RELATIVE NAKAYAMA-ZARISKI DECOMPOSITION AND MINIMAL MODELS OF GENERALIZED PAIRS

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Abstract. We prove some basic properties of the relative Nakayama-Zariski decomposition. We apply them to the study of lc generalized pairs. We prove the existence of log minimal models or Mori fiber spaces for (relative) lc generalized pairs polarized by an ample divisor. This extends a result of Hashizume-Hu to generalized pairs. We also show that, for any lc generalized pair $(X,B+A,\mathbf{M})/Z$ such that $K_X+B+A+\mathbf{M}_X\sim_{\mathbb{R},Z}0$ and $B\geq 0, A\geq 0$, $(X,B,\mathbf{M})/Z$ has either a log minimal model or a Mori fiber space. This is an analogue of a result of Birkar/Hacon-Xu and Hashizume in the category of generalized pairs, and is later shown to be crucial to the proof of the existence of lc generalized flips in full generality.

Contents

1. Introduction

We work over the field of complex numbers $\mathbb{C}$.

The theory of *generalized pairs* (*g-pairs* for short) was introduced by Birkar and Zhang in [BZ16] to tackle the effective Iitaka fibration conjecture. The structure of g-pairs naturally appears in the canonical bundle formula and sub-adjunction formulas [Kaw98, FM00]. This theory has been used in an essential way in the proof of the Borisov-Alexeev-Borisov conjecture [Bir19, Bir21a]. We refer the reader to [Bir21b] for a more detailed introduction to the theory of g-pairs.

Recently, there is significant progress towards the minimal model program theory for generalized pairs. In particular, in [HL21a], Hacon and the first author proved the cone theorem, contraction theorem, and the existence of flips for $\mathbb{Q}$-factorial lc g-pairs. However, some related results on the termination of flips and the existence of log minimal models and good minimal

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models for generalized pairs remain unknown. For example, we have the following results in the setting of usual pairs:

Theorem 1.1 ([HH20, Theorem 1.5])**.** *Let $(X,B)/Z$ be a pair and $A\geq 0$ an ample/Z $\mathbb{R}$-divisor such that $(X,\Delta:=B+A)$ is lc and Z is normal quasi-projective. Then $(X,\Delta)/Z$ has a good minimal model or a Mori fiber space.*

Theorem 1.2 ([Has19, Theorem 1.1]; see [Bir12, HX13] for the $\mathbb{Q}$-coefficient case)**.** *Let $(X,B)/Z$ be a pair and $A\geq 0$ an $\mathbb{R}$-divisor such that $(X,B+A)$ is lc, Z is normal quasi-projective, and $K_X+B+A\sim_{\mathbb{R},Z}0$. Then:*

(1) $(X,B)/Z$ has either a Mori fiber space or a log minimal model $(Y,B_Y)/Z$.
(2) If K_Y+B_Y is nef/Z, then K_Y+B_Y is semi-ample/Z.
(3) If (X,B) is $\mathbb{Q}$-factorial dlt, then any (K_X+B)-MMP/Z with scaling of an ample/Z $\mathbb{R}$-divisor terminates.

In this paper, we further investigate the minimal model program for generalized pairs. We prove the following results, which can be considered as analogues of Theorems 1.1 and 1.2 respectively:

Theorem 1.3. *Let $(X,B,\mathbf{M})/U$ be an NQC lc g-pair and $A\geq 0$ an ample/U $\mathbb{R}$-divisor such that $(X,\Delta:=B+A,\mathbf{M})$ is lc. Then*

(1) $(X,\Delta,\mathbf{M})/U$ has a log minimal model or a Mori fiber space, and
(2) if $\mathbf{M}_X$ is $\mathbb{R}$-Cartier, then $(X,\Delta,\mathbf{M})/U$ has a good minimal model or a Mori fiber space.

Theorem 1.4. *Let $(X,B,\mathbf{M})/U$ be an NQC lc g-pair such that $X\to U$ is a projective morphism between normal quasi-projective varieties, and $A\geq 0$ an $\mathbb{R}$-divisor such that $(X,B+A,\mathbf{M})$ is lc and $K_X+B+A+\mathbf{M}_X\sim_{\mathbb{R},U}0$. Then*

(1) $(X,B,\mathbf{M})/U$ has a log minimal model or a Mori fiber space, and
(2) if $(X,B,\mathbf{M})$ is $\mathbb{Q}$-factorial dlt, then any $(K_X+B+\mathbf{M}_X)$-MMP/U with scaling of an ample/U $\mathbb{R}$-divisor terminates.

Theorems 1.3 and 1.4 have played important roles in the minimal model program theory for lc generalized pairs, especially the existence of generalized lc flips. See the Postscript for details.

Note that when $\mathbf{M}=\mathbf{0}$, Theorem 1.3 is exactly Theorem 1.1 and Theorem 1.4 is exactly Theorem 1.2(1)(3). For technical reasons, at the moment, we cannot remove the "$\mathbf{M}_X$ is $\mathbb{R}$-Cartier" assumption in Theorem 1.3(2).

We still expect the analogue of Theorem 1.2(2) to be true. That is, we expect that any log minimal model of $(X,B,\mathbf{M})/Z$ is a good minimal model (of) a generalized pair $(X,B,\mathbf{M})/Z$ as in Theorem 1.4 is in fact good; see the first paragraph of the Postscript. This is because such $K_X+B+\mathbf{M}_X$ is log abundant/U with respect to $(X,B,\mathbf{M})$ by Theorem 7.3 below. However, the following example shows that the question is very subtle as "log abundance" does not imply semi-ampleness in general for lc g-pairs:

Example 1.5. Let C_0 be a nodal cubic in $\mathbb{P}^2$ and l the hyperplane class on $\mathbb{P}^2$. Let $P_1,P_2,\dots,P_{12}$ be twelve distinct points on C_0 which are different from the nodal point. Let

$$\mu: X=\mathrm{Bl}_{\{P_1,\dots,P_{12}\}}\to\mathbb{P}^2$$

be the blow-up of $\mathbb{P}^2$ at the chosen points with the exceptional divisor $E=\sum_{i=1}^{12}E_i$, where E_i is the prime exceptional divisor over P_i for each i. Let $H:=\mu^*l$ and $C:=\mu^{-1}_*C_0$. Then $C\cong C_0$, $C\in|3H-E|$, and $K_X+C=\mu^*(K_{\mathbb{P}^2}+C_0)=0$. Moreover, we have $C_0^2=9$ and $C^2=-3$.

We consider the big divisor $M=4H-E\sim H+C$. Since H is semi-ample and $M\cdot C=0$, M is nef. Notice that $\mathcal{O}_C(M)=\mathcal{O}_{C_0}(4l-\sum_{i=1}^{12}P_i)$ and $\mathrm{Pic}^0(C)\cong\mathbb{G}_m$, where $\mathbb{G}_m$ is the multiplication group of $\mathbb{C}^*$.

(1) Suppose that $P_1,\dots,P_{12}$ are in general position so that $\mathcal{O}_C(M)$ is a non-torsion line bundle in $\mathrm{Pic}^0(C)$. Then M can never be semi-ample since $M|_C$ is not. However, the

normalization C^n of C is $\mathbb{P}^1$, so $M|_{C^n}$ is semi-ample. This gives an lc g-pair $(X, C, \mathbf{M} := \overline{M})$ such that both M and $K_X + C + M \sim M$ are nef and log abundant with respect to $(X, C, \mathbf{M})$, but $K_X + C + M$ is not semi-ample. One can further take the blow-up of the nodal point and take the crepant pullback to make each lc center normal.

(2) Suppose that $P_1, ..., P_{12}$ are the intersection points of C_0 with a general quartic curve $Q_0 \in |4l|$. Let Q be the birational transform of Q_0 on X. Then $M \sim Q \sim H + C$ is semi-ample and defines a projective birational contraction $f : X \to Y$ which contracts exactly the nodal curve C. Let $M' = H - 3E_1$. Then $M' \cdot C = 0$ and $\mathcal{O}_C(M') = \mathcal{O}_{C_0}(l - 3P_1)$ is a non-torsion line bundle since Q_0 is general. Therefore M' is not $\mathbb{Q}$-linearly equivalent to 0 over Y (which also implies that $f(M')$ is not $\mathbb{Q}$-Cartier). This gives an lc g-pair $(X, C, \mathbf{M}' := \overline{M'})/Y$ such that both M' and $K_X + C + M' \sim M'$ are log abundant and numerically trivial over Y but $K_X + C + M'$ is not semi-ample over Y.

We refer the reader to [BH22] for some other interesting examples on the failure of positivity results for generalized pairs.

To prove our main theorems, the central idea is to combine the methods in [Has22a] (some originated in [Has20, Has22b, HH20]) and [HL21a]. In particular, we need to generalize many results in [Has22a] for projective varieties X to normal quasi-projective varieties X equipped with projective morphisms $\pi : X \to U$. Despite their similarities, a major difficulty is the use of the Nakayama-Zariski decomposition [Nak04, III. §1], which is usually applied to projective varieties only. It is important to remark that the relative Nakayama-Zariski decomposition [Nak04, III. §4] does not always behave as good as the global Nakayama-Zariski decomposition (see [Les16]), and we lack references for even the most basic properties of them. In this note, we will study the behavior and basic properties of the relative Nakayama-Zariski decomposition. We refer the reader to [LT22b] for further applications of the relative Nakayama-Zariski decomposition on the minimal model theory for generalized pairs.

Idea of the proof. It is important to notice that Theorems 1.3 and 1.4 both have some "***b***-log abundant" conditions:

(1) In Theorem 1.3, possibly replacing $(X, B, \mathbf{M})$ with $(X, B, \mathbf{M} + \frac{1}{2}\bar{A})$ and A with $\frac{1}{2}A$, we may assume that $\mathbf{M}$ is ***b***-log abundant with respect to $(X, B, \mathbf{M})$.

(2) In Theorem 1.4, $K_X + B + A + \mathbf{M}_X$ is automatically ***b***-log abundant$/Z$ as it is $\mathbb{R}$-linearly trivial over Z.

Therefore, one important goal of this paper is to study the minimal model program for g-pairs $(X, B, \mathbf{M})$ with ***b***-log abundant nef part $\mathbf{M}$ or with log abundant $K_X + B + \mathbf{M}_X$. Despite the technicality, the condition "***b***-log abundant" is actually a very natural condition as it is preserved under adjunction. The key idea to study the minimal model program for such g-pairs is the following:

- By applying the Iitaka fibration and the generalized canonical bundle formula, we reduce the questions to the cases when either $\kappa_\iota(X/U, K_X + B + \mathbf{M}_X) = 0$ or $\kappa_\iota(X/U, K_X + B + \mathbf{M}_X) = \dim X - \dim U$ (see Section 4).
- When the invariant Iitaka dimension is 0, by abundance, the minimal model program behaves well (cf. Lemma 4.1). So we can reduce the question to the case when $K_X + B + \mathbf{M}_X$ is big$/U$.
- If $(X, B, \mathbf{M})$ is klt then we can apply [BZ16, Lemma 4.4(2)]. Otherwise, by induction on dimension, we can apply special termination results near $\mathrm{Nklt}(X, B, \mathbf{M})$.

Structure of the paper. In Section 2, we introduce some preliminary results. In particular, we will recall some results on the minimal model program for generalized pairs that are already included in [HL21a, Version 2, Version 3] (but may not appear in the published version). In Section 3, we study the basic behavior of the relative Nakayama-Zariski decomposition. In Section 4, we use the Iitaka fibration and the generalized canonical bundle formula to simplify the question. In Section 5,6 and 7, we use the relative Nakayama-Zariski decomposition to prove analogues of most results in [Has22a, Section 3] (Section 5), [Has22a, Theorem 3.14] (Section 6), and [Has22b, Theorem 4.1] (Section 7) respectively. In Section 8, we prove Theorems 1.3 and 1.4.

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Some parts of this note has overlap with results in [HL21a, Version 2 or Version 3]. Since these results are not expected to be published in the final version of [HL21a] due to the simplification of the proofs of the main theorems of [HL21a], for the reader's convenience, we include some of the results of [HL21a, Version 2 or Version 3] in this paper and provide detailed proofs. The authors would like to thank Christopher D. Hacon for granting the text overlap.

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Postscript. After the first version of the paper appeared on the arXiv, the authors proved a stronger version of Theorem 1.4 in [LX22, Theorem 1.1], which shows that the log minimal model $(Y, B_Y, \mathbf{M})/Z$ is essentially a good minimal model of $(X, B, \mathbf{M})/Z$. This is crucial for the complete solution of the existence of flips for lc generalized pairs [LX22, Theorem 1.2], which removes the $\mathbb{R}$-Cartier assumption of $\mathbf{M}_X$ as in [HL21a, Theorem 1.2]. Although the proof of [LX22] heavily relies on this paper, we decided to write and submit them as two separate papers, as this paper contains most technical results that we need while [LX22] mainly focuses on establishing a Kollár-type gluing theory.

We also remark that the second author and N. Tsakanikas proved a stronger version of Theorem 1.3, removing the $\mathbb{R}$-Cartierness assumption of $\mathbf{M}_X$ in Theorem 1.3(2), see [TX23, Theorem F]. The proof of [TX23, Theorem F] relies on [LX22, Xie22] which in turn rely on this paper. Therefore, we will avoid citing results from [LX22, Xie22, TX23] in this paper.

2. Preliminaries

We adopt the same notation as in [KM98, BCHM10]. For g-pairs, we adopt the same notation as in [HL21a], which is the same as [FS20, Has22a] except that we use "$a(E, X, B, \mathbf{M})$ instead of "$a(E, X, B + \mathbf{M}_X)$" to represent log discrepancies. This is because $(X, B + \mathbf{M}_X)$ is a sub-pair and the log discrepancies of this sub-pair may be different from the log discrepancies of the generalized pair $(X, B, \mathbf{M})$.

2.1. Equidimensional reduction.

Theorem 2.1. *Let (X, B) be a dlt pair and $\pi : X \to U$ a projective surjective morphism over a normal variety U. Then there exists a commutative diagram of projective morphisms*

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ V & \xrightarrow{\varphi} & U \end{array}$$

such that

(1) *f, φ are birational morphisms, π' is an equidimensional contraction, Y only has $\mathbb{Q}$-factorial toroidal singularities, and V is smooth, and*
(2) *there exist two $\mathbb{R}$-divisors B_Y and E on Y, such that*
    (a) *$K_Y + B_Y = f^*(K_X + B) + E$,*

(b) $B_Y \geq 0$, $E \geq 0$, *and* $B_Y \wedge E = 0$,

(c) (Y, B_Y) *is lc quasi-smooth, and any lc center of* (Y, B_Y) *on* X *is an lc center of* (X, B).

Proof. This result follows from [AK00], see also [Hu20, Theorem B.6], [Kaw15, Theorem 2] and [Has19, Step 2 of the proof of Lemma 3.1]. □

2.2. **Iitaka dimensions.** We refer the readers to [HH20, Section 2] for the formal definitions and basic properties of $\kappa_\sigma(X/U, D)$ and $\kappa_\iota(X/U, D)$.

Lemma 2.2 (cf. [Nak04, V. 2.6(5) Remark]). *Let* X *be a normal projective variety and* D *an* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor on* X *such that* $\kappa_\sigma(D) \geq 0$. *Then* D *is pseudo-effective.*

Proof. By definition, there exists a Cartier divisor A on X such that $\sigma(D; A) \geq 0$. In particular, there exists a sequence of strictly increasing positive integers m_i, such that $\dim H^0(X, \lfloor m_i D \rfloor + A) > 0$, hence $\lfloor m_i D \rfloor + A$ is effective for any i. Thus $m_i D + A$ is effective for any i, hence $D + \frac{1}{m_i} A$ is effective for any i. Thus D is the limit of the effective $\mathbb{R}$-divisors $D + \frac{1}{m_i} A$, hence D is pseudo-effective. □

Lemma 2.3. *Let* $\pi : X \to U$ *be a projective morphism from a normal variety to a variety, and* D *an* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor on* X. *Then:*

(1) D *is big/*U *if and only if* $\kappa_\sigma(X/U, D) = \dim X - \dim U$.

(2) Let D_1, D_2 *be two* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisors on* X. *Suppose that* $D_1 \sim_{\mathbb{R},U} E_1 \geq 0$ *and* $D_2 \sim_{\mathbb{R},U} E_2 \geq 0$ *for some* $\mathbb{R}$*-divisors* E_1, E_2 *such that* $\operatorname{Supp} E_1 = \operatorname{Supp} E_2$. *Then* $\kappa_\sigma(X/U, D_1) = \kappa_\sigma(X/U, D_2)$ *and* $\kappa_\iota(X/U, D_1) = \kappa_\iota(X/U, D_2)$.

(3) Let $f : Y \to X$ *be a surjective birational morphism and* D_Y *an* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor on* Y *such that* $D_Y = f^*D + E$ *for some* f*-exceptional* $\mathbb{R}$*-divisor* $E \geq 0$. *Then* $\kappa_\sigma(Y/U, D_Y) = \kappa_\sigma(X/U, D)$ *and* $\kappa_\iota(Y/U, D_Y) = \kappa_\iota(X/U, D)$.

(4) Let $g : Z \to X$ *be a surjective morphism from a normal variety such that* Z *is projective over* U. *Then* $\kappa_\sigma(Z/U, g^*D) = \kappa_\sigma(X/U, D)$ *and* $\kappa_\iota(Z/U, g^*D) = \kappa_\iota(X/U, D)$.

(5) Let $\bar{D}$ *be an* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor on* X *such that* $D \equiv_U \bar{D}$. *Then* $\kappa_\sigma(X/U, D) = \kappa_\sigma(X/U, \bar{D})$.

(6) Let $\phi : X \dashrightarrow X'$ *be a partial* D*-MMP/*U *and let* $D' := \phi_* D$. *Then* $\kappa_\sigma(X/U, D) = \kappa_\sigma(X'/U, D')$ *and* $\kappa_\iota(X/U, D) = \kappa_\iota(X'/U, D')$

Proof. For (1)-(5), let F be a very general fiber of the Stein factorization of π. Possibly replacing X with F, U with $\{pt\}$, and $D, D_1, D_2, \bar{D}$ with $D|_F, D_1|_F, D_2|_F, \bar{D}|_F$ respectively, we may assume that X is projective and $U = \{pt\}$. (2) follows from [HH20, Remark 2.8(1)] and (3)(4) follow from [HH20, Remark 2.8(2)].

To prove (1)(5), let $h : \tilde{X} \to X$ be a resolution of X. By (4), we may replace X with $\tilde{X}$, D with h^*D, and $\bar{D}$ with $h^*\bar{D}$, and assume that X is smooth.

If D is big, then $\kappa_\sigma(D) = \dim X$ by definition. If $\kappa_\sigma(D) = \dim X$, then D is pseudo-effective by Lemma 2.2, hence D is big by [Nak04, V. 2.7(3) Proposition]. This gives (1).

To prove (5), notice that D is pseudo-effective if and only if $\bar{D}$ is pseudo-effective. If D is not pseudo-effective, then $\kappa_\sigma(D) = \kappa_\sigma(\bar{D}) = -\infty$ by Lemma 2.2. If D is pseudo-effective, then (5) follows from [Nak04, V. 2.7(1) Proposition].

To prove (6), let $p : W \to X$ and $q : W \to X'$ be a common resolution such that $q = \phi \circ p$. Then $p^*D = q^*D' + F$ for some $F \geq 0$ that is q-exceptional. By (3), we have

$$\kappa_\sigma(X/U, D) = \kappa_\sigma(W/U, p^*D) = \kappa_\sigma(W/U, q^*D' + F) = \kappa_\sigma(X'/U, D')$$

and

$$\kappa_\iota(X/U, D) = \kappa_\iota(W/U, p^*D) = \kappa_\iota(W/U, q^*D' + F) = \kappa_\iota(X'/U, D').$$ □

Lemma 2.4. *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be two contractions between normal quasi-projective varieties such that general fibers of* $Y \to Z$ *are smooth and* Y *is* $\mathbb{Q}$*-Gorenstein. Let* (X, B) *be a* *pair that is lc over a non-empty open subset of Y. Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $D-(K_{X/Y}+B)$ is nef/Z. Then for any $\mathbb{R}$-Cartier $\mathbb{R}$-divisor Q on Y, we have*

$$\kappa_\sigma(X/Z, D+f^*Q) \geq \kappa_\sigma(X/Y, D) + \kappa(Y/Z, Q).$$

Proof. Let $z \in Z$ be a very general point and let $X_z := (g\circ f)^{-1}(z), Y_z := g^{-1}(z)$ be the fibers of X and Y over z respectively. We have an induced contraction $f_z : X_z \to Y_z$. Let F be a very general fiber of f_z. Then F is also a very general fiber of f.

First assume that $\dim Y > \dim Z$. By our assumption, Y_z is smooth, $(X_z, B|_{X_z})$ is lc over a non-empty open subset of Y_z, and

$$D|_{X_z} - (K_{X_z/Y_z} + B|_{X_z}) = (D-(K_{X/Y}+B))|_{X_z}$$

is nef. By [Fuj20, (3.3)],

$$\begin{aligned}\kappa_\sigma(X/Z, D+f^*Q) &= \kappa_\sigma(X_z, D|_{X_z} + f_z^*Q|_{Y_z}) \geq \kappa_\sigma(X_z/Y_z, D|_{X_z}) + \kappa(Y_z, Q|_{Y_z})\\ &= \kappa_\sigma(F, D|_F) + \kappa(Y/Z, Q) = \kappa_\sigma(X/Y, D) + \kappa(Y/Z, Q).\end{aligned}$$

Now assume that $\dim Y = \dim Z$ so that $\kappa(Y/Z, Q) = 0$. If $\dim X = \dim Y$ then there is nothing left to prove, so we may assume that $\dim X > \dim Y$. In this case $f^*Q|_{X_z} = 0$, so we have

$$\begin{aligned}\kappa_\sigma(X/Z, D+f^*Q) &= \kappa_\sigma(X_z, D|_{X_z} + f^*Q|_{X_z}) = \kappa_\sigma(X_z, D|_{X_z}) = \kappa_\sigma(X/Z, D)\\ &= \kappa_\sigma(X/Y, D) = \kappa_\sigma(X/Y, D) + \kappa(Y/Z, Q),\end{aligned}$$

and we are done. □

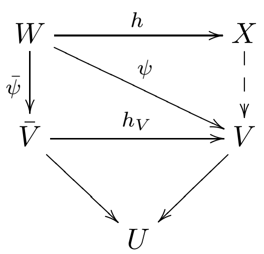
Lemma 2.5. *Let $(X, B, \mathbf{M})/U$ be an lc g-pair such that $K_X+B+\mathbf{M}_X \equiv_U G$ for some $\mathbb{R}$-divisor $G \geq 0$, such that U is quasi-projective and G is abundant over U. Let $X \dashrightarrow V$ be the Iitaka fibration over U associated to G, and $(W, B_W, \mathbf{M})$ a log smooth model of $(X, B, \mathbf{M})$ such that the induced map $\psi : W \to V$ is a morphism over U. Then*

(1) $\kappa_\sigma(W/U, K_W + B_W + \mathbf{M}_W) = \dim V - \dim U$, and
(2) $\kappa_\sigma(W/V, K_W + B_W + \mathbf{M}_W) = 0$.

Proof. Let $h_V : \bar{V} \to V$ be a resolution of V. By Lemmas 2.3(3) and [HL21a, Lemma 3.6] possibly replacing $(W, B_W, \mathbf{M})/U$ with a higher model, we may assume that the induced map $\bar{\psi} : W \to \bar{V}$ is a morphism. Since $(W, B_W, \mathbf{M})$ a log smooth model of $(X, B, \mathbf{M})$, we have

$$K_W + B_W + \mathbf{M}_W = h^*(K_X + B + \mathbf{M}_X) + E$$

where $h : W \to X$ is the induced morphism, $\mathbf{M}$ descends to W, and $E \geq 0$.



