ≤ 3. This result is crucial for the proof of our main theorem, Theorem 1.1, and has potential applications in the study of foliations and their singularities.

Theorem 1.3. *Let $v_1^0,\dots,v_m^0$ be positive numbers and $\boldsymbol{v}_0:=(v_1^0,\dots,v_m^0)$. Then there exists an open set $U\ni\boldsymbol{v}_0$ of the rational envelope of $\boldsymbol{v}_0$ satisfying the following.*

Let $(X,\mathcal{F},B=\sum_{j=1}^m v_j^0B_j)$ be a projective foliated lc triple of dimension ≤ 3, where $B_j\geq 0$ are distinct Weil divisors. Then $(X,\mathcal{F},B=\sum_{j=1}^m v_jB_j)$ is lc for any $(v_1,\dots,v_m)\in U$.

We remark that the projectivity condition in Theorem 1.3 is expected to be unnecessary. We add this condition for consistency with references, e.g. [CS20].

When $\dim X=2$, Theorem 1.3 was proved in [LMX24a]. Theorem 1.3 is an analogue of the theory of uniform rational lc polytopes for usual pairs with functional boundaries [HLS19, Theorem 5.6].

The proof of Theorem 1.1 is an application of the theory of uniform rational polytopes which allows us to reduce Theorem 1.1 to the case of rational coefficients. This reduction greatly simplifies the arguments and allows us to bypass the difficulties arising from the lack of effective birationality for foliations.

As a direct corollary of Theorem 1.3, we obtain the following result on the accumulation points of lc thresholds of foliations. Again, the projectivity condition is added in Corollary 1.4 only for consistency with references.

Corollary 1.4. *Let $\Gamma \subset [0,1]$ be a DCC set such that $\bar{\Gamma} \subset \mathbb{Q}$. We consider the set of foliated lc thresholds in dimension 3*

$$\mathrm{lct}_{\mathrm{fol}}(3,\Gamma) := \{\mathrm{lct}(X,\mathcal{F},B;D) \mid \dim X = 3, X \textit{ is projective, } B \in \Gamma, D \in \mathbb{N}^+\}.$$

Then the accumulation points of $\mathrm{lct}_{\mathrm{fol}}(3,\Gamma)$ are rational numbers.

Idea and sketch of the proof. We start with the proof of Theorem 1.1. Since we work in dimension 3, it is not difficult to reduce Theorem 1.1 to the case when there exists a contraction $\pi : X \to Z$ such that the general fibers of π are tangent to $\mathcal{F}$ and $\dim Z > 0$, i.e. the setting of Proposition 5.2. In this case, the usual global ACC implies that the coefficients of the horizontal/Z part of B belong to a finite set, so we only need to worry about the vertical/Z part of B. We let B_Z be the discriminant part of the canonical bundle formula of $\pi : (X,\mathcal{F},B) \to Z$. Then the coefficients of B_Z belong to a DCC set by the ACC for lc thresholds of foliations.

At this point, we prove by using contradiction: if the coefficients of B do not belong to a finite set, then we may construct a boundary $\bar{B} \geq B$ sufficiently close to B whose coefficients belong to a finite set, and $K_{\mathcal{F}} + \bar{B} \sim_{\mathbb{R},Z} 0$. By the ACC for lc thresholds of foliations, the discriminant part $\bar{B}_Z$ induced by the canonical bundle formula of $\pi : (X,\mathcal{F},\bar{B}) \to Z$ has coefficients which are larger and sufficiently close to the ones of B_Z, and its coefficients also belong to a DCC set. Therefore, the coefficients of B_Z belong to a DCC but not finite set. For the rational coefficient case, we get a contradiction to the global ACC of foliated log Calabi-Yau triples polarized with a semi-ample divisor [LLM23, Lemma 7.2], but this can no longer work for the real coefficient case.

To deal with this issue, the key idea is the following: we would like to find positive real numbers $a_1,\dots,a_k$ depending only on the coefficient set Γ, so that we have a decomposition $K_{\mathcal{F}}+\bar{B} = \sum a_i(K_{\mathcal{F}}+\bar{B}_i)$, where $\sum a_i = 1$, $(X,\mathcal{F},\bar{B}_i)$ is lc, $\bar{B}_i$ is a $\mathbb{Q}$-divisor, and $K_{\mathcal{F}}+\bar{B}_i \sim_{\mathbb{Q},Z} 0$ for each i. With such decomposition established, we may let $\bar{B}_{Z_i}$ be the discriminant part induced by the canonical bundle formula of $\pi : (X,\mathcal{F},\bar{B}_i) \to Z$ for each i. By using similar arguments as the rational coefficient case, we can compare the coefficients of $\sum a_i\bar{B}_{Z_i}$ and B_Z and get a contradiction, which leads to a proof of Theorem 1.1. The remaining difficulty is to find such a decomposition $K_{\mathcal{F}} + \bar{B} = \sum a_i(K_{\mathcal{F}} + \bar{B}_i)$, which is nothing but Theorem 1.3.

Next we turn to the proof of Theorem 1.3. Due to technicality, in the following, we provide the reader with a sketch of the proof of Theorem 1.3 for the following special case: $m = 1$, $\boldsymbol{v}_0 = \left(\frac{\sqrt{2}}{2}\right)$, and $S := B_1$ is a prime divisor. The general case follows from similar lines of proof.

To prove this special case of Theorem 1.3, we only need to show that there exists a rational number $a > \frac{\sqrt{2}}{2}$ (which does not depend on X) such that $(X,\mathcal{F},aS)$ is lc. By passing to a dlt model of $(X,\mathcal{F},\frac{\sqrt{2}}{2}S)$ we may assume that $(X,\mathcal{F},\frac{\sqrt{2}}{2}S)$ is $\mathbb{Q}$-factorial dlt. (Here divisors with coefficient 1 may appear but will not influence the proof. For simplicity, let us ignore them.) Suppose Theorem 1.3 does not hold in this case, then there exists a sequence $(X_i,\mathcal{F}_i,a_iS_i)$ such that a_i is strictly decreasing, $\lim_{i\to+\infty} a_i = \frac{\sqrt{2}}{2}$, and a_i is the lc threshold of S_i with respect to $\mathcal{F}_i$. We suppose that E_i is a prime divisor which achieves the lc threshold. Since $(X_i,\mathcal{F}_i,\frac{\sqrt{2}}{2}S_i)$ is $\mathbb{Q}$-factorial dlt, it can be reduced to the case when E_i is $\mathcal{F}_i$-invariant. Moreover, we may assume that $\operatorname{center}_{X_i} E_i$ is a closed point, as other cases can be reduced to the lower dimensional cases which are proved in [LMX24a].

Now there is a key observation: suppose that there exists an extraction $g_i : Y_i \to X_i$ of E_i. Let $\mathcal{F}_{Y_i} := g_i^{-1}\mathcal{F}_i$ and $S_{Y_i} := (g_i^{-1})_*S_i$, then $K_{\mathcal{F}_{Y_i}} + a_iS_{Y_i} \sim_{\mathbb{R},X_i} 0$ but $K_{\mathcal{F}_{Y_i}} + \frac{\sqrt{2}}{2}S_{Y_i}$ is anti-ample/X_i. Therefore, for the normalization V_i of any non-trivial lc center of $(Y_i,\mathcal{F}_{Y_i},a_iS_{Y_i})$ in E_i (e.g. normalization of E_i), $(K_{\mathcal{F}_{Y_i}} + a_iS_{Y_i})|_{V_i} \equiv 0$ but $(K_{\mathcal{F}_{Y_i}} + \frac{\sqrt{2}}{2}S_{Y_i})|_{V_i} \not\equiv 0$. Therefore, if we can achieve precise adjunction formulas of foliated threefolds to $\mathcal{F}_{Y_i}$-invariant lc centers, the coefficients of $(K_{\mathcal{F}_{Y_i}}+a_iS_{Y_i})|_{V_i}$ and $(K_{\mathcal{F}_{Y_i}}+\frac{\sqrt{2}}{2}S_{Y_i})|_{V_i}$ will be controlled. Now we can use lower dimensional results to get a contradiction in this case. With this observation in mind, there are two things we need to do:

(1) Establish precise adjunction formulas to invariant lc centers. The whole Section 3 is dedicated to it. The proofs generally follow from the same lines of the proofs of the adjunction formulas in [CS20, CS21] but we provide more details. It is important to note that for the rank two case, when the minimal lc center is not a divisor, we need to consider adjunction to the normalization of lc centers of dimension 1 as well (see Theorem 3.5(6)).

(2) Show the existence of the extraction $g_i : Y_i \to X_i$ of E_i. This is our key lemma, see Lemma 4.5, and Subsections 4.1, 4.2 are dedicated to it. More precisely, we shall prove the existence of such $g_i : Y_i \to X_i$ after a suitable substitution of X_i and E_i.

The proof of Lemma 4.5 is as follows: first we take a dlt modification $W_i \to X_i$ of $(X_i, \mathcal{F}_i, a_iS_i)$ which extracts E_i. It may extract some divisors that are lc centers of $(X_i, \mathcal{F}_i, \frac{\sqrt{2}}{2}S_i)$. By replacing X_i with a higher model, we may assume that any divisor extracted by $W_i \to X_i$ is not an lc place of $(X_i, \mathcal{F}_i, \frac{\sqrt{2}}{2}S_i)$. Now $K_{\mathcal{F}_{W_i}}+\frac{\sqrt{2}}{2}S_{W_i}\sim_{\mathbb{R},X_i} F_i$ where $F_i\geq 0$ is supported on the exceptional divisor of $W_i \to X_i$. Now we may run a $(K_{\mathcal{F}_{W_i}}+\frac{\sqrt{2}}{2}S_{W_i})$-MMP$/X_i$ with scaling of an ample divisor. By the general negativity lemma [Bir12, Lemma 3.3] this MMP contracts F_i. Since X_i is $\mathbb{Q}$-factorial, it must terminate with X_i, and the last step of the MMP must be a divisorial contraction. We may let $g_i : Y_i \to X_i$ be the last step of this MMP. By our construction, g_i extracts a prime divisor E_i' which achieves the lc threshold of S_i with respect to $\mathcal{F}_i$ (although it is possible that $E_i'\neq E_i$). This suffices our requirements.

It is worth to mention that our key Lemma, Lemma 4.5, is proved to be important in the study of log canonicity for foliations, although we cannot find any analogue of it in the study of log canonicity for usual pairs. In later works, analogues of Lemma 4.5 have played crucial roles in the proofs of the ACC for lc thresholds and the existence of uniform rational polytopes for algebraically integrable foliations ([DLM23, Lemma 5.1]) and algebraically integrable generalized foliated quadruples ([CHLX23, Lemma 10.2.1]).

Finally, with both (1) and (2) settled, we can get pair (resp. foliated pair) structures of $(K_{\mathcal{F}_{Y_i}}+a_iS_{Y_i})|_{V_i}$ and $(K_{\mathcal{F}_{Y_i}}+\frac{\sqrt{2}}{2}S_{Y_i})|_{V_i}$ with coefficients controlled and good numerical properties when $\operatorname{rank}\mathcal{F}=2$ (resp. $\operatorname{rank}\mathcal{F}=1$). Now we may use [Che23a, Theorem 3.6] and [Nak16, Theorem 3.8, Corollary 3.9] to study the coefficients of $(K_{\mathcal{F}_{Y_i}}+a_iS_{Y_i})|_{V_i}$ and $(K_{\mathcal{F}_{Y_i}}+\frac{\sqrt{2}}{2}S_{Y_i})|_{V_i}$ and conclude our proof.

To be specific, when $\operatorname{rank}\mathcal{F}=2$, [Che23a, Theorem 3.6] and [Nak16, Theorem 3.8, Corollary 3.9] can be applied directly to our setting to check the coefficients of $(K_{\mathcal{F}_{Y_i}}+a_iS_{Y_i})|_{V_i}$ and $(K_{\mathcal{F}_{Y_i}}+\frac{\sqrt{2}}{2}S_{Y_i})|_{V_i}$ and conclude the proof of Theorem 1.3 in this case (see Subsection 4.4). When $\operatorname{rank}\mathcal{F}=1$, it is easy to check that the restricted foliation $\mathcal{F}_{E_i^\nu}$ of $\mathcal{F}_i$ on the normalization E_i^ν of E_i has a non-pseudo-effective foliated canonical divisor, hence it is algebraically integrable. Therefore, we can consider the intersection numbers of $(K_{\mathcal{F}_{Y_i}}+a_iS_{Y_i})|_{E_i^\nu}$ and $(K_{\mathcal{F}_{Y_i}}+\frac{\sqrt{2}}{2}S_{Y_i})|_{E_i^\nu}$ with a general member of the family of leaves of $\mathcal{F}_{E_i^\nu}$ and apply [Che23a, Theorem 3.6] and [Nak16, Theorem 3.8, Corollary 3.9] again to conclude the proof of Theorem 1.3 in this case (see Subsection 4.3).

Structure of this paper. Section 2 introduces some preliminary results for foliations. Section 3 proves the precise adjunction formulas for foliations in dimension ≤ 3 to invariant divisors (Theorems 3.2, 3.3, and 3.5). Section 4 establishes the theory of uniform rational polytopes for foliations with functional boundaries in dimension ≤ 3, and proves Theorem 1.3. Section 5 provides the proofs of Theorem 1.1 and Corollary 1.4.

Acknowledgements. We thank Paolo Cascini, Yen-An Chen, Omprokash Das, Christopher D. Hacon, Jingjun Han, and Yuchen Liu for helpful discussions. We would like to acknowledge the assistance of ChatGPT in polishing the wording. The third author is partially supported by NSF research grants no: DMS-1801851, DMS-1952522 and by a grant from the Simons Foundation; Award Number: 256202.

2. Preliminaries

We will work over the field of complex numbers $\mathbb{C}$. Throughout the paper, we will mainly work with normal quasi-projective varieties to ensure consistency with the references. However, most results should also hold for normal varieties that are not necessarily quasi-projective. Similarly, most results in our paper should hold for any algebraically closed field of characteristic zero. We will adopt the standard notations and definitions in [Sho92, KM98, BCHM10] and use them freely. For foliations and foliated (sub-)triples, we will follow the notations and definitions in [LLM23], which are mostly consistent with those in [CS20, Spi20, ACSS21, CS21]. For generalized pairs and generalized foliated quadruples, we will follow the notations and definitions in [HL23, LLM23].

2.1. Sets.

Definition 2.1. Let $\Gamma\subset\mathbb{R}$ be a set. We say that Γ satisfies the *descending chain condition* (DCC) if any decreasing sequence in Γ stabilizes, and Γ satisfies the *ascending chain condition* (ACC) if any increasing sequence in Γ stabilizes. We define

$$\Gamma_+ := \{0\}\cup\left\{\sum_{i=1}^n \gamma_i \,\middle|\, n\in\mathbb{N}^+,\gamma_1,\dots,\gamma_n\in\Gamma\right\}.$$

and define $\bar\Gamma$ to be the closure of Γ in $\mathbb{R}$.

