VANISHING THEOREMS FOR GENERALIZED PAIRS

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Abstract. We establish the Kodaira vanishing theorem and the Kawamata-Viehweg vanishing theorem for lc generalized pairs. As a consequence, we provide a new proof of the base-point-freeness theorem for lc generalized pairs. This new approach allows us to prove the contraction theorem for lc generalized pairs without using Kollár's gluing theory.

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Contents

1. Introduction 1
2. Preliminaries 3
2.1. Definition of generalized pairs 3
2.2. Universal push-out diagram 4
2.3. Union of lc centers of generalized pairs 6
3. Proof of the vanishing theorems 7
4. Base-point-freeness for lc g-pairs 10
References 11

1. Introduction

We work with the field of complex numbers $\mathbb{C}$. All generalized pairs are assumed to be NQC generalized pairs (cf. [HL22]) in this paper.

The theory of "generalized pairs" (abbreviated as "g-pairs") holds significant importance in modern birational geometry. It was initially introduced by Birkar and Zhang in their study on effective Iitaka fibrations [BZ16]. Since then, this theory has proven to be crucial in various aspects of birational geometry, including the proof of the Borisov-Alexeev-Borisov conjecture [Bir19, Bir21a], the theory of complements [Bir19, Sho20], the connectedness principle [Bir20, FS23], non-vanishing theorems [LMPTX22], the minimal model program for Kähler manifolds [DHY23, HX23], and foliations [LLM23], etc. For a comprehensive overview of the theory of g-pairs, we refer interested readers to [Bir21b].

An important aspect of the study of g-pairs is their minimal model program. The foundations for the minimal model program of klt g-pairs and $\mathbb{Q}$-factorial dlt g-pairs were established in [BZ16, HL22]. Recently, progress has been made towards the minimal model program theory for lc g-pairs. Specifically, a series of recent works [HL21, LX22, Xie22] have established the cone theorem, contraction theorem, base-point-freeness theorem, and the existence of flips for lc g-pairs. This enables us to run the minimal model program for lc g-pairs in a comprehensive manner. For further related works, we refer readers to [Has22a, Has22b, LT22, LX23, TX23].

Apart from the minimal model program, there are numerous other topics within classical birational geometry that are worth discussing in the context of lc g-pairs. For instance, it is known that lc g-pairs have Du Bois singularities [LX22]. In this paper, we establish several vanishing theorems for lc g-pairs. The first main theorem of the paper is the following:

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Theorem 1.1. *Let $(X,B,\mathbf{M})/U$ be an lc generalized pair associated with projective morphism $f:X\to U$, D a Cartier divisor on X such that $D-(K_X+B+\mathbf{M}_X)$ is nef/U and log big/U with respect to $(X,B,\mathbf{M})$ (cf. Definition 2.4), Y a union of lc centers of $(X,B,\mathbf{M})$ such that $Y\neq X$, and $\mathcal{I}_Y$ the defining ideal sheaf of Y on X. Then:*

(1) $R^if_*\mathcal{O}_Y(D)=0$ for any positive integer i.
(2) $R^if_*\mathcal{O}_X(D)=0$ for any positive integer i.
(3) The map $f_*\mathcal{O}_X(D)\to f_*\mathcal{O}_Y(D)$ is surjective.
(4) $R^if_*(\mathcal{I}_Y\otimes\mathcal{O}_X(D))=0$ for any positive integer i.

Remark 1.2. We briefly explain the history on results that are related to Theorem 1.1.

(1) When $\mathbf{M}=\mathbf{0}$ and (X,B) is klt, Theorem 1.1(1)(3) become trivial, and Theorem 1.1(2)(4) are both equivalent to the usual relative Kawamata-Viehweg vanishing theorem (cf. [KMM87, Theorem 1-2-7]).
(2) When $\mathbf{M}=\mathbf{0}$ and $D-(K_X+B+\mathbf{M}_X)$ is ample/U, Theorem 1.1(2) becomes the usual Kodaira vanishing theorem for lc pairs [Fuj09, Theorem 4.4] and Theorem 1.1(4) is [Amb03, Theorem 7.3] and [Fuj11, Theorem 8.1].
(3) When $\mathbf{M}=\mathbf{0}$, Theorem 1.1(2)(3)(4) follow from [Amb03, Theorem 7.3], [Fuj17, Theorem 6.3.4(2)] and Theorem 1.1(1) follows from [Fuj14, Theorem 1.14]. Note that Theorem 1.1(2) becomes the usual Kawamata-Viehweg vanishing theorem for lc pairs.
(4) In fact, [Amb03, Theorem 7.3], [Fuj17, Theorem 6.3.4(2)] prove the qlc case of Theorem 1.1. Since any qlc pair is always an lc g-pair [Fuj22, Remark 1.9], Theorem 1.1 implies [Amb03, Theorem 7.3], [Fuj17, Theorem 6.3.4(2)] for qlc pairs.
(5) There is no previously written result when $\mathbf{M}\neq\mathbf{0}$, but the case when $\mathbf{M}_X$ is $\mathbb{R}$-Cartier and $D-(K_X+B+\mathbf{M}_X)$ is ample/U can be easily deduced from [HL21, Lemma 5.18] and the Kodaira vanishing theorem for lc pairs.

Theorem 1.1 immediately implies the Kodaira vanishing theorem for lc g-pairs and the Kawamata-Viehweg vanishing theorem for lc g-pairs. We provide the precise statement of these results here as they are more useful for direct applications.

Theorem 1.3 (Kodaira vanishing theorem for lc generalized pairs)**.** *Let $(X,B,\mathbf{M})$ be a projective lc generalized pair, and let D be a Cartier divisor on X such that $D-(K_X+B+\mathbf{M}_X)$ is ample. Then $H^i(X,\mathcal{O}_X(D))=0$ for any positive integer i.*

Theorem 1.4 (Relative Kawamata-Viehweg vanishing for lc generalized pairs)**.** *Let $(X,B,\mathbf{M})/U$ be an lc generalized pair associated with morphism $f:X\to U$, and let D be a Cartier divisor on X such that $D-(K_X+B+\mathbf{M}_X)$ is nef/U and log big/U with respect to $(X,B,\mathbf{M})$. Then $R^if_*\mathcal{O}_X(D)=0$ for any positive integer i.*

It was anticipated by Hashizume [Has22b, Page 77, Line 24-25] that Theorem 1.3 would play a pivital role in establishing the base-point-freeness theorem for lc g-pairs. Despite the base-point-freeness theorem's prior proof in [Xie22, Theorem 1.4], we endeavor to explore the viability of Hashizume's approach. Leveraging the implications of Theorem 1.1, we provide a new proof of the base-point-free theorem for lc g-pairs, thereby fulfilling Hashizume's expectation. It is noteworthy that our proof diverges significantly from the one in [Xie22], as the latter relies heavily on Kollár's gluing theory for g-pairs, while our novel approach bypasses this necessity.

