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Probabilities of Causation with Nonbinary Treatment and Effect

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Abstract

This paper deals with the problem of estimating the probabilities of causation when treatment and effect are not binary. Tian and Pearl derived sharp bounds for the probability of necessity and sufficiency (PNS), the probability of sufficiency (PS), and the probability of necessity (PN) using experimental and observational data. In this paper, we provide theoretical bounds for all types of probabilities of causation to multi-valued treatments and effects. We further discuss examples where our bounds guide practical decisions and use simulation studies to evaluate how informative the bounds are for various combinations of data.

Introduction

In many areas of industry, marketing, and health science, the probabilities of causation are widely used to solve decision-making problems. For example, Li and Pearl (Li and Pearl 2019) proposed the "benefit function", which is the payoff/cost associated with selecting an individual with given characteristics to identify a set of individuals who are most likely to exhibit a desired mode of behavior. In Li and Pearl's paper, the benefit function is a linear combination of the probabilities of causation with binary treatment and effect. For another example, Mueller and Pearl (Mueller and Pearl 2022) demonstrated that the probabilities of causation should be considered in personalized decision-making.

Consider the following motivating scenario: an elderly patient with cancer is faced with the choice of treatment to pursue. The options include surgery, chemotherapy, and radiation. The outcomes include ineffective, cured, and death. Given that the elderly patient has a high risk of death from cancer surgery, the patient wants to know the probability that he would be cured if he chose radiation, would die if he chose surgery, and nothing would change if he chose chemotherapy. Let X denotes the treatment, where x_1 denotes surgery, x_2 denotes chemotherapy, and x_3 denotes radiation. Let Y denotes the outcome, where y_1 denotes ineffective, y_2 denotes cured, and y_3 denotes death. The probability that the patient desires is the probability of causation, $P(y_{3_{x_1}}, y_{1_{x_2}}, y_{2_{x_3}})$.

Pearl (Pearl 1999) first defined three binary probabilities of causation (i.e., PNS, PN, and PS) using SCM (Galles and Pearl 1998; Halpern 2000; Pearl 2009). Tian and Pearl (Tian and Pearl 2000) then used observational and experimental data to bound those three probabilities of causation. Li and Pearl (Li and Pearl 2019, 2022b) provided formal proof of those bounds. Mueller, Li, and Pearl (Mueller, Li, and Pearl 2021) recently proposed using covariate information and the causal structure to narrow the bounds of the probability of necessity and sufficiency. Dawid et al. (Dawid, Musio, and Murtas 2017) also proposed using covariate information to narrow the bounds of the probability of necessity.

All the above-mentioned studies are restricted to binary treatment and effect, limiting the application of probabilities of causation. Zhang, Tian, and Bareinboim (Zhang, Tian, and Bareinboim 2022), as well as Li and Pearl (Li and Pearl 2022a), proposed nonlinear programming-based solutions to compute the bounds of nonbinary probabilities of causation numerically. However, the theoretical foundation of nonbinary probabilities of causation is still required, not only because numerical methods are limited by computational power but also because people are interested in the theoretical foundation due to further development and analysis. In this paper, we will introduce the theoretical bounds of any probabilities of causation defined using SCM without restricting them to binary treatment and effect.

Preliminaries

In this section, we review the definitions for the three aspects of binary causation, as defined in (Pearl 1999). We use the language of counterfactuals in SCM, as defined in (Galles and Pearl 1998; Halpern 2000).

We use $Y_x = y$ to denote the counterfactual sentence "Variable Y would have the value y, had X been x". For the remainder of the paper, we use y_x to denote the event $Y_x = y$, $y_{x'}$ to denote the event $Y_{x'} = y$, y'_x to denote the event $Y_x = y'$, and $y'_{x'}$ to denote the event $Y_{x'} = y'$. We assume that experimental data will be summarized in the form of the causal effects such as $P(y_x)$ and observational data will be summarized in the form of the joint probability function such as $P(x, y)$. If not specified, the variable X stands for treatment and the variable Y stands for effect.