Definition 2.2. Let m be a positive integer and $\boldsymbol{v}\in\mathbb{R}^m$. The *rational envelope* of $\boldsymbol{v}$ is the minimal rational affine subspace of $\mathbb{R}^m$ which contains $\boldsymbol{v}$. For example, if $m=2$ and $\boldsymbol{v}=(\frac{\sqrt{2}}{2},1-\frac{\sqrt{2}}{2})$, then the rational envelope of $\boldsymbol{v}$ is $(x_1+x_2=1)\subset\mathbb{R}^2_{x_1x_2}$.

2.2. Foliations.

Definition 2.3 (Special divisors on foliations, cf. [CS21, Definition 2.2])**.** Let X be a normal variety and $\mathcal{F}$ a foliation on X. For any prime divisor C on X, we define $\epsilon_{\mathcal{F}}(C):=1$ if C is not $\mathcal{F}$-invariant, and $\epsilon_{\mathcal{F}}(C):=0$ if C is $\mathcal{F}$-invariant. If $\mathcal{F}$ is clear from the context, then we may use $\epsilon(C)$ instead of $\epsilon_{\mathcal{F}}(C)$. For any $\mathbb{R}$-divisor D on X, we define

$$D^{\mathcal{F}} := \sum_{C\mid C\text{ is a component of }D}\epsilon_{\mathcal{F}}(C)C.$$

Let E be a prime divisor over X and $f:Y\to X$ a projective birational morphism such that E is on Y. We define $\epsilon_{\mathcal{F}}(E):=\epsilon_{f^{-1}\mathcal{F}}(E)$. It is clear that $\epsilon_{\mathcal{F}}(E)$ is independent of the choice of f.

Definition 2.4. Let X be a normal variety, $\mathcal{F}$ a foliation on X, and B an $\mathbb{R}$-divisor on X, such that either $\operatorname{rank}\mathcal{F}=\dim X-1$ or $\dim X=3$ and $\operatorname{rank}\mathcal{F}=1$. A *foliated log resolution* of $(X,\mathcal{F},B)$ is a projective birational morphism $f:Y\to X$ such that $(Y,\mathcal{F}_Y:=f^{-1}\mathcal{F},B_Y:=f^{-1}_*B+E)$ is foliated log smooth (cf. [CS21, Definition 3.1], [LLM23, Definition 4.2]), where E is the reduced exceptional divisor of f. A *foliated resolution* of $\mathcal{F}$ is a foliated log resolution of $(X,\mathcal{F},0)$.

2.3. Dlt models.

Definition 2.5 (Dlt singularities)**.** Let $(X,\mathcal{F},B)$ be a foliated triple.

(1) Suppose that $\operatorname{rank}\mathcal{F}=\dim X-1$. We say that $(X,\mathcal{F},B)$ and $(\mathcal{F},B)$ are *dlt* if
  (a) $(X,\mathcal{F},B)$ is lc,
  (b) every component of B is generically transverse to $\mathcal{F}$, and
  (c) there exists a foliated log resolution of $(X,\mathcal{F},B)$ which only extracts divisors E of discrepancy $>-\epsilon(E)$.

(2) Suppose that $\dim X=3$ and $\operatorname{rank}\mathcal{F}=1$. We say that $(X,\mathcal{F},B)$ and $(\mathcal{F},B)$ are *dlt* if
  (a) $(X,\mathcal{F},B)$ is lc, and
  (b) $\mathcal{F}$ has simple singularities [CS20, Definition 2.32].

We remark that dlt implies non-dicritical (cf. [CS21, Theorem 11.3] and [CS20, Lemma 2.8]).

Definition 2.6 (Dlt modification)**.** Let $(X,\mathcal{F},B)$ be an lc foliated triple. An *dlt modification* of $(X,\mathcal{F},B)$ is a birational morphism $f:Y\to X$ satisfying the following. Let $\mathcal{F}_Y:=f^{-1}\mathcal{F}$, E the reduced exceptional divisor of f, and $B_Y:=f^{-1}_*B+E^{\mathcal{F}_Y}$.

(1) Y is $\mathbb{Q}$-factorial klt,
(2) $K_{\mathcal{F}_Y}+B_Y=f^*(K_{\mathcal{F}}+B)$, and
(3) $(Y,\mathcal{F}_Y,B_Y)$ is dlt.

We say that $(Y,\mathcal{F}_Y,B_Y)$ is a *dlt model* of $(X,\mathcal{F},B)$ and $(\mathcal{F}_Y,B_Y)$ is a *dlt model* of $(\mathcal{F},B)$.

2.4. A perturbation formula.

We recall a weaker version of Theorem 1.3 which will be crucial for the proof of our theorems.

Theorem 2.7 ([LLM23, Theorem 1.7])**.** *Let $v_1^0,\dots,v_m^0$ be positive integers, $\boldsymbol{v}_0:=(v_1^0,\dots,v_m^0)$, and $(X,\mathcal{F},B=\sum_{i=1}^m v_i^0B_i)$ an lc foliated triple of dimension ≤ 3, where $B_i\geq 0$ are distinct Weil divisors. Then there exists an open set $U\ni\boldsymbol{v}_0$ of the rational envelope of $\boldsymbol{v}_0$, such that $(X,\mathcal{F},B=\sum_{i=1}^m v_iB_i)$ is lc for any $(v_1,\dots,v_m)\in U$.*

Note that the major difference between Theorem 2.7 and Theorem 1.3 is that the set U may depend on $(X,\mathcal{F},B)$ in Theorem 2.7.

3. Precise adjunction formulas to invariant divisors

To prove Theorem 1.3, we need several precise adjunction formulas to divisors that are invariant to foliations. Although there are many results on adjunction formulas of foliations in the literature [Bru02, Bru15, CS20, Spi20, CS21, SS22, Che22], the statements of these results are insufficient for our purposes, as we need an accurate description of the coefficients of the different foliated triples. For instance, when applying the adjunction formula to two distinct foliated triples $(X,\mathcal{F},B=\sum b_jB_j)$ and $(X,\mathcal{F},B'=\sum b_j'B_j)$, we need to understand the relationship between the coefficients of the differents of two distinct triples. Additionally, we require adjunction formulas for triples with $\mathbb{R}$-coefficients rather than $\mathbb{Q}$-coefficients. In this section, we provide precise adjunction formulas for invariant divisors, even if they may be well-known to experts. The key differences between these formulas and the traditional adjunction formulas for foliations are the following:

- We have a more accurate control of the coefficients of the foliated different C_i. This will be helpful when applying adjunction to two or more triples and considering their behavior.
- We deal with $\mathbb{R}$-coefficients rather than $\mathbb{Q}$-coefficients.
- We only need to control the singularities of one foliated triple $(X,\mathcal{F},B=\sum b_jB_j)$ to get an adjunction formula for all foliated triples of the form $(X,\mathcal{F},B'=\sum b_j'B_j)$, even if the singularities of the latter triple may not be controlled.

We refer the reader to [CS21, Lemma 3.18], [CS20, Proposition 2.16], [SS22, Lemma 3.11], [Che22, Proposition 3.10, Theorem 4.6] for related references.

Finally, we remark that a recent paper of Cascini and Spicer [CS23b] has established the adjunction formula of foliations to non-invariant divisors in all dimensions. We expect that the ideas in [CS23b] to provide a precise adjunction formula of foliations to non-invariant divisors in any dimensions.

3.1. **Surface case.** In this subsection we prove Theorem 3.2. We actually do not need the precise adjunction formula for surfaces to prove our main theorems, Theorems 1.1 and 1.3, but we include the result for completeness.

Lemma 3.1. *Let $(X \ni x, \mathcal{F}, B)$ be an lc (resp. terminal) foliated germ such that $X \ni x$ is terminal. Let $f : Y \to X$ be the minimal resolution of $X \ni x$ and $I := \det(\mathcal{D}(f))$. Then there exists a unique $\mathcal{F}$-invariant curve C which passes through x, and $(B \cdot C)_x \leq \frac{1}{I}$ (resp. $< \frac{1}{I}$).*

Proof. The existence and uniqueness of C follows from [LMX24a, Theorems 3.19, 4.1]. Moreover, by [LMX24a, Theorem 4.1], $(X \ni x, B + C)$ is lc. Let

$$K_{C^\nu} + B_C := (K_X + B + C)|_{C^\nu},$$

where C^ν is the normalization of C, then $(C^\nu \ni x, B_C)$ is lc. By [LMX24a, Theorem 3.19], I is the local Cartier index of $X \ni x$. Thus

$$1 \geq \text{(resp. } > \text{)} \operatorname{mult}_x B_C = \frac{I-1}{I} + (B \cdot C)_x$$

and the lemma follows. □

Theorem 3.2 (Surface case)**.** *Let $(X, \mathcal{F}, B = \sum_{j=1}^m b_j B_j)$ be a dlt foliated triple such that $\dim X = 2, \operatorname{rank} \mathcal{F} = 1$, and B_j are the irreducible components of B. Assume that*

- *C is an $\mathcal{F}$-invariant curve on X,*
- *C^ν is the normalization of C,*
- *$P_1, \dots, P_n$ are all closed points on C that are singular points of X or contained in $C \cap \operatorname{Supp} B$, and*
- *$Q_1, \dots, Q_l$ are all closed points on C such that $Z(\mathcal{F}, C, Q_i) \in \mathbb{N}^+$. We refer the reader to* [Bru02, Section 2] *for the definition of $Z(\mathcal{F}, C, Q_i)$.*

Then there exist positive integers $w_1, \dots, w_n$ and non-negative integers $\{w_{i,j}\}_{1 \leq i \leq n, 1 \leq j \leq m}$, such that for any real numbers $b'_1, \dots, b'_m$, we have the following.

(1) By identifying Q_i with its inverse image in C^ν under the normalization $C^\nu \to C$, we have

$$\left(K_{\mathcal{F}} + \sum_{j=1}^m b'_j B_j\right)\Bigg|_{C^\nu} = K_{C^\nu} + \sum_{i=1}^n \frac{w_i - 1 + \sum_{j=1}^m w_{i,j} b'_j}{w_i} P_i + \sum_{i=1}^l Z(\mathcal{F}, C, Q_i) Q_i.$$

(2) w_i is the local Cartier index (i.e. order of the local fundamental group) of P_i for each i.
(3) For any i such that $w_i = 1$, $w_{i,j} > 0$ for some j.
(4) If $(X, \mathcal{F}, \sum_{j=1}^m b'_j B_j)$ is lc, then

$$\left(C^\nu, \sum_{i=1}^n \frac{w_i - 1 + \sum_{j=1}^m w_{i,j} b'_j}{w_i} P_i + \sum_{i=1}^l Q_i\right)$$

is lc, i.e. $\sum_{j=1}^m w_{i,j} b'_j \leq 1$ for any i.

Proof. Since $(X, \mathcal{F}, B)$ is dlt, X is klt (cf. [LMX24a, Corollary 3.20]), so X is $\mathbb{Q}$-factorial.

First we prove (1)(2). By definition, for any closed point $Q \notin \operatorname{Supp}(B) \cup \operatorname{Sing}(X)$, the vanishing order of $K_{\mathcal{F}}|_{C^\nu}$ at Q is equal to $Z(\mathcal{F}, C, Q)$. Thus if Q is a non-singular point of C, then we may identify Q with its inverse image in C^ν. If Q is a singular point of C, then since $(X, \mathcal{F}, B)$ is dlt, Q is a nodal singularity of C and $Z(\mathcal{F}, C, Q) = 0$.

Therefore, to prove (1)(2), we only need to show that for any $1 \leq i \leq n$, C is non-singular at P_i and the vanishing order of $(K_{\mathcal{F}}+\sum_{j=1}^m b_j'B_j)|_{C^\nu}$ at P_i is equal to $\frac{w_i-1+\sum_{j=1}^n w_{i,j}b_j'}{w_i}$, and w_i is equal to the index of P_i.

By [LMX24a, Theorem 3.19], each P_i is either a non-singular point of X or a cyclic quotient singularity of X. By [LMX24a, Theorem 4.1], $(X \ni P_i, C)$ is plt. Thus C is non-singular at P_i.

We let $\pi_i : X_i \to \tilde{X}$ be an index 1 cover of $X \ni P_i$, $P_i' := \pi_i^{-1}(P_i)$, $C_i := \pi_i^* C$, and $B_{j,i} := \pi_i^* B_j$ for any j. Then C_i is non-singular at P_i', and

$$\operatorname{mult}_{P_i'}\left(\left(\sum_{j=1}^m b_j' B_{j,i}\right)\Bigg|_{C_i}\right) = \sum_{j=1}^m b_j'(B_{j,i}\cdot C_i)_{P_i'}.$$

Moreover, we have $\deg(\pi_i) = w_i$, the index of P_i, for each i. We let $w_{i,j} := (B_{j,i}\cdot C_i)_{P_i'}$ for any i,j. (1)(2) now follow from the Hurwitz formula (cf. [Spi20, Proposition 3.7]).

Since $P_i \in \operatorname{Sing}(X)\cup(C\cap \operatorname{Supp} B)$, if $w_i = 1$, then $P_i \in C \cap \operatorname{Supp} B$. Thus $(B_{j,i}\cdot C_i)_{P_i'} \neq 0$ for some j, and (3) follows.

Since $w_{i,j} := (B_{j,i}\cdot C_i)_{P_i'}$ for any i,j, by Lemma 3.1,

$$\frac{1}{w_i} \geq (B\cdot C)_{P_i} = \frac{1}{w_i}(\pi_i^* B\cdot C_i)_{P_i'} = \frac{1}{w_i}\left(\sum_{j=1}^m b_j' B_{j,i}\cdot C_i\right)_{P_i'} = \sum_{j=1}^m \frac{w_{i,j}b_j'}{w_i},$$

and (4) follows. $\square$

3.2. **Threefold rank one case.**

Theorem 3.3 (Threefold rank one case)**.** *Let $(X,\mathcal{F},B=\sum_{j=1}^m b_jB_j)$ be a dlt foliated triple such that $\dim X = 3, \operatorname{rank}\mathcal{F} = 1$, and B_j are the irreducible components of B. Let S be an $\mathcal{F}$-invariant surface in X and S^ν the normalization of S. Suppose that B_j is $\mathbb{Q}$-Cartier for any j. Then there exist a rank 1 foliation $\mathcal{F}_S$ on S^ν, prime divisors $C_1,\dots,C_n$ on S^ν, and non-negative integers $\{w_{i,j}\}_{1\leq i\leq n, 0\leq j\leq m}$, such that for any real numbers $b_1',\dots,b_m'$, we have the following adjunction formula*

$$\left(K_{\mathcal{F}}+\sum_{j=1}^m b_j'B_j\right)\Bigg|_{S^\nu} = K_{\mathcal{F}_S}+\sum_{i=1}^n \frac{w_{i,0}+\sum_{j=1}^m w_{i,j}b_j'}{2}C_i.$$

We remark that there is no control of the singularities of $\left(S^\nu,\mathcal{F}_S,\sum_{i=1}^n \frac{w_{i,0}+\sum_{j=1}^m w_{i,j}b_j'}{2}C_i\right)$ even if $(X,\mathcal{F},\sum_{j=1}^m b_j'B_j)$ is dlt.