Theorem 1.5 (Base-point-freeness theorem for lc generalized pairs, cf. [Xie22, Theorem 1.4])**.** *Let $(X,B,\mathbf{M})/U$ be an lc g-pair and D a nef/U Cartier divisor on X, such that $aD-(K_X+B+\mathbf{M}_X)$ is ample/U for some positive real number a. Then $\mathcal{O}_X(mD)$ is globally generated over U for any integer $m\gg 0$.*

As an immediate application, we have the following semi-ampleness theorem for lc g-pairs.

Theorem 1.6 (Semi-ampleness theorem for lc generalized pairs, cf. [Xie22, Theorems 1.2])**.** *Let $(X,B,\mathbf{M})/U$ be an lc g-pair and D a nef/U $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, such that $D-(K_X+B+\mathbf{M}_X)$ is ample/U. Then D is semi-ample/U.*

We remark that [Xie22, Theorem 1.4] is stronger than Theorem 1.5 since [Xie22, Theorem 1.4] only requires that $aD-(K_X+B+\mathbf{M}_X)$ is nef$/U$ and log big$/U$. Nonetheless, Theorem 1.5 is strong enough for us to immediately deduce the contraction theorem for lc g-pairs [Xie22, Theorem 1.5] without using Kollár's gluing theory (see Remark 4.3).

Theorem 1.7 (Contraction theorem for lc generalized pairs, cf. [Xie22, Theorem 1.5])**.** *Let $(X,B,\mathbf{M})/U$ be an lc generalized pair and F a $(K_X+B+\mathbf{M}_X)$-negative extremal face$/U$. Then there exists a contraction$/U$ $\mathrm{cont}_F: X\to Z$ of F satisfying the following.*

(1) For any integral curve C on X such that the image of C in U is a closed point, $\mathrm{cont}_F(C)$ is a point if and only if $[C]\in F$.
(2) $\mathcal{O}_Y=(\mathrm{cont}_F)_*\mathcal{O}_X$. In other words, cont_F is a contraction.
(3) For any Cartier divisor D on Y such that $D\cdot C=0$ for any curve C contracted by cont_F, there exists a Cartier divisor D_Y on Y such that $D=\mathrm{cont}_F^* D_Y$.

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2. Preliminaries

Throughout the paper, we will mainly work with normal quasi-projective varieties to ensure consistency with the references. However, most results should also hold for normal varieties that are not necessarily quasi-projective. Similarly, most results in our paper should hold for any algebraically closed field of characteristic zero. We will adopt the standard notations and definitions in [KM98, BCHM10] and use them freely. For generalized pairs, we will follow the notations as in [HL21].

2.1. Definition of generalized pairs.

Definition 2.1 (***b***-divisors)**.** Let X be a normal quasi-projective variety. We call Y a *birational model* over X if there exists a projective birational morphism $Y\to X$.

Let $X\dashrightarrow X'$ be a birational map. For any valuation ν over X, we define $\nu_{X'}$ to be the center of ν on X'. A ***b**-divisor* $\mathbf{D}$ over X is a formal sum $\mathbf{D}=\sum_\nu r_\nu\nu$ where ν are valuations over X and $r_\nu\in\mathbb{R}$, such that ν_X is not a divisor except for finitely many ν. The *trace* of $\mathbf{D}$ on X' is the $\mathbb{R}$-divisor

$$\mathbf{D}_{X'}:=\sum_{\nu_{X'}\text{ is a divisor}} r_\nu\nu_{X'}.$$

If $\mathbf{D}_{X'}$ is $\mathbb{R}$-Cartier and $\mathbf{D}_Y$ is the pullback of $\mathbf{D}_{X'}$ on Y for any birational model Y over X', we say that $\mathbf{D}$ *descends* to X' and $\mathbf{D}$ is the *closure* of $\mathbf{D}_{X'}$, and write $\mathbf{D}=\overline{\mathbf{D}_{X'}}$.

Let $X\to U$ be a projective morphism and assume that $\mathbf{D}$ is a $\boldsymbol{b}$-divisor over X such that $\mathbf{D}$ descends to a birational model Y over X. If $\mathbf{D}_Y$ is nef$/U$, then we say that $\mathbf{D}$ is *nef$/U$*. If $\mathbf{D}_Y$ is a Cartier divisor, then we say that $\mathbf{D}$ is ***b**-Cartier*. If $\mathbf{D}$ can be written as an $\mathbb{R}_{\geq 0}$-linear combination of nef$/U$ $\boldsymbol{b}$-Cartier $\boldsymbol{b}$-divisors, then we say that $\mathbf{D}$ is *NQC$/U$*.

We let $\mathbf{0}$ be the $\boldsymbol{b}$-divisor $\bar{0}$.

Definition 2.2 (Generalized pairs)**.** A *generalized pair* (*g-pair* for short) $(X,B,\mathbf{M})/U$ consists of a normal quasi-projective variety X associated with a projective morphism $X\to U$, an $\mathbb{R}$-divisor $B\geq 0$ on X, and an NQC$/U$ $\boldsymbol{b}$-divisor $\mathbf{M}$ over X, such that $K_X+B+\mathbf{M}_X$ is $\mathbb{R}$-Cartier.

If $\mathbf{M}=\mathbf{0}$, a g-pair $(X,B,\mathbf{M})/U$ is called a *pair* and is denoted by (X,B) or $(X,B)/U$.

If $U=\{pt\}$, we usually drop U and say that $(X,B,\mathbf{M})$ is *projective*. If U is not important, we may also drop U.

Definition 2.3 (Singularities of generalized pairs)**.** Let $(X,B,\mathbf{M})/U$ be a g-pair. For any prime divisor E and $\mathbb{R}$-divisor D on X, we define $\operatorname{mult}_E D$ to be the *multiplicity* of E along D. Let $h:W\to X$ be any log resolution of $(X,\operatorname{Supp}B)$ such that $\mathbf{M}$ descends to W, and let

$$K_W+B_W+\mathbf{M}_W:=h^*(K_X+B+\mathbf{M}_X).$$

The *log discrepancy* of a prime divisor D on W with respect to $(X,B,\mathbf{M})$ is $1-\operatorname{mult}_D B_W$ and it is denoted by $a(D,X,B,\mathbf{M})$.

