

CONTRACTION THEOREM FOR GENERALIZED PAIRS

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ABSTRACT. We use Kollár’s gluing theory to prove the contraction theorem for generalized pairs.

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1. INTRODUCTION

We work over the field of complex numbers \mathbb{C} .

In recent years, it has become increasingly clear that it is important to generalize results from the MMP for pairs to the MMP for generalized pairs, see [Bir21] and references therein. One of the most important conjectures in the MMP is the abundance conjecture. It is expected that if (X, D) is an lc pair and $K_X + D$ is nef then $K_X + D$ is semi-ample. An important result in this direction is [FG14, HX16], where it is shown that log abundant nef lc pairs are semi-ample. Unluckily, this is false for generalized lc pairs even if we assume log abundance ([LX22a, Example 1.4]). Nonetheless, some weaker semi-ampleness results related to the MMP are still believed to be true for generalized pairs and will lead to many interesting applications. Therefore it is important to understand exactly where the semi-ampleness starts to fail and what assumptions one should add to avoid this failure.

As in the log canonical case (cf. [Kol13, Section 5.5]), a generalized log canonical structure gives a stratification (called glc stratification) of a variety with respect to its glc centers ([LX22b, Section 4]). Thanks to the \mathbb{P}^1 -link techniques developed in [FS20, Theorem 1.4](cf. [Bir20, Theorem 3.5]), the glc stratification turns out to be nice and useful. For instance, we use the glc stratification to prove that any glc singularity is Du Bois ([LX22b, Section 6]). More importantly, we can do adjunction to glc centers via the generalized canonical bundle formula developed in [Fil20, HL21b, J LX22], thus this stratification allows us to use Kollár’s gluing theory to prove semi-ampleness properties by induction on the dimension. The essential difficulty is to show that some induced equivalence relation is finite. In the lc pair case, as explained in [HX13, HX16], these relations arise from some subgroup of $\text{Bir}(V, \Delta_V)$, where V is some hereditary lc center ([Kol13, Definition 5.30]), hence

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the finiteness of \mathbf{B} -representations will imply the required finiteness of relations. In the glc pair case, this finiteness of \mathbf{B} -representations can fail in general, which is the main reason that the abundance conjecture is not true for generalized pairs. However, there are certain situations where we can actually show the finiteness of Kollár's relations regardless of the \mathbf{B} -representations. In these cases, we expect the corresponding semi-ampleness results hold. For example, in [LX22b] Jihao Liu and the author prove the existence of glc flips and an analogue of [HX13, Theorem 1.1] in the setting of generalized pairs.

Except for the semi-ampleness, the Minimal Model Program seems to work pretty well for generalized pairs. For example, termination of flips and existence of minimal models or Mori fiber spaces hold for many generalized pairs with some standard assumptions similar to the case of usual lc pairs ([HL22, Theorem 4.1], [LT22, Theorem 1.1], [Has22, Theorem 3.17], [LX22a, Theorem 1.2, 1.3]). As summarized in [HL21a], many very general results concerning running the MMP for lc/dlt pairs are still true for glc/gdlt pairs. Moreover, [HL21a] established the Cone theorem for glc pairs, and also established the Contraction theorem when \mathbf{M}_X is \mathbb{R} -Cartier. Their approach involves replacing the generalized pairs by some auxiliary usual pairs ([HL21a, Theorem 4.1]) with the help of some ample divisor. However, there are essential differences between glc pairs and the lc pairs when the ambient variety is not \mathbb{Q} -factorial ([LX22b, Example 2.1]). Hence in order to obtain the contraction theorem for glc pairs in full generality, which is equivalent to showing some semi-ampleness, we have to extend the theory for generalized pairs instead of just using theorems developed for lc pairs.

The main purpose of this paper is to use the glc stratification developed in [LX22b] and Kollár's gluing theory to prove the following semi-ampleness theorem:

Theorem 1.1 (=Theorem 4.1). *Let $(X, \Delta, \mathbf{M})/U$ be a glc \mathbb{Q} -pair, and L a nef \mathbb{Q} -divisor such that $L - (K_X + \Delta + \mathbf{M}_X)$ is nef and log big/ U with respect to (X, Δ, \mathbf{M}) . Then L is semi-ample over U .*

Since ample divisors are automatically nef and log big, therefore in particular we have:

Theorem 1.2. *Let $(X, \Delta, \mathbf{M})/U$ be a glc pair and A an ample/ U \mathbb{R} -divisor. Then $K_X + \Delta + \mathbf{M}_X + A$ is nef/ U if and only if it is semi-ample/ U .*

As an easy corollary, we have:

Theorem 1.3. *Let $(X, \Delta + A, \mathbf{M})/U$ be a glc pair such that*

- $\mathbf{B}_+(A/U)$ contains no glc center of $(X, \Delta + A, \mathbf{M})$.
- $K_X + \Delta + A + \mathbf{M}_X$ is nef over U .

Then $K_X + \Delta + A + \mathbf{M}_X$ is semi-ample over U .

By looking at the gluing relations more carefully, we can actually get a stronger result, which is the g-pair analogue of the base point free theorem for lc pairs.

Theorem 1.4 (Base point free theorem for glc pairs). *Let $(X, \Delta, \mathbf{M})/U$ be a glc \mathbb{Q} -pair, and L a nef/ U Cartier divisor such that $aL - (K_X + \Delta + \mathbf{M}_X)$ is nef and log big/ U with respect to (X, Δ, \mathbf{M}) for some real positive number a . Then $\mathcal{O}_X(mL)$ is globally generated over U for all $m \gg 0$.*

The immediate application is the Contraction theorem for g-pairs, which fulfill the last part of [HL21a, Theorem 1.3] when \mathbf{M}_X is not necessarily \mathbb{R} -Cartier:

Theorem 1.5. *Let $(X, \Delta, \mathbf{M})/U$ be a glc pair and R be a $(K_X + \Delta + \mathbf{M}_X)$ -negative extremal ray in $\overline{\text{NE}}(X/U)$. Then R is a rational extremal ray. In particular, there exists a projective morphism $\text{cont}_R : X \rightarrow Y$ over U satisfying the following.*

- For any integral curve C such that the image of C in U is a point, then $\text{cont}_R(C)$ is a point if and only if $[C] \in R$.
- $\mathcal{O}_Y = (\text{cont}_R)_* \mathcal{O}_X$. In other words, cont_R is a contraction.
- Let L be a line bundle on X such that $L \cdot R = 0$, then there exists a line bundle L_Y on Y such that $L = \text{cont}_R^* L_Y$.

The author has been told by Jihao Liu that Theorem 1.5 along with [LX22b, Theorem 1.1] would allow one to run MMP for glc pairs in the non- \mathbb{Q} -factorial setting:

Theorem 1.6. *Let $(X, \Delta, \mathbf{M})/U$ be a glc pair, then we can run a $(K_X + \Delta + \mathbf{M}_X)$ -MMP over U .*

This turns out to be useful for proving some expected semi-ampleness results since log bigness is not preserved when pulling back to \mathbb{Q} -factorial gdl models. Actually N. Tsakanikas and I will pursue the following statement in a forthcoming paper:

Theorem 1.7. *Let $(X, \Delta, \mathbf{M})/U$ be a glc pair and A be an ample/ U \mathbb{R} -divisor such that $(X, \Delta + A, \mathbf{M})$ is also glc. Then we can run a $(K_X + \Delta + \mathbf{M}_X + A)$ -MMP which terminates with a Mori fiber space or a good minimal model (not necessarily \mathbb{Q} -factorial).*

Since the proof of the main theorem relies on showing certain finiteness of relations, we will inevitably run into many technical issues, so we would like to give a sketch here to explain the core ideas in the proof.