Proof. The proof is parallel to [CS20, Propositon 2.16]. For the reader's convenience we give a full proof here.

By [CS23b, Proposition-Definition 3.6, Remark 3.7], there exists a naturally defined restricted foliation $\mathcal{F}_S$ on S^ν of $\mathcal{F}$ such that $\operatorname{rank}\mathcal{F}_S = 1$. We prove the following claim.

Claim 3.4. *For any prime divisor D on S^ν and any prime divisor L on X, $2L$ is Cartier near the generic point of the image of D in S.*

Proof. Since $(X,\mathcal{F},B)$ is dlt, $\mathcal{F}$ has simple singularities. Let η_D be the generic point of the image of D in S. If $\mathcal{F}$ has canonical but not terminal singularity near η_D, then by the definition of simple singularities, $2L$ is Cartier near η_D. Thus we may assume that $\mathcal{F}$ is terminal near η_D. By [CS20, Lemma 2.12], the image of D in S is not $\mathcal{F}$-invariant. By [CS20, Lemma 2.6], η_D is not contained in $\operatorname{Sing}(X)$, so L is Cartier near η_D and we are done. $\square$

Proof of Theorem 3.3 continued. By Noetherian property, we only need to show that for any prime divisor D on S^ν, there exist non-negative integers $d_0,\dots,d_n$ such that

$$\left(K_{\mathcal{F}}+\sum_{j=1}^m b_j'B_j\right)\Bigg|_{S^\nu}=K_{\mathcal{F}_S}+\frac{d_0+\sum_{j=1}^m d_jb_j'}{2}D$$

for any real numbers $b_1',\dots,b_m'$ near the generic point of D. By Claim 3.4, $2K_{\mathcal{F}}$ and $2B_j$ are Cartier near the image of the generic point of D on S. Then we may let $d_j:=2\operatorname{mult}_D(B_j|_{S^\nu})$ for any $j\geq 1$. Moreover, the map $\Omega^1_X\otimes\Omega^1_X\to\mathcal{O}_X(2K_{\mathcal{F}})$ naturally restricts to a map $\Omega^1_X|_S\otimes\Omega^1_X|_S\to\mathcal{O}_S(2K_{\mathcal{F}})$. By [AD14, Lemma 3.7], $\Omega^1_X|_S\otimes\Omega^1_X|_S\to\mathcal{O}_S(2K_{\mathcal{F}})$ extends uniquely to a map $\Omega^1_{S^\nu}\otimes\Omega^1_{S^\nu}\to\mathcal{O}_{S^\nu}(2K_{\mathcal{F}})$. By construction of $\mathcal{F}_S$, $\Omega^1_{S^\nu}\otimes\Omega^1_{S^\nu}\to\mathcal{O}_{S^\nu}(2K_{\mathcal{F}})$ factors through $\mathcal{O}_{S^\nu}(2K_{\mathcal{F}_S})$. Thus there exists a Weil divisor $M\geq 0$ on S^ν such that

$$2K_{\mathcal{F}}|_{S^\nu}=2K_{\mathcal{F}_S}+M.$$

In particular, near the generic point of D, there exists a positive integer d_0 such that $M=d_0D$. Then $d_0,\dots,d_m$ satisfy our requirements. □

3.3. Threefold rank two case.

Theorem 3.5 (Threefold rank two case)**.** *Let $(X,\mathcal{F},B=\sum_{j=1}^m b_jB_j)$ be a dlt foliated triple such that $\dim X=3,\operatorname{rank}\mathcal{F}=2$, and B_j are the irreducible components of B. Assume that*

- *S is an $\mathcal{F}$-invariant surface in X,*
- *S^ν is the normalization of S,*
- *$\widehat{X}$ is the formal completion of X along S,*
- *$D_1,\dots,D_u$ are all $\mathcal{F}$-invariant divisors on $\widehat{X}$ that are not equal to S, and*
- *$K_X,S,\sum_{i=1}^u D_i$ are $\mathbb{Q}$-Cartier, and B_j is $\mathbb{Q}$-Cartier for any j.*

Then there exist prime divisors $C_1,\dots,C_n,T_1,\dots,T_l$ on S^ν, positive integers $w_1,\dots,w_n,\lambda_1,\dots,\lambda_l$, $\{e_{i,k}\}_{1\leq i\leq l,1\leq k\leq n_i}$, and non-negative integers $\{w_{i,j}\}_{1\leq i\leq n,1\leq j\leq m}$, $\{e_{i,k,j}\}_{1\leq i\leq l,1\leq k\leq n_i,1\leq j\leq m}$, such that for any real numbers $b_1',\dots,b_m'$, we have the following.

(1)
$$\left(K_{\mathcal{F}}+\sum_{j=1}^m b_j'B_j\right)\Bigg|_{S^\nu}=K_{S^\nu}+\sum_{i=1}^n\frac{w_i-1+\sum_{j=1}^m w_{i,j}b_j'}{w_i}C_i+\sum_{i=1}^l\lambda_iT_i.$$

(2)
$$\left(K_{\widehat{X}}+S+\sum_{i=1}^u D_i+\sum_{j=1}^m b_j'B_j\right)\Bigg|_{S^\nu}=K_{S^\nu}+\sum_{i=1}^n\frac{w_i-1+\sum_{j=1}^m w_{i,j}b_j'}{w_i}C_i+\sum_{i=1}^l T_i.$$

(3) If $(X,\mathcal{F},\sum_{j=1}^m b_j'B_j)$ is lc, then
$$\left(S^\nu,\sum_{i=1}^n\frac{w_i-1+\sum_{j=1}^m w_{i,j}b_j'}{w_i}C_i+\sum_{i=1}^l T_i\right)$$
is lc.

(4) For any i such that $\lambda_i\geq 2$,
   - *(a) T_i is an lc center of $(X,\mathcal{F},0)$,*
   - *(b) $\mathcal{F}$ has simple singularities at the generic point of T_i, and*
   - *(c) for any general closed point x in T_i, there are exactly two different separatrices of $\mathcal{F}$ containing x. One of these two separatrices is a strong separatrix of $\mathcal{F}$ at x, and the other is S, which is not a strong separtrix of $\mathcal{F}$ at x.*

(5) For any i such that S is a strong separatrix along any general closed point in T_i, $\lambda_i=1$.

(6) *Let T_i^ν be the normalization of T_i for each i. For any i such that $\lambda_i\geq 2$, there exist a Weil divisor $P_i\geq 0$ on T_i^ν and prime divisors (closed points) $P_{i,1},\dots,P_{i,n_i}$ on T_i^ν satisfying the following.*

(a)

$$\left(K_{\mathcal{F}}+\sum_{j=1}^m b_j'B_j\right)\Bigg|_{T_i^\nu}=K_{T_i^\nu}+P_i+\sum_{k=1}^{n_i}\frac{e_{i,k}-1+\sum_{j=1}^m e_{i,k,j}b_j'}{e_{i,k}}P_{i,k}.$$

(b) *Any component of* $\operatorname{Supp}P_i$ *is an lc center of* $(X,\mathcal{F},0)$.

(c) *If* $(X,\mathcal{F},\sum_{j=1}^m b_j'B_j)$ *is lc, then*

$$\left(T_i^\nu,(P_i)_{\mathrm{red}}+\sum_{k=1}^{n_i}\frac{e_{i,k}-1+\sum_{j=1}^m e_{i,k,j}b_j'}{e_{i,k}}P_{i,k}\right)$$

is lc.

Proof. The proof can be done by following the same lines of the proofs of [CS21, Lemma 3.18, Corollary 3.20, Lemma 3.22] and [Che22, Proposition 3.10]. Here we provide a short proof by only using the results in [Spi20, CS21] and not their proofs.

Step 1. We prove (2) and (3). By [Spi20, Lemma 8.14], $(\widehat{X},S+\sum_{i=1}^u D_i+B)$ is lc. We let

$$K_{S^\nu}+D(\Delta):=\left(K_{\widehat{X}}+S+\sum_{i=1}^u D_i+\Delta\right)\Bigg|_{S^\nu}$$

for any $\mathbb{R}$-Cartier $\mathbb{R}$-divisor Δ. Let C be a prime divisor on S^ν and η_C the generic point of C. If $(\widehat{X},S+\sum_{i=1}^u D_i)$ is lc but not klt at η_C, then η_C is not contained in B_j for any j. By [Kol$^+$92, 16.7 Corollary], the coefficient of C in $D(\Delta)$ is either 0 or 1 for any $\mathbb{R}$-divisor Δ such that $\operatorname{Supp}\Delta=\operatorname{Supp}B$. If $(\widehat{X},S+\sum_{i=1}^u D_i)$ is klt at η_C, then by [HLS19, Theorem 3.10], there exist positive integer w_C and non-negative integers $w_{C,j}$, such that for any real numbers $b_1',\dots,b_m'$, the coefficient of C in $D(\sum_{j=1}^m b_j'B_j)$ is equal to

$$\frac{w_C-1+\sum_{j=1}^m w_{C,j}b_j'}{w_C}.$$

This implies (2). (3) follows from (2) and [HLS19, Theorem 3.10] ([Kol$^+$92, 16.9 Proposition] for the $\mathbb{Q}$-coefficient case).

In the following, we let $C_1,\dots,C_n.T_1,\dots,T_l,w_1,\dots,w_n$, and $\{w_{i,j}\}_{1\leq i\leq n,1\leq j\leq m}$ be as in (2).

Step 2. We prove (1).

Claim 3.6. *Suppose that $(X,\mathcal{F},\sum_{j=1}^m b_j'B_j)$ is lc. Then (1) holds.*

Proof. By Theorem 2.7, there exist vectors $\boldsymbol{b}_1,\dots,\boldsymbol{b}_{m+1}\in\mathbb{Q}^m$ and real numbers $a_1,\dots,a_{m+1}\in(0,1]$, such that $\boldsymbol{b}_i=(b_{i,1},\dots,b_{i,m})$, $(X,\mathcal{F},B^i:=\sum_{j=1}^m b_{i,j}B_j)$ is lc for any i, $\sum_{i=1}^{m+1}a_i=1$, and $\sum_{i=1}^{m+1}a_i\boldsymbol{b}_i=(b_1',\dots,b_m')$. By [CS21, Corollary 3.20],

$$\left(K_{\mathcal{F}}+B^k\right)\Big|_{S^\nu}=K_{S^\nu}+\sum_{i=1}^n\frac{w_i-1+\sum_{j=1}^m w_{i,j}b_{k,j}}{w_i}C_i+\sum_{i=1}^l\lambda_{i,k}T_i.$$

where $\lambda_{i,k}$ are positive integers.

Suppose that $\lambda_{r,k}\neq\lambda_{r,k'}$ for some r and some $k\neq k'$. Let N be a sufficiently large positive integer, then by [CS21, Corollary 3.20] and (2), there exist positive integers $\mu_1,\dots,\mu_l$, such that

$$K_{S^\nu}+\sum_{i=1}^n\frac{w_i-1+\sum_{j=1}^m w_{i,j}(\frac{1}{N}b_{k,j}+\frac{N-1}{N}b_{k',j})}{w_i}C_i+\sum_{i=1}^l\mu_iT_i=\left(K_{\mathcal{F}}+\frac{1}{N}B^k+\frac{N-1}{N}B^{k'}\right)\Big|_{S^\nu}$$
$$=K_{S^\nu}+\sum_{i=1}^n\frac{w_i-1+\sum_{j=1}^m w_{i,j}(\frac{1}{N}b_{k,j}+\frac{N-1}{N}b_{k',j})}{w_i}C_i+\sum_{i=1}^l\frac{1}{N}(\lambda_{i,k}+(N-1)\lambda_{i,k'})T_i,$$

which is not possible as μ_r is an integer but $\frac{1}{N}(\lambda_{r,k}+(N-1)\lambda_{r,k'})$ is not. Thus for any i, $\lambda_{i,k}$ is a constant for any k, so we may let $\lambda_i:=\lambda_{i,k}$ for any i. (1) immediately follows in this case. $\square$

Proof of Theorem 3.5 continued. (1) follows immediately from Claim 3.6 and linearity of the coefficients of $B_1,\dots,B_m$.

Step 3. We conclude the proof in this step.

(4) By [CS21, Corollary 3.20], the generic point of T_i is not contained in $\mathrm{Sing}(X)$.

Suppose that T_i is not an lc center of $(X,\mathcal{F},0)$. Then $(X,\mathcal{F},0)$ is terminal at the generic point of T_i. By [CS21, Lemma 3.14], S is the unique $\mathcal{F}$-invariant divisor passing through the generic point of T_i. Thus $\sum_{i=1}^u D_i=0$ near the generic point of T_i. By (2), the generic point of T_i is contained in $\mathrm{Sing}(X)$, a contradiction. Thus T_i is an lc center of $(X,\mathcal{F},0)$, which implies (4.a). Since $(X,\mathcal{F},0)$ is dlt, $\mathcal{F}$ has simple singularities near the generic point of T_i, which is (4.b). (4.c) follows from (4.b), [CS20, Lemma 3.14], and [CS21, Corollary 3.20].

(5) It immediately follows from (4).

(6) By (4), we may let S' be the strong separatrix along T_i and S'^ν the normalization of S'. Then by (1)(3)(5),

$$\left(K_{\mathcal{F}}+\sum_{j=1}^m b'_jB_j\right)\Big|_{S'^\nu}=K_{S'^\nu}+\sum_k\frac{w'_k-1+\sum_{j=1}^m w'_{k,j}b'_j}{w'_k}C'_k+\sum_k\lambda'_kT'_k.$$

for some prime divisors C'_k,T'_k, positive integers w'_k,λ'_k, and non-negative integers $w'_{k,j}$, such that the images of T'_i and T_i in X coincide, $\lambda'_i=1$, and

$$\left(S'^\nu,\sum_k\frac{w'_k-1+\sum_{j=1}^m w'_{k,j}b'_j}{w'_k}C'_k+\sum_kT'_k\right)$$

is lc. In particular, the normalization of T'_i is T_i^ν. By [Kol+92, 16.6.3, 16.7 Corollary] and [HLS19, Theorem 3.10], we have

$$\begin{aligned}(K_{\mathcal{F}}+\sum_{j=1}^m b'_jB_j)|_{T_i^\nu}&=(K_{S'^\nu}+\sum_k\frac{w'_k-1+\sum_{j=1}^m w'_{k,j}b'_j}{w'_k}C'_k+\sum_k\lambda'_kT'_k)|_{T_i^\nu}\\&=K_{T_i^\nu}+P'_i+\sum_{k=1}^{n_i}\frac{e_{i,k}-1+\sum_{j=1}^m e_{i,k,j}b'_j}{e_{i,k}}P_{i,k}+\sum_{\lambda'_k\geq 2}(\lambda'_k-1)T'_k|_{T_i^\nu}.\end{aligned}$$

for some non-negative integer n_i, positive integers $\{e_{i,k}\}_{k=1}^{n_i}$, $\{e_{i,k,j}\}_{1\leq k\leq n_i,1\leq j\leq m}$, and an effective Weil divisor $P_i\geq 0$, such that

- $$\left(T_i^\nu,P'_i+\sum_{k=1}^{n_i}\frac{e_{i,k}-1+\sum_{j=1}^m e_{i,k,j}b'_j}{e_{i,k}}P_{i,k}\right)$$
  is lc, and
- $\mathrm{Supp}\,T'_k|_{T_i^\nu}\subset\mathrm{Supp}\,P'_i$ for any $k\neq i$.