We say that $(X,B,\mathbf{M})$ is *lc* (resp. *klt*) if $a(D,X,B,\mathbf{M})\geq 0$ (resp. >0) for every log resolution $h:W\to X$ as above and every prime divisor D on W. We say that $(X,B,\mathbf{M})$ is *dlt* if $(X,B,\mathbf{M})$ is lc, and there exists a closed subset $V\subset X$, such that

(1) $X\backslash V$ is smooth and $B_{X\backslash V}$ is simple normal crossing, and
(2) for any prime divisor E over X such that $a(E,X,B,\mathbf{M})=0$, $\operatorname{center}_X E\not\subset V$ and $\operatorname{center}_X E\backslash V$ is an lc center of $(X\backslash V,B|_{X\backslash V})$.

We refer the reader to [Has22a, Theorem 6.1] for equivalent definitions of dlt g-pairs.

Suppose that $(X,B,\mathbf{M})$ is lc. An *lc place* of $(X,B,\mathbf{M})$ is a prime divisor E over X such that $a(E,X,B,\mathbf{M})=0$. An *lc center* of $(X,B,\mathbf{M})$ is either X, or the center of an lc place of $(X,B,\mathbf{M})$ on X. The *non-klt locus* $\operatorname{Nklt}(X,B,\mathbf{M})$ of $(X,B,\mathbf{M})$ is the union of all lc centers of $(X,B,\mathbf{M})$ except X itself.

We note that the definitions above are independent of the choice of U

Definition 2.4 (Log big)**.** Let $(X,B,\mathbf{M})/U$ be a g-pair and D an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on X. We say that D is *log big$/U$ with respect to* $(X,B,\mathbf{M})$ if $D|_V$ is big$/U$ for any lc center V of $(X,B,\mathbf{M})$. In particular, D is big$/U$.

2.2. **Universal push-out diagram.**

Definition 2.5. We say a commutative diagram of schemes

$$\begin{tikzcd} \mathcal{C} \arrow[r, "j"] \arrow[d, "q"'] & Y \arrow[d, "p"] \\ \mathcal{D} \arrow[r, "i"] & X \end{tikzcd}$$

is a *universal push-out diagram* if for any scheme T, the induced diagram

$$\begin{tikzcd} \operatorname{Hom}(X,T) \arrow[r, "\circ i"] \arrow[d, "\circ p"'] & \operatorname{Hom}(\mathcal{D},T) \arrow[d, "\circ q"] \\ \operatorname{Hom}(Y,T) \arrow[r, "\circ j"] & \operatorname{Hom}(\mathcal{C},T) \end{tikzcd}$$

is a universal pull-back diagram of sets.

Lemma 2.6. *Let X be a semi-normal variety and let $\pi:X^n\to X$ be the normalization of X. Let Z be a reduced closed subvariety of X such that $X\setminus Z$ is normal. Let $Y:=\pi^{-1}(Z)$ associated with the reduced scheme structure. Denote the induced morphism $Y\to Z$ by π_Y. Then we have the following universal push-out diagram*

$$\begin{tikzcd} Y \arrow[r, hook, "j"] \arrow[d, "\pi_Y"'] & X^n \arrow[d, "\pi"] \\ Z \arrow[r, hook, "i"] & X \end{tikzcd}$$

and a short exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{\pi^* \oplus i^*} \pi_* \mathcal{O}_{X^n} \oplus \mathcal{O}_Z \xrightarrow{j^* - \pi_Y^*} (\pi_Y)_* \mathcal{O}_Y \to 0, \tag{2.1}$$

where i, j are the natural closed immersions.

Proof. Since j is a closed immersion and π_Y is a finite morphism, by [Kol13, Theorem 9.30], [Kol95, 8.1], we have a universal push-out diagram

$$\begin{tikzcd} Y \arrow[r, hook, "j"] \arrow[d, "\pi_Y"'] & X^n \arrow[d, "\pi'"] \\ Z \arrow[r, hook, "i'"] & X' \end{tikzcd}$$

where

$$X' := \operatorname{Spec}_X \operatorname{Ker}[\pi_* \mathcal{O}_{X^n} \oplus \mathcal{O}_Z \xrightarrow{j^* - \pi_Y^*} (\pi_Y)_* \mathcal{O}_Y].$$

Therefore, it suffices to prove the short exact sequence (2.1) exists. Let $\mathcal{J}$ be the conductor ideal sheaf of $\pi : X^n \to X$, which can be regarded as both an $\mathcal{O}_X$-module and an $\mathcal{O}_{X^n}$-module via the inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}_{X^n}$. By [Kol95, 5.5.3], $\mathcal{J}$ is its own radical in $\mathcal{O}_{X^n}$ and hence is its own radical in $\mathcal{O}_X$. Let $\mathcal{I}_Y, \mathcal{I}_Z$ be the ideal sheaves of Y, Z respectively. Since $X \setminus Z$ is normal, $\mathcal{I}_Z \subset \mathcal{J}$.

Claim 2.7. $\mathcal{I}_Z \cdot \mathcal{O}_{X^n} = \mathcal{I}_Z$.

Proof. Let $\mathcal{I}' := \mathcal{I}_Z \cdot \mathcal{O}_{X^n}$, then $\mathcal{I}_Z \subset \mathcal{I}' \subset \mathcal{J}$. Since Z is reduced, we only need to prove that the sub-schemes defined by $\mathcal{I}_Z$ and $\mathcal{I}'$ in X have the same support. Thus we only need to prove that the sub-schemes defined by $\mathcal{I}_Z$ and $\mathcal{I}'$ in X have the same support near any point $x \in X$.

If $x \in \operatorname{Supp} \mathcal{O}_X/\mathcal{J}$, then $x \in \operatorname{Supp} \mathcal{O}_X/\mathcal{I}' \subseteq Z$ and we are done.

If $x \notin \operatorname{Supp} \mathcal{O}_X/\mathcal{J}$, then X is normal at x, hence $\pi : X^n \to X$ is an isomorphism near x. Therefore, $\mathcal{I}_Z = \mathcal{I}'$ and $\operatorname{Supp} \mathcal{O}_X/\mathcal{I}' = Z$ near x. $\square$

Claim 2.8. *$\mathcal{I}_Z = \mathcal{I}_Y$ in $\mathcal{O}_{X^n}$.*

Proof. By definition, $\mathcal{I}_Y$ is the radical of $\mathcal{I}_Z$ in $\mathcal{O}_{X^n}$. Since $\mathcal{I}_Z \subset \mathcal{J}$ and $\mathcal{J}$ is its own radical in $\mathcal{O}_{X^n}$, $\mathcal{I}_Y$ is contained in $\mathcal{J}$ and hence is an ideal sheaf of $\mathcal{O}_X$. Therefore, $\mathcal{I}_Y$ is the radical of $\mathcal{I}_Z$ in $\mathcal{O}_X$. Since Z is reduced, $\mathcal{I}_Z = \mathcal{I}_Y$ in $\mathcal{O}_{X^n}$. $\square$