Three prominent probabilities of causation are as follows:

Definition 1 (Probability of necessity (PN)). *Let X and Y be two binary variables in a causal model M, let x and y stand for the propositions $X = true$ and $Y = true$, respectively, and x' and y' for their complements. The probability of necessity is defined as the expression (Pearl*

1999)

$$PN \triangleq P(Y_{x'} = false | X = true, Y = true) \triangleq P(y'_{x'} | x, y)$$

Definition 2 (Probability of sufficiency (PS))**.** *(Pearl 1999)*

$$PS \triangleq P(y_x | y', x')$$

Definition 3 (Probability of necessity and sufficiency (PNS))**.** *(Pearl 1999)*

$$PNS \triangleq P(y_x, y'_{x'})$$

PNS stands for the probability that y would respond to x both ways, and therefore measures both the sufficiency and necessity of x to produce y.

Tian and Pearl (Tian and Pearl 2000) provided tight bounds for PNS, PN, and PS using Balke's program (Balke 1995) (we will call them Tian-Pearl's bounds). Li and Pearl (Li and Pearl 2019, 2022b) provided theoretical proof of the tight bounds for PNS, PS, PN, and other binary probabilities of causation.

PNS, PN, and PS have the following tight bounds:

$$\max\left\{\begin{array}{c} 0, \\ P(y_x) - P(y_{x'}), \\ P(y) - P(y_{x'}), \\ P(y_x) - P(y) \end{array}\right\} \le PNS$$

$$PNS \le \min\left\{\begin{array}{c} P(y_x), \\ P(y'_{x'}), \\ P(x,y) + P(x',y'), \\ P(y_x) - P(y_{x'}), + \\ +P(x,y') + P(x',y) \end{array}\right\}$$

$$\max\left\{\begin{array}{c} 0, \\ \frac{P(y)-P(y_{x'})}{P(x,y)} \end{array}\right\} \le PN$$

$$PN \le \min\left\{\begin{array}{c} 1, \\ \frac{P(y'_{x'})-P(x',y')}{P(x,y)} \end{array}\right\}$$

Note that we only consider PNS and PN here because the bounds of PS can easily be obtained by exchanging x with x' and y with y' in the bounds of PN. To obtain bounds for a specific population, defined by a set C of characteristics, the expressions above should be modified by conditioning each term on $C = c$.

However, the above three probabilities of causation are unable to answer the query in our motivating example. In this paper, we demonstrate the bounds of any types of probabilities of causation. We illustrate the theorems by the order of the number of hypothetical terms (i.e., number of y_x terms in the probability of causation). For example, the number of hypothetical terms in $P(y_x, {y'}_{x'})$ is 2. The proof of all theorems is provided in the appendix.

Probabilities of Causation with Single Hypothetical Term

We start with four simple probabilities of causation with a single hypothetical term. Let X denotes the treatment with potential values $x_1, ..., x_m$ and Y denotes the effect with potential values $y_1, ..., y_n$. The four probabilities of causation with a single hypothetical term are $P(y_{i_{x_j}}, y_i)$, $P(y_{i_{x_j}}, y_k), s.t., i \neq k$, $P(y_{i_{x_j}}, x_k), s.t., j \neq k$, and $P(y_{i_{x_j}}, y_k, x_m), s.t., m \neq j$. The following theorems define their bounds using observational and experimental data.

Theorem 4. *Suppose variable X has m values $x_1, ..., x_m$ and Y has n values $y_1, ..., y_n$, then the probability of causation $P(y_{i_{x_j}}, y_i)$, where $1 \le i \le n, 1 \le j \le m$, is bounded as following:*

$$\max\left\{\begin{array}{c} P(x_j, y_i), \\ P(y_{i_{x_j}}) + P(y_i) - 1 \end{array}\right\} \le P(y_{i_{x_j}}, y_i)$$

$$P(y_{i_{x_j}}, y_i) \le \min\left\{\begin{array}{c} P(y_{i_{x_j}}), \\ P(y_i) \end{array}\right\}$$