Sketch of the proof of Theorem 1.1: By perturbing the generalized pair and applying Fujino's technique (cf. [Fuj11]), we can easily reduce the question to proving that $L|_V$ is semi-ample, where $V = \text{Ngklt}(X, \Delta, \mathbf{M})$ is the non-gklt locus of (X, Δ, \mathbf{M}) with the reduced scheme structure. The subtle thing here is that the structure of V is somehow complicated (eg. V may not be equi-dimensional or irreducible) so it is usually really hard to tell when a line bundle on V should be semi-ample.

However, V is actually semi-normal and has a good stratification structure (glc stratification) coming from the glc centers (cf. [LX22b]), and if we consider some nicely chosen stratified morphism, for example, the normalization $\pi : V^n \rightarrow V$, then we can do subadjunction to V^n and then by induction on the dimension we know that $L|_{V^n}$ is semi-ample. Notice that $V^n = \coprod V_i$ is a disjoint union of irreducible normal varieties, so for each V_i , $L|_{V_i}$ defines a contraction

$$g_i : V_i \rightarrow Z_i$$

with a so called glc crepant log structure (see Definition 2.5), which induces a glc stratification on Z_i . In order to show that $L|_V$ is semi-ample, we must first find the correct candidate morphism $g : V \rightarrow Z$ that will be defined by $L|_V$ with the information coming from the g_i and π . More precisely, we must consider the relation

$$\{g_i(x) \sim g_j(y) \mid x \in V_i, y \in V_j, \pi(x) = \pi(y)\}$$

between Z_i and Z_j (i, j need not to be distinct). After some extra effort, we can give a nice interpretation on the induced relation between the Z_i 's by relating it with some group actions on the strata induced by the stratification. Fortunately for us, we have a powerful gluing theory introduced by Kollár, with the help of which we only need to show that the above relation generated by g_i and π is finite in some sense ([Kol13, Theorem 9.21]). Moreover, we only need to check that this holds on each glc center $Z_{i,\gamma} \subset Z_i$. This is essentially equivalent to showing that the stabilizer group $\text{stab}(Z_{i,\gamma})$ is finite.

The relation given by π is always finite since π is a finite morphism, but the contractions g_i will create extra gluing information. A more careful computation shows that the extra relations essentially come from the different minimal glc centers on V_i that dominate the same glc center on Z_i . For simplicity, we consider the case that there is only one $g_i : V_i \rightarrow Z_i$ and $g_i|_{V_{i,\alpha}}$ is also a contraction for any glc center $V_{i,\alpha} \subset V_i$. Let V_{i,α_1} and V_{i,α_2} be two different minimal glc center that dominate Z_i . Assume that the only gluing relation coming from π is given by an isomorphism

$$\tau_{12} : V_{i,\alpha_1} \rightarrow V_{i,\alpha_2}.$$

Then we can see that τ_{12} does not generate any automorphism in V_{i,α_1} or V_{i,α_2} . However, τ_{12} induces an automorphism of Z_i , which may not be of finite order in general (see Example 1.8 below).

Nevertheless, the log bigness in our assumption makes the situation much better behaved (see Theorem 3.2). We actually show that for any glc center $Z_{i,\gamma} \subset Z_i$, there is a unique minimal glc center $V_{i,\alpha}$ that controls all the relations concerning $Z_{i,\gamma}$. In particular, any automorphism of $Z_{i,\gamma}$ coming from the relations will lift to an automorphism of $V_{i,\alpha}$ which in turn is induced by the π . Therefore the finiteness of relations between the V_i will ensure the finiteness of relations between the Z_i and we can obtain the geometric quotient Z as we desired.

Applying the same philosophy to the total space of the line bundle $mL|_V$ over V for sufficiently divisible m , we will be able to find a line bundle H on Z such that $g^*H = mL|_V$. We can easily show that H is ample, hence $L|_V$ is semi-ample and we are done.

The following example shows that the uniqueness of minimal glc centers is really necessary when using gluing theory to find the desired Z and H that correspond to the semi-ample L .

Example 1.8 ([LX22b, Example 4.15]). Let $\lambda \in \mathbb{C}^*$ and consider $\mathbb{P}^1 \times \mathbb{A}^1$, which can be regarded as the total space of a trivial line bundle over \mathbb{P}^1 . We define $\phi_\lambda : \{0\} \times \mathbb{A}^1 \simeq \{\infty\} \times \mathbb{A}^1$ by $(0, t) \mapsto (\infty, \lambda t)$ and glue $\{0\} \times \mathbb{A}^1$ and $\{\infty\} \times \mathbb{A}^1$ together using ϕ_λ to get a demi-normal variety M with projection $p : M \rightarrow C$, where C is a nodal cubic. Then M is the total space of a line bundle N on C . Moreover, $N \in \text{Pic}^0(C) \simeq \mathbb{G}_m = \mathbb{C}^*$ and can be canonically identified with $\lambda \in \mathbb{C}^*$.

- (1) Let $W := \mathbf{P}_C(\mathcal{O}_C \oplus N)$ be a \mathbb{P}^1 -bundle over C , and let $C' \subset W$ be the section at infinity, which belongs to $|\mathcal{O}_W(1)|$. Then the normalization $W^n = \mathbb{P}^1 \times \mathbb{P}^1$, and the extended isomorphism

$$\tilde{\phi}_\lambda : \{0\} \times \mathbb{P}^1 \simeq \{\infty\} \times \mathbb{P}^1, (0, [x, y]) \mapsto (\infty, [\lambda x, y])$$

gives the gluing relation of $\pi_W : W^n \rightarrow W$.

Notice that K_W is Cartier since W is a locally complete intersection. Let $L := K_X + 3C'$, then we see that

$$\pi^*L = K_{W^n} + \{0\} \times \mathbb{P}^1 + \{\infty\} \times \mathbb{P}^1 + 3\pi^*C' \sim p_2^*(\{\infty\})$$

is base point free and defines the second projection $p_2 : W^n \rightarrow Y \simeq \mathbb{P}^1$. Since $V_1 = \{0\} \times \mathbb{P}^1$ and $V_2 = \{\infty\} \times \mathbb{P}^1$ both dominate Y under p_2 , $\tilde{\phi}_\lambda$ induces an automorphism of V :

$$\tilde{\sigma}_\lambda : [x, y] \mapsto [\lambda x, y]$$

Thus the relation generated by π and p_2 is given by

$$[x, y] \sim [x', y'] | [x', y'] = [x, \lambda^l y] \text{ for some } l \in \mathbb{Z}$$

and is finite if and only if λ is a root of unity.

- (2) Let $\pi_C : \mathbb{P}^1 \rightarrow C$ be the normalization. Then $\pi_C^*(N) \simeq \mathcal{O}_{\mathbb{P}^1}$ and it defines $g^n : \mathbb{P}^1 \rightarrow \text{Spec } \mathbb{C}$, then the gluing relation $\{0\} \sim \{\infty\}$ from π_C gives no extra relation under g^n , and so we get the morphism:

$$g : C \rightarrow \text{Spec } \mathbb{C}$$

However, if we consider the total space M then $\pi_M : \mathbb{P}^1 \times \mathbb{A}^1 \rightarrow M$ is the normalization. Notice that $\mathbb{P}^1 \times \mathbb{A}^1$ is also the total space of the trivial line bundle over \mathbb{P}^1 , thus there is a canonical morphism

$$g_M^n : \mathbb{P}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$$

between total spaces of corresponding line bundles coming from $g^{n,*}\mathcal{O}_{\text{Spec } \mathbb{C}} = \mathcal{O}_{\mathbb{P}^1}$. Then ϕ_λ induces an automorphism of \mathbb{A}^1 :

$$\sigma_\lambda : t \mapsto \lambda t$$

Thus the relation generated by π_M and g_M^n on \mathbb{A}^1 is given by

$$\{t \sim s \mid t = \lambda^l s \text{ for some } l \in \mathbb{Z}\}$$

and is finite if and only if λ is a root of unity.