Proof of Lemma 2.6 continued. By Claims 2.7 and 2.8, we may consider the question locally and assume that $X = \operatorname{Spec} A$, $X^n = \operatorname{Spec} B$, and $\mathcal{I}_Z = \mathcal{I}_Y = I$. Then the map

$$\phi : B \oplus A/I \to B/I,\ (b, a + I) \mapsto (b - a) + I$$

is surjective and the map

$$\psi : A \to B \oplus A/I,\ a \mapsto (a, a + I)$$

is injective. Thus

$$(b, a + I) \in \operatorname{Ker} \phi \iff b \in A \text{ and } (b, a + I) = (b, b + I) \iff (b, a + I) \in \operatorname{Im}(\psi),$$

so (2.1) is a short exact sequence and we are done. $\square$

2.3. **Union of lc centers of generalized pairs.**

Definition 2.9 (Union of lc centers)**.** Let $(X,B,\mathbf{M})$ be an lc g-pair. A *union of lc centers* of $(X,B,\mathbf{M})$ is a reduced scheme $Y=\cup Y_i$, where each Y_i is an lc center of $(X,B,\mathbf{M})$. We denote by $S(X,B,\mathbf{M})$ the set of all unions of lc centers of $(X,B,\mathbf{M})$. We remark that

(1) $\emptyset$ is also considered as a union of lc centers, and
(2) a union of lc center may be represented in different ways. For example, if Y_1 and Y_2 are two lc centers such that $Y_1\subsetneq Y_2$, then $Y_1\cup Y_2$ and Y_2 are the same union of lc centers.

Definition 2.10 (Adjacent unions of lc centers)**.** Let $(X,B,\mathbf{M})$ be an lc g-pair. For any two unions of lc centers $Y,Y'\in S(X,B,\mathbf{M})$, we say that Y and Y' are *adjacent* in $S(X,B,\mathbf{M})$ if

(1) $Y\subsetneq Y'$ or $Y'\subsetneq Y$, and
(2) there does not exist any $Y''\in S(X,B,\mathbf{M})$ such that $Y\subsetneq Y''\subsetneq Y'$ or $Y'\subsetneq Y''\subsetneq Y$.

An lc center V is called *minimal* in $S(X,B,\mathbf{M})$ if V and $\emptyset$ are adjacent in $S(X,B,\mathbf{M})$.

Lemma 2.11. *Let $(X,B,\mathbf{M})/U$ be an lc g-pair, W a union of lc centers of $(X,B,\mathbf{M})$, and $\pi:W^n\to W$ the normalization of W. Suppose that $\dim W\geq 1$. Then there exists an lc g-pair $(W^n,B_{W^n},\mathbf{M}^{W^n})/U$, such that*

(1) $K_{W^n}+B_{W^n}+\mathbf{M}^{W^n}_{W^n}\sim_{\mathbb{R},U}(K_X+B+\mathbf{M}_X)|_{W^n}$.
(2) For any lc center L of $(W^n,B_{W^n},\mathbf{M}^{W^n})$, $\pi(L)$ is an lc center of $(X,B,\mathbf{M})$.
(3) For any lc center C of $(X,B,\mathbf{M})$, $\pi^{-1}(C)$ is a union of lc centers of $(W^n,B_{W^n},\mathbf{M}^{W^n})$.

Proof. We may assume that W is irreducible and $W\neq X$.

Let $f:Y\to X$ be a dlt modification (cf. [HL22, Proposition 3.10]) of $(X,B,\mathbf{M})$, such that there exists a prime divisor $S\subset\lfloor B_Y\rfloor$ such that $f(S)=W$, where $K_Y+B_Y+\mathbf{M}_Y:=f^*(K_X+B+\mathbf{M}_X)$. Let W_Y be an lc center of $(Y,B_Y,\mathbf{M})$ which is minimal with respect to inclusion under the condition $f(W_Y)=W$. Since $(Y,B_Y,\mathbf{M})$ is dlt, by repeatedly applying adjunction (cf. [HL22, Proposition 2.10]), we get a dlt g-pair $(W_Y,B_{W_Y},\mathbf{M}^{W_Y})/U$ such that

$$K_{W_Y}+B_{W_Y}+\mathbf{M}^{W_Y}_{W_Y}:=(K_Y+B_Y+\mathbf{M}_Y)|_{W_Y}.$$

By construction, there exists a naturally induced projective surjective morphism $f_W:W_Y\to W^n$ such that $K_{W_Y}+B_{W_Y}+\mathbf{M}^{W_Y}_{W_Y}\sim_{\mathbb{R},W^n}0$. By [LX22, Lemma 3.19], [LX23, Theorem 2.14], there exists an lc g-pair $(W^n,B_{W^n},\mathbf{M}^{W^n})/U$, such that

- $(W^n,B_{W^n},\mathbf{M}^{W^n})$ is induced by a canonical bundle formula of $(W_Y,B_{W_Y},\mathbf{M}^{W_Y})\to W^n$,
- any lc center of $(W^n,B_{W^n},\mathbf{M}^{W^n})$ is the image of an lc center of $(W_Y,B_{W_Y},\mathbf{M}^{W_Y})$, and
- the image of any lc center of $(W_Y,B_{W_Y},\mathbf{M}^{W_Y})$ on W^n is an lc center of $(W^n,B_{W^n},\mathbf{M}^{W^n})$.

We show that $(W^n,B_{W^n},\mathbf{M}^{W^n})$ satisfies our requirement.

(1) It immediately follows from our construction.

(2) L is the image of an lc center L_Y of $(W_Y,B_{W_Y},\mathbf{M}^{W_Y})$. By repeatedly applying [LX22, Lemma 3.18], L_Y is an lc center of $(Y,B_Y,\mathbf{M})$. Since $K_Y+B_Y+\mathbf{M}_Y:=f^*(K_X+B+\mathbf{M}_X)$, $f(L_Y)=\pi(L)$ is an lc center of $(X,B,\mathbf{M})$.