Since $G \geq 0$ is abundant over U, by [Cho08, Proposition 2.2.2(1)],

$$\dim V - \dim U = \kappa(X/U, G) = \kappa_\iota(X/U, G) = \kappa_\sigma(X/U, G) \geq 0.$$

Since $X \dashrightarrow V$ is the Iitaka fibration associated to G over U, there exists an effective ample/U $\mathbb{R}$-divisor A on V and an $\mathbb{R}$-divisor $F \geq 0$ on W such that $h^*G = \psi^*A + F$ for some h-exceptional $\mathbb{R}$-divisor $F \geq 0$ on W. Then for any real number k, we have

$$K_W + B_W + \mathbf{M}_W + k\psi^*A \equiv_U (1+k)\psi^*A + E + F.$$

By Lemma 2.3(2)(3)(5), for any $k \geq 0$ we have

$$\begin{aligned}\kappa_\sigma(W/U, K_W + B_W + \mathbf{M}_W + k\psi^*A) &= \kappa_\sigma(W/U, (1+k)\psi^*A + E + F) = \kappa_\sigma(W/U, \psi^*A + E + F)\\ &= \kappa_\sigma(W/U, K_W + B_W + \mathbf{M}_W) = \kappa_\sigma(X/U, K_X + B + \mathbf{M}_X)\\ &= \kappa_\sigma(X/U, G) = \kappa(X/U, G) = \dim V - \dim U.\end{aligned}$$

In particular, we get (1). Since A is ample/U, h_V^*A is big/U, and we may pick a sufficiently large positive integer k such that $K_{\bar{V}} + kh_V^*A$ is big/U.

Since $(W, B_W, \mathbf{M})$ is a log smooth model of $(X, B, \mathbf{M})$, (W, B_W) is lc. Since $\bar{V}$ is smooth, any very general fiber of the induced morphism $\bar{V} \to U$ is smooth. Let $D := K_W + B_W + \mathbf{M}_W - \bar{\psi}^*K_{\bar{V}}$ and $Q := K_{\bar{V}} + kh_V^*A$. Then $D - (K_{W/\bar{V}} + B_W) = \mathbf{M}_W$ is nef/U. By Lemma 2.4 and noticing that the restriction of $\bar{\psi}^*K_{\bar{V}}$ to a general fiber of $\bar{\psi}$ is zero, we have

$$\begin{aligned}\dim V - \dim U &= \kappa_\sigma(W/U, K_W + B_W + \mathbf{M}_W + k\psi^*A) = \kappa_\sigma(W/U, K_W + B_W + \mathbf{M}_W + k\bar{\psi}^*h_V^*A)\\ &= \kappa_\sigma(W/U, D + \bar{\psi}^*Q) \geq \kappa_\sigma(W/\bar{V}, D) + \kappa(\bar{V}/U, Q)\\ &= \kappa_\sigma(W/\bar{V}, K_W + B_W + \mathbf{M}_W - \bar{\psi}^*K_{\bar{V}}) + \kappa(\bar{V}/U, K_{\bar{V}} + kh_V^*A)\\ &= \kappa_\sigma(W/\bar{V}, K_W + B_W + \mathbf{M}_W) + (\dim V - \dim U).\end{aligned}$$

Thus $\kappa_\sigma(W/\bar{V}, K_W + B_W + \mathbf{M}_W) \leq 0$, hence $\kappa_\sigma(W/V, K_W + B_W + \mathbf{M}_W) \leq 0$. Since $K_W + B_W + \mathbf{M}_W \equiv_U h^*G + E \geq 0$, $\kappa_\sigma(W/V, K_W + B_W + \mathbf{M}_W) \geq 0$. Thus $\kappa_\sigma(W/V, K_W + B_W + \mathbf{M}_W) = 0$, and we get (2). □

2.3. Preliminaries on the MMP for generalized pairs.

Lemma 2.6 ([HL21a, Lemma 2.20], cf. [HL22, Proposition 3.9]). *Let $(X, B, \mathbf{M})/U$ be a $\mathbb{Q}$-factorial lc g-pair such that X is klt and $K_X + B + \mathbf{M}_X \equiv_U D_1 - D_2$ (resp. $\sim_{\mathbb{R},U} D_1 - D_2$) where $D_1 \geq 0$, $D_2 \geq 0$ have no common components. Suppose that D_1 is very exceptional over U (see* [Bir12, Definition 3.1]*). Then any $(K_X + B + \mathbf{M}_X)$-MMP/U with scaling of an ample/U $\mathbb{R}$-divisor either terminates with a Mori fiber space or contracts D_1 after finitely many steps. Moreover, if $D_2 = 0$, then this MMP terminates with a model Y such that $K_Y + B_Y + \mathbf{M}_Y \equiv_U 0$ (resp. $\sim_{\mathbb{R},U} 0$), where B_Y is the strict transform of B on Y.*

Lemma 2.7 ([HL21a, Lemma 2.25]). *Let $X \to U$ be a projective morphism such that X is normal quasi-projective. Let D, A be two $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on X and let $\phi : X \dashrightarrow X'$ be a partial D-MMP/U. Then there exists a positive real number t_0, such that for any $t \in (0, t_0]$, ϕ is also a partial $(D + tA)$-MMP/U. Note that A is not necessarily effective.*

Proof. We let

$$X := X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n = X'$$

be this partial MMP, and D_i, A_i the strict transforms of D and A on X_i respectively. Let $X_i \to Z_i$ be the D_i-negative extremal contraction of a D_i-negative extremal ray R_i in this MMP for each i. Then $D_i \cdot R_i < 0$ for each i. Thus there exists a positive real number t_0 such that $(D_i + t_0A_i) \cdot R_i < 0$ for each i. In particular, $(D_i + tA_i) \cdot R_i < 0$ for any i and any $t \in (0, t_0]$. Thus ϕ is a partial $(D + tA)$-MMP/U for any $t \in (0, t_0]$. □

Lemma 2.8 (cf. [LT22a, Lemma 2.17]). *Let $(X, B, \mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC lc g-pair such that X is klt and $K_X + B + \mathbf{M}_X$ is pseudo-effective/U. Let $A \geq 0$ be an ample/U $\mathbb{R}$-divisor on X such that $(X, B + A, \mathbf{M})$ is lc and $K_X + B + A + \mathbf{M}_X$ is nef/U. Let*

$$(X, B, \mathbf{M}) := (X_0, B_0, \mathbf{M}) \dashrightarrow (X_1, B_1, \mathbf{M}) \dashrightarrow \cdots \dashrightarrow (X_i, B_i, \mathbf{M}) \dashrightarrow \dots$$

be a $(K_X + B + \mathbf{M}_X)$-MMP/U with scaling of A, and A_i the strict transform of A on X_i for each i. Then there exists a positive integer n and a positive real number ϵ_0, such that $K_{X_j} + B_j + \epsilon A_j + \mathbf{M}_{X_j}$ is movable/U for any $\epsilon \in (0, \epsilon_0)$ and $j \geq n$. In particular, $K_{X_j} + B_j + \mathbf{M}_{X_j}$ is a movable/U (cf. Definition 3.1) $\mathbb{R}$-divisor for any $j \geq n$.

Proof. Let λ_i be the i-th scaling number of this MMP for each i, i.e.

$$\lambda_i := \inf\{t \geq 0 \mid K_{X_i} + B_i + tA_i + \mathbf{M}_{X_i} \text{ is nef}/U\}.$$

We may assume that this MMP does not terminate. By [HL21a, Theorem 2.24], we have $\lim_{i\to+\infty} \lambda_i = 0$.

Let n be the minimal positive integer such that $X_i \dashrightarrow X_{i+1}$ is a flip for any $i \geq n$. If $\lambda_i < \lambda_{i-1}$, then $X \dashrightarrow X_i$ is a $(K_X + B + tA + \mathbf{M}_X)$-MMP$/U$ with scaling of $(1-t)A$ for any $t \in [\lambda_i, \lambda_{i-1})$. Since X is $\mathbb{Q}$-factorial klt, there exists $\Delta_t \sim_{\mathbb{R},U} B + tA + \mathbf{M}_X$ such that (X, Δ_t) is klt and Δ_t is big for any $t \in (0,1]$. By [BCHM10, Corollary 3.9.2], $K_{X_i} + B_i + tA_i + \mathbf{M}_{X_i}$ is semi-ample$/U$ for any i and any $t \in [\lambda_i, \lambda_{i-1})$. Let $\epsilon_0 := \lambda_n$. Then for any $\epsilon \in (0, \epsilon_0)$, there exists $i \geq n$ such that $\lambda_i < \lambda_{i-1}$ and $\epsilon \in [\lambda_i, \lambda_{i-1})$, and $K_{X_i} + B_i + \epsilon A_i + \mathbf{M}_{X_i}$ is semi-ample$/U$. Since $X_i \dashrightarrow X_j$ is small for any $i, j \geq n$, $K_{X_j} + B_j + \epsilon A_j + \mathbf{M}_{X_j}$ is movable$/U$ for any $j \geq n$ and $\epsilon \in (0, \epsilon_0)$, and $K_{X_j} + B_j + \mathbf{M}_{X_j}$ is a $/U$ $\mathbb{R}$-divisor. □

Lemma 2.9. *Let $X \to U$ be a projective morphism such that X is quasi-projective. Assume that D is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that D is a movable$/U$ $\mathbb{R}$-divisor on X, and let $\phi : X \dashrightarrow X'$ be a partial D-MMP$/U$. Then ϕ only contains flips.*

Proof. Since D is a movable$/U$ $\mathbb{R}$-divisor, D is pseudo-effective$/U$, so ϕ only contains flips and divisorial contractions.

If ϕ contains a divisorial contraction, let $\psi : X_1 \to X_1'$ be the first divisorial contraction in ϕ. Let D_1 be the strict transform of D on X_1. Then $X \dashrightarrow X_1$ only contains flips, hence it is an isomorphism in codimension one, so D_1 is also a movable$/U$ $\mathbb{R}$-divisor on X_1. Let $D_1' := \psi_* D_1$. Then

$$D_1 = \psi^* D_1' + F$$

for some $F \geq 0$ that is exceptional over X_1'.

Since D_1 is a movable$/U$ divisor, D_1' is a movable$/X_1'$ divisor. Thus for any very general ψ-exceptional curve C, $D_1 \cdot C \geq 0$. By the general negativity lemma [Bir12, Lemma 3.3], $-F \geq 0$. Thus $F = 0$, and ψ cannot be a D_1-negative extremal contraction, a contradiction. Thus ϕ only contains flips. □

Lemma 2.10. *Let $(X, B, \mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC lc g-pair. Let $H \geq 0$ be an $\mathbb{R}$-divisor on X such that $(X, B+H, \mathbf{M})$ is lc and $K_X + B + H + \mathbf{M}_X$ is nef$/U$. Assume that $(X, B+\mu H, \mathbf{M})/U$ has a log minimal model for any $\mu \in (0,1]$. Then we can construct a $(K_X + B + \mathbf{M}_X)$-MMP$/U$ with scaling of H:*

$$(X, B, \mathbf{M}) := (X_0, B_0, \mathbf{M}) \dashrightarrow (X_1, B_1, \mathbf{M}) \dashrightarrow \cdots \dashrightarrow (X_i, B_i, \mathbf{M}) \dashrightarrow \dots.$$

Let H_i be the strict transform of H on X_i for each i, and let

$$\lambda_i := \inf\{t \geq 0 \mid K_{X_i} + B_i + tH_i + \mathbf{M}_{X_i} \text{ is nef}/U\}$$

be the i-th scaling number of this MMP for each i. Then this MMP

(1) either terminates after finitely many steps, or
(2) does not terminate and $\lim_{i\to+\infty} \lambda_i = 0$.

Proof. If $\lambda_0 = 0$ then there is nothing left to prove. So we may assume that $\lambda_0 > 0$. By [HL22, Lemma 3.21], we may pick $\lambda_0' \in (0, \lambda_0)$ such that any sequence of the $(K_X + B + \lambda_0' H + \mathbf{M}_X)$-MMP$/U$ is $(K_X + B + \lambda_0 H + \mathbf{M}_X)$-trivial.

By [HL21a, Theorem 2.24], we may run a $(K_X + B + \lambda_0' H + \mathbf{M}_X)$-MMP$/U$ with scaling of a general ample$/U$ divisor, which terminates with a log minimal model. We let

$$(X, B, \mathbf{M}) := (X_0, B_0, \mathbf{M}) \dashrightarrow (X_1, B_1, \mathbf{M}) \dashrightarrow \cdots \dashrightarrow (X_{k_1}, B_{k_1}, \mathbf{M})$$

be this sequence of the MMP$/U$. Then this sequence consists of finitely many steps of a $(K_X + B + \mathbf{M}_X)$-MMP$/U$ with scaling of H, with scaling numbers $\lambda_0 = \lambda_1 = \cdots = \lambda_{k_1 - 1}$. Since

$$K_{X_{k_1}} + B_{k_1} + \lambda_1' H_{k_1} + \mathbf{M}_{X_{k_1}}$$

is nef/U, we have $\lambda_{k_1} \leq \lambda'_1 < \lambda_1$.

We may replace $(X, B, \mathbf{M})/U$ with $(X_{k_1}, B_{k_1}, \mathbf{M})/U$ and continue this process. If this MMP does not terminate, then we may let $\lambda := \lim_{i\to+\infty} \lambda_i$. By our construction, $\lambda \neq \lambda_i$ for any i, and the lemma follows from [LT22b, Theorem 4.1]. □

Lemma 2.11 (cf. [HL22, 3.5 Lifting flips, Page 727-728], [LT22b, 2.5 Lifting a sequence of flips with scaling, Lemma 2.13]). *Let $(X, B, \mathbf{M})/U$ be an NQC lc g-pair, S an lc center of $(X, B, \mathbf{M})$, $(Y, B_Y, \mathbf{M})$ a dlt model of $(X, B, \mathbf{M})$ with induced birational morphism $f : Y \to X$, and S_Y a component of $\lfloor B_Y \rfloor$ such that $f(S_Y) = S$. Let*

$$\phi : (X, B, \mathbf{M}) \dashrightarrow (X', B', \mathbf{M})$$

be a partial $(K_X + B + \mathbf{M}_X)$-MMP/U and S' an lc center of $(X', B', \mathbf{M})$ such that $\phi|_S : S \dashrightarrow S'$ is a birational map. Then there exists a partial $(K_Y + B_Y + \mathbf{M}_Y)$-MMP/$U$

$$\psi : (Y, B_Y, \mathbf{M}) \dashrightarrow (Y', B'_Y, \mathbf{M}),$$

such that

(1) $(Y', B'_Y, \mathbf{M})$ is a dlt model of $(X', B', \mathbf{M})$, and
(2) the strict transform of S_Y on Y' is a component of $\lfloor B'_Y \rfloor$.

Proof. We only need to prove the lemma when ϕ is a divisorial contraction or a flip. If ϕ is a flip, then we let $X \to Z$ be the flipping contraction and let $X' \to Z$ be the flipped contraction. The rest of the proof of (1) is similar to the [LT22b, First paragraph of the proof of Lemma 2.13]: If ϕ is a divisorial contraction, then we let $Z = X'$. Thus $(X', B', \mathbf{M})/Z$ is a log minimal model of $(X, B, \mathbf{M})/Z$ such that $K_{X'} + B' + \mathbf{M}_{X'}$ is ample/Z. By [HL21a, Lemmas 3.9, 3.15] and [HL21a, Theorem 3.14], we may run a $(K_Y + B_Y + \mathbf{M}_Y)$-MMP/$Z$ with scaling of an ample/Z divisor which terminates with a good minimal model $(Y', B'_Y, \mathbf{M})/Z$. By [HL21a, Lemma 3.9], $(Y', B'_Y, \mathbf{M})$ is a dlt model of $(X', B', \mathbf{M})$, and we get (1).

We let $p : W \to Y$ and $q : W \to Y'$ be a resolution of indeterminacies of the induced birational map $\phi_Y : Y \dashrightarrow Y'$. By [HL21a, Lemma 3.8], $p^*(K_Y + B_Y + \mathbf{M}_Y) = q^*(K_{Y'} + B_{Y'} + \mathbf{M}_{Y'}) + F$ where $F \geq 0$ is exceptional/Y', and $\operatorname{Supp} p_*F$ contains all ϕ_Y-exceptional divisors. By (1), $a(S_Y, Y, B_{Y'}, \mathbf{M}) = 0$, hence S_Y is not a component of $\operatorname{Supp} p_*F$, and we get (2). □

2.4. **Proper log smooth models.**

Definition 2.12 (Log smooth model, [HL21a, Definition 3.1]). Let $(X, B, \mathbf{M})/U$ be an lc g-pair and $h : W \to X$ a log resolution of $(X, \operatorname{Supp} B)$ such that $\mathbf{M}$ descends to W. Let $B_W \geq 0$ and $E \geq 0$ be two $\mathbb{R}$-divisors on W such that

(1) $K_W + B_W + \mathbf{M}_W = h^*(K_X + B + \mathbf{M}_X) + E$,
(2) (W, B_W) is log smooth dlt,
(3) E is h-exceptional, and
(4) for any h-exceptional prime divisor D such that $a(D, X, B, \mathbf{M}) > 0$, D is a component of E.

Then $(W, B_W, \mathbf{M})$ is called a *log smooth model* of $(X, B, \mathbf{M})$. If we additionally assume that

(5) for any h-exceptional prime divisor D such that $a(D, X, B, \mathbf{M}) > 0$, D is a component of $\{B_W\}$,

then $(W, B_W, \mathbf{M})$ is called a *proper log smooth model* of $(X, B, \mathbf{M})$.

Lemma 2.13. *Let $(X, B, \mathbf{M})/U$ be an lc g-pair. Then there exists a proper log smooth model $(W, B_W = B^h_W + B^v_W, \mathbf{M})$ of $(X, B, \mathbf{M})$, such that*

(1) $B^h_W \geq 0$ and B^v_W is reduced,
(2) B^v_W is vertical over U, and
(3) for any real number $t \in (0, 1]$, all lc centers of $(W, B_W - tB^v_W, \mathbf{M})$ dominate U.

Proof. By [HL21a, Lemma 3.6], possibly replacing $(X, B, \mathbf{M})$ with a proper log smooth model, we may assume that $(X, \operatorname{Supp} B)$ is log smooth and $\mathbf{M}$ descends to X. By [Has18, Lemma 2.10], there exists a proper log smooth model $(W, B_W = B_W^h + B_W^v)$ of (X, B), such that

- $B_W^h \geq 0$ and B_W^v is reduced,
- B_W^v is vertical over U, and
- for any real number $t \in (0,1]$, all lc centers of $(W, B_W - tB_W^v)$ dominate U.

Since $\mathbf{M}$ descends to X, $(W, B_W, \mathbf{M})$ is a proper log smooth model of $(X, B, \mathbf{M})$, and for any real number $t \in (0,1]$, any lc center of $(W, B_W - tB_W^v, \mathbf{M})$ is an lc center of $(W, B_W - tB_W^v)$ and dominates U. Thus $(W, B_W = B_W^h + B_W^v, \mathbf{M})$ satisfies our requirements. □

2.5. **Canonical bundle formula.**

Theorem 2.14. *Let $(X, B, \mathbf{M})/U$ be an NQC lc g-pair such that U is quasi-projective, and let $\pi : X \to V$ be a surjective morphism over U. Assume that $K_X + B + \mathbf{M}_X \sim_{\mathbb{R},V} 0$. Then there exists an NQC lc g-pair $(V, B_V, \mathbf{M}^V)/U$, such that*

(1) $K_X + B + \mathbf{M}_X \sim_{\mathbb{R}} \pi^*(K_V + B_V + \mathbf{M}^V_V)$,
(2) any lc center of $(V, B_V, \mathbf{M}^V)$ is the image of an lc center of $(X, B, \mathbf{M})$ in V, and
(3) if all lc centers of $(X, B, \mathbf{M})$ dominate V, then $(V, B_V, \mathbf{M}^V)$ is klt.

Proof. By the theory of Shokurov-type rational polytopes (cf. [HL22, Proposition 3.20]) and the theory of uniform rational polytopes (see [HLS19, Lemma 5.3], [Che20, Therem 1.4]), we may assume that $(X, B, \mathbf{M})/U$ is a $\mathbb{Q}$-g-pair.

Step 1. In this step we prove the case when $X \to V$ is a generically finite morphism.

By [HL20, Theorem 4.5, (4.3),(4.4)], there exists an lc $\mathbb{Q}$-g-pair $(V, B_V, \mathbf{M}^V)/U$, such that $K_X + B + \mathbf{M}_X \sim_{\mathbb{Q}} \pi^*(K_V + B_V + \mathbf{M}^V_V)$, and B_V and $\mathbf{M}^V$ are defined in the following way:

Let V^0 be the smooth locus of V, $X^0 := X \times_V V^0$, and $\pi|_{X^0} : X^0 \to V^0$ the restriction of π. Then we have the Hurwitz formula

$$K_{X^0} = (\pi|_{X^0})^* K_{V^0} + R^0$$

where R^0 is the effective ramification divisor of $f|_{X^0}$. Let R be the closure of R^0 in X, and let $B_V := \frac{1}{\deg \pi}\pi_*(R + B)$. For any proper birational morphism $\mu : V' \to V$, let X' be the main component of $X \times_V V'$ with induced birational map $\pi' : X' \to V'$. We let $\mathbf{M}^V_{V'} = \frac{1}{\deg \pi}\pi'_*\mathbf{M}_{X'}$.

(1) follows immediately.

Since $(V, B, \mathbf{M}^V)/U$ is a g-pair, for any prime divisor E over V, there exists a birational morphism $h_V : \tilde{V} \to V$ such that $\mathbf{M}^V$ descends to $\tilde{V}$ and E is on $\tilde{V}$. We let $h : \tilde{X} \to X$ be a birational morphism such that $\mathbf{M}$ descends to $\tilde{X}$ and the induced map $\tilde{\pi} : \tilde{X} \to \tilde{V}$ is a morphism.

$$\begin{array}{ccccc} X' & \longrightarrow & X & \xleftarrow{\ h\ } & \tilde{X} \\ {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\ V' & \xrightarrow{\ \mu\ } & V & \xleftarrow{\ h_V\ } & \tilde{V} \end{array}$$

There are two cases:

Case 1. E is exceptional over V. In this case we let $F \subset \tilde{\pi}^{-1}(E)$ be a prime divisor, and let $r \leq \deg f$ be the ramification index of $\tilde{\pi}$ along F. Near the generic point of F, we have

$$\begin{aligned} K_{\tilde{X}} &= h^*(K_X + B + \mathbf{M}_X) + (a(F, X, B, \mathbf{M}) - 1)F \\ &\sim_{\mathbb{Q}} h^*\pi^*(K_V + B_V + \mathbf{M}_V) + (a(F, X, B, \mathbf{M}) - 1)F \end{aligned}$$

and

$$\begin{aligned} K_{\tilde{X}} &= \tilde{\pi}^*K_{\tilde{V}} + (r-1)F \\ &= \tilde{\pi}^*h_V^*(K_V + B_V + \mathbf{M}_V) + r(a(E,V,B_V,\mathbf{M}^V) - 1)F + (r-1)F \\ &= h^*\pi^*(K_V + B_V + \mathbf{M}_V) + (ra(E,V,B_V,\mathbf{M}^V) - 1)F. \end{aligned}$$

Let $\tilde{X} \to \bar{X} \to V$ be the Stein factorization of $\pi \circ h = h_V \circ \tilde{\pi}$. Since E is exceptional over V, F is exceptional over $\bar{X}$. Therefore $aF \sim_{\mathbb{Q},\bar{X}} 0$ iff $a = 0$ (applying negativity lemma to both aF and $-aF$). By comparing the two expressions of $K_{\tilde{X}}$ above, we have

$$a(F,X,B,\mathbf{M}) - 1 = ra(E,V,B_V,\mathbf{M}^V) - 1,$$

hence $a(F,X,B,\mathbf{M}) \geq 0$ if and only if $a(E,V,B_V,\mathbf{M}^V) \geq 0$ and $a(F,X,B,\mathbf{M}) > 0$ if and only if $a(E,V,B_V,\mathbf{M}^V) > 0$. Moreover, since $F \subset \tilde{\pi}^{-1}(E)$, if E is an lc place of $(V,B_V,\mathbf{M}^V)$, then F is an lc place of $(X,B,\mathbf{M})$ and $\operatorname{center}_V E$ is contained in the image of $\operatorname{center}_X F$ in V.