Even if λ is a root of unity, for example, assume λ generates $\mu_n \subset \mathbb{G}_m$, we have $\mathbb{A}^1/\mu_n \simeq \mathbb{A}^1$ and get $g_M : M \rightarrow \mathbb{A}^1$ as our desired morphism. However, if we look at the equivariant \mathbb{G}_m action under g_M , we see the action on $\mathbb{A}^1 \setminus \{0\}$ is the natural \mathbb{G}_m action on \mathbb{G}_m/μ_n . This corresponds to the fact that N is not a pullback of a line bundle on $\text{Spec } \mathbb{C}$. Actually the $\mathbb{A}^1 \setminus \{0\}$ with the above \mathbb{G}_m action is called a Seifert bundle ([Kol13, Definition 9.50]) and it will become a line bundle if we replace N with nN .

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2. PRELIMINARIES

We adopt the standard notation and definitions in [KM98, BCHM10] and will freely use them. We will first introduce the definition of generalized pairs by using **b**-divisors. Then we will recall the glc crepant log structures and its induced glc stratifications developed in [LX22b].

2.1. Generalized pairs. We will follow the original definitions in [BZ16] but will adopt the same notations as in [HL21a]. Notice that there are some small differences with the definitions in [HL21a]: in this paper, all generalized (sub)-pairs are assumed to be NQC.

Definition 2.1 (b-divisors). Let X be a normal quasi-projective variety. We call Y a *birational model* over X if there exists a projective birational morphism $Y \rightarrow X$.

Let $X \dashrightarrow X'$ be a birational map. For any valuation ν over X , we define $\nu_{X'}$ to be the center of ν on X' . A **b-divisor** \mathbf{D} over X is a formal sum $\mathbf{D} = \sum_\nu r_\nu \nu$ where ν are valuations over X and $r_\nu \in \mathbb{R}$, such that ν_X is not a divisor except for finitely many ν . If in addition, $r_\nu \in \mathbb{Q}$ for every ν , then \mathbf{D} is called a \mathbb{Q} -**b-divisor** over X . The *trace* of \mathbf{D} on X' is the \mathbb{R} -divisor

$$\mathbf{D}_{X'} := \sum_{\nu_{i,X'} \text{ is a divisor}} r_i \nu_{i,X'}.$$

If $\mathbf{D}_{X'}$ is \mathbb{R} -Cartier and \mathbf{D}_Y is the pullback of $\mathbf{D}_{X'}$ on Y for any birational model Y of X' , we say that \mathbf{D} *descends* to X' and write $\mathbf{D} = \overline{\mathbf{D}_{X'}}$.

Let $X \rightarrow U$ be a projective morphism and assume that \mathbf{D} is a \mathbf{b} -divisor over X such that \mathbf{D} descends to some birational model Y over X . If \mathbf{D}_Y is nef/ U (resp. semi-ample/ U), then we say that \mathbf{D} is nef/ U (resp. semi-ample/ U). If \mathbf{D}_Y is a Cartier divisor, then we say that \mathbf{D} is *\mathbf{b} -Cartier*. If \mathbf{D} can be written as an $\mathbb{R}_{\geq 0}$ -linear combination of nef/ U \mathbf{b} -Cartier \mathbf{b} -divisors, then we say that \mathbf{D} is *NQC/ U* .

Definition 2.2 (Generalized pairs). A *generalized sub-pair* (*g-sub-pair* for short) $(X, \Delta, \mathbf{M})/U$ consists of a normal quasi-projective variety X associated with a projective morphism $X \rightarrow U$, an \mathbb{R} -divisor Δ on X , and an NQC/ U \mathbf{b} -divisor \mathbf{M} over X , such that $K_X + \Delta + \mathbf{M}_X$ is \mathbb{R} -Cartier. If Δ is a \mathbb{Q} -divisor and \mathbf{M} is a \mathbb{Q} - \mathbf{b} -divisor, then we say that $(X, \Delta, \mathbf{M})/U$ is a *\mathbb{Q} -g-sub-pair*.

A g-sub-pair (resp. \mathbb{Q} -g-sub-pair) $(X, \Delta, \mathbf{M})/U$ is called a *g-pair* (resp. *\mathbb{Q} -g-pair*) if $\Delta \geq 0$. A sub-pair (X, Δ) is called a *pair* if $\Delta \geq 0$.

Definition 2.3 (Singularities of generalized pairs). Let $(X, \Delta, \mathbf{M})/U$ be a g-(sub-)pair. For any prime divisor E and \mathbb{R} -divisor D on X , we define $\text{mult}_E D$ to be the *multiplicity* of E along D . Let $h : W \rightarrow X$ be any log resolution of $(X, \text{Supp } \Delta)$ such that \mathbf{M} descends to W , and let

$$K_W + \Delta_W + \mathbf{M}_W := h^*(K_X + \Delta + \mathbf{M}_X).$$

The *log discrepancy* of a prime divisor D on W with respect to (X, Δ, \mathbf{M}) is $1 - \text{mult}_D \Delta_W$ and it is denoted by $a(D, X, \Delta, \mathbf{M})$.

We say that (X, Δ, \mathbf{M}) is *(sub-)glc* (resp. *(sub-)gklt*) if $a(D, X, \Delta, \mathbf{M}) \geq 0$ (resp. > 0) for every log resolution $h : W \rightarrow X$ as above and every prime divisor D on W . We say that (X, Δ, \mathbf{M}) is *gdlt* if (X, Δ, \mathbf{M}) is glc, and there exists a closed subset $V \subset X$, such that

- (1) $X \setminus V$ is smooth and $\Delta_{X \setminus V}$ is simple normal crossing, and
- (2) for any prime divisor E over X such that $a(E, X, \Delta, \mathbf{M}) = 0$, $\text{center}_X E \not\subset V$ and $\text{center}_X E \setminus V$ is an lc center of $(X \setminus V, B|_{X \setminus V})$.

Suppose that (X, Δ, \mathbf{M}) is sub-glc. A *glc place* of (X, Δ, \mathbf{M}) is a prime divisor E over X such that $a(E, X, \Delta, \mathbf{M}) = 0$. A *glc center* of (X, Δ, \mathbf{M}) is the center of a glc place of (X, Δ, \mathbf{M}) on X . The *non-gklt locus* $\text{Ngklt}(X, \Delta, \mathbf{M})$ of (X, Δ, \mathbf{M}) is the union of all glc centers of (X, Δ, \mathbf{M}) .

If a \mathbb{Q} -g-pair (X, Δ, \mathbf{M}) is glc, then we will call (X, Δ, \mathbf{M}) a glc \mathbb{Q} -pair for short.

We say that an \mathbb{R} -Cartier divisor D is *log big* over U with respect to (X, Δ, \mathbf{M}) if D is big over U and for any generalized lc center T of $(X, \Delta, \mathbf{M})/U$ with the normalization $T^n \rightarrow T$, the pullback $D|_{T^n}$ is big over U .

The following lemma is important for applying [Fuj11, Theorem 13.1] when we try to prove semi-ampleness by inductions. We refer the readers to [Fuj11, Section 7] for the definitions of non-lc ideal and locus.