For any point $D \subset \operatorname{Supp} P'_i$, by [CS21, Lemma 3.22] (applied to the case when $b'_j = 0$ for every j), D is an lc center of $(X,\mathcal{F},0)$. Since $(X,\mathcal{F},0)$ is dlt, T_i is Cartier near D. Thus $\sum_{\lambda'_k\geq 2}(\lambda'_k-1)T'_k|_{T_i^\nu}$ is an effective Weil divisor and we may let $P_i := P'_i + \sum_{\lambda'_k\geq 2}(\lambda'_k-1)T'_k|_{T_i^\nu}$. (6.a) and (6.b) immediately follow, and (6.c) follows from the adjunction formula for usual pairs. □

4. Uniform rational polytopes

In this section, we establish the theory of uniform rational polytopes and functional divisors for foliations in dimension ≤ 3 and prove Theorem 1.3.

4.1. **Log canonical thresholds.** In this subsection, we define the lc thresholds of foliations. Our definition is slightly different from the traditional definition since we need to consider lc thresholds of $\mathbb{R}$-divisors which are not necessarily effective.

Definition 4.1 (Lc thresholds). Let $(X,\mathcal{F},B)$ be an lc foliated sub-triple and D an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. An *lc threshold* (*lct* for short) of D with respect to $(X,\mathcal{F},B)$ is a real number t_0, such that

(1) $(X,\mathcal{F},B+t_0D)$ is sub-lc, and
(2) for any positive real number δ, either $(X,\mathcal{F},B+(t_0+\delta)D)$ or $(X,\mathcal{F},B+(t_0-\delta)D)$ is not sub-lc.

When $D\geq 0$, the lc threshold of D with respect to $(X,\mathcal{F},B)$ is unique, and we denote the lc threshold of D with respect to $(X,\mathcal{F},B)$ by $\operatorname{lct}(X,\mathcal{F},B;D)$.

The following lemma indicates that lc thresholds can always be achieved in dimension ≤ 3.

Lemma 4.2. *Let $(X,\mathcal{F},B)$ be a sub-lc foliated sub-triple of dimension ≤ 3 and D an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Let $B(t) := B+tD$ for any real number t. Then there exist $t_1,t_2\in\mathbb{R}\cup\{-\infty,+\infty\}$, such that*

(1) $t_1\leq 0\leq t_2$,
(2) for any real number t, $(X,\mathcal{F},B(t))$ is sub-lc if and only if $t_1\leq t\leq t_2$, and
(3) if $t_1\neq t_2$, then for any $i\in\{1,2\}$,
- *either $t_i\in\{-\infty,+\infty\}$, or*
- *there exists a prime divisor E_i over X, such that $a(E_i,X,\mathcal{F},B(t_i))=-\epsilon_{\mathcal{F}}(E_i)$ and $a(E_i,X,\mathcal{F},B(t))>-\epsilon_{\mathcal{F}}(E_i)$ for any $t\in(t_1,t_2)$.*

Proof. We may let $t_1:=\inf\{t\mid (X,\mathcal{F},B(t)) \text{ is sub-lc}\}$ and $t_2:=\sup\{t\mid (X,\mathcal{F},B(t)) \text{ is sub-lc}\}$. (1) immediately follows.

By [LLM23, Theorem 4.5], there exists a foliated log resolution $f: Y\to X$ of $(X,\mathcal{F},B)$. Let $K_{\mathcal{F}_Y}+B_Y(t):=f^*(K_{\mathcal{F}}+B(t))$ for any real number t where $\mathcal{F}_Y:=f^{-1}\mathcal{F}$. Possibly replacing $(X,\mathcal{F},B(t))$ with $(Y,\mathcal{F}_Y,B_Y(t))$, we may assume that $(X,\mathcal{F},\operatorname{Supp} B\cup\operatorname{Supp} D)$ is foliated log smooth. By [LLM23, Lemma 4.3], $(X,\mathcal{F},B(t))$ is lc if and only if for any component T of $\operatorname{Supp} B\cup \operatorname{Supp} D$, $\operatorname{mult}_T B(t)\leq\epsilon_{\mathcal{F}}(T)$. Since the coefficients of $B(t)$ are affine functions, $t_1=\min\{t\mid (X,\mathcal{F},B(t)) \text{ is sub-lc}\}$ or $-\infty$, and $t_2=\max\{t\mid (X,\mathcal{F},B(t)) \text{ is sub-lc}\}$ or $+\infty$. This implies (2). (3) immediately follows from (2). □

4.2. **Special dlt models.** In this subsection we prove some key lemmas related to dlt models of different foliated triples.

Lemma 4.3. *Let c,m be be positive integers, $r_1,\dots,r_c$ real numbers such that $1,r_1,\dots,r_c$ are linearly independent over $\mathbb{Q}$, $\boldsymbol{r}:=(r_1,\dots,r_c)$, and $s_1,\dots,s_m:\mathbb{R}^{c+1}\to\mathbb{R}$ $\mathbb{Q}$-linear functions. Assume that*

- *$(X,\mathcal{F},B(\boldsymbol{r}):=\sum_{i=1}^m s_i(1,\boldsymbol{r})B_i)$ is an lc foliated triple of dimension ≤ 3, and*
- *$B_i\geq 0$ are distinct Weil divisors and $s_i(1,\boldsymbol{r})\geq 0$ for each i.*

Let $f: Y \to X$ be a dlt modification of $(X, \mathcal{F}, B(\boldsymbol{r}))$ with prime exceptional divisors $E_1, \dots, E_n$, $\mathcal{F}_Y := f^{-1}\mathcal{F}$, $B(\boldsymbol{v}) := \sum_{i=1}^m s_i(1, \boldsymbol{v})B_i$ for any $\boldsymbol{v} \in \mathbb{R}^c$, and $B_Y(\boldsymbol{v}) := f_*^{-1}B(\boldsymbol{v}) + \sum_{i=1}^n \epsilon_{\mathcal{F}}(E_i)E_i$. Then for any $\boldsymbol{v} \in \mathbb{R}^c$,

(1) *$K_{\mathcal{F}_Y} + B_Y(\boldsymbol{v}) = f^*(K_{\mathcal{F}} + B_Y(\boldsymbol{v}))$, and*
(2) *$(X, \mathcal{F}, B(\boldsymbol{v}))$ is lc if and only if $(Y, \mathcal{F}_Y, B_Y(\boldsymbol{v}))$ is lc.*

Proof. There exist $\mathbb{Q}$-divisors $M_0, \dots, M_c$, such that $B(1, v_1, \dots, v_c) = M_0 + \sum_{i=1}^c v_i M_i$ for any $v_1, \dots, v_c$. By [HLS19, Lemma 5.3], M_i is $\mathbb{Q}$-Cartier for any $1 \le i \le m$. Then for any j,

$$-\epsilon_{\mathcal{F}}(E_j) = a(E_i, \mathcal{F}, B(\boldsymbol{r})) = a(E_j, \mathcal{F}, M_0) - \sum_{i=1}^m r_i \operatorname{mult}_{E_j} M_i,$$

so $\operatorname{mult}_{E_j} M_i = 0$ for each i. This implies (1) and (2) follows from (1). □

Lemma 4.4. *Let $(X, \mathcal{F}, B)$ be an lc foliated triple and M an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, such that for any positive real number δ, either $(X, \mathcal{F}, B + \delta M)$ is not lc or $(X, \mathcal{F}, B - \delta M)$ is not lc. Assume that*

- *either $\dim X \le 3$ and $\operatorname{rank} \mathcal{F} = \dim X - 1$, or*
- *$\dim X = 3$, $\operatorname{rank} \mathcal{F} = 1$, and X is projective.*

Then one of the following holds.

(1) *There exists a prime divisor D on X, such that $a(D, \mathcal{F}, B) = -\epsilon_{\mathcal{F}}(D)$ and $\operatorname{mult}_D M \neq 0$.*
(2) *There exists a dlt modification $f: Y \to X$ of $(X, \mathcal{F}, B)$ with prime f-exceptional divisors $E_1, \dots, E_n$ satisfying the following.*
   (a) *Let $K_{\mathcal{F}_Y} + B_Y := f^*(K_{\mathcal{F}} + B)$, where $\mathcal{F}_Y := f^{-1}\mathcal{F}$. Then there exists a positive real number δ such that $(Y, \mathcal{F}_Y, B_Y + \delta\{B_Y\})$ is dlt.*
   (b) *There exists an integer $1 \le i \le n$, such that $\operatorname{mult}_{E_i} M \neq 0$.*

Proof. By Lemma 4.2(2), there exists a prime divisor E that is exceptional over X, such that $a(E, X, \mathcal{F}, B) = -\epsilon_{\mathcal{F}}(E)$ and $\operatorname{mult}_E M \neq 0$. If E is on X, then either (1) holds, so we may assume that E is exceptional over X. By [LLM23, Theorem 4.5], we may let $g: Z \to X$ be a log resolution of $(X, \mathcal{F}, \operatorname{Supp} B \cup \operatorname{Supp} M)$ such that E is on Z. Let $F_1, \dots, F_m$ be the prime g-exceptional divisors, $\mathcal{F}_Z := g^{-1}\mathcal{F}$, $B_Z := g_*^{-1}B$, and $M_Z := g_*^{-1}M$. By [LLM23, Lemma 4.3] and the definition of dlt singularities,

$$\left(Z, \mathcal{F}_Z, \operatorname{Supp}(B_Z + M_Z) + \sum_{i=1}^m \epsilon_{\mathcal{F}}(F_i)\right)$$

is dlt, and

$$K_{\mathcal{F}_Z} + B_Z - \sum_{i=1}^m a(F_i, \mathcal{F}, B)F_i = g^*(K_{\mathcal{F}} + B).$$

Let $F := \sum_{i=1}^m (\epsilon_{\mathcal{F}}(F_i) + a(F_i, \mathcal{F}, B))F_i$. Then $F \ge 0$ is exceptional over X and $E \not\subset \operatorname{Supp} F$. Since $K_{\mathcal{F}_Z} + B_Z + \sum_{i=1}^m \epsilon_{\mathcal{F}}(F_i)F_i \sim_{\mathbb{R},X} F$, by [LLM23, Theorem 5.9], we may run a $(K_{\mathcal{F}_Z} + B_Z + \sum_{i=1}^m \epsilon_{\mathcal{F}}(F_i)F_i)$-MMP$/X$ which terminates with a good minimal model (cf. [LLM23, Definition 5.5]) $(Y, \mathcal{F}_Y, B_Y)/X$ of $(Z, \mathcal{F}_Z, B_Z + \sum_{i=1}^m \epsilon_{\mathcal{F}}(F_i)F_i)/X$ such that $K_{\mathcal{F}_Y} + B_Y \sim_{\mathbb{R},X} 0$ and the divisors contracted by $Z \dashrightarrow Y$ are exactly those contained in $\operatorname{Supp} F$. Then $(Y, \mathcal{F}_Y, B_Y)$ is $\mathbb{Q}$-factorial dlt and the induced birational morphism $f: Y \to X$ is a dlt modification of $(X, \mathcal{F}, B)$.

Since $(Z, \mathcal{F}_Z, \operatorname{Supp}(B_Z + M_Z) + \sum_{i=1}^m \epsilon_{\mathcal{F}}(F_i))$ is dlt, there exists a positive real number δ such that $(Z, \mathcal{F}_Z, B_Z + \sum_{i=1}^m \epsilon_{\mathcal{F}}(F_i)F_i + \delta\{B_Z\})$ is dlt and the induced birational map $Z \dashrightarrow Y$ is also a partial $(K_{\mathcal{F}_Z} + B_Z + \sum_{i=1}^m \epsilon_{\mathcal{F}}(F_i)F_i + \delta\{B_Z\})$-MMP for some positive real number δ. By [LLM23, Theorem 5.8], $(Y, \mathcal{F}_Y, B_Y + \delta\{B_Y\})$ is dlt. Thus (2.a) holds.

Let $E_1, \dots, E_n$ be the prime f-exceptional divisors. Since $E \not\subset \operatorname{Supp} F$, we may let i be the index such that E_i is the image of E on Y. Thus (2.b) holds. □

Lemma 4.5 (Key Lemma)**.** *Let $(X,\mathcal{F},B)$ be an lc foliated triple and M an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, such that*

- – *either* $\dim X\leq 3$ *and* $\operatorname{rank}\mathcal{F}=\dim X-1$, *or*
  – $\dim X=3$, $\operatorname{rank}\mathcal{F}=1$, *and X is projective,*
- $(X,\mathcal{F},B+M)$ *is lc,*
- $(X,\mathcal{F},B+(1+\epsilon)M)$ *is not lc for any positive real number ϵ,*
- $\operatorname{Supp}B=\operatorname{Supp}(B+M)$, *and*
- *for any prime divisor D on X such that* $a(D,\mathcal{F},B+M)=-\epsilon_{\mathcal{F}}(D)$, $\operatorname{mult}_D M=0$.

Then there are two projective birational morphisms $h:X'\to X$ and $g:Y'\to X'$ and a real number $t\in(0,1)$ satisfying the following.

(1) h is a dlt modification of $(X,\mathcal{F},B+tM)$.
(2) For any prime h-exceptional divisor D, $a(D,\mathcal{F},B)=-\epsilon_{\mathcal{F}}(D)$. In particular, $\operatorname{mult}_D M=0$ and $a(D,\mathcal{F},B+sM)=-\epsilon_{\mathcal{F}}(D)$ for any real number s.
(3) g extracts a unique prime divisor E. In particular, $-E$ is ample over X'.
(4) $a(E,\mathcal{F},B+M)=-\epsilon_{\mathcal{F}}(E)$ and $a(E,\mathcal{F},B)>-\epsilon_{\mathcal{F}}(E)$. In particular, $\operatorname{mult}_E M>0$ and $a(E,\mathcal{F},B+sM)>-\epsilon_{\mathcal{F}}(E)$ for any real number $s<1$.
(5) Let $B_{Y'},M_{Y'}$ be the strict transforms of B,M on Y' respectively, $\mathcal{F}_{Y'}:=(h\circ g)^{-1}\mathcal{F}$, and

$$F_{Y'}:=\sum_{D\text{ is a prime }h\circ g\text{-exceptional divisor}}\epsilon_{\mathcal{F}}(D)D.$$

Then $(Y',\mathcal{F}_{Y'},B_{Y'}+tM_{Y'}+F_{Y'})$ is $\mathbb{Q}$-factorial dlt.