(3) $f^{-1}(C)$ is a union of lc centers of $(Y,B_Y,\mathbf{M})$. Since $(Y,B_Y,\mathbf{M})$ is dlt, $f^{-1}(C)\cap W_Y$ is a union of lc centers of $(Y,B_Y,\mathbf{M})$. By [LX22, Lemma 3.18], $f^{-1}(C)\cap W_Y$ is a union of lc centers of $(W_Y,B_{W_Y},\mathbf{M}^{W_Y})$. Hence $\pi^{-1}(C)=f_W(f^{-1}(C)\cap W_Y)$ is a union of lc centers of $(W^n,B_{W^n},\mathbf{M}^{W^n})$. $\square$

Lemma 2.12. *Let $(X,B,\mathbf{M})$ be an lc g-pair. Let Y and Y' be two unions of lc centers, such that $Y'\subsetneq Y$, and Y and Y' are adjacent in $S(X,B,\mathbf{M})$. Let $\pi:Y^n\to Y$ be the normalization of Y and let $Y'':=\pi^{-1}(Y')$ with the reduced scheme structure. Denote the induced morphism $Y''\to Y'$ by π''. Then there exist a universal push-out diagram*

$$\begin{tikzcd} Y'' \arrow[r, hook, "j"] \arrow[d, "\pi''"'] & Y^n \arrow[d, "\pi"] \\ Y' \arrow[r, hook, "i"] & Y \end{tikzcd}$$

and a short exact sequence

$$0 \to \mathcal{O}_Y \xrightarrow{\pi^*\oplus i^*} \pi_*\mathcal{O}_{Y^n} \oplus \mathcal{O}_{Y'} \xrightarrow{j^*-\pi''^*} \pi''_*\mathcal{O}_{Y''} \to 0,$$

where i, j are the natural closed immersions.

Proof. By [LX22, Theorem 4.10] and [Kol13, Theorem 9.26], Y is semi-normal. Let L be an lc center contained in Y but not contained in Y'. Since Y' and Y are adjacent in $S(X, B, \mathbf{M})$, we have

$$Y \backslash Y' = L \backslash (L \cap Y'),$$

and $L \cap Y'$ is the union of all lc centers of $(X, B, \mathbf{M})$ that are contained in L but not equal to L. By [LX22, Theorem 4.10], $Y \setminus Y'$ is normal. The lemma follows from Lemma 2.6. $\square$

3. Proof of the vanishing theorems

In this section, we prove Theorem 1.1, which immediately implies Theorems 1.3 and 1.4.

Lemma 3.1. *Let $(X, B, \mathbf{M})/U$ be an lc g-pair associated with morphism $f : X \to U$, and D a Cartier divisor on X such that $D - (K_X + B + \mathbf{M}_X)$ is nef/U and log big/U with respect to $(X, B, \mathbf{M})$. Let $W = \mathrm{Nklt}(X, B, \mathbf{M})$ with the reduced scheme structure, and let $\mathcal{I}_W$ be the defining ideal sheaf of W on X. Then:*

(1) $R^i f_*(\mathcal{I}_W \otimes \mathcal{O}_X(D)) = 0$ for any $i > 0$.
(2) $f_*\mathcal{O}_X(D) \to f_*\mathcal{O}_W(D)$ is surjective.

Proof. By [Xie22, Lemma 2.4], there exists a pair (X, Δ) such that $L - K_X - \Delta$ is ample/U and $W = \mathrm{Nlc}(X, \Delta)$. (1) follows from [Fuj11, Theorem 8.1]. (2) follows from (1) and the long exact sequence

$$0 \to f_*(\mathcal{I}_W \otimes \mathcal{O}_X(D)) \to f_*\mathcal{O}_X(D) \to f_*\mathcal{O}_W(D) \to R^1 f_*(\mathcal{I}_W \otimes \mathcal{O}_X(D)) \to \dots.$$

$\square$

Lemma 3.2. *Let $(X, B, \mathbf{M})/U$ be an lc g-pair associated with morphism $f : X \to U$, and D a Cartier divisor on X such that $D - (K_X + B + \mathbf{M}_X)$ is nef/U and log big/U with respect to $(X, B, \mathbf{M})$. Let Y and Y' be two unions of lc centers, such that $Y' \subsetneq Y$, and Y and Y' are adjacent in $S(X, B, \mathbf{M})$. Let $\pi : Y^n \to Y$ be the normalization of Y, $Y'' := \pi^{-1}(Y')$ with the reduced scheme structure, and $\pi'' := \pi|_{Y''}$.*

$$\begin{tikzcd} Y'' \arrow[r, hook, "j"] \arrow[d, "\pi''"'] & Y^n \arrow[d, "\pi"] \\ Y' \arrow[r, hook, "i"] & Y \end{tikzcd}$$

Then the induced map

$$f_*\pi_*\mathcal{O}_{Y^n}(D|_{Y^n}) \to f_*\pi''_*\mathcal{O}_{Y''}(D|_{Y''})$$

is surjective.

Proof. By Lemma 2.11, there exists an lc g-pair $(Y^n, B_{Y^n}, \mathbf{M}^{Y^n})/U$, such that

- $K_{Y^n} + B_{Y^n} + \mathbf{M}^{Y^n}_{Y^n} \sim_{\mathbb{R},U} (K_X + B + \mathbf{M}_X)|_{Y^n}$,
- for any lc center L of $(Y^n, B_{Y^n}, \mathbf{M}^{Y^n})$, $\pi(L)$ is an lc center of $(X, B, \mathbf{M})$, and
- for any lc center C of $(X, B, \mathbf{M})$, $\pi^{-1}(C \cap Y)$ is a union of lc centers of $(Y^n, B_{Y^n}, \mathbf{M}^{Y^n})$.

Then $D|_{Y^n}-(K_{Y^n}+B_{Y^n}+\mathbf{M}^{Y^n}_{Y^n})$ is nef$/U$ and log big$/U$ with respect to $(Y^n,B_{Y^n},\mathbf{M}^{Y^n})$.

Let Y_0 be a connected component of Y^n and let $Y_0'':=Y''\cap Y_0$. We claim that

$$\text{either } Y_0''=Y_0 \text{ or } Y_0''=\mathrm{Nklt}(Y_0,B_{Y^n}|_{Y^0},\mathbf{M}^{Y^n}|_{Y^0}). \tag{3.1}$$

Indeed, if this is not the case, then there exists an lc center L of $(Y_0,B_{Y^n}|_{Y^0},\mathbf{M}^{Y^n}|_{Y^0})$ such that $L\neq Y_0$ and L is not contained in Y_0''. Then $\tilde Y:=\pi(Y''\cup L)\in S(X,\Delta,\mathbf{M})$ and $Y'\subsetneq\tilde Y\subsetneq Y$, which contradicts the condition that Y',Y are adjacent. By (3.1) and Lemma 3.1(2),

$$f_*\pi_*\mathcal{O}_{Y_0}(D|_{Y_0})\to f_*\pi''_*\mathcal{O}_{Y_0''}(D|_{Y_0''})$$

is surjective. Thus

$$f_*\pi_*\mathcal{O}_{Y^n}(D|_{Y^n})\to f_*\pi''_*\mathcal{O}_{Y''}(D|_{Y''})$$

is surjective. □

Proof of Theorem 1.1. We apply induction on $\dim X$. When $\dim X=1$ the theorem is obvious.