Case 2. E is not exceptional over V. In this case, if E is not a component of B_V, then $a(E,V,B_V,\mathbf{M}^V) = 1 > 0$. If E is a component of B_V, then we may let $B_1,\dots,B_m \subset \pi^{-1}(E)$ be the prime divisors on X lying over V and let d_i be the degree of the induced morphism $\pi|_{B_i} : B_i \to E$. By our construction of B_V,

$$a(E,V,B_V,\mathbf{M}^V) = 1 - \operatorname{mult}_E B_V = 1 - \frac{\sum_{i=1}^m d_i \operatorname{mult}_{B_i} B}{\deg \pi}.$$

Since $\sum_{i=1}^m d_i \leq \deg \pi$, $a(E,V,B_V,\mathbf{M}^V) \geq 0$ if $\operatorname{mult}_{B_i} B \leq 1$ for each i, and $a(E,V,B_V,\mathbf{M}^V) > 0$ if $\operatorname{mult}_{B_i} B < 1$ for each i. Moreover, since $B_i \subset \pi^{-1}(E)$ for each i, if E is an lc place of $(V,B_V,\mathbf{M}^V)$, then B_i is an lc place of $(X,B,\mathbf{M})$ for some i and E is contained in the image of B_i in V.

By our discussions above, we finish the proof in the case when $X \to V$ is a generically finite morphism.

Step 2. In this step we prove the case when $X \to V$ is a contraction.

By [FS20, Theorem 2.20], there exists an lc $\mathbb{Q}$-g-pair $(V,B_V,\mathbf{M}^V)/U$, such that $K_X + B + \mathbf{M}_X \sim_{\mathbb{Q}} \pi^*(K_V + B_V + \mathbf{M}^V_V)$. Moreover, for any birational morphism $h_V : \tilde{V} \to V$, we have an $\mathbb{R}$-divisor $B_{\tilde{V}}$ satisfying $K_{\tilde{V}} + B_{\tilde{V}} + \mathbf{M}^V_{\tilde{V}} = h_V^*(K_V + B_V + \mathbf{M}^V_V)$ and defined in the following way: let $\tilde{X}$ be the main component of $X \times_V \tilde{V}$, and $h : \tilde{X} \to X$ and $\tilde{\pi} : \tilde{X} \to \tilde{V}$ the induced morphisms. Let $K_{\tilde{X}} + \tilde{B} + \mathbf{M}_{\tilde{X}} := h^*(K_X + B + \mathbf{M}_X)$. For any prime divisor E on $\tilde{V}$, $\operatorname{mult}_E B_{\tilde{V}} = 1 - t_E$, where

$$t_E := \sup\{s \mid (\tilde{X}, \tilde{B} + s\tilde{\pi}^*E, \mathbf{M}) \text{ is lc over the generic point of } E\}.$$

Note that E may not be $\mathbb{Q}$-Cartier but $\tilde{\pi}^*E$ is always defined over the generic point of E.

(1) follows immediately.

If E is an lc place of $(V,B_V,\mathbf{M}^V)$ on $\tilde{V}$, then $t_E = 0$, hence $\tilde{\pi}^*E$ contains an lc center F of $(\tilde{X},\tilde{B},\mathbf{M})$ over the generic point of E. We have $F \subset \operatorname{Supp} \tilde{\pi}^*E$ and $\tilde{\pi}(F) \subset E$, hence $\tilde{\pi}(F) = E$. Thus E is the image of an lc center of $(\tilde{X},\tilde{B},\mathbf{M})$ on $\tilde{V}$, hence $\operatorname{center}_V E$ is the image of an lc center of $(X,B,\mathbf{M})$ in V.

By our discussions above, we finish the proof in the case when $X \to V$ is a contraction.

Step 3. In this step we prove the general case.

We let $X \xrightarrow{f} Y \xrightarrow{g} V$ be the Stein factorization of π. Then $K_X + B + \mathbf{M}_X \sim_{\mathbb{Q},Y} 0$, $f : X \to Y$ is a contraction and $g : Y \to V$ is a finite morphism. By Step 2, $K_X + B + \mathbf{M}_X \sim_{\mathbb{Q}} f^*(K_Y + B_Y + \mathbf{M}^Y_Y)$ for some lc $\mathbb{Q}$-g-pair $(Y,B_Y,\mathbf{M}^Y)/U$ such that any lc center of $(Y,B_Y,\mathbf{M}^Y)$ is the image of an lc center of $(X,B,\mathbf{M})$ in Y. Moreover, $K_Y + B_Y + \mathbf{M}^Y_Y \sim_{\mathbb{Q},V} 0$. By Step 1, $K_Y + B_Y + \mathbf{M}^Y_Y \sim_{\mathbb{Q}} g^*(K_V + B_V + \mathbf{M}^V_V)$ for some lc g-pair $(V,B_V,\mathbf{M}^V)/U$ such that any lc center of $(V,B_V,\mathbf{M}^V)$ is the image of an lc center of $(Y,B_Y,\mathbf{M}^Y)$ in V, hence the image of an lc center of $(X,B,\mathbf{M})$ in V. We immediately get (1)(2) and (3) follows from (2). □

2.6. Special termination.

Definition 2.15. Let $\mathcal{I}\subset[0,1]$ and $\mathcal{I}'\subset[0,+\infty)$ be two sets. We define

$$\mathbb{S}(\mathcal{I},\mathcal{I}'):=\{1-\frac{1}{m}+\sum_j\frac{r_jb_j}{m}+\sum_i\frac{s_i\mu_i}{m}\mid m\in\mathbb{N}^+,r_i,s_i\in\mathbb{N},b_j\in\mathcal{I},\mu_j\in\mathcal{I}'\}\cap(0,1].$$

Proposition 2.16 ([HL22, Proposition 2.10])**.** *Let $\mathcal{I}\subset[0,1]$ and $\mathcal{I}'\subset[0,+\infty)$ be two sets. Let $(X,B,\mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC dlt g-pair such that $B\in\mathcal{I}$ and $\mathbf{M}=\sum\mu_i\mathbf{M}_i$, where $\mu_i\in\mathcal{I}'$ for each i and each $\mathbf{M}_i$ is nef/U **b**-Cartier. Then for any lc center S of $(X,B,\mathbf{M})$, the g-pair $(S,B_S,\mathbf{M}^S)/U$ given by the adjunction*

$$K_S+B_S+\mathbf{M}^S_S:=(K_X+B+\mathbf{M}_X)|_S$$

is dlt, and $B_S\in\mathbb{S}(\mathcal{I},\mathcal{I}')$.

Definition 2.17 (Difficulty, [HL22, Definition 4.5])**.** Let $\mathcal{I}$ and $\mathcal{I}'$ be two finite sets of non-negative real numbers. Let $(X,B,\mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC dlt g-pair such that $B\in\mathcal{I}$ and $\mathbf{M}=\sum\mu_i\mathbf{M}_i$, where $\mu_i\in\mathcal{I}'$ for each i and each $\mathbf{M}_i$ is nef/U **b**-Cartier. For any lc center S of $(X,B,\mathbf{M})$ of dimension ≥ 1, let $(S,B_S,\mathbf{M}^S)$ be the g-pair given by the generalized adjunction

$$K_S+B_S+\mathbf{M}^S_S:=(K_X+B+\mathbf{M}_X)|_S,$$

then we define

$$\begin{aligned}d_{\mathcal{I},\mathcal{I}'}(S,B_S,\mathbf{M}^S):=&\sum_{\alpha\in\mathbb{S}(\mathcal{I},\mathcal{I}')}\#\{E\mid a(E,B_S,\mathbf{M}^S)<1-\alpha,\operatorname{center}_S E\not\subset\lfloor B_S\rfloor\}\\&+\sum_{\alpha\in\mathbb{S}(\mathcal{I},\mathcal{I}')}\#\{E\mid a(E,B_S,\mathbf{M}^S)\leq 1-\alpha,\operatorname{center}_S E\not\subset\lfloor B_S\rfloor\}.\end{aligned}$$

The following special termination result is similar to [Fuj07, LMT20, HL22]. The proofs are also similar. For the reader's convenience, we provide a full proof here.

Lemma 2.18. *Let $(X,B,\mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC dlt g-pair and let*

$$(X,B,\mathbf{M}):=(X_0,B_0,\mathbf{M})\dashrightarrow(X_1,B_1,\mathbf{M})\dashrightarrow\cdots\dashrightarrow(X_i,B_i,\mathbf{M})\dashrightarrow\dots$$

be a $(K_X+B+\mathbf{M}_X)$-MMP/U. Let $\phi_{i,j}:X_i\dashrightarrow X_j$ be the induced birational maps for each i. For any $i\geq 0$ and any lc center S_i of $(X_i,B_i,\mathbf{M})$ of dimension ≥ 1, we let $(S_i,B_{S_i},\mathbf{M}^{S_i})/U$ be the generalized pair given by adjunction

$$K_{S_i}+B_{S_i}+\mathbf{M}^{S_i}_{S_i}:=(K_{X_i}+B_i+\mathbf{M}_{X_i})|_{S_i}.$$

Then we have the following.

(1) *For any $i\gg 0$, $j\geq i$, and any lc center S_i of $(X_i,B_i,\mathbf{M})$, $\phi_{i,j}$ induces an isomorphism near the generic point of S_i. In particular, for any $i,j\gg 0$ and any lc center S_i of $(X_i,B_i,\mathbf{M})$, we may let $S_{i,j}$ be the strict transform of S_i on X_j.*

(2) *Fix $i\gg 0$ and an lc center S_i of $(X_i,B_i,\mathbf{M})$ such that $\phi_{i,j}$ induces an isomorphism for every lc center of $(S_i,B_{S_i},\mathbf{M}^{S_i})/U$ for any $j\geq i$. Then*
   - (a) *$\phi_{j,k}|_{S_{i,j}}:S_{i,j}\dashrightarrow S_{i,k}$ is an isomorphism in codimension 1 for any $j,k\gg i$, and*
   - (b) *$B_{S_{i,j}}$ is the strict transform of $B_{S_{i,k}}$ for any $j,k\gg i$.*

(3) *Suppose that this $(K_X+B+\mathbf{M}_X)$-MMP/U is a MMP with scaling of an $\mathbb{R}$-divisor $A\geq 0$ on X. Let*

$$\lambda_j:=\inf\{t\mid t\geq 0,K_{X_j}+B_j+tA_j+\mathbf{M}_{X_j}\text{ is nef}/U\}$$

be the scaling numbers, where A_j is the strict transform of A on X_j for each j. Fix $i\gg 0$ and an lc center S_i of $(X_i,B_i,\mathbf{M})$ such that $\phi_{j,k}|_{S_{i,j}}:S_{i,j}\dashrightarrow S_{i,k}$ is an isomorphism *in codimension* 1 *and* $B_{S_{i,j}}$ *is the strict transform of* $B_{S_{i,k}}$ *for any* $k,j\geq i$. *Let* T *be the normalization of the image of* S_i *on* U, $(S'_i, B_{S'_i}, \mathbf{M}^{S_i})$ *a dlt model of* $(S_i, B_{S_i}, \mathbf{M}^{S_i})$, *and* $A_{S'_i}$ *the pullback of* A_i *on* S'_i. *Then this* $(K_X+B+\mathbf{M}_X)$*-MMP/*U *with scaling of* A *induces the following commutative diagram/*T

$$\begin{array}{ccccccc}(S'_i, B_{S'_i}, \mathbf{M}^{S_i}) & \dashrightarrow & (S'_{i,i+1}, B_{S'_{i,i+1}}, \mathbf{M}^{S_i}) & \dashrightarrow \dots \dashrightarrow & (S'_{i,j}, B_{S'_{i,j}}, \mathbf{M}^{S_i}) & \dashrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ (S_i, B_{S_i}, \mathbf{M}^{S_i}) & \dashrightarrow & (S_{i,i+1}, B_{S_{i,i+1}}, \mathbf{M}^{S_i}) & \dashrightarrow \dots \dashrightarrow & (S_{i,j}, B_{S_{i,j}}, \mathbf{M}^{S_i}) & \dashrightarrow & \dots\end{array}$$

such that

(a)

$$(S'_i, B_{S'_i}, \mathbf{M}^{S_i}) \dashrightarrow (S'_{i,i+1}, B_{S'_{i,i+1}}, \mathbf{M}^{S_i}) \dashrightarrow \cdots \dashrightarrow (S'_{i,j}, B_{S'_{i,j}}, \mathbf{M}^{S_i}) \dashrightarrow \dots$$

is a $(K_{S'_i}+B_{S'_i}+\mathbf{M}^{S_i}_{S'_i})$*-MMP/*$T$ *with scaling of* $A_{S'_i}$. *Note that it is possible that* $(S'_{i,j}, B_{S'_{i,j}}, \mathbf{M}^{S_i}) \dashrightarrow (S'_{i,j+1}, B_{S'_{i,j+1}}, \mathbf{M}^{S_i})$ *is the identity morphism or a composition of several steps of the* $(K_{S'_{i,j}}+B_{S'_{i,j}}+\mathbf{M}^{S_i}_{S'_{i,j}})$*-MMP/*$T$ *for some* j,

(b) for any $j\geq i$, $(S'_{i,j}, B_{S'_{i,j}}, \mathbf{M}^{S_i})$ *is a dlt model of* $(S_{i,j}, B_{S_{i,j}}, \mathbf{M}^{S_i})$, *and*

(c) let

$$\mu_j := \inf\{t \mid t\geq 0, K_{S'_{i,j}}+B_{S'_{i,j}}+tA_{S'_{i,j}}+\mathbf{M}^{S_i}_{S'_{i,j}} \text{ is nef}/T\}$$

for each $j\geq i$, *where* $A_{S'_{i,j}}$ *is the pullback of* A_j *on* $S'_{i,j}$. *Then* $\mu_j\leq\lambda_j$ *for each* $j\geq i$.

Proof. Let $\mathcal{I}\subset[0,1]$ be a finite set such that $B\in\mathcal{I}$, and let $\mathcal{I}'\subset[0,+\infty)$ be a finite set such that $\mathbf{M}=\sum\mu_i\mathbf{M}_i$, where each $\mathbf{M}_i$ is nef/U $\boldsymbol{b}$-Cartier and each $\mu_i\in\mathcal{I}'$. Let $\phi_i:=\phi_{i,i+1}$ for each i.

We may assume that the MMP does not terminate, otherwise there is nothing left to prove. Possibly replacing X with X_i for $i\gg 0$, we may assume that each ϕ_i is a flip. Since the number of lc centers of $(X,B,\mathbf{M})$ is finite, possibly replacing X with X_i for $i\gg 0$, we may assume that the flipping locus of ϕ_i does not contain any lc centers. This proves (1).

We prove (2). We let $S:=S_i$. By (1), we may let $S_j:=S_{i,j}$ for any $j\geq i$. Possibly replacing X we X_i, we may assume that $i=0$. By [HL22, Proposition 2.10], for any j, the g-pair $(S_j, B_{S_j}, \mathbf{M}^S)$ given by the adjunction

$$K_{S_j}+B_{S_j}+\mathbf{M}^S_{S_j} := (K_{X_j}+B_j+\mathbf{M}_{X_j})|_{S_j}$$

is dlt, and $B_{S_j}\in\mathbb{S}(\mathcal{I},\mathcal{I}')$. By assumption, $\phi_{j,k}$ induces an isomorphism on $\lfloor B_{S_j}\rfloor$ for any j,k. Thus for any j and any prime divisor E over S_j, $\operatorname{center}_{S_j}E\subset\lfloor B_{S_j}\rfloor$ if and only if $\operatorname{center}_{S_{j+1}}E\subset\lfloor B_{S_{j+1}}\rfloor$. By the negativity lemma, $a(E,S_j,B_{S_j},\mathbf{M}^S)\leq a(E,S_{j+1},B_{S_{j+1}},\mathbf{M}^S)$ for each j and any prime divisor E over S_j. Thus

$$d_{\mathcal{I},\mathcal{I}'}(S_j,B_{S_j},\mathbf{M}^S)\geq d_{\mathcal{I},\mathcal{I}'}(S_{j+1},B_{S_{j+1}},\mathbf{M}^S)$$

for each j. Moreover, for any j such that S_j and S_{j+1} are not isomorphic in codimension 1, if there exists a prime divisor E on S_{j+1} that is exceptional over S_j, then

$$1-\alpha=a(E,S_{j+1},B_{S_{j+1}},\mathbf{M}^S)>a(E,S_j,B_{S_j},\mathbf{M}^S)$$

for some $\alpha\in\mathbb{S}(\mathcal{I},\mathcal{I}')$, and hence

$$d_{\mathcal{I},\mathcal{I}'}(S_j,B_{S_j},\mathbf{M}^S)>d_{\mathcal{I},\mathcal{I}'}(S_{j+1},B_{S_{j+1}},\mathbf{M}^S).$$

By [HL22, Remark 4.6], $d_{\mathcal{I},\mathcal{I}'}(S_j,B_{S_j},\mathbf{M}^S)<+\infty$. Thus possibly replacing X with X_j for some $j\gg 0$, we may assume that $d_{\mathcal{I},\mathcal{I}'}(S_j,B_{S_j},\mathbf{M}^S)=d_{\mathcal{I},\mathcal{I}'}(S_k,B_{S_k},\mathbf{M}^S)$ for any j,k. Thus $S_j \dashrightarrow$

S_{j+1} does not extract any divisor for any j. In particular, $\rho(S_{j+1})\leq\rho(S_j)$, and $\rho(S_{j+1})<\rho(S_j)$ if $S_j\dashrightarrow S_{j+1}$ contracts a divisor. Thus possibly replacing X with X_j for some $i\gg 0$, we may assume that S_j and S_{j+1} are isomorphic in codimension 1 for each j, which implies (2.a). Since $d_{\mathcal{I},\mathcal{I}'}(S_j,B_{S_j},\mathbf{M}^S)=d_{\mathcal{I},\mathcal{I}'}(S_k,B_{S_k},\mathbf{M}^S)$ for any j,k, (2.b) follows from (2.a).

We prove (3). Since $i\gg 0$, possibly replacing X with X_i, we may assume that $i=0$ and ϕ_j is a flip for every j. We let $S:=S_0, S':=S'_0, S_j:=S_{0,j}$, and $S'_j:=S'_{0,j}$ for every j. We let $X_j\to Z_j\leftarrow X_{j+1}$ be each flip and let T_j be the normalization of the image of S_j on Z_j for each j. Then we have an induced birational map $S_j\dashrightarrow S_{j+1}$ for each j.

Since ϕ_0 is a $(K_{X_0}+B_0+\mathbf{M}_{X_0})$-flip$/U$, $X_1\to Z_0$ is $(K_{X_1}+B_1+\mathbf{M}_{X_1})$-positive and $K_{S_1}+B_{S_1}+\mathbf{M}^S_{S_1}$ is ample$/T_0$. In particular, $(S_1,B_{S_1},\mathbf{M}^S)/T_0$ is a weak lc model of $(S_0,B_{S_0},\mathbf{M}^S)$. By [HL21a, Lemmas 3.9, 3.15] and [HL21a, Theorem 3.14], we may run a $(K_{S'_0}+B_{S'_0}+\mathbf{M}^S_{S'_0})$-MMP$/T_0$ with scaling of an ample$/T_0$ divisor, which terminates with a good minimal model of $(S'_0,B_{S'_0},\mathbf{M}^S)/T_0$. By [HL21a, Lemma 3.9], $(S'_0,B_{S'_0},\mathbf{M}^S)$ is a dlt model of $(S_1,B_{S_1},\mathbf{M}^S)$. Since

$$K_{S'_0}+B_{S'_0}+\lambda_0A_{S'_0}+\mathbf{M}^S_{S'_0}\equiv_{T_0}0,$$

this MMP is also a $(K_{S'_0}+B_{S'_0}+\mathbf{M}^S_{S'_0})$-MMP$/T_0$ with scaling of $\lambda_0A_{S'_0}$. We may replace $(S_0,B_{S_0},\mathbf{M}^S)/T$ with $(S_1,B_{S_1},\mathbf{M}^S)/T$ and continue this process. This gives us the desired $(K_{S'_0}+B_{S'_0}+\mathbf{M}^S_{S'_0})$-MMP$/T$ with scaling of $A_{S'_0}$, which gives the commutative diagram, and proves (3.a) and (3.b). For each j, since $K_{S'_j}+B_{S'_j}+\lambda_jA_{S'_j}+\mathbf{M}^S_{S'_j}\equiv_{T_j}0$, $K_{S'_j}+B_{S'_j}+\lambda_jA_{S'_j}+\mathbf{M}^S_{S'_j}$ is nef, hence $\mu_j\leq\lambda_j$, and we get (3.c). □

3. Relative Nakayama-Zariski decomposition

Definition 3.1. Let $\pi:X\to U$ be a projective morphism from a normal variety to a variety, A an ample$/U$ $\mathbb{R}$-divisor on X, D a pseudo-effective$/U$ $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, and P a prime divisor on X. For any big$/U$ $\mathbb{R}$-Cartier $\mathbb{R}$-divisor B, we define

$$\sigma_P(X/U,B):=\inf\{\operatorname{mult}_P B'\mid 0\leq B'\sim_{\mathbb{R},U}B\}.$$

We define

$$\sigma_P(X/U,D):=\lim_{\epsilon\to 0^+}\sigma_P(X/U,D+\epsilon A),$$

where we allow $+\infty$ as a limit as well. As in [Nak04, III §1.], we can easily check that $\sigma_P(X/U,D)$ is well-defined and does not depend on the choice of A (we left the proof for the readers). We let

$$N_\sigma(X/U,D):=\sum_{C\text{ is a prime divisor on }X}\sigma_C(X/U,D)\cdot C$$

be a formal sum of divisors with coefficients in $\mathbb{R}_{\geq 0}\cup\{+\infty\}$. We say that D is movable$/U$ if $N_\sigma(X/U,D)=0$, and this coincides with the original definition when D is big$/U$.

For any divisor D' on X, we say $D'\leq N_\sigma(X/U,D)$ if $\operatorname{mult}_C(D')\leq\sigma_C(X/U,D)$ for any prime divisor C on X. We can naturally define the addition of D' and $N_\sigma(X/U,D)$ as

$$N_\sigma(X/U,D)+D':=\sum_{C\text{ is a prime divisor on }X}(\sigma_C(X/U,D)+\operatorname{mult}_C(D'))\cdot C,$$

by noticing that $+\infty+a=+\infty$ for any $a\in\mathbb{R}$. If $f:X\to Y$ is a projective birational morphism over U, then we can define the pushforward

$$f_*N_\sigma(X/U,D):=\sum_{C\text{ is a prime divisor on }X}\sigma_C(X/U,D)\cdot f_*C$$

as a formal sum of divisors with coefficients in $\mathbb{R}_{\geq 0}\cup\{+\infty\}$.