Proof. By Lemma 4.4, there exists a dlt modification $f:Y\to X$ of $(X,\mathcal{F},B+M)$ satisfying the following: let $\mathcal{F}_Y:=f^{-1}\mathcal{F}$, $B_Y:=f^{-1}_*B,M_Y:=f^{-1}_*M$, $K_{\mathcal{F}_Y}+B_Y:=f^*(K_{\mathcal{F}}+B)$, and $E_1,\dots,E_n$ the prime f-exceptional divisors, then

- $(Y,\mathcal{F}_Y,B_Y+M_Y+\delta\{B_Y+M_Y\}+\sum_{i=1}^n\epsilon_{\mathcal{F}}(E_i)E_i)$ is dlt for some positive real number δ, and
- $\operatorname{mult}_{E_i}M\neq 0$ for some $1\leq i\leq n$.

Since $\operatorname{Supp}B=\operatorname{Supp}(B+M)$, $(Y,\mathcal{F}_Y,B_Y+tM_Y+\sum_{i=1}^n\epsilon_{\mathcal{F}}(E_i)E_i)$ is dlt for some $t\in(0,1)$. Since $(X,\mathcal{F},B)$ and $(X,\mathcal{F},B+M)$ are lc, $(X,\mathcal{F},B+tM)$ is lc. Thus

$$K_{\mathcal{F}_Y}+B_Y+tM_Y+\sum_{i=1}^n\epsilon_{\mathcal{F}}(E_i)E_i\sim_{\mathbb{R},X}\sum_{i=1}^n(\epsilon_{\mathcal{F}}(E_i)+a(E_i,\mathcal{F},B_Y+tM_Y))E_i\geq 0.$$

By [LLM23, Theorem 5.9], we may run a $(K_{\mathcal{F}_Y}+B_Y+tM_Y+\sum_{i=1}^n\epsilon_{\mathcal{F}}(E_i)E_i)$-MMP$/X$ which terminates with a good minimal model $(X',\mathcal{F}',B'+tM'+F')/X$ of $(Y,\mathcal{F}_Y,B_Y+tM_Y+\sum_{i=1}^n\epsilon_{\mathcal{F}}(E_i)E_i)/X$ such that $K_{\mathcal{F}'}+B'+tM'+F'\sim_{\mathbb{R},X}0$, where B',M',F' are the images of $B_Y,M_Y,\sum_{i=1}^n\epsilon_{\mathcal{F}}(E_i)E_i$ on X' respectively. Then $(X',\mathcal{F}',B'+tM'+F')$ is $\mathbb{Q}$-factorial dlt and the induced morphism $h:X'\to X$ is a dlt modification of $(X,\mathcal{F},B+tM)$.

By construction, the divisors contracted by the induced birational map $Y\dashrightarrow X'$ are all divisors E_i such that $\epsilon_{\mathcal{F}}(E_i)>a(E_i,\mathcal{F},B_Y+tM_Y)$. Since $\operatorname{mult}_{E_i}M\neq 0$, $Y\dashrightarrow X'$ contracts E_i, hence $Y\dashrightarrow X'$ contains a divisorial contraction. We let $g:Y'\dashrightarrow X'$ be the last step of the $(K_{\mathcal{F}_Y}+B_Y+tM_Y+\sum_{i=1}^n\epsilon_{\mathcal{F}}(E_i)E_i)$-MMP$/X$. Since X' is $\mathbb{Q}$-factorial and $K_{\mathcal{F}'}+B'+tM'+F'\sim_{\mathbb{R},X}0$, g is a divisorial contraction of a prime divisor E.

We show that h,g and t satisfy our requirements.

(1)(5) immediately follow from our construction.

For any prime divisor D on X' that is exceptional over X, the center of D on Y is a divisor that is exceptional over X. Since $f:Y\to X$ is a dlt modification of $(X,\mathcal{F},B+M)$, $a(D,\mathcal{F},B+M)=-\epsilon_{\mathcal{F}}(D)$. By (1), h is a dlt modification of $(X,\mathcal{F},B+tM)$, so $a(D,\mathcal{F},B+tM)=-\epsilon_{\mathcal{F}}(D)$. Thus $\operatorname{mult}_D M=0$, so $a(D,\mathcal{F},B)=-\epsilon_{\mathcal{F}}(D)$. This implies (2).

Since g is a divisorial contraction of a prime divisor E, by the negativity lemma, $-E$ is ample$/X'$. This implies (3).

Since the center of E on Y is a divisor that is exceptional over X, $a(E,\mathcal{F},B+M)=-\epsilon_{\mathcal{F}}(E)$. Since E is contracted by the $(K_{\mathcal{F}_Y}+B_Y+tM_Y+\sum_{i=1}^n\epsilon_{\mathcal{F}}(E_i)E_i)$-MMP$/X$: $Y\dashrightarrow X'$,

$$a(E,\mathcal{F},B+tM)=a(E,\mathcal{F}',B'+tM'+F')>-\epsilon_{\mathcal{F}}(E).$$

Thus $a(E,\mathcal{F},B)>-\epsilon_{\mathcal{F}}(E)$. This implies (4). $\square$

4.3. **Threefold rank one case.** In this subsection, we prove a slightly weaker version of Theorem 1.3 when $\dim X=3$ and $\operatorname{rank}\mathcal{F}=1$.

Proposition 4.6. *Let c,m be positive integers, $r_1,\dots,r_c$ real numbers such that $1,r_1,\dots,r_c$ are linearly independent over $\mathbb{Q}$, $\boldsymbol{r}:=(r_1,\dots,r_c)$, and $s_1,\dots,s_m:\mathbb{R}^{c+1}\to\mathbb{R}$ $\mathbb{Q}$-linear functions. Then there exists a positive real number δ depending only on $\boldsymbol{r}$ and $s_1,\dots,s_m$ satisfying the following. Assume that*

(1) $(X,\mathcal{F},B=\sum_{i=1}^m s_i(1,\boldsymbol{r})B_i)$ is a projective lc foliated triple such that $\dim X=3$ and $\operatorname{rank}\mathcal{F}=1$,
(2) $B_i\geq 0$ are distinct Weil divisors (possibly 0) and $s_i(1,\boldsymbol{r})\geq 0$, and
(3) $B(t):=\sum_{i=1}^m s_i(1,r_1,\dots,r_{c-1},t)B_i$ for any $t\in\mathbb{R}$.

Then $(X,\mathcal{F},B(t))$ is lc for any $t\in(r_c-\delta,r_c+\delta)$.

Proof. We let $s_i(t):=s_i(1,r_1,\dots,r_{c-1},t)$ for any $t\in\mathbb{R}$. If $s_i(r_c)=0$, then $s_i(t)=0$ for any t, so we may assume that $s_i(r_c)\neq 0$ for any i. By Lemma 4.3 and [LLM23, Theorem 5.7], possibly replacing $(X,\mathcal{F},B(r_c))$ with a dlt model, we may assume that $(X,\mathcal{F},B(r_c))$ is $\mathbb{Q}$-factorial dlt. In particular, $\mathcal{F}$ is non-dicritical.

We only need to prove that there exists a positive real number ϵ depending only on $\boldsymbol{r}$ and $s_1,\dots,s_m$, such that for any lc threshold t_0 of $(X,\mathcal{F},B(t))$, $|t_0-r_c|>\epsilon$. Thus we may assume that $(X,\mathcal{F},B(t))$ has an lc threshold t_0. Since $1,r_1,\dots,r_c$ are linearly independent over $\mathbb{Q}$, $r_c\neq t_0$. Moreover, there exists a positive real number δ_1 depending only on $\boldsymbol{r}$ and $s_1,\dots,s_m$, such that for any $t\in(r_c-\delta_1,r_c+\delta_1)$ and any i, $s_i(t)\geq\frac{1}{2}s_i(r_c)>0$. In particular, for any $t\in(r_c-\delta_1,r_c+\delta_1)$, $\operatorname{Supp}B(t)=\operatorname{Supp}B(r_c)$, $\operatorname{Supp}\lfloor B(t)\rfloor=\operatorname{Supp}\lfloor B(r_c)\rfloor$, and $B(t)\geq\frac{1}{2}B(r_c)$.

We may assume that $t_0\in(r_c-\delta_1,r_c+\delta_1)$. In particular, we may assume that $(X,\mathcal{F},B(t_0))$ does not have an lc center in codimension 1 that is also an lc center of $(X,\mathcal{F},B(r_c))$. Thus $(X,\mathcal{F},B(t_0))$ has an lc center x such that $\dim\bar{x}\leq 1$, and x is not an lc center of $(X,\mathcal{F},B(r_c))$. In particular, $x\in\operatorname{Supp}B(r_c)$.

By Lemma 4.5 (more precisely, $X,\mathcal{F},B,M$ in Lemma 4.5 correspond to our $X,\mathcal{F},B(r_c),B(t_0)-B(r_c)$ respectively), possibly replacing $(X,\mathcal{F},B(t))$, we may assume that there exists a divisorial contraction $g:Y\to X$ of a prime divisor E and a real number s satisfying the following. Let $B_Y(t)$ be the strict transform of $B(t)$ on Y for any t and $\mathcal{F}_Y:=g^{-1}\mathcal{F}$, then:

- $s\in(r_c,t_0)$ if $r_c<t_0$, and $s\in(t_0,r_c)$ if $t_0<r_c$,
- $(X,\mathcal{F},B(s))$ is $\mathbb{Q}$-factorial dlt, $(X,\mathcal{F},B(r_c))$ is lc, and $(X,\mathcal{F},B(t_0))$ is lc. In particular, since $\operatorname{rank}\mathcal{F}=1$, by definition, $(X,\mathcal{F},B(r_c))$ and $(X,\mathcal{F},B(t_0))$ are dlt.
- $-E$ is ample over X,
- $(Y,\mathcal{F}_Y,B_Y(s)+\epsilon_{\mathcal{F}}(E)E)$ is $\mathbb{Q}$-factorial dlt, and
- $a(E,\mathcal{F},B(t_0))=-\epsilon_{\mathcal{F}}(E)$ and $a(E,\mathcal{F},B(r_c))>-\epsilon_{\mathcal{F}}(E)$. In particular, $(Y,\mathcal{F}_Y,B_Y(t_0)+\epsilon_{\mathcal{F}}(E)E)$ is lc.

Since $(X,\mathcal{F},B(s))$ is $\mathbb{Q}$-factorial dlt, $\mathcal{F}$ is non-dicritical, so E is $\mathcal{F}_Y$-invariant and $\epsilon_{\mathcal{F}}(E)=0$. Let V be the normalization of $\operatorname{center}_X E$, E^ν the normalization of E, $g|_{E^\nu}:E^\nu\to V$ the induced projective surjective morphism with Stein factorization

$$E^\nu\xrightarrow{\pi}W\xrightarrow{\tau}V,$$

and F a general fiber of π. Then since $-E$ is ample over X,

$$K_{\mathcal{F}_Y}+B_Y(r_c)=g^*(K_{\mathcal{F}}+B(r_c))+a(E,\mathcal{F},B(r_c))E\sim_{\mathbb{R},X}a(E,\mathcal{F},B(r_c))E$$

is anti-ample$/X$. Thus

$$(K_{\mathcal{F}_Y}+B_Y(r_c))|_F$$

is anti-ample. Moreover, since

$$K_{\mathcal{F}_Y}+B_Y(t_0)=g^*(K_{\mathcal{F}}+B(t_0))\sim_{\mathbb{R},X}0,$$

we have that

$$(K_{\mathcal{F}_Y}+B_Y(t_0))|_F\sim_{\mathbb{R}}0.$$

We let x be the generic point of $\operatorname{center}_X E$ and let E^ν be the normalization of E. Since $(Y,\mathcal{F}_Y,B_Y(s))$ is $\mathbb{Q}$-factorial dlt, by Theorem 3.3, there exist prime divisors $C_1,\dots,C_n$ on E^ν, non-negative integers $\{w_{i,j}\}_{0\le i\le n,1\le j\le m}$, and a rank 1 foliation $\mathcal{F}_E$ on E^ν, such that

$$(K_{\mathcal{F}}+B_Y(t))|_{E^\nu}=K_{\mathcal{F}_E}+\sum_{i=1}^n\frac{w_{i,0}+\sum_{j=1}^m w_{i,j}s_j(t)}{2}C_i$$

for any real number t. There are two cases.

Case 1. $\dim\bar{x}=1$. We let

$$h(t):=\sum_{i=1}^n\frac{w_{i,0}+\sum_{j=1}^m w_{i,j}s_j(t)}{2}(C_i\cdot F)$$

for any real number t. Then $K_{\mathcal{F}_E}\cdot F+h(t)\neq 0$ when $t\neq t_0$ and $K_{\mathcal{F}_E}\cdot F+h(t_0)=0$. In particular, $h(t)$ is not a constant function. Since F is an irreducible component of a general fiber, $C_i\cdot F\ge 0$ for any i and $F^2=0$. In particular, $h(t_0)\ge 0$, and $h(t_0)=0$ if and only if $C_i\cdot F=0$ for any i. Recall that $s_j(t_0)\ge\frac{1}{2}s_j(r_c)>0$ for any j. Therefore, since $h(t)$ is not a constant function, $h(t_0)>0$. There are two cases.

Case 1.1. F is not $\mathcal{F}_E$-invariant. In this case, since E^ν is smooth near F,

$$0=K_{\mathcal{F}_E}\cdot F+h(t_0)>K_{\mathcal{F}_E}\cdot F=(K_{\mathcal{F}_E}+F)\cdot F=\operatorname{tang}(\mathcal{F}_E,F)\ge 0,$$

a contradiction.

Case 1.2. F is $\mathcal{F}_E$-invariant. In this case, since E^ν is smooth near F,

$$0=K_{\mathcal{F}_E}\cdot F+h(t_0)>K_{\mathcal{F}_E}\cdot F=Z(\mathcal{F}_E,F)-\chi(F),$$

so $K_{\mathcal{F}_E}\cdot F=Z(\mathcal{F}_E,F)-\chi(F)\in\{-1,-2\}$. Thus

$$h(t_0)=\sum_{i=1}^n\frac{c_{i,0}+\sum_{j=1}^m c_{i,j}s_j(t_0)}{2}(C_i\cdot F)\in\{1,2\}.$$

Since $s_j(t_0)\ge\frac{1}{2}s_j(r_c)>0$ for any j, there are finitely many possibilities of $c_{i,j}$ (which do not depend on t_0) for any i such that $C_i\cdot F\neq 0$. Thus there are only finitely many possibilities of $h(t)$. Since $h(t_0)=0$ and h is not a constant function, there are only finitely many possibilities of t_0. The proposition follows in this case.