For any union of lc centers Z of $(X,B,\mathbf{M})$, we define $m(Z)$ to be the number of lc centers of $(X,B,\mathbf{M})$ that are contained in Z. We let $W:=\mathrm{Nklt}(X,B,\mathbf{M})$, associated with the reduced scheme structure.

Step 1. In this step we prove (1) when Y is minimal in $S(X,B,\mathbf{M})$.

By [LX22, Theorem 4.10], Y is normal. If $\dim Y=0$ then we are done. Otherwise, by Lemma 2.11, there exists a klt g-pair $(Y,B_Y,\mathbf{M}^Y)/U$ such that $K_Y+B_Y+\mathbf{M}^Y_Y\sim_{\mathbb{R},U}(K_X+B+\mathbf{M}_X)|_Y$. Hence $D|_Y-(K_Y+B_Y+\mathbf{M}^Y_Y)$ is nef$/U$ and big$/U$. By [Xie22, Lemma 2.4], there exists a klt pair (Y,Δ_Y) such that $D|_Y-(K_Y+\Delta_Y)$ is ample$/U$. (1) follows from the usual Kawamata-Viehweg vanishing theorem (cf. [KMM87, Theorem 1-2-7]).

Step 2. In this step we prove (1).

We apply induction on $m(Y)$. When $m(Y)=1$, Y is minimal in $S(X,B,\mathbf{M})$ and we are done by **Step 1**. Thus we may assume that $m(Y)>1$. Then there exists a union of lc centers Y' such that $Y'\subsetneq Y$, and Y and Y' are adjacent in $S(X,B,\mathbf{M})$. Since $m(Y')<m(Y)$, by induction on $m(Y)$, we have

$$R^if_*\mathcal{O}_{Y'}(D)=0 \tag{3.2}$$

for any positive integer i.

Let $\pi:Y^n\to Y$ be the normalization of Y, and let $Y'':=\pi^{-1}(Y')$ with the reduced scheme structure. Let $i:Y'\hookrightarrow Y$ and $j:Y''\hookrightarrow Y^n$ be the natural inclusions, and let $\pi'':=\pi|_{Y''}$. By Lemma 2.12, there exists a universal push-out diagram

$$\begin{tikzcd} Y'' \arrow[r, hook, "j"] \arrow[d, "\pi''"'] & Y^n \arrow[d, "\pi"] \\ Y' \arrow[r, hook, "i"] & Y \end{tikzcd}$$

and a short exact sequence

$$0\to\mathcal{O}_Y\xrightarrow{\pi^*\oplus i^*}\pi_*\mathcal{O}_{Y^n}\oplus\mathcal{O}_{Y'}\xrightarrow{j^*-\pi''^*}\pi''_*\mathcal{O}_{Y''}\to 0. \tag{3.3}$$

By Lemma 2.11, there exists an lc g-pair $(Y^n,B_{Y^n},\mathbf{M}^{Y^n})/U$, such that

- $K_{Y^n}+B_{Y^n}+\mathbf{M}^{Y^n}_{Y^n}\sim_{\mathbb{R},U}(K_X+B+\mathbf{M}_X)|_{Y^n}$,
- for any lc center L of $(Y^n,B_{Y^n},\mathbf{M}^{Y^n})$, $\pi(L)$ is an lc center of $(X,B,\mathbf{M})$, and
- for any lc center C of $(X,B,\mathbf{M})$, $\pi^{-1}(C\cap Y)$ is a union of lc centers of $(Y^n,B_{Y^n},\mathbf{M}^{Y^n})$.

Then $D|_{Y^n}-(K_{Y^n}+\Delta_{Y^n}+\mathbf{M}^{Y^n}_{Y^n})$ is nef$/U$ and log big$/U$ with respect to $(Y^n,B_{Y^n},\mathbf{M}^{Y^n})$, and Y'' is a union of lc centers of $(Y^n,B_{Y^n},\mathbf{M}^{Y^n})$. Since $\dim Y^n<\dim X$ and π is a finite morphism, by induction on $\dim X$ we have

$$R^i(f\circ\pi)_*\mathcal{O}_{Y^n}(D|_{Y^n})=R^if_*\big(\pi_*(\mathcal{O}_{Y^n}(D|_{Y^n})\big)=0 \tag{3.4}$$

and

$$R^i(f\circ\pi'')_*\mathcal{O}_{Y''}(D|_{Y''}) = R^if_*\big(\pi''_*\mathcal{O}_{Y''}(D|_{Y''})\big) = 0. \tag{3.5}$$

By the short exact sequence (3.3), we have a short exact sequence

$$0\to \mathcal{O}_Y(D)\xrightarrow{\pi^*\oplus i^*}\pi_*\mathcal{O}_{Y^n}(D|_{Y^n})\oplus\mathcal{O}_{Y'}(D)\xrightarrow{j^*-\pi''^*}\pi''_*\mathcal{O}_{Y''}(D|_{Y''})\to 0,$$

which induces a long exact sequence

$$\begin{aligned}0\to f_*\mathcal{O}_Y(D)\to f_*\pi_*\mathcal{O}_{Y^n}(D|_{Y^n})\oplus f_*\mathcal{O}_{Y'}(D)&\xrightarrow{j^*-\pi''^*} f_*\pi''_*\mathcal{O}_{Y''}(D|_{Y''})\to\cdots\\ \cdots\to R^if_*\mathcal{O}_Y(D)\to R^if_*\big(\pi_*(\mathcal{O}_{Y^n}(D|_{Y^n})\big)\oplus R^if_*\mathcal{O}_{Y'}(D)&\to R^if_*\big(\pi''_*\mathcal{O}_{Y''}(D|_{Y''})\big)\to\cdots.\end{aligned}$$

Hence, it follows from (3.2), (3.4), (3.5) and Lemma 3.2 that $R^if_*\mathcal{O}_Y(D)=0$ for any positive integer i.

Step 3. In this step we prove (2) and prove (3)(4) when $Y=W=\mathrm{Nklt}(X,B,\mathbf{M})$.

We have the long exact sequence

$$\begin{aligned}0\to f_*(\mathcal{I}_W\otimes\mathcal{O}_X(D))\to f_*\mathcal{O}_X(D)\to f_*\mathcal{O}_W(D)\to\dots\\ \dots\to R^if_*(\mathcal{I}_W\otimes\mathcal{O}_X(D))\to R^if_*\mathcal{O}_X(D)\to R^if_*\mathcal{O}_W(D)\to\dots\end{aligned}$$

By (1), $R^if_*\mathcal{O}_W(D)=0$ for any positive integer i. By Lemma 3.1(1), $R^i(\mathcal{I}_W\otimes f_*\mathcal{O}_X(D))=0$ for any positive integer i. This implies (2), and also implies (3)(4) when $Y=W$.