We define the support of $N_\sigma(X/U,D)$ as

$$\operatorname{Supp} N_\sigma(X/U,D) := \bigcup_{\sigma_C(X/U,D)>0} C.$$

If there are only finitely many prime divisors C on X such that $\sigma_C(X/U,D) > 0$ and $\sigma_C(X/U,D) < +\infty$ (e.g $D \geq 0$), then (we say) $N_\sigma(X/U,D)$ is well-defined as a divisor and we let

$$P_\sigma(X/U,D) := D - N_\sigma(X/U,D).$$

Definition 3.1 is the same as the one adopted in [HX13, HMX18]. The following lemma shows that the relative Nakayama-Zariski decomposition defined in Definition 3.1 is the same as the σ-decomposition defined in [Nak04, III. §4.a]:

Lemma 3.2. *Notation as in Definition 3.1. If X is smooth, then $\sigma_P(X/U,D)$ is the same as $\sigma_P(D,X/U)$, where the latter is the value defined as in Nakayama's original relative σ-decomposition* [Nak04, III. §4.a].

Proof. By definition, we only need to deal with the case when D is big. We may pick an affine open subset U^0 of U such that P intersects $X^0 := X \times_U U^0$. Let $P^0 := P \times_U U^0$ and $D^0 := D \times_U U^0$. Then

$$\sigma_P(X/U,D) = \sigma_{P^0}(X^0/U^0,D^0).$$

Possibly replacing $(X/U,D)$ and P with $(X^0/U^0,D^0)$ and P^0 respectively, we may assume that U is affine. Thus for any Cartier divisor Q on U, there exists a principal divisor Q' on U such that $Q' = Q$ in a neighborhood of the generic point of $\pi(P)$. In particular, we have

$$\sigma_P(X/U,D) = \inf\{\operatorname{mult}_{P^0} B' \mid 0 \leq B' \sim_{\mathbb{R}} B^0\}.$$

For any Cartier divisor F on X, let

$$m_F := \inf\{+\infty, \operatorname{mult}_P F' \mid 0 \leq F' \sim F\}.$$

If $m_F < +\infty$, then by definition,

$$m_F = \max\{m \in \mathbb{N} \mid H^0(X, F - mP) \hookrightarrow H^0(X,F) \text{ is an isomorphism}\}.$$

Moreover, since U is affine and $H^0(X,\mathcal{O}_X(F)) = H^0(U,\pi_*\mathcal{O}_X(F))$, if $m_F < +\infty$, then

$$m_F = \max\{m \in \mathbb{N} \mid \pi_*\mathcal{O}_X(F - mP) \hookrightarrow \pi_*\mathcal{O}_X(F) \text{ is an isomorphism}\}.$$

Now the lemma follows from the construction in [Nak04, III. §4.a]. □

Lemma 3.3. *Let $\pi : X \to U$ be a projective morphism from a normal variety to a variety, D, D' two pseudo-effective/U $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on X, and P a prime divisor on X.*

(1) If D is nef/U, then $\sigma_P(X/U,D) = 0$.
(2) $\sigma_P(X/U,D+D') \leq \sigma_P(X/U,D) + \sigma_P(X/U,D')$.
(3) If $\sigma_P(X/U,D') < +\infty$, then $\lim_{\epsilon\to 0^+} \sigma_P(X/U,D+\epsilon D') = \sigma_P(X/U,D)$.

Proof. Let A be an ample/U divisor on X.

(1) is straightforward from the definition.

(2) follows from the fact that $\sigma_P(X/U,D+D'+\epsilon A) \leq \sigma_P(X/U,D+\frac{\epsilon}{2}A)+\sigma_P(X/U,D'+\frac{\epsilon}{2}A)$.

There exists $a > 0$ such that $A - aD'$ is ample/U. Thus, by (1) and (2), we have

$$\sigma_P(X/U,D) + \sigma_P(X/U,a\epsilon D') \geq \sigma_P(X/U,D+a\epsilon D') \geq \sigma_P(X/U,D+\epsilon A),$$

and (3) follows after taking $\epsilon \to 0^+$. □

Lemma 3.4. *Let $\pi : X \to U$ be a projective morphism from a normal variety to a variety and D a pseudo-effective/U $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Let $f : Y \to X$ be a projective birational morphism. Then:*

(1) For any prime divisor P on X, we have

$$\sigma_P(X/U,D) = \sigma_{f_*^{-1}P}(Y/U,f^*D).$$

(2) *For any exceptional/X $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $E \geq 0$ and any prime divisor P on Y, we have*
$$\sigma_P(Y/U, f^*D + E) = \sigma_P(Y/U, f^*D) + \operatorname{mult}_P E.$$
(3) *For any exceptional/X $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $E \geq 0$ on Y we have*
$$N_\sigma(X/U, D) = f_*N_\sigma(Y/U, f^*D + E)$$
as a formal sum of divisors with coefficients in $\mathbb{R}_{\geq 0} \cup \{+\infty\}$. In particular, if $N_\sigma(Y/U, f^*D + E)$ is well-defined, then $N_\sigma(X/U, D)$ is well-defined.
(4) *If $D' \geq 0$ is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $D' \leq N_\sigma(X/U, D)$, then $f^*D' \leq N_\sigma(Y/U, f^*D)$ and $D - D'$ is pseudo-effective/U.*

Proof. Set $g = \pi \circ f$ and let A (resp. A') be an ample/U divisor on X (resp. Y). Fix a real number $a > 0$ such that $aA' + f^*A$ is ample/U. Notice that f^{-1}_*P is a prime divisor on Y. Since f^*A is semi-ample/U, by Lemma 3.3 we have $\lim_{\epsilon\to 0^+} \sigma_{f^{-1}_*P}(Y/U, f^*D + \epsilon f^*A) = \sigma_{f^{-1}_*P}(Y/U, f^*D)$.

Since $\pi_*\mathcal{O}_X(F) = g_*\mathcal{O}_Y(f^*F)$ for any Cartier divisor F on X, by definition we have $\sigma_P(X/U, D + \epsilon A) = \sigma_{f^{-1}_*P}(Y/U, f^*D + \epsilon f^*A)$ for any $\epsilon > 0$. Thus we have
$$\sigma_P(X/U, D) = \lim_{\epsilon\to 0^+} \sigma_{f^{-1}_*P}(Y/U, f^*D + \epsilon f^*A) = \sigma_{f^{-1}_*P}(Y/U, f^*D)$$
which is (1).

Since $\lim_{\epsilon\to 0^+} \sigma_P(Y/U, f^*D + \epsilon f^*A) = \sigma_P(Y/U, f^*D)$, we may assume that D is a big/U. (2) follows from the fact that $g_*\mathcal{O}_Y(f^*F + E) = \pi_*\mathcal{O}_X(F)$ for any Cartier divisor F on X and any exceptional/X divisor $E \geq 0$.

We have
$$N_\sigma(Y/U, f^*D + E) = N_\sigma(Y/U, f^*D) + E$$
by (2) and
$$f_*N_\sigma(Y/U, f^*D) = N_\sigma(X/U, D)$$
by (1), which imply (3).

For (4), since there are only finitely many prime divisors P on X such that $\operatorname{mult}_P D' > 0$, by assumption and by the definition of $\sigma_P(X/U, D)$ we know that $D' \leq D''_\epsilon$ for any element $D''_\epsilon \sim_{\mathbb{R},U} D + \epsilon A$ and any $1 \gg \epsilon > 0$. Then $[D - D'] = \lim_{\epsilon\to 0^+}[D''_\epsilon - D']$ is indeed pseudo-effective/U. Moreover, $f^*D' \leq f^*D''_\epsilon$ for any $1 \gg \epsilon > 0$ and Lemma 3.3(3) implies that $\operatorname{mult}_P f^*D' \leq \sigma_P(Y/U, f^*D)$ for any prime divisor P on Y by the same argument as in the proof of (2) above. □

Lemma 3.5. *Let $\pi : X \to U$ be a projective morphism from a normal variety to a variety and D a pseudo-effective/U $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then there are only finitely many prime divisors P on X such that $\sigma_P(X/U, D) > 0$. In particular,* $\operatorname{Supp} N_\sigma(X/U, D)$ *can be regarded as a reduced divisor. If furthermore $\sigma_P(X/U, D) < +\infty$ for any prime divisor P on X, then $N_\sigma(X/U, D)$ and $P_\sigma(X/U, D)$ are well-defined as divisors.*

Proof. Let $f : Y \to X$ be a resolution of X. By Lemma 3.4(1), for any prime divisor P on X such that $\sigma_P(X/U, D) \neq 0$, $\sigma_{f^{-1}_*P}(Y/U, f^*D) \neq 0$. Therefore, we only need to show that there are finitely many prime divisors P_Y on Y such that $\sigma_{P_Y}(Y/U, f^*D) \neq 0$. Possibly replacing X with Y and D with f^*D, we may assume that X is smooth. In the following, we will show that there are at most $\rho(X/U) = \dim N^1(X/U)_{\mathbb{R}}$ prime divisors P on X such that $\sigma_P(X/U, D) \neq 0$.

Let $P_1, P_2, ..., P_l$ be distinct prime divisors of X such that $\sigma_{P_i}(X/U, D) > 0$ for each i. If $l \leq \dim N^1(X/U)_{\mathbb{R}}$ then we are done. Otherwise, $P_1, P_2, ..., P_l$ are not linearly independent in $N^1(X/U)_{\mathbb{R}}$ and possibly reordering indices, we have
$$\sum_{i=1}^{s} x_iP_i \equiv_U \sum_{j=s+1}^{l} x_jP_j \in N^1(X/U)$$

for some $x_1, x_2, ..., x_l \in \mathbb{R}_{\geq 0}$ and $1 \leq s \leq l$, we may also assume that $x_1 \neq 0$. By Lemma 3.2 and [Nak04, III, Lemma 4.2(2)], we have

$$\sigma_{P_i}(X/U, \sum_{j=1}^{l} x_j P_j) = x_i$$

for any $x_1, x_2, ..., x_l \in \mathbb{R}_{\geq 0}$. Since $\sigma_P(X/U, D)$ depends only on the numerical equivalence class of D over U, by Lemma 3.2 and [Nak04, III, Lemma 4.2(2)] again, we obtain

$$x_1 = \sigma_{P_1}(X/U, \sum_{i=1}^{s} x_i P_i) = \sigma_{P_1}(X/U, \sum_{j=s+1}^{l} x_j P_j) = 0,$$

a contradiction. □

Definition 3.6. Let $\pi : X \to U$ be a projective morphism from a normal variety to a variety, D a pseudo-effective/U $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, and P a prime divisor over X. Let $f : Y \to X$ be a projective birational morphism such that P descends to Y. We define

$$\sigma_P(X/U, D) := \sigma_P(Y/U, f^*D).$$

By Lemma 3.4, $\sigma_P(X/U, D)$ is independent of the choice of Y. Also notice that f^*D is pseudo-effective/U iff D is.

Lemma 3.7. *Let $\pi : X \to U$ be a projective morphism from a normal variety to a variety, D, D' two pseudo-effective/U $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on X, and P a prime divisor over X. Then*

(1) $\sigma_P(X/U, D + D') \leq \sigma_P(X/U, D) + \sigma_P(X/U, D')$.
(2) If $\sigma_P(X/U, D') < +\infty$, then $\lim_{\epsilon \to 0^+} \sigma_P(X/U, D + \epsilon D') = \sigma_P(X/U, D)$.
(3) If D is a movable/U $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, then $N_\sigma(X/U, D) = 0$ and $P_\sigma(X/U, D) = D$ is movable.
(4) $\operatorname{Supp} N_\sigma(X/U, D)$ coincides with the divisorial part of $\mathbf{B}_-(D/U)$.
(5) If $0 \leq D' \leq N_\sigma(X/U, D)$, then $N_\sigma(X/U, D - D') + D' = N_\sigma(X/U, D)$.
(6) If $D' \geq 0$ and $\operatorname{Supp} D' \subset \operatorname{Supp} N_\sigma(X/U, D)$, then $N_\sigma(X/U, D + D') = N_\sigma(X/U, D) + D'$.

Proof. Let A be an ample/U $\mathbb{R}$-divisor on X.

(1) and (2) follow directly from Lemma 3.3(2)(3).

For (3), if this is not true, then we have $\sigma_P(X/U, D) > 0$ for some P. By definition, there exist an $\epsilon > 0$ such that $\sigma_P(X/U, D + \epsilon A) > 0$. Assume $[D] = \lim_{i \to \infty}[D_i]$, where D_i is a movable divisor for each $i \geq 1$. Then $\epsilon A - (D_i - D)$ is ample for any $i \gg 0$, and we have

$$0 < \sigma_P(X/U, D + \epsilon A) = \sigma_P(X/U, D_i + \epsilon A - (D_i - D)) \leq \sigma_P(X/U, D_i) = 0,$$

which is a contradiction.

For (4), from the definition of $\mathbf{B}_-(D/U)$ we know that $\operatorname{Supp} N_\sigma(X/U, D) \subset \mathbf{B}_-(D/U)$. For any divisorial component P of $\mathbf{B}_-(D/U)$, there exist $\epsilon > 0$ such that $P \subset \mathbf{B}(D + \epsilon A/U)$, so $\sigma_P(X/U, D) \geq \sigma_P(X/U, D + \epsilon A) > 0$.

By Lemma 3.4, possibly replacing X with a resolution, we may assume that X is smooth and P is a prime divisor on X. Then (5) and (6) follow from Lemma 3.2 and [Nak04, III, Lemma 4.2]. Notice that $N_\sigma(X/U, D - D')$ makes sense by Lemma 3.4(4). □

Lemma 3.8 (cf. [Has20, Lemma 2.4])**.** *Let $\pi : X \to U$ be a projective morphism from a normal variety to a variety, D (resp. D') a pseudo-effective/U $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $N_\sigma(X/U, D)$ (resp. $N_\sigma(X/U, D')$) is well-defined as a divisor. Then there exists $t_0 > 0$ such that $\operatorname{Supp} N_\sigma(X/U, D + tD')$ is independent of t for any $t \in (0, t_0]$.*

Proof. The proof is exactly the same as that of [Has20, Lemma 2.4]. □

Lemma 3.9. *Let $(X, B, \mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC dlt g-pair. Then for any partial $(K_X + B + \mathbf{M}_X)$-MMP/U $\phi : X \dashrightarrow \bar{X}$,*

(1) *the divisors contracted by* ϕ *are contained in* $\operatorname{Supp} N_\sigma(X/U, K_X + B + \mathbf{M}_X)$*, and*
(2) *let* $\bar{B}$ *be the strict transform of* B *on* $\bar{X}$*. If* $K_{\bar{X}} + \bar{B} + \mathbf{M}_{\bar{X}}$ *is movable/*U*, then* $\operatorname{Supp} N_\sigma(X/U, K_X + B + \mathbf{M}_X)$ *is the set of all* ϕ*-exceptional divisors.*

Proof. Let $p : W \to X$ and $q : W \to \bar{X}$ be a resolution of indeterminacies of ϕ. Then

$$p^*(K_X + B + \mathbf{M}_X) = q^*(K_{\bar{X}} + \bar{B} + \mathbf{M}_{\bar{X}}) + E$$

for some $E \geq 0$ that is exceptional/$\bar{X}$ and $\operatorname{Supp} E$ contains the strict transforms on W of all ϕ-exceptional divisors. By Lemma 3.4(2) we have

$$\operatorname{Supp} E \subset \operatorname{Supp} N_\sigma(W/U, q^*(K_{\bar{X}} + \bar{B} + \mathbf{M}_{\bar{X}}) + E) = \operatorname{Supp} N_\sigma(W/U, p^*(K_X + B + \mathbf{M}_X)),$$

and by Lemma 3.4(3) we know that $\operatorname{Supp} p_*E \subset \operatorname{Supp} N_\sigma(X/U, K_X + B + \mathbf{M}_X)$. Therefore, any ϕ-exceptional divisor is contained in $\operatorname{Supp} N_\sigma(X/U, K_X + B + \mathbf{M}_X)$.

If $K_{\bar{X}} + \bar{B} + \mathbf{M}_{\bar{X}}$ is movable/U, then by Lemma 3.4(3) we have $q_*N_\sigma(W/U, q^*(K_{\bar{X}} + \bar{B} + \mathbf{M}_{\bar{X}}) + E) = 0$ so $\operatorname{Supp} N_\sigma(W/U, q^*(K_{\bar{X}} + \bar{B} + \mathbf{M}_{\bar{X}}) + E)$ (viewed as a reduced divisor) is q-exceptional. By Lemma 3.4(3) again we have $\operatorname{Supp} N_\sigma(X/U, K_X + B + \mathbf{M}_X) = \operatorname{Supp} p_*N_\sigma(W/U, q^*(K_{\bar{X}} + \bar{B} + \mathbf{M}_{\bar{X}}) + E)$, whose components are all ϕ-exceptional. $\square$

Lemma 3.10. *Let* $(X, B, \mathbf{M})/U$ *be an NQC lc g-pair such that* $K_X + B + \mathbf{M}_X$ *is pseudo-effective/*U*. Let* $\phi : X \dashrightarrow X'$ *be a birational map/*U *which does not extract any divisor and* B' *the strict transform of* B *on* X'*, such that*

(1) $K_{X'} + B' + \mathbf{M}_{X'}$ *is nef/*U*, and*
(2) ϕ *only contracts divisors contained in* $\operatorname{Supp} N_\sigma(X/U, K_X + B + \mathbf{M}_X)$*.*

Then $(X', B', \mathbf{M})/U$ *is a log minimal model (not necessarily* $\mathbb{Q}$*-factorial) of* $(X, B, \mathbf{M})/U$*.*

Proof. Let $p : W \to X$ and $q : W \to X'$ be a resolution of indeterminacies of ϕ such that

$$p^*(K_X + B + \mathbf{M}_X) + E = q^*(K_{X'} + B' + \mathbf{M}_{X'}) + F$$

where $E \geq 0, F \geq 0$, and $E \wedge F = 0$. Then E and F are q-exceptional. By Lemma 3.7(3) and 3.4(2)(3), $F = N_\sigma(W/U, q^*(K_{X'} + B' + \mathbf{M}_{X'}) + F)$.

We may write $E = E_1 + E_2$ such that E_1 is p-exceptional and every component of E_2 is not p-exceptional. Then p_*E_2 is ϕ-exceptional and therefore by assumption (2), we obtain $\operatorname{Supp} p_*E_2 \subset \operatorname{Supp} N_\sigma(X/U, K_X + B + \mathbf{M}_X)$. By Lemma 3.4(3), we know

$$\operatorname{Supp} E_2 \subset \operatorname{Supp} N_\sigma(W/U, p^*(K_X + B + \mathbf{M}_X)).$$

By Lemma 3.4(2), we have

$$N_\sigma(W/U, p^*(K_X + B + \mathbf{M}_X) + E_1) = N_\sigma(W/U, p^*(K_X + B + \mathbf{M}_X)) + E_1.$$

Therefore,

$$\operatorname{Supp}(E_1 + E_2) \subset \operatorname{Supp} N_\sigma(W/U, p^*(K_X + B + \mathbf{M}_X) + E_1),$$

and then by Lemma 3.7(6), we have

$$\begin{aligned} N_\sigma(W/U, p^*(K_X + B + \mathbf{M}_X) + E_1 + E_2) &= N_\sigma(W/U, p^*(K_X + B + \mathbf{M}_X) + E_1) + E_2 \\ &= N_\sigma(W/U, p^*(K_X + B + \mathbf{M}_X)) + E_1 + E_2, \end{aligned}$$

which immediately implies that

$$\operatorname{Supp} E = \operatorname{Supp}(E_1 + E_2) \subset \operatorname{Supp} N_\sigma(W/U, p^*(K_X + B + \mathbf{M}_X) + E_1 + E_2) = \operatorname{Supp} F,$$

and hence E must be zero. Now by Lemma 3.4(3) again $\operatorname{Supp} p_*F = \operatorname{Supp} N_\sigma(X/U, K_X + B + \mathbf{M}_X)$, which contains all ϕ-exceptional divisors and we are done. $\square$

4. Reduction via Iitaka fibration

Lemma 4.1. *Let $(X,B,\mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC lc g-pair with X klt and $\pi : X \to U$ the induced morphism, such that*

(1) π is an equidimensional contraction,
(2) U is quasi-projective and $\mathbb{Q}$-factorial, and
(3) $\kappa_\sigma(X/U, K_X+B+\mathbf{M}_X) = \kappa_\iota(X/U, K_X+B+\mathbf{M}_X) = 0$.

Let $A \geq 0$ be an ample/U $\mathbb{R}$-divisor on X such that $(X, B+A, \mathbf{M})$ is lc and $K_X+B+A+\mathbf{M}_X$ is nef/U, and run a $(K_X+B+\mathbf{M}_X)$-MMP/U with scaling of A. Then this MMP terminates with a good minimal model $(X', B', \mathbf{M})/U$ of $(X,B,\mathbf{M})/U$. Moreover, $K_{X'}+B'+\mathbf{M}_{X'} \sim_{\mathbb{R},U} 0$.

Proof. If $\dim X = \dim U$, then π is the identity map since π is an equidimensional contraction and there is nothing left to prove. In the following, we assume that $\dim X > \dim U$.

Since $\kappa_\iota(X/U, K_X+B+\mathbf{M}_X) = 0$, $K_X+B+\mathbf{M}_X \sim_{\mathbb{R},U} E \geq 0$ for some $\mathbb{R}$-divisor E on X. We may write $E = E^h + E^v$, such that $E^h \geq 0, E^v \geq 0$, each component of E^h is horizontal over U, and E^v is vertical over U. Since π is equidimensional, the image of any component of E^v on U is a divisor. Since U is $\mathbb{Q}$-factorial, for any prime divisor P on U, we may define

$$\nu_P := \sup\{\nu \geq 0 \mid E^v - \nu\pi^*P \geq 0\}.$$

Then $\nu_P > 0$ for only finitely many prime divisors P on U. Possibly replacing E^v with $E^v - \pi^*(\sum_P \nu_P P)$, we may assume that E^v is very exceptional over U.