Case 2. $\dim\bar{x}=0$. In this case $F=E^\nu$ and x is a closed point. We let

$$C(t):=\sum_{i=1}^n\frac{w_{i,0}+\sum_{j=1}^m w_{i,j}s_j(t)}{2}C_i.$$

Then $K_{\mathcal{F}_E}+C(t_0)\sim_{\mathbb{R}}0$, and $K_{\mathcal{F}_E}+C(r_c)$ is anti-ample. Therefore, $C(t_0)\neq 0$, so $K_{\mathcal{F}_E}$ is not pseudo-effective. By [CP19, Theorem 1.1], [LLM23, Theorem 3.1], $\mathcal{F}_E$ is algebraically integrable. By [Dru21, 3.5], there exists a projective birational morphism $f:V\to E^\nu$ and a contraction $\tau:V\to Z$ to a curve Z, such that $\mathcal{F}_V:=f^{-1}\mathcal{F}_E$ is induced by τ, i.e. τ is the family of leaves of $\mathcal{F}$. Let L_V be a general fiber of τ, $L:=f_*L_V$, and L_Y the image of L on Y.

Claim 4.7. *For any i such that $w_{i,j}\neq 0$ for some j, $C_i\cdot L$ is well-defined and is an integer.*

Proof. We only need to show that L is Cartier near any closed point $y \in C_i \cap L$. Since $(Y, \mathcal{F}_Y, B_Y(r_c))$ is lc and $s_j(r_c) > 0$ for any j, $\operatorname{Supp} B_Y(t)$ does not contain any lc center of $(Y, \mathcal{F}_Y, 0)$ for any real number t. Since $w_{i,j} \neq 0$ for some j, C_i does not contain any lc center of $(Y, \mathcal{F}_Y, 0)$. In particular, the image of y in Y is not an lc center of $(Y, \mathcal{F}_Y, 0)$. Since $\mathcal{F}_Y$ has simple singularities, $\mathcal{F}_Y$ is terminal near y. By [CS20, Lemma 2.12(1)], $\mathcal{F}_Y$ is induced by a fibration near y up to a quasi-étale cover. Therefore, there are only finitely many $\mathcal{F}_Y$-invariant curves passing through the image of y in Y. Thus, there are only finitely many $\mathcal{F}_E$-invariant curves passing through y, so y is a non-dicritical singularity of $\mathcal{F}_E$. Since L_V is a general fiber of τ, f is an isomorphism near a neighborhood of y, and L_V is smooth near $f^{-1}(y)$. Thus L is smooth near y. Therefore, L is Cartier near y and we are done. $\square$

Proof of Proposition 4.6 continued. We let

$$h(t) := \sum_{i=1}^{n} \frac{\sum_{j=1}^{m} w_{i,j} s_j(t)}{2} (C_i \cdot L)$$

for any real number t, then $h(r_c) \geq 0$, $h(t_0) \geq 0$, and $h(t) = B_Y(t) \cdot L_Y$ for any t. Since

$$0 > (K_{\mathcal{F}_Y} + B_Y(r_c)) \cdot L_Y = K_{\mathcal{F}_Y} \cdot L_Y + h(r_c),$$

we have that

$$\left(K_{\mathcal{F}_E} + \sum_{i=1}^{n} \frac{w_{i,0}}{2} C_i\right) \cdot L = K_{\mathcal{F}_Y} \cdot L_Y < 0.$$

Since $\mathcal{F}_Y$ has simple singularities, for any closed point $y \in L_Y$ such that $\mathcal{F}_Y$ is not terminal at y, $2K_{\mathcal{F}_Y}$ is Cartier near y. By [CS20, Proposition 2.16(3)(4), Proposition 3.3],

$$K_{\mathcal{F}_Y} \cdot L_Y = -2 + \frac{1}{2}\lambda + \sum_{i=1}^{u} \frac{\mu_i - 1}{\mu_i}$$

where λ, u are non-negative integers and μ_i are positive integers. Therefore,

$$0 > (K_{\mathcal{F}_Y} + B_Y(r_c)) \cdot L_Y = -2 + \frac{1}{2}\lambda + \sum_{i=1}^{u} \frac{\mu_i - 1}{\mu_i} + \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{w_{i,j}(C_i \cdot L)}{2} s_j(r_c)$$

and

$$0 = (K_{\mathcal{F}_Y} + B_Y(t_0)) \cdot L_Y = -2 + \frac{1}{2}\lambda + \sum_{i=1}^{u} \frac{\mu_i - 1}{\mu_i} + \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{w_{i,j}(C_i \cdot L)}{2} s_j(t_0).$$

We consider the g-pair

$$\left(\mathbb{P}^1, B_{\mathbb{P}^1}(t) := \sum_{i=1}^{u} \frac{\mu_i - 1}{\mu_i} P_i, \mathbf{M}(t) := \frac{\lambda}{2} \bar{Q}_0 + \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{w_{i,j}(C_i \cdot L)}{2} s_j(t) \bar{Q}_{i,j}\right)$$

where $P_i, Q_0, Q_{i,j}$ are distinct points on $\mathbb{P}^1$. Then $(\mathbb{P}^1, B_{\mathbb{P}^1}(t_0), \mathbf{M}(t_0))$ is lc, $K_{\mathbb{P}^1} + B_{\mathbb{P}^1}(t_0) + \mathbf{M}(t_0)_{\mathbb{P}^1} \equiv 0$, and $K_{\mathbb{P}^1} + B_{\mathbb{P}^1}(r_c) + \mathbf{M}(r_c)_{\mathbb{P}^1} \not\equiv 0$. The proposition now follows from [Che23a, Theorem 3.6] (which is essentially [Nak16, Theorem 3.8, Corollary 3.9] but the latter does not discuss the dimension 1 case). $\square$

4.4. **Threefold rank two case.** In this subsection, we prove a slightly weaker version of Theorem 1.3 when $\dim X = 3$ and $\operatorname{rank} \mathcal{F} = 2$.

Proposition 4.8. *Let c, m be positive integers, $r_1, \dots, r_c$ real numbers such that $1, r_1, \dots, r_c$ are linearly independent over $\mathbb{Q}$, $\boldsymbol{r} := (r_1, \dots, r_c)$, and $s_1, \dots, s_m : \mathbb{R}^{c+1} \to \mathbb{R}$ $\mathbb{Q}$-linear functions. Then there exists a positive real number δ depending only on $\boldsymbol{r}$ and $s_1, \dots, s_m$ satisfying the following. Assume that*

(1) $(X, \mathcal{F}, B = \sum_{i=1}^{m} s_i(1, \boldsymbol{r}) B_i)$ is an lc foliated triple such that $\dim X = 3$ and $\operatorname{rank} \mathcal{F} = 2$,
(2) $B_i \geq 0$ are distinct Weil divisors (possibly 0) and $s_i(1, \boldsymbol{r}) \geq 0$, and

(3) $B(t) := \sum_{i=1}^{m} s_i(1, r_1, \dots, r_{c-1}, t)B_i$ for any $t \in \mathbb{R}$.
Then $(X, \mathcal{F}, B(t))$ is lc for any $t \in (r_c - \delta, r_c + \delta)$.

Proof. **Step 1**. In this step we apply Lemma 4.5 and reduce to the case when $(X, \mathcal{F}, B)$ is $\mathbb{Q}$-factorial dlt with additional good properties. This step is very similar to the beginning of the proof of Proposition 4.6.

We let $s_i(t) := s_i(1, r_1, \dots, r_{c-1}, t)$ for any $t \in \mathbb{R}$. If $s_i(r_c) = 0$, then $s_i(t) = 0$ for any i, so we may assume that $s_i(r_c) \neq 0$ for any i. By Lemma 4.3 and [LLM23, Theorem 5.7], possibly replacing $(X, \mathcal{F}, B(r_c))$ with a dlt model, we may assume that $(X, \mathcal{F}, B(r_c))$ is $\mathbb{Q}$-factorial lc and $(X, \mathcal{F}, 0)$ is dlt. In particular, $\mathcal{F}$ is non-dicritical. (We remark that $(X, \mathcal{F}, B(r_c))$ is actually dlt here, but later we will replace the foliated triple $(X, \mathcal{F}, B(r_c))$ again and it may no longer be dlt).

We only need to prove that there exists a positive real number ϵ depending only on Γ and r, such that for any lc threshold t_0 of $(X, \mathcal{F}, B(t))$, $|t_0 - r_c| > \epsilon$. Thus we may assume that $(X, \mathcal{F}, B(t))$ has an lc threshold t_0. Since $1, r_1, \dots, r_c$ are linearly independent over $\mathbb{Q}$, $r_c \neq t_0$. Moreover, there exists a positive real number δ_1, such that for any $t \in (r_c - \delta_1, r_c + \delta_1)$ and any i, $s_i(t) \geq \frac{1}{2}s_i(r_c) > 0$. In particular, for any $t \in (r_c - \delta_1, r_c + \delta_1)$, $\operatorname{Supp} B(t) = \operatorname{Supp} B(r_c)$, $\operatorname{Supp}\lfloor B(t) \rfloor = \operatorname{Supp}\lfloor B(r_c) \rfloor$, and $B(t) \geq \frac{1}{2}B(r_c)$.

We may assume that $t_0 \in (r_c - \delta_1, r_c + \delta_1)$. In particular, we may assume that $(X, \mathcal{F}, B(t_0))$ does not have an lc center in codimension 1 that is also an lc center of $(X, \mathcal{F}, B(r_c))$. Thus $(X, \mathcal{F}, B(t_0))$ has an lc center x such that $\dim \bar{x} \leq 1$, and x is not an lc center of $(X, \mathcal{F}, B(r_c))$. In particular, $x \in \operatorname{Supp} B(r_c)$.

By Lemma 4.5 (more precisely, $X, \mathcal{F}, B, M$ in Lemma 4.5 correspond to our $X, \mathcal{F}, B(r_c), B(t_0) - B(r_c)$ respectively), possibly replacing $(X, \mathcal{F}, B(t))$, we may assume that there exist a divisorial contraction $g : Y \to X$ of a prime divisor E and a real number s satisfying the following: let $B_Y(t)$ be the strict transform of $B(t)$ on Y for any t and $\mathcal{F}_Y := g^{-1}\mathcal{F}$, then

- $s \in (r_c, t_0)$ if $r_c > t_0$, and $s \in (t_0, r_c)$ if $t_0 < r_c$,
- $(X, \mathcal{F}, B(s))$ is $\mathbb{Q}$-factorial dlt, $(X, \mathcal{F}, B(r_c))$ is lc, and $(X, \mathcal{F}, B(t_0))$ is lc,
- $-E$ is ample over X,
- $(Y, \mathcal{F}_Y, B_Y(s) + \epsilon_{\mathcal{F}}(E)E)$ is $\mathbb{Q}$-factorial dlt, and
- $a(E, \mathcal{F}, B(t_0)) = -\epsilon_{\mathcal{F}}(E)$ and $a(E, \mathcal{F}, B(r_c)) > -\epsilon_{\mathcal{F}}(E)$. In particular, $(Y, \mathcal{F}_Y, B_Y(t_0) + \epsilon_{\mathcal{F}}(E)E)$ is lc.

Step 2. We deal with the case when $\epsilon_{\mathcal{F}}(E) = 1$.

Suppose that $\epsilon_{\mathcal{F}}(E) = 1$. Since $(Y, \mathcal{F}_Y, B_Y(s) + \epsilon_{\mathcal{F}}(E))$ is $\mathbb{Q}$-factorial dlt, $\mathcal{F}_Y$ is non-dicritical. By [CS21, Remark 2.16] and [Spi20, Lemma 3.11], over the generic point of $\operatorname{center}_X E$,

- $(X, B(r_c))$ is lc, and $(X, B(t_0))$ is lc, and
- $a(E, X, B(t_0)) = -1$ and $a(E, X, B(r_c)) > -1$.

The proposition follows from [Nak16, Theorem 1.6] (see also [HLS19, Theorem 5.6]) in this case.

Step 3. From now on we can assume that $\epsilon_{\mathcal{F}}(E) = 0$. We summarize some known properties in this step.

Let V be the normalization of $\operatorname{center}_X E$, E^ν the normalization of E, $g|_{E^\nu} : E^\nu \to V$ the induced projective surjective morphism with Stein factorization

$$E^\nu \xrightarrow{\pi} W \xrightarrow{\tau} V,$$

and F a general fiber of π. Then since $-E$ is ample over X,

$$K_{\mathcal{F}_Y} + B_Y(r_c) = g^*(K_{\mathcal{F}} + B(r_c)) + a(E, \mathcal{F}, B(r_c))E \sim_{\mathbb{R},X} a(E, \mathcal{F}, B(r_c))E$$

is anti-ample$/X$. Thus

$$(K_{\mathcal{F}_Y} + B_Y(r_c))|_F$$

is anti-ample. Moreover, since

$$K_{\mathcal{F}_Y} + B_Y(t_0) = g^*(K_{\mathcal{F}} + B(t_0)) \sim_{\mathbb{R},X} 0,$$

we have that

$$(K_{\mathcal{F}_Y}+B_Y(t_0))|_F\sim_{\mathbb{R}}0.$$

By Theorem 3.5, there exist positive integers $w_1,\dots,w_n$, $\lambda_1,\dots,\lambda_k$, non-negative integers $\{w_{i,j}\}_{1\le i\le n,1\le j\le m}$, and prime divisors $C_1,\dots,C_n,T_1,\dots,T_l$, such that for any real number t, we have

$$(K_{\mathcal{F}_Y}+B_Y(t))\,|_{E^\nu}=K_{E^\nu}+\sum_{i=1}^n\frac{w_i-1+\sum_{j=1}^m w_{i,j}s_j(t)}{w_i}C_i+\sum_{i=1}^l\lambda_iT_i.$$

We let

$$B_E(t):=\sum_{i=1}^n\frac{w_i-1+\sum_{j=1}^m w_{i,j}s_j(t)}{w_i}C_i+\sum_{i=1}^l\lambda_iT_i$$

and

$$\tilde{B}_E(t):=\sum_{i=1}^n\frac{w_i-1+\sum_{j=1}^m w_{i,j}s_j(t)}{w_i}C_i+\sum_{i=1}^lT_i$$

for any real number t. By Theorem 3.5(3), $(E^\nu,\tilde{B}_E(t_0))$ and $(E^\nu,\tilde{B}_E(r_c))$ are lc.

Step 4. We deal with the case when $\dim F=1$.

Suppose that $\dim F=1$. We let

$$h(t):=B_E(t)\cdot F=\sum_{i=1}^n\frac{w_i-1+\sum_{j=1}^m w_{i,j}s_j(t)}{w_i}(C_i\cdot F)+\sum_{i=1}^l\lambda_i(T_i\cdot F)$$

for any real number t. Then $h(t)$ is an affine function, $h(r_c)\ge 0$, and $h(t_0)\ge 0$.