Lemma 2.4. *Let $(X, \Delta, \mathbf{M})/U$ be a glc pair, and L a nef \mathbb{Q} -divisor such that $L - (K_X + \Delta + \mathbf{M}_X)$ is nef and big/ U . Then there exists an effective \mathbb{R} -divisor B such that $L - K_X - B$ is ample over U and $\text{Nlc}(X, B) = \text{Ngklt}(X, \Delta, \mathbf{M})$ (as subschemes).*

Proof. Let $f : Y \rightarrow X$ be a log resolution such that $\text{Exc}(f) \cup f^{-1}\Delta$ is snc and \mathbf{M} descends on Y . Let $K_Y + \Delta_Y + \mathbf{M}_Y = f^*(K_X + \Delta + \mathbf{M}_X)$, then $f^*L - (K_Y + \Delta_Y) + \mathbf{M}_Y$ is nef and big so there is an effective \mathbb{R} -divisor E on Y such that

$$f^*L - (K_Y + \Delta_Y) + \mathbf{M}_Y \sim_{\mathbb{R}, U} A_n + \frac{1}{n}E,$$

where A_n can be chosen to be a sufficiently general effective ample \mathbb{R} -divisor. By perturbing A_n a little bit we can also assume that $[\Delta_Y] \subset \text{Supp } E$. Since $\frac{1}{2}(L - K_X - \Delta - \mathbf{M}_X)$ is nef and big, there is an effective \mathbb{R} -divisor E' on X such that

$$\frac{1}{2}(L - K_X - \Delta - \mathbf{M}_X) \sim_{\mathbb{R}, U} A'_n + \frac{1}{n}E',$$

where A'_n is ample. Then $B := f_*(\Delta_Y) + \frac{1}{2n}f_*(E + A_n) + \frac{1}{n}E'$ would work for $n \gg 0$ \square

2.2. Crepant log structures.

Definition 2.5. A *glc crepant log structure* is of the form $f : (X, \Delta, \mathbf{M}) \rightarrow Z$, where

- (1) $(X, \Delta, \mathbf{M})/Z$ is a glc g-pair,
- (2) $K_X + \Delta + \mathbf{M}_X \sim_{\mathbb{R}, Z} 0$, and
- (3) f is a contraction. In particular, $f_*\mathcal{O}_X = \mathcal{O}_Z$.

In addition, if

- (4) (X, Δ, \mathbf{M}) is gdlt,

then we say that $f : (X, \Delta, \mathbf{M}) \rightarrow Z$ is a *gdlt crepant log structure*.

For any irreducible subvariety $W \subset Z$, we say that W is a *glc center* of a glc crepant log structure $f : (X, \Delta, \mathbf{M}) \rightarrow Z$, if there exists a glc center W_X of (X, Δ, \mathbf{M}) such that $W = f(W_X)$. For any (not necessarily closed) point $z \in Z$, we say that z is a *glc center* of $f : (X, \Delta, \mathbf{M}) \rightarrow Z$ if \bar{z} is a glc center of $f : (X, \Delta, \mathbf{M}) \rightarrow Z$.

Let us recall two important Lemmas in [LX22b]:

Lemma 2.6 ([LX22b, Lemma 3.17]). *Let $f : (X, \Delta, \mathbf{M}) \rightarrow Z$ be a glc crepant log structure and $z \in Z$ a (not necessarily closed) point. Let*

$$\mathcal{S}_z := \{V \mid V \text{ is a glc center of } f : (X, \Delta, \mathbf{M}) \rightarrow Z, z \in V\}.$$

Then:

- (1) *There exists a unique element $W \in \mathcal{S}_z$ that is minimal with respect to inclusion.*
- (2) *W is unibranch at z , i.e. the completion \hat{W}_z is irreducible.*
- (3) *Any intersection of glc centers of $f : (X, \Delta, \mathbf{M}) \rightarrow Z$ is also a union of glc centers.*

Lemma 2.7 ([LX22b, Lemma 3.19]). *Let $f : (X, \Delta, \mathbf{M}) \rightarrow Z$ be a gdlt crepant log structure and $Y \subset X$ a glc center. Let*

$$f|_Y : Y \xrightarrow{f_Y} Z_Y \xrightarrow{\pi} Z$$

be the Stein factorization of $f|_Y$, and $(Y, \Delta_Y, \mathbf{M}^Y)/Z$ the gdlt g-pair induced by repeated adjunctions

$$K_Y + \Delta_Y + \mathbf{M}_Y^Y := (K_X + \Delta + \mathbf{M}_X)|_Y.$$

Then:

- (1) *$f_Y : (Y, \Delta_Y, \mathbf{M}^Y) \rightarrow Z_Y$ is a gdlt crepant log structure.*
- (2) *For any glc center $W_Y \subset Z_Y$ of $f_Y : (Y, \Delta_Y, \mathbf{M}^Y) \rightarrow Z_Y$, $\pi(W_Y)$ is a glc center of $f : (X, \Delta, \mathbf{M}) \rightarrow Z$.*
- (3) *For any glc center $W \subset Z$ of $f : (X, \Delta, \mathbf{M}) \rightarrow Z$, every irreducible component of $\pi^{-1}(W)$ is a glc center of $f_Y : (Y, \Delta_Y, \mathbf{M}^Y) \rightarrow Z_Y$.*

The following theorem is an analogue of [Kol13, Theorem-Definition 4.45].

Definition-Theorem 2.8. Let $(X, \Delta, \mathbf{M}) \rightarrow Y$ be a gdlt crepant log structure and $Z \subset Y$ a glc center with normalization $n : Z^n \rightarrow Z$. Let $S \subset X$ be glc center of (X, Δ, \mathbf{M}) which dominates Z and is minimal with respect to inclusion. Let $(S, \Delta_S, \mathbf{M}^S)$ be the gdlt g-pair induced by the adjunction

$$K_S + \Delta_S + \mathbf{M}_S^S := (K_X + \Delta + \mathbf{M}_X)|_S$$

and let $f_S^n : S \rightarrow Z_S \rightarrow Z^n$ be the Stein factorization. Then:

- (1) (Uniqueness of springs) The isomorphism class of Z_S does not depend on the choice of S . It is called the spring of Z and denoted by $\text{Spr}(Z, X, \Delta, \mathbf{M})$, or $\text{Spr}(Z, X)$ for short if there is no confusion.
- (2) (Uniqueness of sources) The equivalence class of S does not depend on the choice of S . This birational class of S will be called as the *source* of Z and is denoted by $\text{Src}(Z, X, \Delta, \mathbf{M})$, or $\text{Src}(Z, X)$ for short if there is no possible confusion.
- (3) (Crepant log structure) $(S, \Delta_S, \mathbf{M}^S)$ is gdlt, $K_S + \Delta_S + \mathbf{M}_S^S \sim_{\mathbb{Q}, Z} 0$, and $(S, \Delta_S, \mathbf{M}^S)$ is gklt on the generic fiber of $f|_S$.
- (4) (Galois property) The extension $Z_S \rightarrow Z$ is Galois and $\text{Bir}_Z(S) \rightarrow \text{Gal}(Z_S/Z)$ is surjective.
- (5) (Adjunction) Let $W \subset X$ be a glc center, $f_W : W \rightarrow W_Y$ and $n_W : W_Y \rightarrow Y$ be the Stein factorization of $f|_W$. Let $Z_W \subset W_Y$ be an irreducible subvariety such that $n_W(Z_W) = Z$. Then Z_W is a glc center of $(W, \Delta_W, \mathbf{M}^W)$ and

$$\text{Src}(Z, X, \Delta, \mathbf{M}) \sim_{\text{Bir}} \text{Src}(Z_W, W, \Delta_W, \mathbf{M}^W)$$

$$\text{Spr}(Z, X, \Delta, \mathbf{M}) \simeq \text{Spr}(Z_W, W, \Delta_W, \mathbf{M}^W)$$

Proof. By [LX22b, Theorem 3.16](cf. [FS20, Theorem 1.4]), different choices of S are \mathbb{P}^1 -linked to each other, hence they are birational, proving (1). This implies (2) while (3) is clear. For (5), note that Z_W is a glc center by Lemma 2.7 and we can actually choose representatives such that

$$\text{Src}(Z_W, W, \Delta_W, \mathbf{M}^W) = \text{Src}(Z, X, \Delta, \mathbf{M}),$$

the rest then follows from (1) and (2). Finally, (4) follows from the proof of [Kol13, Lemma 4.46] \square

2.3. Generalized log canonical stratification. We refer to [Kol13, Section 5, Section 9] and [LX22b, Section 4] for more details.