Let F be a very general fiber of π. Let

$$(X,B,\mathbf{M}) := (X_0,B_0,\mathbf{M}) \dashrightarrow (X_1,B_1,\mathbf{M}) \dashrightarrow \cdots \dashrightarrow (X_i,B_i,\mathbf{M}) \dashrightarrow \dots$$

be a $(K_X+B+\mathbf{M}_X)$-MMP/U with scaling of A, and let A_i, E_i^h, E_i^v, F_i be the strict transforms of A, E^h, E^v, F on X_i respectively. Then we have

$$\kappa_\sigma(X_i/U, K_{X_i}+B_i+\mathbf{M}_{X_i}) = \kappa_\iota(X_i/U, K_{X_i}+B_i+\mathbf{M}_{X_i}) = 0$$

by Lemma 2.3(6) and hence

$$\kappa_\sigma(E_i^h|_{F_i}) = \kappa_\sigma((K_{X_i}+B_{X_i}+\mathbf{M}_{X_i})|_{F_i}) = \kappa_\iota((K_{X_i}+B_{X_i}+\mathbf{M}_{X_i})|_{F_i}) = \kappa_\iota(E_i^h|_{F_i}) = 0,$$

since $E_i^h|_{F_i} = (E_i^h+E_i^v)|_{F_i} \sim_{\mathbb{R}} (K_{X_i}+B_i+\mathbf{M}_{X_i})|_{F_i}$. As in the proof of [Bir12, Theorem 3.4], there exists a positive integer n such that $K_{X_n}+B_n+\mathbf{M}_{X_n}$ is a movable/U $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. Therefore the restriction $(K_{X_n}+B_n+\mathbf{M}_{X_n})|_{F_n} \sim_{\mathbb{R}} E_n^h|_{F_n}$ is also a movable $\mathbb{R}$-Cartier $\mathbb{R}$-divisor since F is a very general fiber. In particular, $N_\sigma(E_n^h|_{F_n}) = 0$ by Lemma 3.7(3). Notice that now F_n is a normal projective variety and let $g : F_n' \to F_n$ be a resolution of singularities. Then $\kappa_\sigma(g^*(E_n^h|_{F_n})) = \kappa_\iota(g^*(E_n^h|_{F_n})) = 0$. By [Nak04, V, 1.12 Corollary] we have

$$N_\sigma(g^*(E_n^h|_{F_n})) \equiv g^*(E_n^h|_{F_n}).$$

Since $g^*(E_n^h|_{F_n}) \geq 0$, we have $N_\sigma(g^*(E_n^h|_{F_n})) \leq g^*(E_n^h|_{F_n})$ by the definition. Hence we must have

$$N_\sigma(g^*(E_n^h|_{F_n})) = g^*(E_n^h|_{F_n}).$$

Therefore by Lemma 3.4(3) we get

$$E_n^h|_{F_n} = g_*(g^*(E_n^h|_{F_n})) = g_*N_\sigma(g^*(E_n^h|_{F_n})) = N_\sigma(E_n^h|_{F_n}) = 0.$$

This immediately implies that $E_n^h = 0$ since $E_n^h \geq 0$ is horizontal over U. Notice that our $(K_X+B+\mathbf{M}_X)$-MMP/U is also a (E^h+E^v)-MMP/U and E^v is very exceptional over U. By Lemma 2.6, this MMP terminates with a log minimal model $(X', B', \mathbf{M})/U = (X_m, B_m, \mathbf{M})/U$ of $(X,B,\mathbf{M})/U$ for some positive integer m, such that $K_{X'}+B'+\mathbf{M}_{X'} \sim_{\mathbb{R},U} 0$. In particular, $(X', B', \mathbf{M})/U$ is a good minimal model of $(X,B,\mathbf{M})/U$. $\square$

Theorem 4.2. *Let $(X,B,\mathbf{M})/U$ be an NQC lc g-pair and $\pi: X\to V$ a contraction over U such that V is quasi-projective. Assume that $\kappa_\sigma(X/V,K_X+B+\mathbf{M}_X)=\kappa_\iota(X/V,K_X+B+\mathbf{M}_X)=0$. Then there exists a $\mathbb{Q}$-factorial NQC dlt g-pair $(X',B',\mathbf{M})/U$, a contraction $\pi': X'\to V'$ over U, and a birational projective morphism $\varphi: V'\to V$ over U satisfying the following:*

$$\begin{array}{ccc} X' & \dashrightarrow & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ V' & \xrightarrow{\varphi} & V \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

1. *X' is birational to X and V' is smooth,*
2. *$K_{X'}+B'+\mathbf{M}_{X'}\sim_{\mathbb{R},V'}0$.*
3. *$(X,B,\mathbf{M})/U$ has a good minimal model if and only if $(X',B',\mathbf{M})/U$ has a good minimal model.*
4. *Any weak lc model of $(X,B,\mathbf{M})/U$ is a weak lc model of $(X',B',\mathbf{M})/U$, and any weak lc model of $(X',B',\mathbf{M})/U$ is a weak lc model of $(X,B,\mathbf{M})/U$.*
5. *If all lc centers of $(X,B,\mathbf{M})$ dominate V, then all lc centers of $(X',B',\mathbf{M})$ dominate V'.*
6. *$\kappa_\sigma(X/U,K_X+B+\mathbf{M}_X)=\kappa_\sigma(X'/U,K_{X'}+B'+\mathbf{M}_{X'})$ and $\kappa_\iota(X/U,K_X+B+\mathbf{M}_X)=\kappa_\iota(X'/U,K_{X'}+B'+\mathbf{M}_{X'})$*

Proof. Let $h: W\to X$ be a log resolution of $(X,\operatorname{Supp}B)$ such that $\mathbf{M}$ descends to W. By [HL21a, Lemma 3.6], $(X,B,\mathbf{M})$ has a proper log smooth model $(W,B_W,\mathbf{M})$ for some $\mathbb{R}$-divisor B_W on W. By Lemmas 2.3(3) and [HL21a, Lemma 3.7], [HL21a, Theorem 3.14], and [HL21a, Lemmas 3.10, 3.17], we may replace $(X,B,\mathbf{M})$ with $(W,B_W,\mathbf{M})$, and assume that (X,B) is log smooth and $\mathbf{M}$ descends to X.

By Theorem 2.1, there exists a commutative diagram of projective morphisms

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle \pi} \\ V' & \xrightarrow{\varphi} & V \end{array}$$

such that

- f,φ are birational morphisms, π_Y is an equidimensional contraction, Y only has $\mathbb{Q}$-factorial toroidal singularities, and V' is smooth, and
- there exist two $\mathbb{R}$-divisors B_Y and E on Y, such that
  - $K_Y+B_Y+\mathbf{M}_Y=f^*(K_X+B+\mathbf{M}_X)+E$,
  - $B_Y\geq 0$, $E\geq 0$, and $B_Y\wedge E=0$,
  - (Y,B_Y) is lc quasi-smooth, and any lc center of $(Y,B_Y,\mathbf{M})$ on X is an lc center of $(X,B,\mathbf{M})$.

In particular, $(Y,B_Y,\mathbf{M})$ is $\mathbb{Q}$-factorial NQC lc and Y is klt. Since φ is birational, by Lemma 2.3(3) we obtain

$$\kappa_\sigma(Y/V',K_Y+B_Y+\mathbf{M}_Y)=\kappa_\sigma(Y/V,K_Y+B_Y+\mathbf{M}_Y)=\kappa_\sigma(X/V,K_X+B+\mathbf{M}_X)=0$$

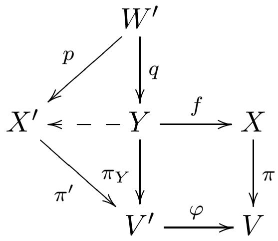
and

$$\kappa_\iota(Y/V',K_Y+B_Y+\mathbf{M}_Y)=\kappa_\iota(Y/V,K_Y+B_Y+\mathbf{M}_Y)=\kappa_\iota(X/V,K_X+B+\mathbf{M}_X)=0.$$

By Lemma 4.1, we may run a $(K_Y+B_Y+\mathbf{M}_Y)$-MMP$/V'$ with scaling of a general ample$/V'$ divisor A on Y, which terminates with a good minimal model $(X',B',\mathbf{M})/V'$ of $(Y,B_Y,\mathbf{M})/V'$ such that $K_{X'}+B'+\mathbf{M}_{X'}\sim_{\mathbb{R},V'}0$. Let $\pi': X'\to V'$ be the induced contraction.

We show that $(X', B', \mathbf{M})/U, \pi', \varphi$ satisfy our requirements. (1)(2) follow from our construction.

Let $p: W' \to Y$ and $q: W' \to X'$ be a resolution of indeterminacies of the induced map $Y \dashrightarrow X'$ such that p is a log resolution of (Y, B_Y).



Then we have

$$p^*(K_Y + B_Y + \mathbf{M}_Y) = q^*(K_{X'} + B' + \mathbf{M}_{X'}) + F$$

for some $F \geq 0$ that is exceptional over X'. Let $B_{W'} := p_*^{-1}B_Y + \operatorname{Exc}(p)$, then $(W', B_{W'}, \mathbf{M})$ is a log smooth model of $(Y, B_Y, \mathbf{M})$ and $(X', B', \mathbf{M})$.

Since $K_Y + B_Y + \mathbf{M}_Y = f^*(K_X + B + \mathbf{M}_X) + E$, by [HL21a, Theorem 3.14], $(X, B, \mathbf{M})/U$ has a good minimal model if and only if $(Y, B_Y, \mathbf{M})/U$ has a good minimal model, if and only if $(W', B_{W'}, \mathbf{M})/U$ has a good minimal model, if and only if $(X', B', \mathbf{M})/U$ has a good minimal model, hence (3).

By [HL21a, Lemmas 3.10, 3.17], a g-pair $(X'', B'', \mathbf{M})/U$ is a weak lc model of $(X, B, \mathbf{M})/U$ if and only if $(X'', B'', \mathbf{M})/U$ is a weak lc model of $(W', B_{W'}, \mathbf{M})/U$, if and only if $(X'', B'', \mathbf{M})/U$ is a weak lc model of $(X', B', \mathbf{M})/U$, hence (4).

Let D be an lc place of $(X', B', \mathbf{M})$. Since $Y \dashrightarrow X'$ is a $(K_Y + B_Y + \mathbf{M}_Y)$-MMP$/V'$, D is an lc place of $(Y, B_Y, \mathbf{M})$, hence an lc place of $(X, B, \mathbf{M})$. Thus if all lc centers of $(X, B, \mathbf{M})$ dominate V, then all lc centers of $(X', B', \mathbf{M})$ dominate V, hence all lc centers of $(X', B', \mathbf{M})$ dominate V' as φ is birational, and we have (5).

Finally, by Lemma 2.3(3) we obtain

$$\begin{aligned}\kappa_\sigma(X/U, K_X + B + \mathbf{M}_X) &= \kappa_\sigma(Y/U, K_Y + B_Y + \mathbf{M}_Y) = \kappa_\sigma(W'/U, p^*(K_Y + B_Y + \mathbf{M}_Y)) \\ &= \kappa_\sigma(W'/U, q^*(K_{X'} + B' + \mathbf{M}_{X'}) + F) \\ &= \kappa_\sigma(X'/U, K_{X'} + B' + \mathbf{M}_{X'})\end{aligned}$$

and

$$\begin{aligned}\kappa_\iota(X/U, K_X + B + \mathbf{M}_X) &= \kappa_\iota(Y/U, K_Y + B_Y + \mathbf{M}_Y) = \kappa_\iota(W'/U, p^*(K_Y + B_Y + \mathbf{M}_Y)) \\ &= \kappa_\iota(W'/U, q^*(K_{X'} + B' + \mathbf{M}_{X'}) + F) \\ &= \kappa_\iota(X'/U, K_{X'} + B' + \mathbf{M}_{X'}),\end{aligned}$$

and we get (6). □

Proposition 4.3 (cf. [Has22a, Lemma 3.10]). *Let $(X, B, \mathbf{M})/U$ be an NQC lc g-pair and $\pi: X \to V$ a contraction over U, such that*

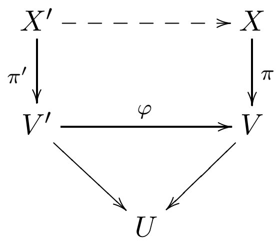
- *V is normal quasi-projective,*
- *$\kappa_\sigma(X/V, K_X + B + \mathbf{M}_X) = \kappa_\iota(X/V, K_X + B + \mathbf{M}_X) = 0$ and $\kappa_\sigma(X/U, K_X + B + \mathbf{M}_X) = \dim V - \dim U$, and*
- *all lc centers of $(X, B, \mathbf{M})$ dominate V.*

Then:

(1) $(X, B, \mathbf{M})/U$ has a good minimal model, and
(2) Let $(\bar{X}, \bar{B}, \mathbf{M})/U$ be a good minimal model of $(X, B, \mathbf{M})/U$ and $\bar{X} \to \bar{V}$ is the contraction over U induced by $K_{\bar{X}} + \bar{B} + \mathbf{M}_{\bar{X}}$. Then all lc centers of $(\bar{X}, \bar{B}, \mathbf{M})$ dominate $\bar{V}$.

Proof. By Theorem 4.2, there exists a $\mathbb{Q}$-factorial NQC dlt g-pair $(X', B', \mathbf{M})/U$, a contraction $\pi' : X' \to V'$ over U, and a birational projective morphism $\varphi : V' \to V$ over U such that

- X' is birational to X and V' is smooth,
- $K_{X'} + B' + \mathbf{M}_{X'} \sim_{\mathbb{R},V'} 0$. In particular, $\kappa_\sigma(X'/V', K_{X'} + B' + \mathbf{M}_{X'}) = 0$ by Lemma 2.3(5),
- $(X, B, \mathbf{M})/U$ has a good minimal model if and only if $(X', B', \mathbf{M})/U$ has a good minimal model,
- any weak lc model of $(X, B, \mathbf{M})/U$ is a weak lc model of $(X', B', \mathbf{M})/U$, and any weak lc model of $(X', B', \mathbf{M})/U$ is a weak lc model of $(X, B, \mathbf{M})/U$,
- all lc centers of $(X', B', \mathbf{M})$ dominate V', and
- $\kappa_\sigma(X'/U, K_{X'}+B'+\mathbf{M}_{X'}) = \kappa_\sigma(X/U, K_X+B+\mathbf{M}_X) = \dim V - \dim U = \dim V' - \dim U$.



Claim 4.4. *Assume that $(X', B', \mathbf{M})/U$ has a good minimal model $(\bar{X}', \bar{B}', \mathbf{M})/U$, $\bar{X}' \to \bar{V}'$ is the contraction over U induced by $K_{\bar{X}'} + \bar{B}' + \mathbf{M}_{\bar{X}'}$, and all lc centers of $(\bar{X}', \bar{B}', \mathbf{M})$ dominate $\bar{V}'$. Then Proposition 4.3(2) holds for $(X, B, \mathbf{M})/U$.*

Proof. Let $(\bar{X}, \bar{B}, \mathbf{M})/U$ be a good minimal model of $(X, B, \mathbf{M})/U$. Then $(\bar{X}, \bar{B}, \mathbf{M})/U$ is a weak lc model of $(X', B', \mathbf{M})/U$. Since $(\bar{X}', \bar{B}', \mathbf{M})/U$ is also a weak lc model of $(X', B', \mathbf{M})/U$, by [HL21a, Lemma 3.9(1)], we may take a resolution of indeterminacies $p : W \to \bar{X}$ and $q : W \to \bar{X}'$ of the induced birational map $\bar{X} \dashrightarrow \bar{X}'$ such that

$$p^*(K_{\bar{X}} + \bar{B} + \mathbf{M}_{\bar{X}}) = q^*(K_{\bar{X}'} + \bar{B}' + \mathbf{M}_{\bar{X}'}).$$

Then:

- $K_{\bar{X}} + \bar{B} + \mathbf{M}_{\bar{X}}$ is semi-ample$/U$, and if we let $\bar{X} \to \bar{V}$ be the contraction over U induced by $K_{\bar{X}} + \bar{B} + \mathbf{M}_{\bar{X}}$, then $\bar{V} = \bar{V}'$ since they are defined by the same linear series.
- Any lc center of $(\bar{X}, \bar{B}, \mathbf{M})$ is an lc center of $(\bar{X}', \bar{B}', \mathbf{M})$, and any lc center of $(\bar{X}', \bar{B}', \mathbf{M})$ is an lc center of $(\bar{X}, \bar{B}, \mathbf{M})$. In particular, since all lc centers of $(\bar{X}', \bar{B}', \mathbf{M})$ dominate $\bar{V}' = \bar{V}$, all lc centers of $(\bar{X}, \bar{B}, \mathbf{M})$ dominate $\bar{V}$.

The claim is proved. □

Proof of Proposition 4.3 continued. By Claim 4.4, we may replace $(X, B, \mathbf{M}), V$ and π with $(X', B', \mathbf{M}), V'$ and π' respectively, and assume that V is smooth and $K_X + B + \mathbf{M}_X \sim_{\mathbb{R},V} 0$. By Theorem 2.14, there exists an NQC klt g-pair $(V, B_V, \mathbf{M}^V)/U$ such that

$$K_X + B + \mathbf{M}_X \sim_{\mathbb{R}} \pi^*(K_V + B_V + \mathbf{M}^V_V).$$

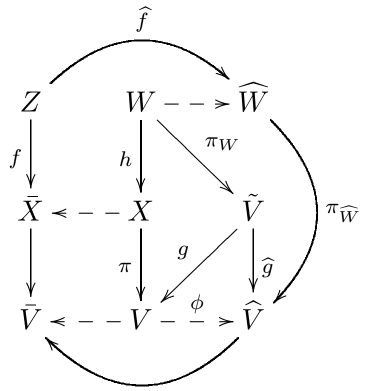
By Lemma 2.3(4)(5), we have

$$\kappa_\sigma(V/U, K_V + B_V + \mathbf{M}^V_V) = \kappa_\sigma(X/U, K_X + B + \mathbf{M}_X) = \dim V - \dim U.$$

By Lemma 2.3(1), $K_V + B_V + \mathbf{M}^V_V$ is big$/U$. By [BZ16, Lemma 4.4(2)], we may run a $(K_V + B_V + \mathbf{M}^V_V)$-MMP$/U$ with scaling of some general ample$/U$ divisor A, which terminates with a good minimal model $(\widehat{V}, B_{\widehat{V}}, \mathbf{M}^V)/U$ of $(V, B_V, \mathbf{M}^V)/U$. Let $\phi : V \dashrightarrow \widehat{V}$ be the induced morphism, and let $g : \tilde{V} \to V$ and $\widehat{g} : \tilde{V} \to \widehat{V}$ be a common resolution such that $\widehat{g} = \phi \circ g$. Then

$$g^*(K_V + B_V + \mathbf{M}^V_V) = \widehat{g}^*(K_{\widehat{V}} + B_{\widehat{V}} + \mathbf{M}^V_{\widehat{V}}) + F.$$

for some $\widehat{g}$-exceptional $\mathbb{R}$-divisor $F \geq 0$ on $\tilde{V}$. Let $h : W \to X$ be a log resolution of $(X, \operatorname{Supp} B)$ such that $\mathbf{M}$ descends to W and the induced map $\pi_W : W \to \tilde{V}$ is a morphism.



By [HL21a, Lemma 3.6], there exists a proper log smooth model $(W, B_W, \mathbf{M})$ of $(X, B, \mathbf{M})$. In particular,

$$K_W + B_W + \mathbf{M}_W = h^*(K_X + B + \mathbf{M}_X) + E$$

for some h-exceptional $\mathbb{R}$-divisor $E \geq 0$. Then

$$\begin{aligned} K_W + B_W + \mathbf{M}_W &= h^*(K_X + B + \mathbf{M}_X) + E \sim_{\mathbb{R}} (\pi \circ h)^*(K_V + B_V + \mathbf{M}^V_V) + E \\ &= \pi_W^* g^*(K_V + B_V + \mathbf{M}^V_V) + E = \pi_W^* \widehat{g}^*(K_{\widehat{V}} + B_{\widehat{V}} + \mathbf{M}^V_{\widehat{V}}) + \pi_W^* F + E. \end{aligned}$$

Since E is exceptional over X, E is very exceptional over V (see [Bir12, Paragraph after Definition 3.1]). Since ϕ is a birational contraction, E is very exceptional over $\widehat{V}$. Since F is exceptional over $\widehat{V}$, $\pi_W^* F$ is very exceptional over $\widehat{V}$. Therefore

$$K_W + B_W + \mathbf{M}_W \sim_{\mathbb{R},\widehat{V}} \pi_W^* F + E$$

is very exceptional over $\widehat{V}$. By Lemma 2.6, we may run a $(K_W+B_W+\mathbf{M}_W)$-MMP$/\widehat{V}$ with scaling of a general ample$/\widehat{V}$ divisor which terminates with a good minimal model $(\widehat{W}, B_{\widehat{W}}, \mathbf{M})/\widehat{V}$ such that $K_{\widehat{W}} + B_{\widehat{W}} + \mathbf{M}_{\widehat{W}} \sim_{\mathbb{R},\widehat{V}} 0$ and the induced birational map $W \dashrightarrow \widehat{W}$ exactly contracts $\operatorname{Supp}(\pi_W^* F + E)$. In particular, let $\pi_{\widehat{W}} : \widehat{W} \to \widehat{V}$ be the induced morphism, then

$$K_{\widehat{W}} + B_{\widehat{W}} + \mathbf{M}_{\widehat{W}} \sim_{\mathbb{R}} \pi_{\widehat{W}}^*(K_{\widehat{V}} + B_{\widehat{V}} + \mathbf{M}^V_{\widehat{V}}).$$

Since $(\widehat{V}, B_{\widehat{V}}, \mathbf{M}^V)/U$ is a good minimal model of $(V, B_V, \mathbf{M}^V)/U$, $K_{\widehat{V}} + B_{\widehat{V}} + \mathbf{M}^V_{\widehat{V}}$ is semi-ample$/U$, hence $K_{\widehat{W}} + B_{\widehat{W}} + \mathbf{M}_{\widehat{W}}$ is semi-ample$/U$. Thus $(\widehat{W}, B_{\widehat{W}}, \mathbf{M})/U$ is a good minimal model of $(W, B_W, \mathbf{M})/U$. By [HL21a, Lemma 3.10], $(\widehat{W}, B_{\widehat{W}}, \mathbf{M})/U$ is a good minimal model of $(X, B, \mathbf{M})/U$, which implies (1).

Let $(\bar{X}, \bar{B}, \mathbf{M})/U$ be a good minimal model of $(X, B, \mathbf{M})/U$. By [HL21a, Lemma 3.9(1)], there exists a resolution $f : Z \to \bar{X}$ and $\widehat{f} : Z \to \widehat{W}$ of indeterminacies of the induced birational map $\bar{X} \dashrightarrow \widehat{W}$ such that

$$f^*(K_{\bar{X}} + \bar{B} + \mathbf{M}_{\bar{X}}) = \widehat{f}^*(K_{\widehat{W}} + B_{\widehat{W}} + \mathbf{M}_{\widehat{W}}).$$

In particular, any lc place of $(\bar{X}, \bar{B}, \mathbf{M})$ is an lc place of $(\widehat{W}, B_{\widehat{W}}, \mathbf{M})$, hence an lc place of $(W, B_W, \mathbf{M})$, and thus an lc place of $(X, B, \mathbf{M})$ by [HL21a, Lemma 3.7]. Therefore, any lc place of $(\bar{X}, \bar{B}, \mathbf{M})$ dominates V. Moreover, since

$$f^*(K_{\bar{X}} + \bar{B} + \mathbf{M}_{\bar{X}}) \sim_{\mathbb{R}} \widehat{f}^* \circ \pi_{\widehat{W}}^*(K_{\widehat{V}} + B_{\widehat{V}} + \mathbf{M}^V_{\widehat{V}}),$$

the contraction $Z \to \bar{V}$ induced by $f^*(K_{\bar{X}} + \bar{B} + \mathbf{M}_{\bar{X}})$ factors through $\widehat{V}$, and the induced morphism $\widehat{V} \to \bar{V}$ is actually given by the big$/U$ semi-ample$/U$ $\mathbb{R}$-divisor $K_{\widehat{V}} + B_{\widehat{V}} + \mathbf{M}^V_{\widehat{V}}$. In particular, the induced map $V \dashrightarrow \bar{V}$ is birational. Thus all lc places of $(\bar{X}, \bar{B}, \mathbf{M})$ dominate $\bar{V}$, hence all lc centers of $(\bar{X}, \bar{B}, \mathbf{M})$ dominate $\bar{V}$, which implies (2). □

5. Applications of the Nakayama-Zariski decomposition

This section is similar to [Has22a, Section 3, before Theorem 3.14].

Lemma 5.1 (cf. [Has22a, Lemma 3.5]). *Let $(X,B,\mathbf{M})/U$ and $(X',B',\mathbf{M})/U$ be NQC dlt g-pairs with a birational map $\phi: X \dashrightarrow X'$ over U such that $\phi_*\mathbf{M}=\mathbf{M}$. Let S and S' be lc centers of $(X,B,\mathbf{M})$ and $(X',B',\mathbf{M})$ respectively, such that ϕ is an isomorphism near the generic point of S, and $\phi|_S: S \dashrightarrow S'$ defines a birational map$/U$. Suppose that*

(1) $K_X+B+\mathbf{M}_X$ is pseudo-effective$/U$,
(2) for any prime divisor D' on X', $a(D',X',B',\mathbf{M})\leq a(D',X,B,\mathbf{M})$, and
(3) for every prime divisor P over X such that $a(P,X,B,\mathbf{M})<1$ and $\operatorname{center}_X(P)\cap S\neq\emptyset$, then $\sigma_P(X/U,K_X+B+\mathbf{M}_X)=0$.