In this case, we have $K_{E^\nu}\cdot F+h(t_0)=0$ and $K_{E^\nu}\cdot F+h(r_c)<0$. Thus $K_{E^\nu}\cdot F<0$, so F is a smooth rational curve. Since $F^2=0$, $K_{E^\nu}\cdot F=-2$. Thus

$$-2+\sum_{i=1}^n\frac{w_i-1+\sum_{j=1}^m w_{i,j}s_j(t)}{w_i}(C_i\cdot F)+\sum_{i=1}^l\lambda_i(T_i\cdot F)=0.$$

We consider the pair

$$\left(\mathbb{P}^1,B_{\mathbb{P}^1}(t):=\sum_{i=1}^n\frac{w_i-1+\sum_{j=1}^m w_{i,j}s_j(t)}{w_i}\sum_{k=1}^{(C_i\cdot F)}P_{i,k}+\sum_{i=1}^l\sum_{k=1}^{\lambda_i(T_i\cdot F)}Q_{i,k}\right)$$

for any real number t, where $P_{i,k}$ are $Q_{i,k}$ are distinct points on $\mathbb{P}^1$. Then $(\mathbb{P}^1,B_{\mathbb{P}^1}(t_0))$ is lc, $K_{\mathbb{P}^1}+B_{\mathbb{P}^1}(t_0)\equiv 0$, and $K_{\mathbb{P}^1}+B_{\mathbb{P}^1}(r_c)\not\equiv 0$, The proposition now follows from [Che23a, Theorem 3.6].

Step 5. We conclude the proof in this step. Now we may assume that $F=E^\nu$ and $\dim F=2$. Then $K_{E^\nu}+B_E(t_0)\sim_{\mathbb{R}}0$, and $K_{E^\nu}+B_E(r_c)$ is anti-ample. By [Nak16, Theorem 3.8, Corollary 3.9], we may assume that $B_E(t)\neq\tilde{B}_E(t)$ for any t. Thus $c_i\ge 2$ for some i. We let T_i^ν be the normalization of T_i.

By Theorem 3.5, there exist a non-negative integer n_i, positive integers $e_{i,1},\dots,e_{i,n_i}$, non-negative integers $\{e_{i,k,j}\}_{1\le k\le n_i,1\le j\le m}$, a Weil divisor $P_i\ge 0$ on T_i^ν, and prime divisors $P_{i,1},\dots,P_{i,n_i}$ on T_i^ν, such that for any real number t,

$$(K_{\mathcal{F}_Y}+B_Y(t))|_{T_i^\nu}=K_{T_i^\nu}+B_{T_i}(t):=K_{T_i^\nu}+P_i+\sum_{k=1}^{n_i}\frac{e_{i,k}-1+\sum_{j=1}^m e_{i,k,j}s_j(t)}{e_{i,k}}P_{i,k},$$

and

$$\left(T_i^\nu,\operatorname{Supp}P_i+\sum_{k=1}^{n_i}\frac{e_{i,k}-1+\sum_{j=1}^m e_{i,k,j}s_j(t_0)}{e_{i,k}}P_{i,k}\right)$$

is lc. Moreover, we have $K_{T_i^\nu}+B_{T_i}(t_0)\sim_{\mathbb{R}}0$ and $K_{T_i^\nu}+B_{T_i}(r_c)$ is anti-ample. Thus $T_i^\nu=\mathbb{P}^1$. We consider the g-pair

$$\left(\mathbb{P}^1, B'_{\mathbb{P}^1}(t):=\sum_{k=1}^{n_i}\frac{e_{i,k}-1+\sum_{j=1}^m e_{i,k,j}s_j(t)}{e_{i,k}}P_{i,k}, \mathbf{M}:=\bar{P}_i\right)$$

for any real number t. Then $(\mathbb{P}^1,B'_{\mathbb{P}^1}(t_0),\mathbf{M})$ is lc, $K_{\mathbb{P}^1}+B'_{\mathbb{P}^1}(t_0)+\mathbf{M}_{\mathbb{P}^1}\equiv 0$, and $K_{\mathbb{P}^1}+B'_{\mathbb{P}^1}(r_c)+\mathbf{M}_{\mathbb{P}^1}\not\equiv 0$. The proposition now follows from [Che23a, Theorem 3.6]. □

Remark 4.9. The same arguments as of Proposition 4.6 can also be helpful to simplify the proof of the rank 2 case of the ACC for foliated lc thresholds for threefolds [Che22, Theorem 3.11]. By applying Lemma 4.5, we do not need the argument from [Che22, Page 15] to [Che22, End of Section 3].

4.5. **Proof of Theorem 1.3.**

Theorem 4.10. *Let c,m be positive integers, $r_1,\dots,r_c$ real numbers such that $1,r_1,\dots,r_c$ are linearly independent over $\mathbb{Q}$, $\boldsymbol{r}:=(r_1,\dots,r_c)$, and $s_1,\dots,s_m:\mathbb{R}^{c+1}\to\mathbb{R}$ $\mathbb{Q}$-linear functions. Then there exists a positive real number δ depending only on $\boldsymbol{r}$ and $s_1,\dots,s_m$ satisfying the following. Assume that*

(1) $(X,\mathcal{F},B=\sum_{i=1}^m s_i(1,\boldsymbol{r})B_i)$ is an lc foliated triple such that $\operatorname{rank}\mathcal{F}<\dim X\leq 3$,
(2) $B_i\geq 0$ are distinct Weil divisors (possibly 0) and $s_i(1,\boldsymbol{r})\geq 0$,
(3) $B(t):=\sum_{i=1}^m s_i(1,r_1,\dots,r_{c-1},t)B_i$ for any $t\in\mathbb{R}$, and
(4) if $\dim X=3$ and $\operatorname{rank}\mathcal{F}=1$, then X is projective.

Then $(X,\mathcal{F},B(t))$ is lc for any $t\in(r_c-\delta,r_c+\delta)$.

Proof. If $\operatorname{rank}\mathcal{F}=0$ then $B(t)=0$ for any t and there is nothing left to prove, so we may assume that $\operatorname{rank}\mathcal{F}>0$. The theorem follows from Propositions 4.6 and 4.8 and [LMX24a, Theorem 1.8]. □

Proof of Theorem 1.3. If $\mathcal{F}=T_X$, then the theorem is [HLS19, Theorem 5.6]. So we may assume that $\operatorname{rank}\mathcal{F}<\dim X$.

We apply induction on c. When $c=1$, Theorem 1.3 directly follows from Theorem 4.10. When $c\geq 2$, by Theorem 4.10, there exists a positive integer δ depending only on $r_1,\dots,r_c,s_1,\dots,s_m$, such that for any $t\in(r_c-\delta,r_c+\delta)$, $(X,\mathcal{F},\sum_{i=1}^m s_i(1,r_1,\dots,r_{c-1},t)B_i)$ is lc. We pick rational numbers $r_{c,1}\in(r_c-\delta,r_c)$ and $r_{c,2}\in(r_c,r_c+\delta)$ depending only on $r_1,\dots,r_c,s_1,\dots,s_m$. By induction on c, there exists an open subset $U_0\ni(r_1,\dots,r_{c-1})$ of $\mathbb{R}^{c-1}$, such that for any $\boldsymbol{v}\in U_0$, $(X,\mathcal{F},\sum_{i=1}^m s_i(1,\boldsymbol{v},r_{c,1})B_i)$ and $(X,\mathcal{F},\sum_{i=1}^m s_i(1,\boldsymbol{v},r_{c,2})B_i)$ are lc. We may pick $U:=U_0\times(r_{c,1},r_{c,2})$. □

5. Proofs of Theorem 1.1 and Corollary 1.4

The following result is a variation of the ACC for lc thresholds [Che22, Theorem 0.5] for foliations in dimension ≤ 3, which is more useful in some scenarios.

Proposition 5.1. *Let $\Gamma\subset[0,1]$ be a DCC set of real numbers and α a positive real number. Then there exists a function $g:\bar{\Gamma}\to\bar{\Gamma}$ satisfying the following.*

(1) $g\circ g=g$ and $g(\bar{\Gamma})$ is a finite set.
(2) $\gamma+\alpha\geq g(\gamma)\geq\gamma$ for any $\gamma\in\bar{\Gamma}$.
(3) $g(\gamma)\leq g(\gamma')$ for any $\gamma,\gamma'\in\bar{\Gamma}$ such that $\gamma\leq\gamma'$.
(4) For any non-negative integer m and lc foliated triple $(X,\mathcal{F},\sum_{i=1}^m b_iB_i)$, such that $\dim X\leq 3$, $b_i\in\Gamma$ for any i, and B_i are effective $\mathbb{Q}$-Cartier Weil divisors, we have that

$$\left(X,\mathcal{F},\sum_{i=1}^m g(b_i)B_i\right)$$

is lc.

Proof. We may assume that $\Gamma=\bar\Gamma$. Let

$$\Gamma':=\overline{\{\operatorname{lct}(X,\mathcal{F},B;D)\mid \dim X\leq 3, X \text{ is projective}, B\in\bar\Gamma, D\in\mathbb{N}^+\}}.$$

By [Che22, Theorem 0.5], Γ' is an ACC set. By [HLS19, Lemma 5.17], there exists a function $g:\bar\Gamma\to\bar\Gamma$, such that (1-3) hold, and for any $\beta\in\Gamma'$ and $\gamma\in\bar\Gamma$ such that $\beta\geq\gamma$, we have $\beta\geq g(\gamma)$.

Suppose that $(X,\mathcal{F},\sum_{i=1}^m g(b_i)B_i)$ is not lc. Then $m\geq 1$, and there exists $0\leq j\leq m-1$, such that $(X,\mathcal{F},\sum_{i=1}^j g(b_i)B_i+\sum_{i=j+1}^m b_iB_i)$ is lc and $(X,\mathcal{F},\sum_{i=1}^{j+1} g(b_i)B_i+\sum_{i=j+2}^m b_iB_i)$ is not lc. Let

$$b:=\operatorname{lct}\left(X,\mathcal{F},\sum_{i\neq j+1} g(b_i)B_i;B_{j+1}\right),$$

then $b_{j+1}\leq b<g(b_{j+1})$ and $b\in\Gamma'$, which is not possible. Thus $(X,\mathcal{F},\sum_{i=1}^m g(b_i)B_i)$ is lc and we are done. □

Proposition 5.2. *Let $\Gamma\subset[0,1]$ be a DCC set. Then there exist a finite set $\Gamma_0\subset\Gamma$ depending only on Γ satisfying the following. Assume that*

(1) $(X,\mathcal{F},B)$ is a projective lc foliated triple of dimension ≤ 3,
(2) $B\in\Gamma$,
(3) $\pi:X\to Z$ is a contraction such that $0<\dim Z<\dim X$,
(4) there exists a foliation $\mathcal{F}_Z$ on Z such that $\mathcal{F}=\pi^{-1}\mathcal{F}_Z$,
(5) $K_{\mathcal{F}}+B\sim_{\mathbb{R}}0$, and
(6) if $\dim X=3,\operatorname{rank}\mathcal{F}=2$, and $\dim Z=2$, then Z is klt.

Then $B\in\Gamma_0$.

In the proof of Proposition 5.2, we will introduce a lot of sets of coefficients depending only on Γ. For the reader's convenience, we note that the sets with/without the subscript "0" are finite/DCC sets in the following proof.

Proof. By [HMX14, Theorem 1.5], we may assume that $\operatorname{rank}\mathcal{F}<\dim X$. If $\operatorname{rank}\mathcal{F}=0$, then $B=0$ and there is nothing left to prove. So we may assume that $1\leq\operatorname{rank}\mathcal{F}\leq 2$.

By [LLM23, Theorem 5.7], possibly replacing Γ with $\Gamma\cup\{1\}$, we may assume that $(X,\mathcal{F},B)$ is a $\mathbb{Q}$-factorial projective dlt foliated triple of dimension ≤ 3.

Let F be a general fiber of π and $B_F:=B|_F$. Then

$$K_F+B_F=(K_X+B)|_F=(K_{\mathcal{F}}+B)|_F\sim_{\mathbb{R}}0,$$

and $B_F\in\Gamma$. Since $(X,\mathcal{F},B)$ is lc, (F,B_F) is lc. By [HMX14, Theorem 1.5], there exists a finite set $\Gamma_0'\subset\Gamma$ such that $B_F\in\Gamma_0'$.

By [Che22, Theorem 2.5], we may assume that $\dim X=3$. By [LLM23, Proposition 6.4(2)], there exists a projective lc generalized foliated quadruple $(Z,\mathcal{F}_Z,B_Z,\mathbf{M})$ induced by a canonical bundle formula $\pi:(X,\mathcal{F},B)\to Z$ (see [LLM23, Definition 1.2] for the definition) Then $K_{\mathcal{F}_Z}+B_Z+\mathbf{M}_Z\sim_{\mathbb{R}}0$.

We let B^h be the horizontal/Z part of B and B^v the vertical/Z part of B. Then $B^h\in\Gamma_0'$. Let $\bar\Gamma$ be the closure of Γ. By Proposition 5.1, there exist a finite set $\Gamma_0''\subset\bar\Gamma$ depending only on Γ and an $\mathbb{R}$-divisor $\bar B\in\Gamma_0''$, such that

- $\Gamma_0'\subset\Gamma_0''$,
- $\bar B\geq B$ and $\operatorname{Supp}\bar B=\operatorname{Supp}B$,
- $(X,\mathcal{F},\bar B)$ is lc, and
- $\bar B^h=B^h$, where $\bar B^h$ is the horizontal/Z part of $\bar B$.

In particular, $0\leq\bar B-B$ is vertical/Z.

Claim 5.3. *Proposition 5.2 holds if $\dim Z=1$.*

Proof. In this case, $\operatorname{rank}\mathcal{F}=2$ and $\mathcal{F}_Z$ is the trivial foliation, so $K_{\mathcal{F}_Z}+B_Z+\mathbf{M}_Z\sim_{\mathbb{R}}0$ implies that $B_Z=0$. Suppose that there exists a component D of B that is vertical over Z. We let $P=\pi(D)$, then

$$\sup\{t\mid (X,\mathcal{F},B+t\pi^*P)\text{ is sub-lc over }P\}<1.$$

By [LLM23, Proposition 6.4(2)], $\operatorname{mult}_P B_Z>0$, a contradiction. Therefore, all components of B are horizontal over Z. Since $B^h\in\Gamma_0'$, Proposition 5.2 follows in this case. □

Proof of Proposition 5.2 continued. By Claim 5.3, we may assume that $\dim Z=2$. Since $(X,\mathcal{F},\bar{B})$ is lc, $(X,\mathcal{F},B)$ is dlt, and $\operatorname{Supp}\bar{B}=\operatorname{Supp}B$, we have that $(X,\mathcal{F},\frac{1}{2}(\bar{B}+B))$ is dlt. By [LLM23, Theorem 5.8], we may run a $(K_{\mathcal{F}}+\frac{1}{2}(\bar{B}+B))$-MMP$/Z$ which terminates with a log minimal model (cf. [LLM23, Definition 5.5]) of $(X,\mathcal{F},\frac{1}{2}(\bar{B}+B))/Z$. Since

$$\frac{1}{2}(K_{\mathcal{F}}+\bar{B})\sim_{\mathbb{R}}K_{\mathcal{F}}+\frac{1}{2}(\bar{B}+B),$$

this MMP is also a $(K_{\mathcal{F}}+\bar{B})$-MMP which terminates with a weak lc model (cf. [LLM23, Definition 5.5]) $(X',\mathcal{F}',\bar{B}')/Z$ of $(X,\mathcal{F},\bar{B})/Z$. We let $\pi':X'\to Z$ be the induced contraction and B' the image of B on X'.