Definition 2.9 ([Kol13, Definition 9.15]). Let X be a scheme. A *stratification* of X is a decomposition of X into a finite disjoint union of reduced locally closed subschemes. We will consider stratifications where the strata are of pure dimensions and are indexed by their dimensions. We write $X = \cup_i S_i X$ where $S_i X \subset X$ is the i -th dimensional stratum. Such a stratified scheme is denoted by (X, S_*) . We also assume that $\cup_{i \leq j} S_i X$ is closed for every j . The *boundary* of (X, S_*) is the closed subscheme

$$B(X, S_*) := \cup_{i < \dim X} S_i X = X \setminus S_{\dim X} X,$$

and is denoted by $B(X)$ if the stratification S_* is clear. We call $S_{\dim X}$ the *open stratum*.

Let (X, S_*) and (Y, S_*) be stratified schemes. We say that $f : X \rightarrow Y$ is a *stratified morphism* if $f(S_i X) \subset S_i Y$ for every i . Since $S_i X$ are disjoint with each other, $f : X \rightarrow Y$ is a stratified morphism if and only if $S_i X = f^{-1}(S_i Y)$.

Let (Y, S_*) be a stratified scheme and $f : X \rightarrow Y$ a quasi-finite morphism such that $f^{-1}(S_i Y)$ has pure dimension i for every i . Then $S_i X := f^{-1}(S_i Y)$ defines a stratification of X . We denote it by $(X, f^{-1} S_*)$, and we say that $f : X \rightarrow (Y, S_*)$ is *stratifiable*.

Definition 2.10. Let (X, S_*) be a stratified variety. A relation $(\sigma_1, \sigma_2) : R \rightrightarrows (X, S_*)$ is *stratified* if each σ_i is stratifiable and $\sigma_1^{-1}S_* = \sigma_2^{-1}S_*$. Equivalently, there exists a stratification $(R, \sigma^{-1}S_i)$, such that $r \in \sigma^{-1}S_i R$ if and only if $\sigma_1(r) \in S_i X$ and if and only if $\sigma_2(r) \in S_i X$.

Next we give a special stratification that is induced by the glc crepant log structure.

Definition 2.11 (Glc stratification). Let $f : (X, \Delta, \mathbf{M}) \rightarrow Z$ be a glc crepant log structure. Let $S_i^*(Z, X, \Delta, \mathbf{M}) \subset Z$ be the union of all $\leq i$ -dimensional glc centers of $f : (X, \Delta, \mathbf{M}) \rightarrow Z$, and

$$S_i(Z, X, \Delta, \mathbf{M}) := S_i^*(Z, X, \Delta, \mathbf{M}) \setminus S_{i-1}^*(Z, X, \Delta, \mathbf{M}).$$

If the glc crepant log structure $f : (X, \Delta, \mathbf{M}) \rightarrow Z$ is clear from the context, we will use $S_i(Z)$ for abbreviation. It is clear that each $S_i(Z)$ is a locally closed subspace of Z of pure dimension i , and Z is the disjoint union of all $S_i(Z)$.

The stratification of Z induced by $S_i(Z)$ is called the *generalized log canonical stratification* (glc stratification for short) of Z induced by $f : (X, \Delta, \mathbf{M}) \rightarrow Z$. Since this is the only stratification we are going to use in the rest of this paper, we usually will not emphasize the glc crepant structure $f : (X, \Delta, \mathbf{M}) \rightarrow Z$, and we will denote the corresponding stratified scheme by (Z, S_*) . The *boundary* of (Z, S_*) is the closed subspace

$$B(Z, S_*) := Z \setminus S_{\dim Z}(Z) = \cup_{i < \dim Z} S_i(Z).$$

3. CREPANT LOG STRUCTURE WITH LOG BIGNESS

In this section we show that there will be no \mathbb{P}^1 -link in a glc crepant log structure if some mild log bigness assumptions are posed on the g-pair.

Lemma 3.1. *Let $f : (X, \Delta, \mathbf{M}) \rightarrow Z$ be a glc crepant log structure and V be a glc center on Z . Let W and W' be two minimal glc centers on X that dominates V . Let $\tilde{W} \subset X$ be another glc center such that $V \subset f(\tilde{W})$ (We allow X itself to be a glc center when $Z = V$). Then the followings hold:*

- (1) *There exist glc centers W_0, W_1, \dots, W_n and $\hat{W}_1, \dots, \hat{W}_n$ on X such that $W = W_0 \subset \hat{W}_1 \supset W_1 \subset \dots \subset \hat{W}_n \supset W_n = W'$ and $f(W_i) = f(\hat{W}_i) = V$.*
- (2) *There exists a glc center $W'' \subset X$ such that $W'' \subset \tilde{W}$ and $f(W'') = V$, hence we can also choose such W'' to be minimal.*

Proof. If (X, Δ, \mathbf{M}) is \mathbb{Q} -factorial gdl, then (1) follows directly from [LX22b, Theorem 3.16] (cf. [FS20, Theorem 1.4]) and (2) follows from Lemma 2.7 (3). In the general case we can take a gdl modification $g : (Y, \Delta_Y, \mathbf{M}) \rightarrow (X, \Delta, \mathbf{M})$ and replace \tilde{W} (resp. W, W') by any (resp. minimal) glc centers on Y that dominates W (resp. W, W'). Then the statements follow by looking at the images of those glc centers under g . \square

Theorem 3.2. *Let $(X, \Delta, \mathbf{M})/U$ be a glc pair, and L a \mathbb{Q} -Cartier \mathbb{Q} divisor such that $A := L - (K_X + \Delta + \mathbf{M}_X)$ is nef and log big/ U with respect to (X, Δ, \mathbf{M}) . Assume that L is semi-ample and defines a projective contraction $\phi : X \rightarrow Z$ over U . Then ϕ can also be regarded as a glc crepant log structure $(X, \Delta, \bar{A} + \mathbf{M}) \rightarrow Z$. Let V be any glc center on Z , then there exists a unique minimal glc center W on X such that*

- *W dominates V , or in other words, $\phi(W) = V$.*
- *For any glc center \tilde{W} on X such that $V \subset \phi(\tilde{W})$, we have $W \subset \tilde{W}$.*

Moreover, let $\tilde{W} \subset X$ be any glc center that dominates Z and \tilde{W}^n be the normalization, then $\tilde{W}^n \rightarrow Z$ is a contraction, or equivalently, the Stein factorization of $\tilde{W}^n \rightarrow Z$ is trivial.

Proof. We use induction on the dimension. If $\dim X = 1$, then ϕ is an isomorphism unless $X = \mathbb{P}^1$, for which case the statements are also straightforward.

By shrinking Z we can assume that V is the unique minimal glc center on Z . Let $W \subset X$ be a minimal glc center over Z , which implies $\phi(W) = V$ by our assumption, then it suffices to prove that any other glc centers $\tilde{W} \subset X$ intersects with W , which is equivalent to $W \subset \tilde{W}$ since W is minimal. By Lemma 3.1 (2) we only need to show this for those minimal \tilde{W} that $\phi(\tilde{W}) = V$. We first claim that there is no glc centers \tilde{W} and \hat{W} such that

- $\phi(\tilde{W}) = \phi(\hat{W}) = V$.
- \tilde{W} intersects with \hat{W} .
- $W \subset \hat{W}$ but $\tilde{W} \cap W = \emptyset$.