Step 4. We prove (3)(4) in this step, hence conclude the proof of the theorem.

We apply induction on $m(W)-m(Y)$. When $m(W)-m(Y)=0$, $Y=W$ and we are done by **Step 3**. Thus we may assume that $m(W)-m(Y)>0$. Then there exists a union of lc centers $\tilde{Y}$ such that $Y\subsetneq\tilde{Y}\subset W$, and Y and $\tilde{Y}$ are adjacent in $S(X,B,\mathbf{M})$.

Let $\tilde{\pi}:\tilde{Y}^n\to\tilde{Y}$ be the normalization of $\tilde{Y}$, and let $\widehat{Y}:=\tilde{\pi}^{-1}(Y)$ with the reduced scheme structure. Let $\tilde{i}:Y\hookrightarrow\tilde{Y}$ and $\tilde{j}:\widehat{Y}\hookrightarrow\tilde{Y}^n$ be the natural inclusions, and let $\widehat{\pi}:=\tilde{\pi}|_{\widehat{Y}}$. By Lemma 2.12, there exists a universal push-out diagram

$$\begin{tikzcd}\widehat{Y} \arrow[r, hook, "\tilde{j}"] \arrow[d, "\widehat{\pi}"'] & \tilde{Y}^n \arrow[d, "\tilde{\pi}"] \\ Y \arrow[r, hook, "\tilde{i}"] & \tilde{Y}\end{tikzcd}$$

and a short exact sequence

$$0\to\mathcal{O}_{\tilde{Y}}\xrightarrow{\tilde{\pi}^*\oplus\tilde{i}^*}\tilde{\pi}_*\mathcal{O}_{\tilde{Y}^n}\oplus\mathcal{O}_Y\xrightarrow{\tilde{j}^*-\widehat{\pi}^*}\widehat{\pi}_*\mathcal{O}_{\widehat{Y}}\to 0.$$

which induces a short exact sequence

$$0\to\mathcal{O}_{\tilde{Y}}(D)\xrightarrow{\tilde{\pi}^*\oplus\tilde{i}^*}\tilde{\pi}_*\mathcal{O}_{\tilde{Y}^n}(D|_{\tilde{Y}^n})\oplus\mathcal{O}_Y(D)\xrightarrow{\tilde{j}^*-\widehat{\pi}^*}\widehat{\pi}_*\mathcal{O}_{\widehat{Y}}(D|_{\widehat{Y}})\to 0.$$

So we have the left exact sequence

$$0\to f_*\mathcal{O}_{\tilde{Y}}(D)\xrightarrow{\tilde{\pi}^*\oplus\tilde{i}^*}f_*\tilde{\pi}_*\mathcal{O}_{\tilde{Y}^n}(D|_{\tilde{Y}^n})\oplus f_*\mathcal{O}_Y(D)\xrightarrow{\tilde{j}^*-\widehat{\pi}^*}f_*\widehat{\pi}_*\mathcal{O}_{\widehat{Y}}(D|_{\widehat{Y}}). \tag{3.6}$$

By Lemma 3.2,

$$\tilde{j}^*: f_*\tilde{\pi}_*\mathcal{O}_{\tilde{Y}^n}(D|_{\tilde{Y}^n})\to f_*\widehat{\pi}_*\mathcal{O}_{\widehat{Y}}(D|_{\widehat{Y}})$$

is surjective. Thus by an easy map tracing of (3.6) we have that

$$\tilde{i}^*: f_*\mathcal{O}_{\tilde{Y}}(D)\to f_*\mathcal{O}_Y(D)$$

is also surjective. Since $m(W)-m(\tilde{Y})<m(W)-m(Y)$, by induction on $m(W)-m(Y)$,

$$f_*\mathcal{O}_X(D)\to f_*\mathcal{O}_{\tilde{Y}}(D)$$

is surjective. This implies (3).

We have the long exact sequence

$$0 \to f_*(\mathcal{I}_Y \otimes \mathcal{O}_X(D)) \to f_*\mathcal{O}_X(D) \to f_*\mathcal{O}_Y(D) \to \dots$$
$$\dots \to R^i f_*(\mathcal{I}_Y \otimes \mathcal{O}_X(D)) \to R^i f_*\mathcal{O}_X(D) \to R^i f_*\mathcal{O}_Y(D) \to \dots,$$

so (4) follows immediately from (1)(2)(3). □

Proof of Theorem 1.3. It immediately follows from Theorem 1.1(2) by letting $U = \{pt\}$. □

Proof of Theorem 1.4. It immediately follows from Theorem 1.1(2). □

4. Base-point-freeness for lc g-pairs

In this section, we prove Theorems 1.5, 1.6, and 1.7.

Lemma 4.1. *Let a be a positive real number, $(X,B,\mathbf{M})/U$ an lc g-pair, and D a nef/U Cartier divisor on X such that $aD-(K_X+B+\mathbf{M}_X)$ is ample/U. Let Y be a minimal lc center of $(X,B,\mathbf{M})$ if $(X,B,\mathbf{M})$ is not klt, and let $Y:=X$ if $(X,B,\mathbf{M})$ is klt. Let $D_Y:=D|_Y$. Then for any integer $m\gg 0$,*

1. *$\mathcal{O}_Y(mD_Y)$ is globally generated over U,*
2. *$|mD/U|\neq\emptyset$, and*
3. *Y is not contained in $\operatorname{Bs}|mD/U|$.*

Proof. When $(X,B,\mathbf{M})$ is klt, by [Xie22, Lemma 2.4], there exists a klt pair (X,Δ) such that $D-(K_X+\Delta)$ is ample/U. By the usual base-point-freeness theorem (cf. [KMM87, Theorem 3-1-1]), the lemma follows.

When $(X,B,\mathbf{M})$ is not klt, by [LX22, Theorem 4.10], Y is normal. By Theorem 1.1(3), the map $f_*\mathcal{O}_X(mD)\to f_*\mathcal{O}_Y(mD_Y)$ is surjective for any positive integer $m\geq a$. Thus (2)(3) follow from (1) and we only need to prove (1). If $\dim Y=0$ then there is nothing left to prove. If $\dim Y>0$, then by Lemma 2.11, there exists a klt g-pair $(Y,B_Y,\mathbf{M}^Y)/U$ such that $K_Y+B_Y+\mathbf{M}^Y_Y\sim_{\mathbb{R},U}(K_X+B+\mathbf{M}_X)|_Y$ and $\operatorname{Nklt}(Y,B_Y,\mathbf{M}^Y)=\operatorname{Nklt}(X,B,\mathbf{M})|_Y$. Thus $D_Y-(K_Y+B_Y+\mathbf{M}^Y_Y)$ is nef/U and big/U with respect to $(Y,B_Y,\mathbf{M}^Y)$. By [Xie22, Lemma 2.4], there exists a klt pair (Y,Δ_Y) such that $D_Y-(K_Y+\Delta_Y)$ is ample/U. By the usual base-point-freeness theorem (cf. [KMM87, Theorem 3-1-1]), the lemma follows. □

Proof of Theorem 1.5. By Lemma 4.1, we may let m_0 be the minimal positive integer such that $|mD|\neq\emptyset$ for any integer $m\geq m_0$.