Let $(S,B_S,\mathbf{M}^S)/U$ and $(S',B_{S'},\mathbf{M}^S)/U$ be the dlt g-pairs induced by adjunction

$$K_S+B_S+\mathbf{M}^S_S:=(K_X+B+\mathbf{M}_X)|_S\ ,\ K_{S'}+B_{S'}+\mathbf{M}^S_{S'}:=(K_{X'}+B'+\mathbf{M}_{X'})|_{S'}.$$

Then

$$a(Q,S',B_{S'},\mathbf{M}^S)\leq a(Q,S,B_S,\mathbf{M}^S)$$

for all prime divisors Q on S'.

Proof. The proof follows exactly the same lines as [Has22a, Proof of Lemma 3.5] except the following two places:

- [Has22a, Page 13, Line 30] cites [Has20, Remark 2.3(1)]. We shall replace [Has20, Remark 2.3(1)] with Lemma 3.7(1).
- [Has22a, Page 13, Line 32] cites [Has20, Remark 2.3(3)]. We shall replace [Has20, Remark 2.3(3)] with Lemma 3.4(2).

Therefore, we shall omit the details of the proof to avoid redundance. □

Lemma 5.2 (cf. [HMX18, Lemma 5.3]). *Let $(X,B_1,\mathbf{M})/U$ and $(X,B_2,\mathbf{M})/U$ be $\mathbb{Q}$-factorial NQC dlt g-pairs such that $K_X+B_1+\mathbf{M}_X$ is pseudo-effective$/U$ and*

$$0\leq B_1-B_2\leq N_\sigma(X/U,K_X+B_1+\mathbf{M}_X).$$

Then $(X,B_1,\mathbf{M})/U$ has a log minimal model (resp. good minimal model) if and only if $(X,B_2,\mathbf{M})/U$ has a log minimal model (resp. good minimal model).

Proof. First we assume that $(X,B_1,\mathbf{M})/U$ has a log minimal model (resp. good minimal model). By [HL21a, Theorem 2.24], we may run a $(K_X+B_1+\mathbf{M}_X)$-MMP$/U$ which terminates with a log minimal model (resp. good minimal model) $(X',B',\mathbf{M})/U$ with induced birational map $\phi: X\dashrightarrow X'$ over U. By Lemmas 3.3(1) and 3.9(2), ϕ contracts every component of $\operatorname{Supp} N_\sigma(X/U,K_X+B_1+\mathbf{M}_X)$. Thus B' is also the strict transform of B_2 on X'.

Let $p: W\to X$ and $q: W\to X'$ be a resolution of indeterminacies of ϕ, and write

$$p^*(K_X+B_1+\mathbf{M}_X)=q^*(K_{X'}+B'+\mathbf{M}_{X'})+E$$

for some effective q-exceptional $\mathbb{R}$-divisor E on W. Then by Lemmas 3.3(1) and 3.4(2)(3), $N_\sigma(X/U,K_X+B_1+\mathbf{M}_X)=p_*E$ is well-defined as a divisor. Let $F:=E-p^*(B_1-B_2)$. Then

$$F\geq E-p^*N_\sigma(X/U,K_X+B_1+\mathbf{M}_X)=E-p^*p_*E$$

and

$$p^*(K_X+B_2+\mathbf{M}_X)=q^*(K_{X'}+B'+\mathbf{M}_{X'})+F.$$

Since $E-p^*p_*E$ is p-exceptional, $p_*F\geq 0$. By the negativity lemma, $F\geq 0$. Thus $(X',B',\mathbf{M})/U$ is a weak lc model of $(X,B_2,\mathbf{M})/U$. By [HL21a, Lemmas 3.9(2), 3.15], $(X,B_2,\mathbf{M})/U$ has a log minimal model (resp. good minimal model).

Now we assume that $(X,B_2,\mathbf{M})/U$ has a log minimal model (resp. good minimal model). By [HL21a, Theorem 2.24], we may run a $(K_X+B_2+\mathbf{M}_X)$-MMP$/U$ which terminates with a log minimal model (resp. good minimal model) $(X',B',\mathbf{M})/U$ with induced birational map

$\phi : X \dashrightarrow X'$ over U. Let $C := B_1 - B_2$. Then ϕ is also a $(K_X + B_2 + \epsilon C + \mathbf{M}_X)$-MMP/$U$ for any $0 < \epsilon \ll 1$ by Lemma 2.7. Let C' be the strict transform of C on X'. By Lemma 3.7(5), we obtain

$$N_\sigma(X/U, K_X + B_2 + \epsilon C + \mathbf{M}_X) + (1-\epsilon)C = N_\sigma(X/U, K_X + B_1 + \mathbf{M}_X),$$

and

$$N_\sigma(X/U, K_X + B_2 + \mathbf{M}_X) + C = N_\sigma(X/U, K_X + B_1 + \mathbf{M}_X)$$

for any $\epsilon \in [0, 1]$. Therefore, we have

$$N_\sigma(X/U, K_X + B_2 + \epsilon C + \mathbf{M}_X) = N_\sigma(X/U, K_X + B_2 + \mathbf{M}_X) + \epsilon C$$

for any $\epsilon \in [0, 1]$. Hence, if $\epsilon \in (0, 1]$, then

$$\operatorname{Supp} N_\sigma(X/U, K_X + B_2 + \epsilon C + \mathbf{M}_X) = \operatorname{Supp} N_\sigma(X/U, K_X + B_1 + \mathbf{M}_X),$$

since they are both equal to $\operatorname{Supp} N_\sigma(X/U, K_X + B_2 + \mathbf{M}_X) \cup \operatorname{Supp} C$. Moreover, by [HL22, Lemma 3.21], we may pick $0 < \epsilon \ll 1$ such that any partial $(K_{X'} + B_2' + \epsilon C' + \mathbf{M}_{X'})$-MMP/$U$ is $(K_{X'} + B' + \mathbf{M}_{X'})$-trivial/$U$. We run a $(K_{X'} + B' + \epsilon C' + \mathbf{M}_{X'})$-MMP/$U$ with scaling of an ample/U $\mathbb{R}$-divisor. By Lemma 2.8, after finitely many steps we get a birational map $\psi : X' \dashrightarrow X''$ such that $K_{X''} + B'' + \epsilon C'' + \mathbf{M}_{X''}$ is a movable/U $\mathbb{R}$-divisor, where B'' and C'' are the strict transforms of B' and C' on X'' respectively. By Lemma 3.9(2), the set of $(\psi \circ \phi)$-exceptional divisors is exactly $\operatorname{Supp} N_\sigma(X/U, K_X + B_2 + \epsilon C + \mathbf{M}_X) = \operatorname{Supp} N_\sigma(X/U, K_X + B_1 + \mathbf{M}_X)$. Thus, by assumption, $\operatorname{Supp} C = \operatorname{Supp}(B_1 - B_2)$ is also $(\psi \circ \phi)$-exceptional. Then $C'' = 0$ since it is the pushforward of C to X'', hence B'' is also the strict transform of B_1 on X'' and $K_{X''} + B'' + \mathbf{M}_{X''}$ is nef/U (resp. semi-ample/U) by construction. By Lemma 3.10, $(X'', B'', \mathbf{M})/U$ is a log minimal model of $(X, B_1, \mathbf{M})/U$. The lemma follows from [HL21a, Lemma 3.9(2)]. □

Lemma 5.3 (cf. [Has22a, Lemma 3.6])**.** *Let $(X, B, \mathbf{M})/U$ and $(Y, B_Y, \mathbf{M})/U$ be NQC lc g-pairs and $f : Y \to X$ a projective birational morphism such that*

(1) $K_X + B + \mathbf{M}_X$ is pseudo-effective/U, and
(2) for any prime divisor D on Y,

$$0 \le a(D, Y, B_Y, \mathbf{M}) - a(D, X, B, \mathbf{M}) \le \sigma_D(X/U, K_X + B + \mathbf{M}_X).$$

Then $K_Y + B_Y + \mathbf{M}_Y$ is pseudo-effective/U. Moreover, $(X, B, \mathbf{M})/U$ has a log minimal model (resp. good minimal model) if and only if $(Y, B_Y, \mathbf{M})/U$ has a log minimal model (resp. good minimal model).

Proof. The assumptions imply that

$$0 \le f^*(K_X + B + \mathbf{M}_X) - (K_Y + B_Y + \mathbf{M}_Y) \le N_\sigma(Y/U, f^*(K_X + B + \mathbf{M}_X))$$

and then $K_Y + B_Y + \mathbf{M}_Y$ is pseudo-effective/U by Lemma 3.4(4).

Let $g : W \to Y$ be a log resolution of $(Y, \operatorname{Supp} B_Y)$ such that $\mathbf{M}$ descends to W and $h := f \circ g$ is a log resolution of $(X, \operatorname{Supp} B)$. Let $B_W := h_*^{-1}B + \operatorname{Supp}\operatorname{Exc}(h)$ and $B_W' := g_*^{-1}B_Y + \operatorname{Supp}\operatorname{Exc}(g)$. Then we have

$$K_W + B_W + \mathbf{M}_W = h^*(K_X + B + \mathbf{M}_X) + E$$

for some $E_W \ge 0$ that is exceptional/X. By Lemma 3.4(1)(2),

$$\sigma_P(W/U, K_W + B_W + \mathbf{M}_W) = \sigma_P(X/U, K_X + B + \mathbf{M}_X) + \operatorname{mult}_P E$$

for any prime divisor P on W.

Claim 5.4. *For any prime divisor P on W, we have*

$$0 \le a(P, W, B_W', \mathbf{M}) - a(P, W, B_W, \mathbf{M}) \le \sigma_P(W/U, K_W + B_W + \mathbf{M}_W).$$

Grant Claim 5.4 for the time being. By Claim 5.4 and [HL21a, Theorem 3.14], possibly replacing $(X,B,\mathbf{M})/U$ and $(Y,B_Y,\mathbf{M})/U$ with $(W,B_W,\mathbf{M})/U$ and $(W,B'_W,\mathbf{M})/U$ respectively, we may assume that $(X,B,\mathbf{M})$ and $(Y,B_Y,\mathbf{M})$ are $\mathbb{Q}$-factorial dlt and $X=Y$. The lemma follows from Lemma 5.2. □

Proof of Claim 5.4. For any prime divisor P on W, one of the following cases holds:

Case 1. P is not exceptional over X. In this case

$$a(P,W,B'_W,\mathbf{M})-a(P,W,B_W,\mathbf{M})=a(P,Y,B_Y,\mathbf{M})-a(P,X,B,\mathbf{M})$$

and the claim follows.

Case 2. P is exceptional over X but not exceptional over Y. In this case $a(P,W,B_W,\mathbf{M})=0$, $a(P,W,B'_W,\mathbf{M})=a(P,Y,B_Y,\mathbf{M})$, and $a(P,X,B,\mathbf{M})=\operatorname{mult}_P E$, so

$$0\le a(P,Y,B_Y,\mathbf{M})=a(P,W,B'_W,\mathbf{M})-a(P,W,B_W,\mathbf{M}),$$

and

$$\begin{aligned}a(P,Y,B_Y,\mathbf{M})&\le\sigma_P(X/U,K_X+B+\mathbf{M}_X)+a(P,X,B,\mathbf{M})\\&=\sigma_P(X/U,K_X+B+\mathbf{M}_X)+\operatorname{mult}_P E=\sigma_P(W/U,K_W+B_W+\mathbf{M}_W)\end{aligned}$$

and the claim follows.

Case 3. P is exceptional over Y. In this case $a(P,W,B_W,\mathbf{M})=a(P,W,B'_W,\mathbf{M})=0$, and the claim follows. □

Lemma 5.5 (cf. [Has22a, Lemma 3.8]). *Let $(X,B,\mathbf{M})/U$ be an NQC lc g-pair with induced morphism $\pi:X\to U$ such that U is quasi-projective. Let S be a subvariety of X, and*

$$(X,B,\mathbf{M}):=(X_0,B_0,\mathbf{M})\dashrightarrow(X_1,B_1,\mathbf{M})\dashrightarrow\cdots\dashrightarrow(X_n,B_n,\mathbf{M})\dashrightarrow\dots$$

a $(K_X+B+\mathbf{M}_X)$-MMP/U with scaling of an $\mathbb{R}$-divisor $A\ge 0$. Let

$$\lambda_i:=\inf\{t\ge 0\mid K_{X_i}+B_i+\mathbf{M}_{X_i}+tA_i\text{ is nef}/U\}$$

be the scaling numbers, where A_i is the strict transform of A on X_i. Suppose that

- *each step of this MMP is an isomorphism on a neighborhood of S, and*
- $\lim_{i\to+\infty}\lambda_i=0$.

Then

(1) for any π-ample $\mathbb{R}$-divisor H on X and any closed point $x\in S$, there exists an $\mathbb{R}$-divisor E such that $0\le E\sim_{\mathbb{R},U}K_X+B+\mathbf{M}_X+H$ and $x\notin\operatorname{Supp}E$, and

(2) for any prime divisor P over X such that $\operatorname{center}_X P\cap S\neq\emptyset$, $\sigma_P(X/U,K_X+B+\mathbf{M}_X)=0$.

Proof. (1) follows from [Has22a, Lemma 3.8] and (2) follows from (1) and Lemma 3.7(4). □

Lemma 5.6 (cf. [Has22a, Lemma 3.9]). *Let $(X,B,\mathbf{M})/U$ and $(X',B',\mathbf{M})/U$ be two NQC lc g-pairs and $\phi:X\dashrightarrow X'$ a birational map such that $\phi_*\mathbf{M}=\mathbf{M}$. Suppose that*

- *$a(P,X,B,\mathbf{M})\le a(P,X',B',\mathbf{M})$ for any prime divisor P on X, and*
- *$a(P',X',B',\mathbf{M})\le a(P',X,B,\mathbf{M})$ for any prime divisor P' on X'.*

Then

(1) $K_X+B+\mathbf{M}_X$ is abundant/U if and only if $K_{X'}+B'+\mathbf{M}_{X'}$ is abundant/U, and

(2) $(X,B,\mathbf{M})/U$ has a log minimal model (resp. good minimal model) if and only if $(X',B',\mathbf{M})/U$ has a log minimal model (resp. good minimal model).

Proof. Let $p:W\to X$ and $q:W\to X'$ be a resolution of indeterminacies such that $\mathbf{M}$ descends to W, p is a log resolution of $(X,\operatorname{Supp}B)$, and q is a log resolution of $(X',\operatorname{Supp}B')$. Let

$$B_W:=\sum_{D\text{ is a prime divisor on }W}\max\{1-a(D,X,B,\mathbf{M}),1-a(D,X',B',\mathbf{M}),0\}D.$$

Then $(W, B_W, \mathbf{M})$ is lc and (W, B_W) is log smooth. By construction, there exists a p-exceptional $\mathbb{R}$-divisor $E \geq 0$ and a q-exceptional $\mathbb{R}$-divisor $F \geq 0$ such that

$$E + p^*(K_X + B + \mathbf{M}_X) = K_W + B_W + \mathbf{M}_W = q^*(K_{X'} + B' + \mathbf{M}_{X'}) + F.$$

(1) follows from Lemma 2.3(3) and (2) follows from [HL21a, Theorem 3.14]. ☐

6. A special log minimal model

The purpose of this section is to prove Theorem 6.1 and Theorem 6.6, which are analogues of [Has22a, Theorem 3.14 and Theorem 3.15] in the relative setting.

Theorem 6.1 (cf. [Has22a, Theorem 3.14])**.** *Let $(X, B, \mathbf{M})/U$ be an NQC dlt g-pair such that*

- *$K_X + B + \mathbf{M}_X$ is pseudo-effective/U and abundant/U,*
- *for any lc center S of $(X, B, \mathbf{M})$, $(K_X + B + \mathbf{M}_X)|_S$ is nef/U, and*
- *for any prime divisor P over X such that $a(P, X, B, \mathbf{M}) < 1$ and $\operatorname{center}_X P \cap \operatorname{Nklt}(X, B, \mathbf{M}) \neq \emptyset$, $\sigma_P(X/U, K_X + B + \mathbf{M}_X) = 0$.*

Then $(X, B, \mathbf{M})/U$ has a log minimal model.

Proof. We divide the proof in six steps.

Step 1. In this step we show that we may replace $(X, B, \mathbf{M})$ with a $\mathbb{Q}$-factorial dlt model and find two $\mathbb{R}$-divisors $G \geq 0, H \geq 0$, and a real number $1 > t_0 > 0$ such that

(I) $K_X + B + \mathbf{M}_X \sim_{\mathbb{R},U} G + H$,
(II) $\operatorname{Supp} G \subset \operatorname{Supp}\lfloor B\rfloor$, and
(III) for any $t \in (0, t_0]$, the following hold:
 (III.1) $(X, B + tH, \mathbf{M})/U$ is dlt, $N_\sigma(X/U, K_X + B + tH + \mathbf{M}_X)$ is well-defined as a divisor and $\operatorname{Supp} N_\sigma(X/U, K_X + B + tH + \mathbf{M}_X)$ does not depend on t, and
 (III.2) $(X, B - tG, \mathbf{M})/U$ has a good minimal model.

Since $K_X + B + \mathbf{M}_X$ is pseudo-effective/U and abundant/U, $K_X + B + \mathbf{M}_X \sim_{\mathbb{R},U} D \geq 0$ for some $\mathbb{R}$-divisor D on X. Let $X \dashrightarrow V$ be the Iitaka fibration/U associated to D. Then $\dim V - \dim U = \kappa_\sigma(X/U, K_X + B + \mathbf{M}_X)$. Let $h : W \to X$ be a log resolution of $(X, \operatorname{Supp} B)$ such that $\mathbf{M}$ descends to W and the induced map $\psi : W \dashrightarrow V$ is a morphism. Then we may write

$$K_W + B_W + \mathbf{M}_W = h^*(K_X + B + \mathbf{M}_X) + E$$

such that $B_W \geq 0, E \geq 0$, and $B_W \wedge E = 0$. Notice that $(W, B_W, \mathbf{M})$ is a log smooth model of $(X, B, \mathbf{M})$. By Lemma 2.5,

(i) $\kappa_\sigma(W/U, K_W + B_W + \mathbf{M}_W) = \dim V - \dim U$ and $\kappa_\sigma(W/V, K_W + B_W + \mathbf{M}_W) = 0$.

Thus by construction $K_W + B_W + \mathbf{M}_W$ is $\mathbb{R}$-linearly equivalent/U to the sum of an effective $\mathbb{R}$-divisor and the pullback of an ample/U $\mathbb{R}$-divisor on V. In particular, we may find $0 \leq D_W \sim_{\mathbb{R},U} K_W + B_W + \mathbf{M}_W$ such that $\operatorname{Supp} D_W$ contains all lc centers of $(W, B_W, \mathbf{M})$ that are vertical over V.

Let $(\bar{X}, \bar{B}, \mathbf{M})$ be a proper log smooth model of $(W, B_W, \mathbf{M})$ with induced morphism $g : \bar{X} \to W$ such that g is a log resolution of $(W, B_W + D_W)$, and

$$K_{\bar{X}} + \bar{B} + \mathbf{M}_{\bar{X}} = g^*(K_W + B_W + \mathbf{M}_W) + \bar{E}$$

for some $\bar{E} \geq 0$. By Lemma 2.13, possibly replacing $(\bar{X}, \bar{B}, \mathbf{M})$ with a higher model, we may assume that there is a decomposition $\bar{B} = \bar{B}^h + \bar{B}^v$ such that

(ii) $\bar{B}^h \geq 0$ and $\bar{B}^v$ is reduced,
(iii) $\bar{B}^v$ is vertical over V, and
(iv) for any $t \in (0, 1]$, all lc centers of $(\bar{X}, \bar{B} - t\bar{B}^v, \mathbf{M})$ dominate V.

Let $\bar{D} := g^*D_W + \bar{E}$. Then $(\bar{X}, \bar{B} + \bar{D})$ is log smooth and $\bar{D} \sim_{\mathbb{R},U} K_{\bar{X}} + \bar{B} + \mathbf{M}_{\bar{X}}$. Since $\operatorname{Supp} D_W$ contains any vertical lc center of $(W, B_W, \mathbf{M})$, by [HL21a, Lemma 3.7] we have $\operatorname{Supp} \bar{B}^v \subset \operatorname{Supp} \bar{D}$. Thus we may find $\bar{G}, \bar{H} \geq 0$ and write $\bar{D} = \bar{G} + \bar{H}$ such that

(v) $K_{\bar{X}}+\bar{B}+\mathbf{M}_{\bar{X}}\sim_{\mathbb{R},U}\bar{G}+\bar{H}$,
(vi) $\operatorname{Supp}\bar{B}^v\subset\operatorname{Supp}\bar{G}\subset\operatorname{Supp}\lfloor\bar{B}\rfloor$, and
(vii) no component of $\bar{H}$ is contained in $\lfloor\bar{B}\rfloor$ and $(\bar{X},\bar{B}+\bar{H})$ is log smooth.

We fix a real number $t_1\in(0,1)$ such that $\bar{B}-t_1\bar{G}\geq 0$. For any $t\in(0,t_1]$, by (ii)(iii)(iv)(vi), any lc center of $(\bar{X},\bar{B}-t\bar{G},\mathbf{M})$ dominates V. By (i)(v) and Lemma 2.3(2), we have

$$\kappa_\sigma(\bar{X}/U,K_{\bar{X}}+\bar{B}-t\bar{G}+\mathbf{M}_{\bar{X}})=\dim V-\dim U$$

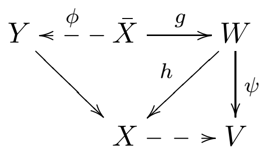
and

$$\kappa_\sigma(\bar{X}/V,K_{\bar{X}}+\bar{B}-t\bar{G}+\mathbf{M}_{\bar{X}})=\kappa_\iota(\bar{X}/V,K_{\bar{X}}+\bar{B}-t\bar{G}+\mathbf{M}_{\bar{X}})=0.$$

Then by Proposition 4.3 we obtain

(viii) $(\bar{X},\bar{B}-t\bar{G},\mathbf{M})/U$ has a good minimal model for any $t\in(0,t_1]$.

Since $(\bar{X},\bar{B},\mathbf{M})$ is a log smooth model of $(X,B,\mathbf{M})$, we may run a $(K_{\bar{X}}+\bar{B}+\mathbf{M}_{\bar{X}})$-MMP$/X$ which terminates with a dlt model $(Y,B_Y,\mathbf{M})$ of $(X,B,\mathbf{M})$ with induced morphism $f:Y\to X$ and birational map $\phi:\bar{X}\dashrightarrow Y$. Let G_Y and H_Y be the strict transforms of $\bar{G}$ and $\bar{H}$ on Y respectively. Then $K_Y+B_Y+\mathbf{M}_Y\sim_{\mathbb{R},U}G_Y+H_Y$. By (vii) and Lemma 2.7, there exists $0<t_2<t_1$ such that $(\bar{Y},\bar{B}+t_2\bar{H},\mathbf{M})$ is dlt and ϕ is a $(K_{\bar{X}}+\bar{B}+t\bar{H}+\mathbf{M}_{\bar{X}})$-MMP$/X$ as well as a $(K_{\bar{X}}+\bar{B}-t\bar{G}+\mathbf{M}_{\bar{X}})$-MMP$/X$ for any $t\in(0,t_2]$. Then $(Y,B_Y+t_2H_Y,\mathbf{M})$ is dlt, and by (viii) and [HL21a, Theorem 2.24, Lemma 3.9(2)], $(Y,B_Y-tG,\mathbf{M}))/U$ has a good minimal model for any $t\in(0,t_2]$. $N_\sigma(Y/U,K_Y+B_Y+tH_Y+\mathbf{M}_Y)$ is well-defined as a divisor since $K_Y+B_Y+tH_Y+\mathbf{M}_Y\sim_{\mathbb{R},U}G_Y+(1+t)H_Y$ is effective for any $t\geq 0$. By Lemma 3.8, we may pick $0<t_0<t_2$ such that $\operatorname{Supp}N_\sigma(Y/U,K_Y+B_Y+tH_Y+\mathbf{M}_Y)$ does not depend on t for any $t\in(0,t_0]$.