Claim 5.4. $K_{\mathcal{F}'}+\bar{B}'\sim_{\mathbb{R},Z}0$.

Proof. Since $K_{\mathcal{F}'}+B'\sim_{\mathbb{R}}0$, we only need to show that $\bar{B}'-B'\sim_{\mathbb{R},Z}0$. By assumption, $\bar{B}'-B'$ is nef$/Z$. There are two cases.

Case 1. $\operatorname{rank}\mathcal{F}=1$. In this case, $\mathcal{F}_Z=0$, so $K_{\mathcal{F}_Z}+B_Z+\mathbf{M}_Z\sim_{\mathbb{R}}0$ implies that $B_Z=0$. Therefore, for any component D of B that is vertical over Z, D is very exceptional over Z. Thus $\bar{B}-B$ is very exceptional over Z, so $\bar{B}'-B'$ is very exceptional over Z. By [Bir12, Lemma 3.3], $\bar{B}-B=0$ and we are done.

Case 2. $\operatorname{rank}\mathcal{F}=2$. In this case, by our assumption, Z is klt. Thus Z is $\mathbb{Q}$-factorial. For any prime divisor D on Z, we define

$$\nu_D:=\sup\{t\mid \bar{B}'-B'-\pi^*D\geq 0\}$$

and let

$$L':=\bar{B}'-B'-\sum_{D\text{ is a prime divisor on }Z}\nu_D\pi'^*D.$$

Then $L'\geq 0$ and L' is very exceptional over Z. By [Bir12, Lemma 3.3], $L'=0$. Thus

$$\bar{B}'-B'=\sum_{D\text{ is a prime divisor on }Z}\nu_D\pi'^*D\sim_{\mathbb{R},Z}0$$

and we are done. □

Proof of Proposition 5.2 continued. By Claim 5.4, $K_{\mathcal{F}'}+\bar{B}'\sim_{\mathbb{R},Z}0$. Since $K_{\mathcal{F}'}+B'\sim_{\mathbb{R}}0$, $\bar{B}'-B'\sim_{\mathbb{R},Z}0$. By Theorem 1.3 and [HLS19, Lemma 5.3], there exist real numbers $a_1,\dots,a_k\in(0,1]$, a finite set $\Gamma_0'''\subset\mathbb{Q}\cap[0,1]$ depending only on Γ, and $\mathbb{Q}$-divisors $\bar{B}_1',\dots,\bar{B}_k'\in\Gamma_0'''$ on X', such that

- $\sum_{i=1}^k a_i=1$ and $\sum_{i=1}^k a_i\bar{B}_i'=\bar{B}'$,
- $(X',\mathcal{F}',\bar{B}_i')$ is lc for each i,
- $(X',\mathcal{F}',B_i':=\bar{B}_i'-(\bar{B}'-B'))$ is lc for each i, and
- $K_{\mathcal{F}'}+\bar{B}_i'\sim_{\mathbb{Q},Z}0$ and $K_{\mathcal{F}'}+B_i'\sim_{\mathbb{R},Z}0$ for each i.

By [LLM23, Proposition 6.4(2.a)(3)], we may let $(Z,\mathcal{F}_Z,B_{i,Z},\mathbf{M}_i)$ and $(Z,\mathcal{F}_Z,\bar{B}_{i,Z},\mathbf{M}_i)$ be projective lc generalized foliated quadruples induces by canonical bundle formulas of $(X',\mathcal{F}',B_i')\to Z$ and $(X',\mathcal{F}',\bar{B}_i')\to Z$ respectively. We let $\mathbf{M}':=\sum_{i=1}^k a_i\mathbf{M}_i$ (note that it is possible that $\mathbf{M}'\neq\mathbf{M}$), and let $B_Z':=\sum_{i=1}^k a_iB_{i,Z}$. Then $K_{\mathcal{F}_Z}+B_Z'+\mathbf{M}_Z'\sim_{\mathbb{R}}0$. Since $\dim Z=2$, by [LLM23, Proposition 6.4(4)], there exists a positive integer I depending only on

Γ, such that for each i, we may choose $\mathbf{M}_i$ such that $I\mathbf{M}_i$ is base-point-free, $I(K_{\mathcal{F}'}+B_i')\sim I\pi'^*(K_{\mathcal{F}_Z}+B_{i,Z}+\mathbf{M}_{i,Z})$, and $I(K_{\mathcal{F}'}+\bar{B}_i')\sim I\pi'^*(K_{\mathcal{F}_Z}+\bar{B}_{i,Z}+\mathbf{M}_{i,Z})$.

By [Che22, Theorem 0.5] and [LLM23, Proposition 6.4(2)], there exists a DCC set $\Gamma'\subset[0,1]$ depending only on Γ such that $\bar{B}_{i,Z},B_{i,Z}\in\Gamma'$. Since $K_{\mathcal{F}_Z}+B_{i,Z}+\mathbf{M}_{i,Z}\sim_{\mathbb{R}}0$, by [LLM23, Lemma 7.2], there exists a finite set Γ_0'''' depending only on Γ such that $B_{i,Z}'\in\Gamma_0''''$ for any i. Possibly replacing Γ_0'''', we may assume that $B_Z'\in\Gamma_0''''$.

We let $\bar{B}_Z:=\sum_{i=1}^k a_i\bar{B}_{i,Z}$. Then there exists a DCC set $\Gamma''\subset(0,1]$ depending only on Γ such that for any component D of $\bar{B}_Z-B_Z'$, $\operatorname{mult}_D(\bar{B}_Z-B_Z')\in\Gamma''$. Since the coefficients of $\bar{B}'$ belongs to the finite set Γ_0'', the coefficients of

$$B'=\bar{B}'-\pi'^*(\bar{B}_Z-B_Z')$$

belong to an ACC set depending only on Γ. Since $B'\in\Gamma$, the coefficients of B' belong to a finite set $\tilde{\Gamma}_0$ depending only on Γ. In this case, by Theorem 1.3 and [HLS19, Lemma 5.3], there exist real numbers $c_1,\dots,c_k\in(0,1]$ and a finite set $\tilde{\Gamma}_0'\subset\mathbb{Q}\cap[0,1]$ depending only on Γ, and $\mathbb{Q}$-divisors $\tilde{B}_1',\dots,\tilde{B}_k'\in\tilde{\Gamma}_0'$ on X', such that

- $\sum_{i=1}^k c_i=1$ and $\sum_{i=1}^k c_i\tilde{B}_i'=B'$,
- $(X',\mathcal{F}',\tilde{B}_i')$ is lc for each i, and
- $K_{\mathcal{F}'}+\tilde{B}_i'\sim_{\mathbb{Q}}0$ for each i.

By [LMX24a, Theorem 1.3], there exists a positive integer I depending only on Γ such that $I(K_{\mathcal{F}'}+\tilde{B}_i')\sim 0$ for each i. Thus for any prime divisor E over X', $a(E,\mathcal{F}',\tilde{B}_i')$ belong to the discrete set

$$\left\{\frac{k}{I}\,\middle|\, k\geq -I,k\in\mathbb{Z}\right\}.$$

In particular, for any component D of B,

$$\operatorname{mult}_D B=-a(D,\mathcal{F},B)=-a(D,\mathcal{F}',B')=-\sum_{i=1}^k c_ia(D,\mathcal{F}',\tilde{B}_i')$$

belongs to a discrete set. Since $\operatorname{mult}_D B\in[0,1]$, $\operatorname{mult}_D B$ belongs to a finite set Γ_0. The theorem follows. $\square$

Proof of Theorem 1.1. By [Che22, Theorem 2.5], we may assume that $\dim X=3$. By [LLM23, Theorem 5.7], possibly replacing Γ with $\Gamma\cup\{1\}$ and $(X,\mathcal{F},B)$ with its dlt model, we may assume that $(X,\mathcal{F},B)$ is $\mathbb{Q}$-factorial dlt. If $B=0$, then there is nothing left to prove, so we may assume that $B\neq 0$, hence $K_{\mathcal{F}}$ is not pseudo-effective. By [CP19, Theorem 1.1],[LLM23, Theorem 3.1], there exists an algebraically integrable foliation $0\neq\mathcal{E}\subset\mathcal{F}$. If $\mathcal{F}$ is algebraically integrable, then by [ACSS21, Theorem 3.10], possibly replacing $(X,\mathcal{F},B)$, we may assume that there exists a contraction $\pi:X\to Z$ such that $\mathcal{F}$ is induced by π and Z is smooth, and the theorem follows from Proposition 5.2. Thus we may assume that $\mathcal{F}$ is not algebraically integrable.

Let S be a component of B and let $b:=\operatorname{mult}_S B$. By [LLM23, Lemma 8.2], there exists a birational map $f:X\dashrightarrow X'$ and a contraction $\pi':X'\to Z$ such that f does not extract any divisor, S is not contracted by f, $\mathcal{F}':=f_*\mathcal{F}$ is induced by a foliation $\mathcal{F}_Z$ on Z (i.e., $\mathcal{F}'=\pi'^{-1}\mathcal{F}_Z$), $\dim Z=2$, and Z is klt. Let $B':=f_*B$ and $S':=f_*S$. Then $S'\neq 0$. Since $K_{\mathcal{F}}+B\sim_{\mathbb{R}}0$, $K_{\mathcal{F}'}+B'\sim_{\mathbb{R}}0$ and $(X',\mathcal{F}',B')$ is lc. By Proposition 5.2, there exists a finite set Γ_0 depending only on Γ such that $b\in\Gamma_0$. Since S can be any component of B, $B\in\Gamma_0$, and we are done. $\square$

Theorem 5.5. *Let c be a non-negative integer, $r_1,\dots,r_c$ real numbers, and $\Gamma\subset[0,1]$ a DCC set, such that $\bar{\Gamma}\subset\operatorname{Span}_{\mathbb{Q}}(1,r_1,r_2,\dots,r_c)$. The the accumulation points of*

$$\{\operatorname{lct}(X,\mathcal{F},B;D)\mid\dim X\leq 3,(X,\mathcal{F},B)\text{ is lc, }B\in\Gamma,D\in\mathbb{N}^+\}$$

belong to $\operatorname{Span}_{\mathbb{Q}}(1,r_1,r_2,\dots,r_c)$.

Proof. Suppose the theorem does not hold. By [Che22, Theorem 0.5], there exist a sequence of lc foliated triples $(X_i,\mathcal{F}_i,B_i)$ of dimension ≤ 3 and effective $\mathbb{Q}$-Cartier Weil divisors D_i on X_i, such that $t_i:=\operatorname{lct}(X_i,\mathcal{F}_i,B_i;D_i)$ is strictly decreasing, $B_i\in\Gamma$, and $t:=\lim_{i\to+\infty}t_i\notin\operatorname{Span}_{\mathbb{Q}}(1,r_1,r_2,\dots,r_c)$. We write $B_i=\sum_{j=1}^{m_i}b_{i,j}B_{i,j}$, where $B_{i,j}$ are the irreducible components of B_i. Possibly replacing $(X_i,\mathcal{F}_i,B_i)$ with a dlt model and replacing D_i with its pullback, we may assume that $(X_i,\mathcal{F}_i,B_i)$ is $\mathbb{Q}$-factorial dlt.

Since $(X_i,\mathcal{F}_i,B_i+t_iD_i)$ is lc, $(X_i,\mathcal{F}_i,B_i+tD_i)$ is lc, and the coefficients of B_i+tD_i belong to a DCC set depending only on Γ. By Proposition 5.1, there exists a function $g:\bar\Gamma\to\bar\Gamma$, such that

(1) $g\circ g=g$ and $\Gamma_0:=g(\bar\Gamma)$ is a finite set,
(2) $g(\gamma)\geq\gamma$ for any $\gamma\in\bar\Gamma$,
(3) $g(\gamma)\leq g(\gamma')$ for any $\gamma,\gamma'\in\bar\Gamma$ such that $\gamma\leq\gamma'$, and
(4)
$$\left(X_i,\mathcal{F}_i,\sum_{j=1}^{m_i}g(b_{i,j})B_{i,j}+tD_i\right)$$
is lc for any i.

Since Γ_0 is a finite set, we have $\Gamma_0=\{\bar b_1,\dots,\bar b_m\}$ for some non-negative integer m, and we may write $\sum_{j=1}^{m_i}g(b_{i,j})B_{i,j}=\sum_{j=1}^{m}\bar b_jC_{i,j}$ where $C_{i,j}$ are effective Weil divisors.

By our assumption, $\bar b_j\in\operatorname{Span}_{\mathbb{Q}}(1,r_1,\dots,r_c)$ for each j and $t\notin\operatorname{Span}_{\mathbb{Q}}(1,r_1,\dots,r_c)$. We let V be the rational envelope of $(\bar b_1,\dots,\bar b_m,t)$ in $\mathbb{R}^{m+1}$, then $V=V'\times\mathbb{R}$, where V' is the rational envelope of $(\bar b_1,\dots,\bar b_m)$ in $\mathbb{R}^m$. By Theorem 1.3, there exist an open subset $U'\ni(\bar b_1,\dots,\bar b_m)$ of V', and an open subset $W\ni t$ of $\mathbb{R}$, such that $(X_i,\mathcal{F}_i,\sum_{j=1}^m v_jC_{i,j}+wD_i)$ is lc for any $(v_1,\dots,v_m)\in U'$ and $w\in W$. In particular,
$$\left(X_i,\mathcal{F}_i,\sum_{j=1}^{m}\bar b_jC_{i,j}+wD_i\right)=\left(X_i,\mathcal{F}_i,\sum_{j=1}^{m_i}g(b_{i,j})B_{i,j}+wD_i\right)$$
is lc for any $w\in W$. Possibly passing to a subsequence, we may assume that there exists a real number $w_0\in W$ such that $w_0>t_i$ for any i. Then $(X_i,\mathcal{F}_i,\sum_{j=1}^{m_i}g(b_{i,j})B_{i,j}+w_0D_i)$ is lc for any i, so $(X_i,\mathcal{F}_i,B_i+w_0D_i)$ is lc for any i, so
$$t_i=\operatorname{lct}(X_i,\mathcal{F}_i,B_i;D_i)\geq w_0>t_i,$$
a contradiction. □

Proof of Corollary 1.4. It immediately follows from Theorem 5.5 by taking $c=0$. □

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