Claim 4.2. *Let $\{p_i\}_{i=1}^{+\infty}$ be a strictly increasing sequence of positive integers. There exist a non-negative integer M and integers $i_1<i_2<\dots<i_{M+1}$ satisfying the following. Let $s_k:=\prod_{l=1}^k p_{i_l}$ for any $1\leq k\leq M+1$, then*

1. *$|s_1D/U|\neq\emptyset$,*
2. *$\operatorname{Bs}|s_kD/U|\supsetneq\operatorname{Bs}|s_{k+1}D/U|$ for any $1\leq k\leq M$, and*
3. *$\operatorname{Bs}|s_{M+1}D/U|=\emptyset$.*

Proof. We may take i_1 to be any integer such that $p_{i_1}\geq m_0$, then (1) holds.

Suppose that we have already found $i_1,\dots,i_k$ for some positive integer k. Let $d:=\dim X$, let $H_1,\cdots,H_{d+1}$ be $d+1$ be general elements in $|s_kD/U|$, and let $H:=H_1+\dots+H_{d+1}$. Then $(X,B+H,\mathbf{M})$ is lc outside $\operatorname{Bs}|s_kD/U|$. If $\operatorname{Bs}|s_kD/U|=\emptyset$, then we may let $M:=k-1$ and we are done. Thus we may assume that $\operatorname{Bs}|s_kD/U|\neq\emptyset$.

Since every H_j contains $\operatorname{Bs}|s_kD/U|$, by [Kol+92, Theorem 18.22], $(X,B+H,\mathbf{M})$ is not lc near $\operatorname{Bs}|s_kD/U|$. Let

$$c:=\sup\{t\mid t\geq 0,(X,B+tH,\mathbf{M})\text{ is lc}\},$$

then $c\in[0,1)$, and there exists at least one lc center of $(X,B+cH,\mathbf{M})$ which is contained in $\operatorname{Bs}|s_kD/U|$. Let $\mathcal{S}$ be the set of all lc centers of $(X,B+cH,\mathbf{M})$ that are contained in $\operatorname{Bs}|s_kD/U|$, and let Y be a minimal lc center in $\mathcal{S}$. Since

$$(a+s_k(d+1))D-(K_X+B+cH+\mathbf{M}_X)\sim_{\mathbb{R}} s_k(d+1)(1-c)D+(aD-(K_X+B+\mathbf{M}_X))$$

is ample$/U$, by Lemma 4.1, there exists a positive integer n, such that for any integer $m\geq n$, $|ms_kD/U|\neq\emptyset$ and $\mathrm{Bs}\,|ms_kD/U|$ does not contain Y. In particular, $\mathrm{Bs}\,|ms_kD/U|\subsetneq\mathrm{Bs}\,|s_kD/U|$. We may let i_{k+1} be any integer such that $i_{k+1}>i_k$ and $p_{i_{k+1}}\geq n$. This construction implies (2). (3) follows from (2) and the Noetherian property. $\square$

Proof of Theorem 1.5 continued. We let $\{p_i\}_{i=1}^{+\infty}$ and $\{q_j\}_{j=1}^{+\infty}$ be two strictly increasing sequence of prime numbers, such that $p_i\neq q_j$ for any i,j. By Claim 4.2, there exist two non-negative integers M,N and positive integers $i_1<i_2<\cdots<i_{M+1}$ and $j_1<j_2<\cdots<j_{N+1}$, such that $\mathcal{O}_X(\prod_{l=1}^{M+1}p_{i_l}D)$ and $\mathcal{O}_X(\prod_{l=1}^{N+1}q_{i_l}D)$ are globally generated$/U$. Let $p:=\prod_{l=1}^{M+1}p_{i_l}$ and $q:=\prod_{l=1}^{N+1}q_{i_l}$, then p and q are coprime. Therefore, for any integer $m\gg 0$, we may write $m=bp+cq$ for some non-negative integers b,c, hence

$$\mathrm{Bs}\,|mD/U|\subset\mathrm{Bs}\,|pD/U|\cup\mathrm{Bs}\,|qD/U|=\emptyset.$$

Therefore, $\mathcal{O}_X(mD)$ is globally generated over U for any integer $m\gg 0$. $\square$

Proof of Theorem 1.6. By the theory of Shokurov-type rational polytopes (cf. [HL22, Proposition 3.20]) and the theory of uniform rational polytopes (cf. [HLS19, Lemma 5.3], [Che20, Theorem 1.4]), we may assume that D is a $\mathbb{Q}$-divisor. The theorem immediately follows from Theorem 1.5. $\square$

Proof of Theorem 1.7. (1)(2) By the cone theorem [HL21, Theorem 1.1(1-4)], F is a finitely dimensional rational $(K_X+B+\mathbf{M}_X)$-negative extremal face$/U$. Thus there exists a nef Cartier divisor L on X that is the supporting function of F. Then $L-(K_X+B+\mathbf{M}_X)$ is ample. By Theorem 1.5, mL is base-point-free$/U$, hence defines a contraction$/U$. Denote this contraction by cont_F. Then cont_F satisfies (1) and (2).

(3) Since $D-(K_X+B+\mathbf{M}_X)$ is ample$/Z$, by Theorem 1.5, $\mathcal{O}_X(mD)$ is globally generated over Z for any integer $m\gg 0$. Since $D\cdot C$ for any curve C contracted by cont_F, cont_F is defined by $|mD|$ for any integer $m\gg 0$. Thus $mD=f^*D_{Y,m}$ and $(m+1)D=f^*D_{Y,m+1}$ for any integer $m\gg 0$. We may let $D_Y:=D_{Y,m+1}-D_{Y,m}$. $\square$

Remark 4.3. Kollár's gluing theory for generalized pairs was originally established in [LX22, Construction 4.12] to glue glc crepant structures. This theory was further developed in [Xie22]. Although we have extensively referenced both [LX22] and [Xie22], it is important to note that we have only cited results before [LX22, Theorem 4.10] from [LX22] and only cited [Xie22, Lemma 2.4] from [Xie22]. None of these cited results are dependent on Kollár's gluing theory for generalized pairs (although [LX22, Theorem 4.10] used the idea of stratification). Consequently, the proofs of our main theorems are independent of Kollár's gluing theory for generalized pairs.

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