We may replace $(X,B,\mathbf{M})$ with $(Y,B_Y,\mathbf{M})$ and let $G:=G_Y$ and $H:=H_Y$, and assume that $(X,B,\mathbf{M}),G,H$ and t_0 satisfy (I)(II)(III). In what follows, we forget all other auxiliary varieties and divisors constructed in this step.

Step 2. For any $t\in(0,t_0]$, by (III.2), $(X,B-\frac{t}{1+t}G,\mathbf{M})/U$ has a good minimal model. Since

$$K_X+B+tH+\mathbf{M}_X\sim_{\mathbb{R},U}(1+t)(K_X+B-\frac{t}{1+t}G+\mathbf{M}_X),$$

by (III.1) and [HL21a, Theorem 2.24, Lemma 3.9(2), 4.2], we may run a $(K_X+B+tH+\mathbf{M}_X)$-MMP$/U$ $\phi_t:X\dashrightarrow X_t$ which terminates with a good minimal model $(X_t,B_t+tH_t,\mathbf{M})/U$ of $(X,B+tH,\mathbf{M})/U$. By Lemma 3.9(2), the divisors contracted by ϕ_t are precisely the components of $\operatorname{Supp}N_\sigma(X/U,K_X+B+tH+\mathbf{M}_X)$. Since $\operatorname{Supp}N_\sigma(X/U,K_X+B+tH+\mathbf{M}_X)$ does not depend on $t\in(0,t_0]$ by (III.1), each MMP ϕ_t contracts precisely the components of $\operatorname{Supp}N_\sigma(X/U,K_X+B+t_0H+\mathbf{M}_X)$. We let $X_0:=X_{t_0},B_0:=B_{t_0}$, and $H_0:=H_{t_0}$. Then X_0 and X_t are isomorphic in codimension 1, and $K_{X_0}+B_0+\mathbf{M}_{X_0}$ is a movable$/U$ $\mathbb{R}$-divisor. By the negativity lemma, $(X_t,B_t+tH_t,\mathbf{M})/U$ is a good minimal model of $(X_0,B_0+tH_0,\mathbf{M})/U$ for any $t\in(0,t_0]$.

Claim 6.2. *We may run a $(K_{X_0}+B_0+\mathbf{M}_{X_0})$-MMP$/U$ with scaling of H_0*

$$(X_0,B_0,\mathbf{M})\dashrightarrow(X_1,B_1,\mathbf{M})\dashrightarrow\cdots\dashrightarrow(X_i,B_i,\mathbf{M})\dashrightarrow\dots$$

with scaling numbers

$$\lambda_i:=\inf\{t\geq 0\mid K_{X_i}+B_i+tH_i+\mathbf{M}_{X_i}\text{ is nef}/U\},$$

where H_i is the strict transform of H on X_i, which consists only of flips such that

(1) either the MMP$/U$ terminates with a minimal model, or $\lim_{i\to+\infty}\lambda_i=0$,

(2) for any $i \geq 1$ and $\lambda \in [\lambda_i, \lambda_{i-1}]$, $(X_i, B_i + \lambda H_i, \mathbf{M})/U$ is a good minimal model of both $(X, B + \lambda H, \mathbf{M})$ and $(X_0, B_0 + \lambda H_0, \mathbf{M})/U$, and

(3) the MMP only contracts sub-varieties of $\operatorname{Supp}\lfloor B_0 \rfloor$.

Proof. Since $K_{X_0} + B_0 + \mathbf{M}_{X_0}$ is a movable/U $\mathbb{R}$-divisor, by Lemma 2.9, any $(K_{X_0} + B_0 + \mathbf{M}_{X_0})$-MMP/$U$ only contains flips. (1) follows from Lemma 2.10. For any $i \geq 1$ and $\lambda \in [\lambda_i, \lambda_{i-1}]$, $(X_i, B_i + \lambda H_i, \mathbf{M})$ is dlt and $K_{X_i} + B_i + \lambda H_i + \mathbf{M}_{X_i}$ is nef/U. Since the induced birational maps $X_0 \dashrightarrow X_\lambda$ and $X_i \to X_\lambda$ are both small, by Lemma 3.10 and [HL21a, Lemma 3.9(2)], we get (2).

Let $X_i \to Z_i \leftarrow X_{i+1}$ be the i-th step of the MMP where $X_i \to Z_i$ the flipping contraction. Then for any flipping curve C_i of $X_i \to Z_i$, we have $(K_{X_i} + B_i + \mathbf{M}_{X_i}) \cdot C_i < 0$ and $H_i \cdot C_i > 0$. Let G_i be the strict transform of G on X_i. Then $0 > (K_{X_i} + B_i + \mathbf{M}_{X_i} - H_i) \cdot C_i = G_i \cdot C_i$. Thus $C_i \subset \operatorname{Supp} G_i$. Since $\operatorname{Supp} G \subset \operatorname{Supp}\lfloor B \rfloor$, $\operatorname{Supp} G_i \subset \operatorname{Supp}\lfloor B_i \rfloor$, and we get (3). □

Claim 6.3. *Let*

$$(X_0, B_0, \mathbf{M}) \dashrightarrow (X_1, B_1, \mathbf{M}) \dashrightarrow \cdots \dashrightarrow (X_i, B_i, \mathbf{M}) \dashrightarrow \dots,$$

λ_i, and H_i be the MMP/U, the scaling numbers, and the strict transform of H on X_i for each i as in Claim 6.2 respectively. If the MMP/U terminates, then Theorem 6.1 holds.

Proof. Let $\lambda_{-1} := t_0$. If the MMP/U terminates, then $\lambda_{l-1} > \lambda_l = 0$ for some $l \in \mathbb{N}$. By Claim 6.2(2), for any $t \in (0, \lambda_{l-1}]$, $K_{X_l} + B_l + tH_l + \mathbf{M}_{X_l}$ is nef/U, and $a(P, X, B + tH, \mathbf{M}) \leq a(P, X_l, B_l + tH_l, \mathbf{M})$ for any prime divisor P on X that is exceptional/X_l. Letting $t \to 0$, we have that $K_{X_l} + B_l + \mathbf{M}_{X_l}$ is nef/U and $a(P, X, B, \mathbf{M}) \leq a(P, X_l, B_l, \mathbf{M})$ for any prime divisor P on X that is exceptional/X_l. Thus $(X_l, B_l, \mathbf{M})/U$ is a weak lc model of $(X, B, \mathbf{M})/U$. The Claim follows from [HL21a, Lemma 3.15]. □

Proof of Theorem 6.1 continued. In the following, we let

$$(X_0, B_0, \mathbf{M}) \dashrightarrow (X_1, B_1, \mathbf{M}) \dashrightarrow \cdots \dashrightarrow (X_i, B_i, \mathbf{M}) \dashrightarrow \dots,$$

λ_i, and H_i be the MMP/U, the scaling numbers, and the strict transform of H on X_i for each i as in Claim 6.2 respectively.

Step 3. For every i and lc center S_i of $(X_i, B_i, \mathbf{M})$, we let $(S_i, B_{S_i}, \mathbf{M}^{S_i})$ be the g-pair induced by adjunction

$$K_{S_i} + B_{S_i} + \mathbf{M}^{S_i}_{S_i} := (K_{X_i} + B_i + \mathbf{M}_{X_i})|_{S_i},$$

and let $H_{S_i} := H_i|_{S_i}$. For every lc center S of $(X, B, \mathbf{M})$ we let $(S, B_S, \mathbf{M}^S)$ be the g-pair induced by adjunction

$$K_S + B_S + \mathbf{M}^S_S := (K_X + B + \mathbf{M}_X)|_S$$

and let $H_S := H|_S$. Since $X_0 \dashrightarrow X_i$ is a $(K_{X_0} + B_0 + \mathbf{M}_{X_0})$-MMP/$U$, $X_0 \dashrightarrow X_i$ is an isomorphism near the generic point of S_i, the strict transform S_0 of S_i on X_0 is an lc center of $(X_0, B_0, \mathbf{M})$, hence also an lc center of $(X_0, B_0 + t_0 H_0, \mathbf{M})$. By the same argument, $\phi_{t_0} : X \dashrightarrow X_0$ is an isomorphism near the generic point of S_0 and the strict transform S of S_0 on X is an lc center of $(X, B + t_0 H, \mathbf{M})$. Since $(X, B, \mathbf{M})$ and $(X, B + tH, \mathbf{M})$ are both dlt and have the same lc centers, S is also an lc center of $(X, B, \mathbf{M})$. In particular, the induced maps $\phi^S_i : S \dashrightarrow S_i$ and $\phi^S_{j,i} : S_j \dashrightarrow S_i$ are birational for any $j \leq i$.

By Lemma 2.18(1), we may find $m \gg 0$ such that $X_m \dashrightarrow X_i$ is an isomorphism near the generic point of any lc center S_m of $(X_m, B_m, \mathbf{M})$ and any $i \geq m$. By Lemma 2.18(2.a), possibly replacing m, we may assume that the induced $\phi^S_{m,i} : S_m \to S_i$ is small for any lc center S_m of $(X_m, B_m, \mathbf{M})$ and any $i \geq m$.

Then we only need to show that for any lc center S_m of $(X_m, B_m, \mathbf{M})$ of dimension ≥ 1, $(S_m, B_{S_m}, \mathbf{M}^S)/U$ has a log minimal model. Indeed, if this is the case, then by Lemma 2.18, [HL22, Remark 3.25, Theorem 4.1], Claim 6.2(1), and induction on the dimension of lc centers, the $(K_{X_0} + B_0 + \mathbf{M}_{X_0})$-MMP/$U$ above will induce isomorphisms on any lc center S_m

of $(X_m, B_m, \mathbf{M})$ for $m \gg 0$. But by Claim 6.2(3) the $(K_{X_0}+B_0+\mathbf{M}_{X_0})$-MMP$/U$ only contracts sub-varieties of $\operatorname{Supp}\lfloor B_0\rfloor$, so it must terminate.

Step 4. We prove the following claim.

Claim 6.4. *There exists a $\mathbb{Q}$-factorial lc g-pair $(T, B_T, \mathbf{M}^S)/U$ and a birational morphism $\psi : T \to S_m$ satisfying the following:*

(1) For any prime divisor D on S such that $a(D, S_m, B_{S_m}, \mathbf{M}^S) < a(D, S, B_S, \mathbf{M}^S)$, D is on T and is a ψ-exceptional.

(2)
$$B_T = \sum_{D \text{ is a prime divisor on } T} (1 - a(D, S, B_S, \mathbf{M}^S))D.$$

(3) For any $i \geq m$ and any prime divisor Q over S, we have
$$a(Q, S, B_S + \lambda_i H_S, \mathbf{M}^S) \leq a(Q, S_i, B_{S_i} + \lambda_i H_{S_i}, \mathbf{M}^S).$$

(4) For any prime divisor Q' over S_m, we have
$$a(Q', S_m, B_{S_m}, \mathbf{M}^S) \leq a(Q', T, B_T, \mathbf{M}^S).$$

Proof. By Claim 6.2(2), $(X_i, B_i + \lambda_i H_i, \mathbf{M})/U$ is a good minimal model of $(X, B + \lambda_i H, \mathbf{M})$, hence (3) holds.

Since ϕ_i does not extract any divisor, $a(P, X_i, B_i, \mathbf{M}) \leq a(P, X, B, \mathbf{M})$ for any prime divisor P on X_i. Since $\sigma_P(X/U, K_X + B + \mathbf{M}_X) = 0$ for any prime divisor P over X such that $a(P, X, B, \mathbf{M}) < 1$ and $\operatorname{center}_X P \cap \operatorname{Nklt}(X, B, \mathbf{M}) \neq \emptyset$, by Lemma 5.1 and since $\phi_{m,i}$ is small for any $i \geq m$, $a(D, S_i, B_{S_i}, \mathbf{M}^S) \leq a(D, S, B_S, \mathbf{M}^S)$ for any prime divisor D on S_m and $i \geq m$. Thus $a(D, S_i, B_{S_i} + \lambda_i H_{S_i}, \mathbf{M}^S) \leq a(D, S, B_S, \mathbf{M}^S)$ for any prime divisor D on S_m and $i \geq m$. By (3), for any $i \geq m$ we have
$$\begin{aligned} a(D, S, B_S + \lambda_i H_S, \mathbf{M}^S) &\leq a(D, S_i, B_{S_i} + \lambda_i H_{S_i}, \mathbf{M}^S) \\ &= a(D, S_m, B_{S_m} + \lambda_i H_{S_m}, \mathbf{M}^S) \leq a(D, S, B_S, \mathbf{M}^S). \end{aligned}$$

Letting $i \to +\infty$, we have
$$a(D, S, B_S, \mathbf{M}^S) = a(D, S_m, B_{S_m}, \mathbf{M}^S)$$
for any prime divisor D on S_m. We define
$$\mathcal{C} := \{D \mid D \text{ is a prime divisor on } S, a(D, S_m, B_{S_m}, \mathbf{M}^S) < a(D, S, B_S, \mathbf{M}^S)\}.$$
Then any element of $\mathcal{C}$ is exceptional over S_m. Thus for any $D \in \mathcal{C}$, by (3), we have
$$\begin{aligned} a(D, S, B_S + \lambda_m H_S, \mathbf{M}^S) &\leq a(D, S_m, B_{S_m} + \lambda_m H_{S_m}, \mathbf{M}^S) \\ &\leq a(D, S_m, B_{S_m}, \mathbf{M}^S) < a(D, S, B_S, \mathbf{M}^S) \leq 1. \end{aligned}$$
Since any element of $\mathcal{C}$ is a prime divisor on S, any element of $\mathcal{C}$ is a component of H_S. Thus $\mathcal{C}$ is a finite set, and for every $D \in \mathcal{C}$, since $\lambda_m < t_0$, we have
$$\begin{aligned} 0 \leq &a(D, S, B_S + t_0 H_S, \mathbf{M}^S) < a(D, S, B_S + \lambda_m H_S, \mathbf{M}^S) \\ \leq &a(D, S_m, B_{S_m} + \lambda_m H_{S_m}, \mathbf{M}^S) \leq a(D, S_m, B_{S_m}, \mathbf{M}^S) < a(D, S, B_S, \mathbf{M}^S) \leq 1. \end{aligned}$$
Thus $0 < a(D, S_m, B_{S_m}, \mathbf{M}^S) < 1$ for any $D \in \mathcal{C}$. By [Has22a, Lemma 3.4], there exists a birational morphism $\psi : T \to S_m$ from a $\mathbb{Q}$-factorial variety T which extracts exactly divisors contained in $\mathcal{C}$. (1) follows immediately from the construction of $\mathcal{C}$. Since $(S, B_S, \mathbf{M}^S)$ is lc, there are only finitely many divisors D on T such that $a(D, S, B_S, \mathbf{M}^S) \neq 1$, hence $B_T \geq 0$ is well-defined, and we get (2).

For any prime divisor D on T, if D is ψ-exceptional, then
$$a(D, S_m, B_{S_m}, \mathbf{M}^S) < a(D, S, B_S, \mathbf{M}^S) \leq 1$$

as $D \in \mathcal{C}$, and if D is not ψ-exceptional, then $\operatorname{center}_{S_m} D$ is a divisor, hence $a(D, S, B_S, \mathbf{M}^S) = a(D, S_m, B_{S_m}, \mathbf{M}^S) \leq 1$. In either case,

$$a(D, S_m, B_{S_m}, \mathbf{M}^S) \leq a(D, S, B_S, \mathbf{M}^S) \leq 1.$$

Since T is $\mathbb{Q}$-factorial, $K_T + B_T + \mathbf{M}^S_T$ is $\mathbb{R}$-Cartier, and

$$K_T + B_T + \mathbf{M}^S_T \leq \psi^*(K_{S_m} + B_{S_m} + \mathbf{M}^S_{S_m}).$$

Thus

$$0 \leq a(Q', S_m, B_{S_m}, \mathbf{M}^S) \leq a(Q', T, B_T, \mathbf{M}^S)$$

for any prime divisor Q' over S_m, and we get (4). In particular, $(T, B_T, \mathbf{M}^S)$ is lc, and the proof is concluded. □

Step 5. In this step we show that $(T, B_T, \mathbf{M}^S)/U$ has a log minimal model. We first prove the following claim:

Claim 6.5. *For any prime divisor D over S,*

(1) if D is on S, then $a(D, S, B_S, \mathbf{M}^S) \leq a(D, T, B_T, \mathbf{M}^S)$, and
(2) if D is on T, then $a(D, T, B_T, \mathbf{M}^S) = a(D, S, B_S, \mathbf{M}^S)$.

Proof. By Claim 6.4(2), we only need to show that for any prime divisor D on S that is exceptional over T, $a(D, S, B_S, \mathbf{M}^S) \leq a(D, T, B_T, \mathbf{M}^S)$. By Claim 6.4(1)(4),

$$a(D, S, B_S, \mathbf{M}^S) \leq a(D, S_m, B_m, \mathbf{M}^S) \leq a(D, T, B_T, \mathbf{M}^S),$$

and we get (1). □

By our assumption, $(S, B_S, \mathbf{M}^S)/U$ is a log minimal model of itself, then by Lemma 5.6 $(T, B_T, \mathbf{M}^S)/U$ also has a log minimal model.

Step 6. We conclude the proof in this step. Recall that we only need to show that $(S_m, B_{S_m}, \mathbf{M}^S)/U$ has a log minimal model.

For any $i \geq m$, since $K_{X_i} + B_i + \lambda_i H_i + \mathbf{M}_{X_i}$ is nef$/U$, $K_{S_i} + B_{S_i} + \lambda_i H_{S_i} + \mathbf{M}^S_{S_i} = (K_{X_i} + B_i + \lambda_i H_i + \mathbf{M}_{X_i})|_{S_i}$ is nef$/U$. Since $\phi^S_{m,i}$ is small, $(S_i, B_{S_i} + \lambda_i H_{S_i}, \mathbf{M}^S)/U$ is a weak lc model of $(S_m, B_{S_m} + \lambda_i H_{S_m}, \mathbf{M}^S)/U$. Let $h_m : W \to S_m$ and $h_i : W \to S_i$ be a resolution of indeterminacies of $\phi^S_{m,i}$. By Lemmas 3.3(1), 3.4(2), 3.7(3) and the negativity lemma, for any prime divisor D on T we have

$$\begin{aligned}0 &\leq a(D, S_i, B_{S_i} + \lambda_i H_{S_i}, \mathbf{M}^S) - a(D, S_m, B_{S_m} + \lambda_i H_{S_m}, \mathbf{M}^S)\\ &= \sigma_D(S_m/U, K_{S_m} + B_{S_m} + \lambda_i H_{S_m} + \mathbf{M}^S_{S_m}).\end{aligned}$$

By Claim 6.4(3), we have

$$\sigma_D(S_m/U, K_{S_m} + B_{S_m} + \lambda_i H_{S_m} + \mathbf{M}^S_{S_m}) \geq a(D, S, B_S + \lambda_i H_S, \mathbf{M}^S) - a(D, S_m, B_{S_m} + \lambda_i H_{S_m}, \mathbf{M}^S).$$

By Claim 6.4(2), $a(D, S, B_S, \mathbf{M}^S) = a(D, T, B_T, \mathbf{M}^S)$. By Lemma 3.7(2) and Claims 6.2(1) and 6.4(4), for any prime divisor D on T,

$$\begin{aligned}&\sigma_D(S_m/U, K_{S_m} + B_{S_m} + \mathbf{M}^S_{S_m})\\ =&\lim_{i\to+\infty} \sigma_D(S_m/U, K_{S_m} + B_{S_m} + \lambda_i H_{S_m} + \mathbf{M}^S_{S_m})\\ \geq&\lim_{i\to+\infty} (a(D, S, B_S + \lambda_i H_S, \mathbf{M}^S) - a(D, S_m, B_{S_m} + \lambda_i H_{S_m}, \mathbf{M}^S))\\ =&a(D, S, B_S, \mathbf{M}^S) - a(D, S_m, B_{S_m}, \mathbf{M}^S)\\ =&a(D, T, B_T, \mathbf{M}^T) - a(D, S_m, B_{S_m}, \mathbf{M}^S) \geq 0.\end{aligned}$$

Since $(T, B_T, \mathbf{M}^S)/U$ has a log minimal model by Step 5, by Lemma 5.3, $(S_m, B_{S_m}, \mathbf{M}^S)/U$ has a log minimal model, and we are done. □

Theorem 6.6 (cf. [Has22a, Theorem 3.15])**.** *Let $(X,B,\mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC dlt g-pair and $A\geq 0$ an $\mathbb{R}$-divisor on X such that $(X,B+A,\mathbf{M})/U$ is lc and $K_X+B+\mathbf{M}_X+A$ is nef/U. Then for any $(K_X+B+\mathbf{M}_X)$-MMP/U with scaling of A*

$$(X,B,\mathbf{M})=:(X_0,B_0,\mathbf{M})\dashrightarrow(X_1,B_1,\mathbf{M})\dashrightarrow\cdots\dashrightarrow(X_i,B_i,\mathbf{M})\dashrightarrow\dots,$$

with scaling numbers

$$\lambda_i:=\inf\{t\geq 0\mid K_{X_i}+B_i+tA_i+\mathbf{M}_{X_i}\text{ is nef}/U\},$$

where A_i is the strict transform of A on X_i, if $\lambda_i>0$ for each i and $\lim_{i\to+\infty}\lambda_i=0$, then there are only finitely many i such that $K_{X_i}+B_i+\mathbf{M}_{X_i}$ is log abundant/U with respect to $(X_i,B_i,\mathbf{M})$.

Proof. We apply induction on the dimension. Suppose that the theorem holds in dimension $\leq\dim X-1$ but the theorem does not hold. Then there exists a $(K_X+B+\mathbf{M}_X)$-MMP/U with scaling of A as in the statement of the theorem such that $K_{X_i}+B_i+\mathbf{M}_{X_i}$ is log abundant/U with respect to $(X_i,B_i,\mathbf{M})$ for infinitely many i. Let $\phi_{i,j}:X_i\dashrightarrow X_j$ be the induced birational map. Possibly replacing $(X,B,\mathbf{M})$ with $(X_m,B_m,\mathbf{M})$ for some $m\gg 0$, we may assume that the maps $\phi_{i,j}$ are small for any i,j.

We prove the following claim.