Indeed, if there is such a \tilde{W} , then by Lemma 2.6 we can assume $\tilde{W} \subset \hat{W}$. Let \hat{W}^n be the normalization of \hat{W} , then by [HL21b, Theorem 1.2] there is a glc structure $(\hat{W}^n, \Delta_{\hat{W}^n}, \mathbf{M}^{\hat{W}^n})$ induced by the subadjunction

$$(K_X + \Delta + \mathbf{M}_X)|_{\hat{W}^n} = K_{\hat{W}^n} + \Delta_{\hat{W}^n} + \mathbf{M}_{\hat{W}^n}^{\hat{W}^n}$$

and the glc centers on \hat{W}^n exactly comes from the pullbacks of glc centers contained in \hat{W} . Let $\hat{W}^n \rightarrow V' \rightarrow V$ be the Stein factorization of $\hat{W}^n \rightarrow V$, then the glc centers dominating V' are exactly those dominating V . Therefore by the induction hypothesis \tilde{W} should contains W (after pullback to \hat{W}^n), which is a contradiction.

Now by Lemma 3.1 (1) we have glc centers W_0, W_1, \dots, W_n and $\hat{W}_1, \dots, \hat{W}_n$ on X such that $W = W_0 \subset \hat{W}_1 \supset W_1 \subset \dots \subset \hat{W}_n \supset W_n = W'$ and $f(W_i) = f(\hat{W}_i) = V$. Thus by keep using the claim above we see that W_i and \hat{W}_i all contain W .

Moreover, we can consider

$$K_F + \Delta_F + \mathbf{M}_F^F = (K_X + \Delta + \mathbf{M}_X)|_F$$

for the general fiber F of ϕ and apply the above statements to $(F, \Delta_F, \mathbf{M}^F)$. Then we can see that $\tilde{W}|_F$ is connected since any irreducible component of it is a glc center, which will contain a unique common minimal glc center. Therefore $\phi|_{\tilde{W}^n} : \tilde{W}^n \rightarrow Z$ must be a contraction. \square

Actually Theorem 3.2 holds in a much more general setting:

Theorem 3.3. *Let $(X, \Delta_1, \mathbf{M}^1)/U$ and $(X, \Delta_2, \mathbf{M}^2)/U$ be two glc \mathbb{Q} -pairs with exactly the same glc places (centers). Assume that $(K_X + \Delta_1 + \mathbf{M}_X^1) - (K_X + \Delta_2 + \mathbf{M}_X^2)$ is log big/ U with respect to $(X, \Delta_1, \mathbf{M}^1)$. Furthermore, $(K_X + \Delta_1 + \mathbf{M}_X^1)$ is semi-ample over U and defines a glc crepant log structure $f : X \rightarrow Z$. Let V be any glc center on Z , then there exists a unique minimal glc center W on X such that*

- W dominates V , or in other words, $\phi(W) = V$.
- For any glc center \tilde{W} on X such that $V \subset \phi(\tilde{W})$, we have $W \subset \tilde{W}$.

Moreover, let $\tilde{W} \subset X$ be any glc center that dominates Z and \tilde{W}^n be the normalization, then $\tilde{W}^n \rightarrow Z$ is a contraction, or equivalently, the Stein factorization of $\tilde{W}^n \rightarrow Z$ is trivial.

Proof. The assumptions preserve under adjunctions, so we can use induction on dimension and the proof now is exactly the same as that of Theorem 3.2. \square

4. GLUING RELATIONS ON GLC CREPANT LOG STRUCTURES

Before giving the proof of Theorem 1.1, we need to make some preparations in order to describe the relations on those glc centers in a clear way. We will keep using the notations in the whole section.

Let $(X, \Delta, \mathbf{M})/U$ be a glc pair and let $f : (Y, \Delta_Y, \mathbf{M}) \rightarrow (X, \Delta, \mathbf{M})$ be a \mathbb{Q} -factorial gdl modification. Let (X, S_*) and (Y, S_*) be the natural glc stratifications. Let $W := \lfloor \Delta_Y \rfloor$ be the boundary $B(Y)$ of (Y, S_*) . Then W is demi-normal and let $\pi : W^n \rightarrow W$ be the normalization. Let D be the double-normal locus on W^n , D^n the normalization of D and $\tilde{\tau} : D^n \rightarrow D^n$ the induced involution. We see that $\tilde{\tau}$ induces a gluing relation $R(\tilde{\tau})$ on (W^n, S_*) , this relation is finite and $W^n/R(\tilde{\tau}) = W$.

Write $W^n = \coprod W_i$, where W_i is irreducible for any i . Let $W_i \xrightarrow{f_i} V_i \xrightarrow{n_i} X$ be the Stein factorization of $f|_{W_i}$. Then we have the contraction:

$$\coprod f_i : \coprod W_i \rightarrow \coprod V_i$$

For any glc center $V_{i,\alpha} \subset V_i$, we define $\tilde{V}_{i,\alpha} := \text{Spr}(V_{i,\alpha}, W_i)$ and let $h_{i,\alpha} : \tilde{V}_{i,\alpha} \rightarrow V_{i,\alpha}$ be the corresponding finite morphism, which is stratified by Lemma 2.7.

Let $W_{i,\alpha,k} \subset W_i$ be a minimal glc center that dominates $V_{i,\alpha}$ and if there is a gluing relation

$$\tilde{\tau}_{i,\alpha,k,T} : W_{i,\alpha,k} \rightarrow T$$

where T is a glc center of W_j for some j . Let $f_j(T) = V_{j,\beta}$, since we have the following commutative diagram:

$$\begin{array}{ccc} W_{i,\alpha,k} & & \\ \tilde{\tau}_{i,\alpha,k,T} \downarrow & \searrow & \\ T & \xrightarrow{\quad} & Y \end{array}$$

We see that $n_j|_{V_{j,\beta}} \circ f_j|_T = f|_T$, $n_i|_{V_{i,\alpha}} \circ f_i|_{W_{i,\alpha,k}} = f|_{W_{i,\alpha,k}}$ and $f|_{W_{i,\alpha,k}} = f|_T \circ \tilde{\tau}_{i,\alpha,k,T}$, thus T is also a minimal glc center that dominates $V_{j,\beta}$ since $W_{i,\alpha,k}$ is a minimal glc center that dominates $V_{i,\alpha}$. Hence $\tilde{\tau}_{i,\alpha,k,T}$ induces an isomorphism $\tau_{i,\alpha,k,T} : \tilde{V}_{i,\alpha} \rightarrow \tilde{V}_{j,\beta}$ as follows:

$$\begin{array}{ccccc} W_{i,\alpha,k} & \xrightarrow{\quad} & \tilde{V}_{i,\alpha} & \xrightarrow{h_{i,\alpha}} & V_{i,\alpha} \\ \tilde{\tau}_{i,\alpha,k,T} \downarrow & & \tau_{i,\alpha,k,T} \downarrow & & \\ T & \xrightarrow{\quad} & \tilde{V}_{j,\beta} & \xrightarrow{h_{j,\beta}} & V_{j,\beta} \end{array}$$

Notice that $\tau_{i,\alpha,k,T}$ is isomorphic on glc centers and gives a relation $\{h_{i,\alpha}(x) \sim h_{j,\beta}(\tau_{i,\alpha,k,T}(x)) \mid x \in \tilde{V}_{i,\alpha}\}$ between $V_{i,\alpha}$ and $V_{j,\beta}$, which is the same as

$$\{f_i(x) \sim f_j(\tilde{\tau}_{i,\alpha,k,T}(x)) \mid x \in W_{i,\alpha,k}\}.$$

If $T' \subset W_i$ is another glc center such that $V_{i,\alpha} \subset f_i(T')$ with a gluing relation $\tilde{\tau}_{T',T''} : T' \rightarrow T''$, where T'' is a glc center on W_j , then by Lemma 3.1(2) we know that there is an minimal glc center $W_{i,\alpha,k} \subset T'$ that dominates $V_{i,\alpha}$. Since $\tilde{\tau}_{T',T''}$ is isomorphic on glc centers, we get a gluing relation $\tilde{\tau}_{i,\alpha,k,T} : W_{i,\alpha,k} \rightarrow T$ by restriction. Let $f_j(T) = V_{j,\beta}$. Then we can easily see that the relation

$$\{f_i(x) \sim f_j(\tilde{\tau}_{T',T''}(x)) \mid x \in f_i^{-1}(V_{i,\alpha}) \cap T'\}$$

between $V_{i,\alpha}$ and $V_{j,\beta}$ is the same as the one given by the $\tau_{i,\alpha,k,T}$ above.

Therefore it suffices to consider all possible $\tau_{i,\alpha,k,T}$ and let $R(\tau)$ be the induced pro-finite relation on $\coprod V_i$. Then $R(\tau)$ will reflect the relation generated by $\tilde{\tau}$ and

$\coprod f_i$. Notice that $W^n/R(\tilde{\tau}) = W$, and all f_i come from f , the relation actually equals to

$$\{f_i(x) \sim f_j(y) \mid x \in W_i, y \in W_j, \pi(x) = \pi(y)\}$$

Since $-(K_Y + \Delta_Y + \mathbf{M}_Y)$ is nef and big over X , we also have $f_*\mathcal{O}_W = \mathcal{O}_{f(W)}$. Then it is not hard to see the pro-finite relation $R(\tau)$ is actually a finite relation (by looking at the geometric points of a fiber) and $f(W) = \coprod V_i/R(\tau)$ is the geometric quotient (see for example [HX13, Propostion 3.12]).

Theorem 4.1. *Let $(X, \Delta, \mathbf{M})/U$ be a glc \mathbb{Q} -pair, and L a nef \mathbb{Q} -divisor such that $A := L - (K_X + \Delta + \mathbf{M}_X)$ is nef and log big/ U with respect to (X, Δ, \mathbf{M}) . Then L is semi-ample over U .*

Proof. There is a natural glc stratification (X, S_*) on X . Then by Lemma 2.4 and [Fuj11, Theorem 13.1] it suffices to prove that $L|_{B(X)}$ is semi-ample. Now we consider the new g-pair $(X, \Delta, \bar{A} + \mathbf{M})$, which induces the same glc stratification, and fix a \mathbb{Q} -factorial gdl modification $f : (Y, \Delta_Y, \bar{A} + \mathbf{M}) \rightarrow (X, \Delta, \bar{A} + \mathbf{M})$.

We use the notations above, by induction on the dimension we can assume that $L|_{V_i}$ is semi-ample and defines a contraction $g_i : V_i \rightarrow Z_i$, so we have

$$\coprod g_i : \coprod V_i \rightarrow \coprod Z_i.$$

Notice that the composition $W_i \rightarrow V_i \rightarrow Z_i$ is a gdl crepant log structure induced by $K_Y + \Delta_Y + \mathbf{M}_Y + f^*A$. Then for any glc center $Z_{i,\gamma} \subset Z_i$, we define $\tilde{Z}_{i,\gamma} := \text{Spr}(Z_{i,\gamma}, W_i)$ and let $p_{i,\gamma} : \tilde{Z}_{i,\gamma} \rightarrow Z_{i,\gamma}$ be the canonical finite stratified morphism.

By Theorem 3.2, there exists a unique minimal glc center $V_{i,\alpha} \subset V_i$ that dominates $Z_{i,\gamma}$. And we have the following diagram:

$$\begin{array}{ccccc} \tilde{V}_{i,\alpha} & \xrightarrow{h_{i,\alpha}} & V_{i,\alpha} & \hookrightarrow & V_i \\ \tilde{g}_{i,\gamma} \downarrow & & \downarrow & & \downarrow g_i \\ \tilde{Z}_{i,\gamma} & \xrightarrow{p_{i,\gamma}} & Z_{i,\gamma} & \hookrightarrow & Z_i \end{array}$$

Then it is easy to see that $\tilde{g}_{i,\gamma}$ is a contraction by considering the gdl crepant log structure $W_i \rightarrow Z_i$. Assume that there is a gluing relation $\tau_{i,\alpha,k,T} : \tilde{V}_{i,\alpha} \rightarrow \tilde{V}_{j,\beta}$. Since the pullback of $L|_W$ from W is indeed $\tau_{i,\alpha,k,T}$ invariant, $V_{j,\beta} \subset V_j$ is also the unique minimal glc center that dominates $Z_{j,\theta} = g_j(V_{j,\beta})$. Hence we have a induced gluing relation $\sigma_{i,\alpha,k,T} : \tilde{Z}_{i,\gamma} \rightarrow \tilde{Z}_{j,\theta}$ as follows:

$$\begin{array}{ccccccc} W_{i,\alpha,k} & \longrightarrow & \tilde{V}_{i,\alpha} & \longrightarrow & \tilde{Z}_{i,\gamma} & \xrightarrow{p_{i,\gamma}} & Z_{i,\gamma} \\ \tilde{\tau}_{i,\alpha,k,T} \downarrow & & \tau_{i,\alpha,k,T} \downarrow & & \sigma_{i,\alpha,k,T} \downarrow & & \\ S & \longrightarrow & \tilde{V}_{j,\beta} & \longrightarrow & \tilde{Z}_{j,\theta} & \xrightarrow{p_{j,\theta}} & Z_{j,\theta} \end{array}$$

This gives a relation

$$\{p_{i,\gamma}(x) \sim p_{j,\theta}(\sigma_{i,\alpha,k,T}(x)) \mid x \in \tilde{Z}_{i,\gamma}\}$$

between $Z_{i,\gamma}$ and $Z_{j,\theta}$. As we have discussed above, the relation induced by $\coprod g_i$ and $R(\tau)$ is generated by all the $\sigma_{i,\alpha,k,T}$, which we use $R(\sigma)$ to denote. Now we want to show that $R(\sigma)$ is a finite relation. It suffices to check on each open stratum of $\tilde{Z}_{i,\gamma}$. Notice that any relation on the open stratum of $\tilde{Z}_{i,\gamma}$ itself is induced by the following form

$$\sigma_{i_0,\gamma_0,i_1,\gamma_1} \circ \cdots \circ \sigma_{i_{n-1},\gamma_{n-1},i_n,\gamma_n},$$

where $\sigma_{i_l, \gamma_l, i_{l+1}, \gamma_{l+1}} : \tilde{Z}_{i_l, \gamma_l} \rightarrow \tilde{Z}_{i_{l+1}, \gamma_{l+1}}$ is the isomorphism induced by some $\tau_{i, \alpha, k, T}$ in $R(\tau)$ and $(i_0, \gamma_0) = (i_n, \gamma_n) = (i, \gamma)$. Then by Theorem 3.2 for each Z_{i_l, γ_l} , there is a unique minimal glc center $V_{i_l, \alpha_l} \subset V_{i_l}$ that dominates Z_{i_l, γ_l} . Therefore this relation lifts to a relation $\tau_{i_0, \alpha_0, i_1, \alpha_1} \circ \cdots \circ \tau_{i_{n-1}, \alpha_{n-1}, i_n, \alpha_n}$ in $R(\tau)$:

$$\begin{array}{ccccccc} \tilde{V}_{i, \alpha} & \xlongequal{\quad} & \tilde{V}_{i_0, \alpha_0} & \xrightarrow{\tau_{i_0, \alpha_0, i_1, \alpha_1}} & \tilde{V}_{i_1, \alpha_1} & \cdots \rightarrow & \tilde{V}_{i_n, \alpha_n} \xlongequal{\quad} \tilde{V}_{i, \alpha} \\ & & \downarrow & & \downarrow & & \downarrow \\ \tilde{Z}_{i, \gamma} & \xlongequal{\quad} & \tilde{Z}_{i_0, \gamma_0} & \xrightarrow{\sigma_{i_0, \gamma_0, i_1, \gamma_1}} & \tilde{Z}_{i_1, \gamma_1} & \cdots \rightarrow & \tilde{Z}_{i_n, \gamma_n} \xlongequal{\quad} \tilde{Z}_{i, \gamma} \end{array}$$