Claim 6.7. *If there exists $m\gg 0$ such that $\phi_{m,i}|_S$ is an isomorphism for any lc center S of $(X_m,B_m,\mathbf{M})$ and $i\geq m$, then Theorem 6.6 holds.*

Proof. Possibly replacing $(X,B,\mathbf{M})$ with $(X_m,B_m,\mathbf{M})$ we may assume that $\phi_{i,j}|_{\mathrm{Nklt}(X_i,B_i,\mathbf{M})}$ is an isomorphism for any i,j and $K_X+B+\mathbf{M}_X$ is abundant/U. Since $\lim_{i\to+\infty}\lambda_i=0$ and $\phi_{i,j}$ are small for any i,j, $K_X+B+\mathbf{M}_X$ is a movable/U $\mathbb{R}$-divisor, hence $K_X+B+\mathbf{M}_X$ is pseudo-effective/U. Notice that $(X_i,B_i,\mathbf{M})$ are all dlt and $\mathbb{Q}$-factorial. Let D be a component of $\lfloor B_i\rfloor$. Then $\phi_{i,i+1}|_D$ being an isomorphism implies that the flip $\phi_{i,i+1}$ is an isomorphism near D. Therefore $\phi_{i,i+1}$ is an isomorphism on a neighborhood of $\lfloor B_i\rfloor$. By Lemma 5.5, $\mathbf{B}_-(K_X+B+\mathbf{M}_X/U)$ does not intersect $\operatorname{Supp}\lfloor B\rfloor$, and $\sigma_P(X/U,K_X+B+\mathbf{M}_X)=0$ for any prime divisor P over X such that $\operatorname{center}_X P\cap\operatorname{Supp}\lfloor B\rfloor\neq\emptyset$. In particular, $(K_X+B+\mathbf{M}_X)|_S$ is nef/U for any lc center S of $(X,B,\mathbf{M})$. By Theorem 6.1, $(X,B,\mathbf{M})/U$ has a log minimal model, but this contradicts [HL22, Theorem 4.1] so we are done. □

Proof of Theorem 6.6 continued. We let $\phi_i:=\phi_{i,i+1}$ for every i and $X_i\to Z_i\leftarrow X_{i+1}$ the flip defined by ϕ_i. By Claim 6.7, we only need to show that for any lc center S of $(X,B,\mathbf{M})$, the MMP terminates along S after finitely many steps. By induction on the dimension of lc centers, we may assume that ϕ_i induces an isomorphism for every k-dimensional lc centers and $i\gg 0$, where $k<d=\dim S$. Let S_i be the strict transform of S on X_i and $(S_i,B_{S_i},\mathbf{M}^S)$ the g-pair given by adjunction

$$K_{S_i}+B_{S_i}+\mathbf{M}^{S_i}_S:=(K_{X_i}+B_i+\mathbf{M}_{X_i})|_{S_i}.$$

Let $(S_i',B_{S_i'},\mathbf{M}^S)$ be a dlt model of $(S_i,B_{S_i},\mathbf{M}^S)$. By Lemma 2.18, for $i\gg 0$, the $(K_X+B+\mathbf{M}_X)$-MMP/U with scaling of A induces a $(K_{S_i'}+B_{S_i'}+\mathbf{M}^{S_i'}_S)$-MMP/$T$ with scaling of $A_{S_i'}$ such that the limit of the scaling numbers is 0, where $A_{S_i'}$ is the pullback of A_i on S_i' and T is the normalization of the image of S_i in U. Since $K_{X_j}+B_j+\mathbf{M}_{X_j}$ is log abundant/U with respect to $(X_j,B_j,\mathbf{M})$ for infinitely many j, $K_{S_j'}+B_{S_j'}+\mathbf{M}^S_{S_j'}$ is log abundant/T with respect to $(S_j',B_{S_j'},\mathbf{M}^S)$ for infinitely many j. By Theorem 6.6 in dimension $<\dim X$, the $(K_{S_j'}+B_{S_j'}+\mathbf{M}^S_{S_j'})$-MMP/$T$ terminates, i.e the top horizontal maps in Lemma 2.18(3) are isomorphisms for $i\gg 0$. Therefore the bottom horizontal maps in Lemma 2.18(3) must also be isomorphisms for $i\gg 0$ by the contructions in the proof. Thus there exists $m\gg 0$ such that $\phi_{m,i}|_S$ is an isomorphism for any lc center S_m of $(X_m,B_m,\mathbf{M})$ and $i\geq m$ and we are done. □

7. Log abundance under the MMP

This section is similar to [Has22b, Section 3 and Theorem 4.1].

Theorem 7.1 (cf. [Has22b, Theorem 3.5])**.** *Let $(X,B,\mathbf{M})/U$ be an NQC lc g-pair and $\pi: X \to Z$ a projective morphism$/U$ such that Z is normal quasi-projective. Let $C \geq 0$ be an $\mathbb{R}$-divisor on X, A_Z an ample$/U$ $\mathbb{R}$-divisor on Z, and $0 \leq A \sim_{\mathbb{R},U} \pi^*A_Z$ an $\mathbb{R}$-divisor on X, such that*

(1) C does not contain any lc center of $(X,B,\mathbf{M})$,
(2) $K_X+B+C+\mathbf{M}_X \sim_{\mathbb{R},Z} 0$, and
(3) $(X,B+A,\mathbf{M})$ is lc.

Then $K_X+B+A+\mathbf{M}_X$ is abundant$/U$.

Proof. Possibly replacing π with the contraction in the Stein factorization of π, we may assume that π is a contraction. Possibly replacing $Z \to U$ with the Stein factorization of $Z \to U$, we may assume that $Z \to U$ is a contraction. Let F be a very general fiber of $X \to U$ and $F_Z := \pi(F)$. Then F_Z is a very general fiber of $Z \to U$. Possibly replacing $(X,B,\mathbf{M}),A,C,Z,A_Z,\pi,U$ with $(F,B|_F,\mathbf{M}|_F),A|_F,C|_F,F_Z,A_Z|_{F_Z},\pi|_F,\{pt\}$, we may assume that $U=\{pt\}$. The theorem follows from [Has22b, Theorem 3.5]. Note that we remove the $\mathbb{R}$-Cartier assumption of C as it is immediate from (2). □

Lemma 7.2 (cf. [Has22b, Lemma 3.6])**.** *Let $(X,B,\mathbf{M})/U$ be an NQC lc g-pair and $\pi: X \to Z$ a projective morphism$/U$ such that Z is normal quasi-projective. Let $C \geq 0$ be an $\mathbb{R}$-divisor on X, A_Z an ample$/U$ $\mathbb{R}$-divisor on Z, and $0 \leq A \sim_{\mathbb{R},U} \pi^*A_Z$ an $\mathbb{R}$-divisor on X, such that*

(1) C does not contain any lc center of $(X,B,\mathbf{M})$,
(2) $K_X+B+C+\mathbf{M}_X \sim_{\mathbb{R},Z} 0$, and
(3) $(X,B+A,\mathbf{M})$ is lc.

Let $h: W \to X$ be a log resolution of $(X,\operatorname{Supp} B)$ such that $\mathbf{M}$ descends to W, and $B_W \geq 0$ an $\mathbb{R}$-divisor on W such that (W,B_W+h^*A) is lc and $(K_W+B_W+\mathbf{M}_W-h^*(K_X+B+\mathbf{M}_X))^{\geq 0}$ is h-exceptional. Then $K_W+B_W+h^*A+\mathbf{M}_W$ is abundant$/U$.

Proof. Possibly replacing π with the contraction in the Stein factorization of π, we may assume that π is a contraction. Possibly replacing $Z \to U$ with the Stein factorization of $Z \to U$, we may assume that $Z \to U$ is a contraction. Let F_w be a very general fiber of $W \to U$, $F := h(F_W)$, and $F_Z := \pi(F)$. Then F and F_Z are very general fibers of $X \to U$ and $Z \to U$ respectively. Possibly replacing $(X,B,\mathbf{M}),A,C,Z,A_Z,\pi,U,W,h,B_W$ with $(F,B|_F,\mathbf{M}|_F),A|_F,C|_F,F_Z,A_Z|_{F_Z},\pi|_F,\{pt\},F_W,h|_{F_W},B_W|_{F_W}$, we may assume that $U=\{pt\}$. The theorem follows from [Has22b, Theorem 3.5, Lemma 3.6]. Note that we remove the $\mathbb{R}$-Cartier assumption of C as it is immediate from (2). □

Theorem 7.3 (cf. [Has22b, Theorem 4.1])**.** *Let $(X,B,\mathbf{M})/U$ be an NQC lc g-pair and $\pi: X \to Z$ a projective morphism$/U$ such that Z is normal quasi-projective. Let $C \geq 0$ be an $\mathbb{R}$-divisor on X, A_Z an ample$/U$ $\mathbb{R}$-divisor on Z, and $0 \leq A \sim_{\mathbb{R},U} \pi^*A_Z$ an $\mathbb{R}$-divisor on X, such that*

(1) C does not contain any lc center of $(X,B,\mathbf{M})$,
(2) $K_X+B+C+\mathbf{M}_X \sim_{\mathbb{R},Z} 0$, and
(3) $(X,\Delta:=B+A,\mathbf{M})$ is lc and $\operatorname{Nklt}(X,B,\mathbf{M})=\operatorname{Nklt}(X,\Delta,\mathbf{M})$.

Then for any $(K_X+\Delta+\mathbf{M}_X)$-MMP$/U$

$$(X,\Delta,\mathbf{M}) := (X_0,\Delta_0,\mathbf{M}) \dashrightarrow (X_1,\Delta_1,\mathbf{M}) \dashrightarrow \cdots \dashrightarrow (X_i,\Delta_i,\mathbf{M}) \dashrightarrow \dots,$$

$K_{X_i}+\Delta_i+\mathbf{M}_{X_i}$ is log abundant$/U$ with respect to $(X_i,\Delta_i,\mathbf{M})$ for every i.

Proof. For each i, we let $\phi_i: X \dashrightarrow X_i$ be the induced birational map.

By Theorem 7.1, $K_X+B+A+\mathbf{M}_X$ is abundant$/U$. By Lemma 2.3(6), $K_{X_i}+\Delta_i+\mathbf{M}_{X_i}$ is abundant$/U$ for any i. Thus we only need to prove that $(K_{X_i}+\Delta_i+\mathbf{M}_{X_i})|_{S_i}$ is abundant$/U$ for any lc center S_i of $(X_i,\Delta_i,\mathbf{M})$.

Fix i and an lc center S_i of $(X_i,\Delta_i,\mathbf{M})$. Then there exists an lc center S of $(X,\Delta,\mathbf{M})$ such that $\phi_i|_S: S \dashrightarrow S_i$ is a birational map. Let $(X',B',\mathbf{M})$ be a dlt model of $(X,B,\mathbf{M})$ with induced birational morphism $f: X'\to X$ such that there exists a component S' of $\lfloor\Delta'\rfloor$ which dominates S. Let $C':=f^*C, A':=f^*A$, and $\Delta':=B'+A'$. Since $\mathrm{Nklt}(X,B,\mathbf{M})=f(\mathrm{Nklt}(X',\Delta',\mathbf{M}))$, $(X',\Delta',\mathbf{M})$ is a dlt model of $(X,\Delta,\mathbf{M})$. By Lemma 2.11, we may run a $(K_{X'}+\Delta'+\mathbf{M}_{X'})$-MMP$/U$ and get a dlt model $(X_i',\Delta_i',\mathbf{M})$ of $(X_i,\Delta_i,\mathbf{M})$ such that the strict transform S_i' of S' on X_i' is a component of $\lfloor\Delta_i'\rfloor$. Then $(K_{X_i}+\Delta_i+\mathbf{M}_{X_i})|_{S_i}$ is abundant$/U$ if and only if $(K_{X_i'}+\Delta_i'+\mathbf{M}_{X_i'})|_{S_i'}$ is abundant$/U$. Moreover, we have

- C' does not contain any lc center of $(X',B',\mathbf{M})$,
- $K_{X'}+B'+C'+\mathbf{M}_{X'}\sim_{\mathbb{R},Z}0$,
- $(X',\Delta',\mathbf{M})$ is lc, and
- $\mathrm{Nklt}(X',B',\mathbf{M})=\mathrm{Nklt}(X',\Delta',\mathbf{M})$.

Thus possibly replacing $(X,\Delta,\mathbf{M})\dashrightarrow(X_i,\Delta_i,\mathbf{M})$ with $(X',\Delta',\mathbf{M})\dashrightarrow(X_i',\Delta',\mathbf{M})$ and A,B,C with A',B',C', we may assume that $(X,\Delta,\mathbf{M})$ is $\mathbb{Q}$-factorial dlt, S is a component of $\lfloor\Delta\rfloor=\lfloor B\rfloor$, and S_i is a component of $\lfloor\Delta_i\rfloor=\lfloor B_i\rfloor$.

Let $(S,B_S,\mathbf{M}^S)/U$ and $(S_i,B_{S_i},\mathbf{M}^S)/U$ be the dlt g-pairs induced by the adjunction formulas

$$K_S+B_S+\mathbf{M}^S:=(K_X+B+\mathbf{M}_X)|_S$$

and

$$K_{S_i}+B_{S_i}+\mathbf{M}_{S_i}:=(K_{X_i}+B_i+\mathbf{M}_{X_i})|_{S_i}.$$

Let $p: W\to S$ and $q: W\to S_i$ be a resolution of indeterminacies of the induced birational map $S\dashrightarrow S_i$ such that $\mathbf{M}^S$ descends to W, p is a log resolution of $(S,\operatorname{Supp}B_S)$, and q is a log resolution of $(S_i,\operatorname{Supp}B_{S_i})$. Since A is semi-ample$/U$, possibly replacing A with a general member of $|A/U|_{\mathbb{R}}$* and setting $A_S:=A|_S$ and $A_{S_i}:=((\phi_i)_*A)|_{S_i}$, we may assume that

- $A_S\geq 0, A_{S_i}\geq 0$,
- $(S,\Delta_S:=B_S+A_S,\mathbf{M})$ and $(S_i,\Delta_{S_i}:=B_{S_i}+A_{S_i},\mathbf{M})$ are dlt, and
- p is a log resolution of $(S,\operatorname{Supp}\Delta_S)$ and q is a log resolution of $(S_i,\operatorname{Supp}\Delta_{S_i})$.

Moreover, since A is general in $|A/U|_{\mathbb{R}}$, $p^*A_S=p^{-1}_*A_S$, hence $A_W:=p^*A\leq q^*A_{S_i}$. We may write

$$K_W+B_W+A_W+\mathbf{M}^S_W=q^*(K_{S_i}+\Delta_{S_i}+\mathbf{M}^S_{S_i})+E$$

and let $\Delta_W:=B_W+A_W$, such that $B_W\geq 0$, $E\geq 0$, and $\Delta_W\wedge E=0$. Then (W,Δ_W) is log smooth. We may write

$$K_W+B_W+\mathbf{M}^S_W=p^*(K_S+\Delta_S+\mathbf{M}^S_S)+G_+-G_-,$$

where $G_+\geq 0, G_-\geq 0$, and $G_+\wedge G_-=0$.

For any component D of G_+,

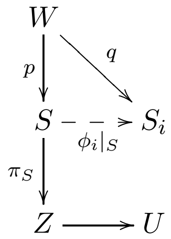
$$a(D,S,\Delta_S,\mathbf{M}^S)>a(D,W,\Delta_W,\mathbf{M}^S)=\min\{a(D,S_i,\Delta_{S_i},\mathbf{M}^S),1\}.$$

Since ϕ_i is $(K_X+\Delta+\mathbf{M}_X)$-non-positive, by [Fuj07, Lemma 4.2.10],

$$a(D,S_i,\Delta_{S_i},\mathbf{M}^S)\leq a(D,S,\Delta_S,\mathbf{M}^S).$$

Thus $a(D,S,\Delta_S,\mathbf{M}^S)>1$, hence D is p-exceptional (since any divisor on S has log discrepancy ≤ 1.). We conclude that G_+ is p-exceptional.

*The "general member" of the $\mathbb{R}$-linear system $|A/U|_{\mathbb{R}}$ is constructed in the following way: we write $A=\sum r_iA_i$ where $r_i\in(0,1)$ are real numbers and A_i are base-point-free$/U$ Cartier divisors. We replace A with $\sum r_iH_i$ where $H_i\in|A_i/U|$ are general members.



Let $\pi_S := \pi|_S$ and $C_S := C|_S$. Since C does not contain any lc center of $(X, B, \mathbf{M})$, S is not a component of C, hence $C_S \geq 0$. Then $(S, B_S, \mathbf{M}^S)/U$ is an NQC lc g-pair, $\pi_S : S \to Z$ is a projective morphism$/U$, Z is normal quasi-projective, $C_S \geq 0$ is an $\mathbb{R}$ divisor on X, $0 \leq A_S \sim_{\mathbb{R},U} \pi_S^* A_Z$, such that

- C_S does not contain any lc center of $(S, B_S, \mathbf{M}^S)$,
- $K_S + B_S + C_S + \mathbf{M}^S_S \sim_{\mathbb{R},Z} 0$,
- $(S, \Delta_S = B_S + A_S, \mathbf{M}^S)$ is lc and $\mathrm{Nklt}(S, B_S, \mathbf{M}^S) = \mathrm{Nklt}(S, \Delta_S, \mathbf{M}^S)$,
- $p : W \to S$ is a log resolution of $(S, \operatorname{Supp} B_S)$ such that $\mathbf{M}^S$ descends to W, $B_W \geq 0$ is an $\mathbb{R}$-divisor on W such that $(W, B_W + p^*A_S)$ is lc and
$$(K_W + B_W + \mathbf{M}^S_W - p^*(K_S + B_S + \mathbf{M}^S_S))^{\geq 0} = G_+$$
is p-exceptional.

By Lemma 7.2, $K_W + \Delta_W + \mathbf{M}^S_W$ is abundant$/U$. By Lemma 2.3(3), $K_{S_i} + \Delta_{S_i} + \mathbf{M}^S_{S_i} = (K_{X_i} + \Delta_i + \mathbf{M}_{X_i})|_{S_i}$ is abundant$/U$ and we are done. □

Theorem 7.4 (cf. [Has22b, Theroem 1.3])**.** *Let $(X, B, \mathbf{M})/U$ be an NQC lc g-pair and $A \geq 0$ an ample$/U$ $\mathbb{R}$-divisor such that $(X, \Delta := B + A, \mathbf{M})$ is lc and $\mathrm{Nklt}(X, B, \mathbf{M}) = \mathrm{Nklt}(X, B + A, \mathbf{M})$. Let $(Y, \Delta_Y, \mathbf{M})$ be a dlt model of $(X, \Delta, \mathbf{M})$. Then for any partial $(K_Y + \Delta_Y + \mathbf{M}_Y)$-MMP$/U$*
$$\phi : (Y, \Delta_Y, \mathbf{M}) \dashrightarrow (Y', \Delta'_Y, \mathbf{M}),$$
$K_{Y'} + \Delta'_Y + \mathbf{M}_{Y'}$ is log abundant$/U$ with respect to $(Y', \Delta'_Y, \mathbf{M})$.

Proof. It follows from Theorem 7.3. □

8. Proof of the main theorems

Proof of Theorem 1.3. By [HL21a, Lemma 4.2], possibly replacing A by a sufficiently general member, we may assume that $\mathrm{Nklt}(X, B, \mathbf{M}) = \mathrm{Nklt}(X, \Delta, \mathbf{M})$.

First we prove (1). Let $(Y, \Delta_Y, \mathbf{M})$ be a dlt model of $(X, \Delta, \mathbf{M})$. By [HL21a, Theorem 3.14], we only need to show that $(Y, \Delta_Y, \mathbf{M})/U$ has a log minimal model. We run a $(K_Y + \Delta_Y + \mathbf{M}_Y)$-MMP$/U$
$$(Y, \Delta_Y, \mathbf{M}) := (Y_0, \Delta_0, \mathbf{M}) \dashrightarrow (Y_1, \Delta_1, \mathbf{M}) \dashrightarrow \cdots \dashrightarrow (Y_i, \Delta_i, \mathbf{M}) \dashrightarrow \dots$$
with scaling of a general ample$/U$ divisor $H \geq 0$, and let
$$\lambda_i := \inf\{t \mid t \geq 0, K_{Y_i} + \Delta_i + \lambda_i H_i + \mathbf{M}_{Y_i} \text{ is nef}/U\}$$
be the scaling numbers. If $\lambda_i = 0$ for some i then $(Y_i, \Delta_i, \mathbf{M})/U$ is a log minimal model of $(Y, \Delta, \mathbf{M})$ and we are done. Thus we may assume that $\lambda_i > 0$ for each i. By [HL21a, Theorem 2.24], $\lim_{i \to +\infty} \lambda_i = 0$. By Theorem 7.4, $K_{Y_i} + \Delta_i + \mathbf{M}_{Y_i}$ is log abundant$/U$ for each i, which contradicts Theorem 6.6.

Now we prove (2). We may pick $0 < \epsilon \ll 1$ such that $\frac{1}{2}A + \epsilon \mathbf{M}_X$ is ample$/U$. Possibly replacing $(X, B, \mathbf{M})$ with $(X, B, (1-\epsilon)\mathbf{M})$ and A with a general member in $|A + \epsilon \mathbf{M}_X|_{\mathbb{R}}$, we may assume that $\mathrm{Nklt}(X, B) = \mathrm{Nklt}(X, B, \mathbf{M}) = \mathrm{Nklt}(X, \Delta, \mathbf{M})$. By [HL21a, Lemma 5.18], there exists a birational morphism $h : W \to X$ such that $\mathbf{M}$ descends to W and $\operatorname{Supp}(h^*\mathbf{M}_X - \mathbf{M}_W) = \operatorname{Exc}(h)$. Let $F := h^*\mathbf{M}_X - \mathbf{M}_W$, then $F \geq 0$ and F is exceptional over X. In particular, $\operatorname{Supp} F$ does not contain any lc place of (X, B). Thus we may pick $E \geq 0$ on W such that $-E$ is ample$/X$.

Let $K_W + B_W := h^*(K_X + B)$. Since $\mathrm{Nklt}(X, B) = \mathrm{Nklt}(X, B, \mathbf{M})$, there exists $0 < \delta \ll 1$ such that $(W, B_W + \delta E)$ is sub-lc and $\frac{1}{2}h^*A - \delta E$ is ample$/U$. Thus $\mathbf{M}_W + \frac{1}{2}h^*A - \delta E$ is ample$/U$, and we may pick $0 \leq H_W \sim_{\mathbb{R},U} \mathbf{M}_W + \frac{1}{2}h^*A - \delta E$ such that $(W, B_W + H_W + \delta E)$ is sub-lc. Let $\Delta' := h_*(B_W + H_W + \delta E)$. Then (X, Δ') is lc and $\Delta' \sim_{\mathbb{R},U} B + \mathbf{M}_X + \frac{1}{2}A$. Possibly replacing A we may assume that $(X, \Delta' + \frac{1}{2}A)$ is lc. By [HH20, Theorem 1.5], $(X, \Delta' + \frac{1}{2}A)$ has a good minimal model. By [HL21a, Lemma 4.2], we get (2). □

Proof of Theorem 1.4. If $K_X + B + \mathbf{M}_X$ is not pseudo-effective$/U$, then the theorem follows from [BZ16, Lemma 4.4(1)] after passing to a dlt model of $(X, B, \mathbf{M})$. So we may assume that $K_X + B + \mathbf{M}_X$ is pseudo-effective$/Z$. By [HL21a, Theorem 3.14], we only need to prove (2), so we may assume that $(X, B, \mathbf{M})$ is $\mathbb{Q}$-factorial dlt. We run a $(K_X + B + \mathbf{M}_X)$-MMP$/U$ with scaling of an ample$/U$ $\mathbb{R}$-divisor $H \geq 0$:

$$(X, B, \mathbf{M}) := (X_0, B_0, \mathbf{M}) \dashrightarrow (X_1, B_1, \mathbf{M}) \dashrightarrow \cdots \dashrightarrow (X_i, B_i, \mathbf{M}) \dashrightarrow \ldots.$$

By Theorem 7.3 (U and Z in Theorem 7.3 both correspond to our U, A_Z and A of Theorem 7.3 correspond to 0, and C corresponds to our A), $K_{X_i} + B_i + \mathbf{M}_{X_i}$ is log abundant$/U$ with respect to $(X_i, B_i, \mathbf{M})$ for every i. By [HL21a, Theorem 2.24] and Theorem 6.6, this MMP terminates with a log minimal model of $(X, B, \mathbf{M})/U$. □

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