Since the relation $R(\tau)$ is finite, so does $R(\tau)|_{\tilde{V}_{i, \alpha}}$, thus the relation $R(\sigma)$ on the open stratum of $\tilde{Z}_{i, \gamma}$ is also finite. Then by the construction we can see that $R(\sigma)$ is actually a finite, set theoretic, stratified equivalence relation (cf. [LX22b, Lemma 4.14]). Therefore by [Kol13, Theorem 9.21] we know that the geometric quotient $Z := \coprod Z_i / R(\sigma)$ exists and there is a morphism $g : f(W) \rightarrow Z$ over U .

Let m be sufficiently divisible such that mL is Cartier and $M_i := mL|_{V_i}$ defines $g_i : V_i \rightarrow Z_i$ for each i . Then as in [LX22b, Construction 4.13], we can consider the total spaces V_{M_i} of the line bundles M_i over V_i , and the total spaces of Z_{H_i} of the line bundles H_i over Z_i , where H_i is very ample and $g_i^* H_i = M_i$. Similarly, we can define the corresponding relation $R(\tau_M)$ on $\coprod V_{M_i}$ and the induced $R(\sigma_H)$ on $\coprod Z_{H_i}$. Then by applying the same statements above we can deduce that $R(\sigma_H)$ is also a finite, stratified equivalence relation. Possibly by replacing m with a multiple, the geometric quotient $\coprod Z_{H_i} / R(\sigma_H)$ exists by [Kol13, Theorem 9.21] and is the total space of an ample line bundle H_Z over Z , where $g^* H_Z = mL|_{f(W)}$. Therefore $L|_{f(W)}$ is semi-ample over U and we are done. \square

Proof of Theorem 1.4. We use induction on the dimension. By Lemma 2.4 and [Fuj11, Theorem 13.1] (cf. [Amb03, Theorem 7.2]) we only need to show that $\mathcal{O}_V(mL)$ is globally generated over U for $m \gg 0$, where $V = \text{Ngklt}(X, \Delta, \mathbf{M})$. We will use the notations in the proof of Theorem 4.1, by which we know that L is semi-ample over U .

By the induction hypothesis, we know that for any $m \gg 0$, $mL|_{V_i}$ induces the contraction $g_i : V_i \rightarrow Z_i$, thus $L_i := L|_{V_i} = g_i^* K_i$ for some ample line bundle on Z_i . Let V_{L_i} (resp. Z_{K_i}) be total space of the line bundle L_i (resp. K_i) over V_i (resp. Z_i), and $R(\tau_L)$ (resp. $R(\sigma_K)$) the corresponding gluing relation. Then $\coprod V_{L_i} / R(\tau_L)$ is just the total space of the line bundle $L|_V$ over V . Since all the relation $R(\sigma_K)$ could lift to a relation on $R(\tau_L)$, it implies that $\coprod Z_{K_i} / R(\sigma_K)$ is also the total space of an ample line bundle H_Z over Z , where $g^* H_Z = L|_V$. Therefore mH_Z is very ample for $m \gg 0$ by Serre Vanishing and Castelnuovo–Mumford regularity (cf. [Laz04, 1.8.22]), which shows that $mL|_V = g^*(mH_Z)$ is globally generated for all $m \gg 0$. \square

Proof of Theorem 1.2. Since ampleness is an open condition, we can assume that (X, Δ, \mathbf{M}) is a glc \mathbb{Q} -pair by using Shokurov-type rational polytopes (cf. [HL22, Proposition 3.16] and [HLS19, Lemma 5.3]). Then the statement follows by Theorem 4.1. \square

Proof of Theorem 1.3. By the definition of $\mathbf{B}_+(A/U)$ ([BCHM10, Definition 3.5.1]), we can write

$$A \sim_{\mathbb{R}, U} H + E$$

such that H is ample and $E \geq 0$ contains no glc center of $(X, \Delta + A, \mathbf{M})$. Then for any sufficiently small $\epsilon > 0$, $(X, \Delta + A + \epsilon E, \mathbf{M})$ is still glc and its glc centers are

the same as those of $(X, \Delta + A, \mathbf{M})$. Therefore we can let $L = (K_X + \Delta + A + \mathbf{M}_X)$ and consider the glc pair

$$(X, \Delta + (1 - \epsilon)A + \epsilon E, \mathbf{M}),$$

then the statement follows from Theorem 1.2. \square

Proof of Theorem 1.5. By using Shokurov-type rational polytopes (cf. [HL22, Proposition 3.16] and [HLS19, Lemma 5.3]) we can assume that (X, Δ, \mathbf{M}) is a glc \mathbb{Q} -pair. Since R is an $(K_X + \Delta + \mathbf{M}_X)$ -negative extremal ray, there exists an ample \mathbb{Q} -divisor A such that $H := K_X + \Delta + \mathbf{M}_X + A$ is nef and $R = \overline{\text{NE}}(X/U) \cap H^\perp$. Then by Theorem 4.1 H is semi-ample and defines a contraction $\varphi_H : X \rightarrow Z$ over U . Let L be a line bundle on X such that $L \cdot R = 0$. Then both $H - (K_X + \Delta + \mathbf{M}_X)$ and $L + H - (K_X + \Delta + \mathbf{M}_X)$ is ample over Z . Hence by Theorem 1.4 $\mathcal{O}_X(mL + mH)$ and $\mathcal{O}_X(mH)$ are both globally generated over Z for $m \gg 0$. Then it is not hard to see that

$$\mathcal{O}_X(mL + mH) \sim_{\varphi_H} \mathcal{O}_X(mH) \sim_{\varphi_H} \mathcal{O}_X$$

for all $m \gg 0$, so $\mathcal{O}_X(mL) \sim_{\varphi_H} \mathcal{O}_X$ for all $m \gg 0$ and therefore

$$\mathcal{O}_X(L) = \mathcal{O}_X((m+1)L) \otimes \mathcal{O}_X(-mL)$$

is a pullback of a line bundle on Z . \square

Proof of Theorem 1.6. If $K_X + \Delta + \mathbf{M}_X$ is not nef, then by [HL21a, Theorem 1.3] there exists a $(K_X + \Delta + \mathbf{M})$ -negative extremal ray R in $\overline{\text{NE}}(X/U)$. Now by Theorem 1.5 we have a projective contraction $f : X \rightarrow Z$ contracting R .

If $\dim X > \dim Z$, then $f : X \rightarrow Z$ is a Mori fiber space and we are done. If f is birational then by [LX22b, Theorem 1.1] we know there exist a canonical model

$$X' := \text{Proj}_Z \oplus_{m \geq 0} f_* \mathcal{O}_X(m(K_X + \Delta + \mathbf{M}_X))$$

over Z . In particular, $K_{X'} + \Delta' + \mathbf{M}_{X'}$ is \mathbb{R} -Cartier, where Δ' is the birational transform. Then we can do the same thing to $(X', \Delta', \mathbf{M})$ and keep running the MMP (see [Fuj17, Section 4.9] for more details). \square

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