Theorem 5. *Suppose variable X has m values $x_1, ..., x_m$ and Y has n values $y_1, ..., y_n$, then the probability of causation $P(y_{i_{x_j}}, y_k)$, where $1 \le i, k \le n, 1 \le j \le m, i \neq k$, is bounded as following:*

$$\max\left\{\begin{array}{c} 0, \\ P(y_{i_{x_j}}) + P(y_k) - 1, \\ \sum_{1\le p\le m, p\neq j} \max\left\{\begin{array}{c} 0, \\ P(y_{i_{x_j}}) \\ +P(x_p, y_k) \\ -1 + P(x_j) \\ -P(x_j, y_i) \end{array}\right\} \end{array}\right\} \le P(y_{i_{x_j}}, y_k)$$

$$P(y_{i_{x_j}}, y_k) \le \min\left\{\begin{array}{c} P(y_{i_{x_j}}) - P(x_j, y_i), \\ P(y_k) - P(y_k, x_j) \end{array}\right\}$$

Theorem 6. *Suppose variable X has m values $x_1, ..., x_m$ and Y has n values $y_1, ..., y_n$, then the probability of causation $P(y_{i_{x_j}}, x_k)$, where $1 \le i \le n, 1 \le j, k \le m, j \neq k$, is bounded as following:*

$$\max\left\{\begin{array}{c} 0, \\ P(y_{i_{x_j}}) - P(x_j, y_i) \\ -1 + P(x_j) + P(x_k) \end{array}\right\} \le P(y_{i_{x_j}}, x_k)$$

$$P(y_{i_{x_j}}, x_k) \le \min\left\{\begin{array}{c} P(y_{i_{x_j}}) - P(x_j, y_i), \\ P(x_k) \end{array}\right\}$$

Theorem 7. *Suppose variable X has m values $x_1, ..., x_m$ and Y has n values $y_1, ..., y_n$, then the probability of causation $P(y_{i_{x_j}}, y_k, x_p)$, where $1 \le i, k \le n, 1 \le j, p \le m, j \neq p$, is bounded as following:*

$$\max\left\{\begin{array}{c} 0, \\ P(y_{i_{x_j}}) + P(x_p, y_k) \\ -1 + P(x_j) - P(x_j, y_i) \end{array}\right\} \le P(y_{i_{x_j}}, y_k, x_p)$$

$$P(y_{i_{x_j}}, y_k, x_p) \le \min\left\{\begin{array}{c} P(y_{i_{x_j}}) - P(x_j, y_i), \\ P(x_p, y_k) \end{array}\right\}$$

Note that, there is no theorem for the probability of causation $P(y_{i_{x_j}}, x_j)$ because $P(y_{i_{x_j}}, x_j)$ simply equals $P(y_i, x_j)$. Moreover, Theorem 7 is the general form of PS and PN. Besides, we do not have any theorem about conditional probabilities, because conditioning on observations does not change the properties of the bounds. For example, $P(y_{i_{x_j}}|y_k, x_p) = P(y_{i_{x_j}}, y_k, x_p)/P(y_k, x_p)$.

Probabilities of Causation with Multi Hypothetical Terms

In this section, we deal with four complicated probabilities of causation with multi hypothetical terms. They are $P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}})$, $P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, x_p)$, $P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, y_q)$, and $P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, x_p, y_q)$, s.t.,$j_1 \neq ... \neq j_k \neq p$. Unlike the bounds in single hypothetical term cases, the bounds in this section are bounded recursively with cases with a smaller number of hypothetical terms. The following theorems provide the bounds using observational and experimental data.

Theorem 8. *Suppose variable X has m values $x_1, ..., x_m$ and Y has n values $y_1, ..., y_n$, then the probability of causation $P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}})$, where $1 \leq i_1, ..., i_k \leq n, 1 \leq j_1, ..., j_k \leq m, j_1 \neq ... \neq j_k$, is bounded as following:*

$$\max\left\{\begin{array}{c} 0, \\ \sum_{1\leq t\leq k} P(y_{i_t x_{j_t}}) - k + 1, \\ \max_{1\leq t\leq k}(LB(P(y_{i_1 x_{j_1}}, ..., y_{i_{t-1} x_{j_{t-1}}}, \\ y_{i_{t+1} x_{j_{t+1}}}, ..., y_{i_k x_{j_k}})) \\ +P(y_{i_t x_{i_t}}) - 1), \\ \sum_{1\leq p\leq m, s.t., \exists r, 1\leq r\leq k, p=j_r} \\ LB(P(y_{i_1 x_{j_1}}, ..., y_{i_{r-1} x_{j_{r-1}}}, \\ y_{i_{r+1} x_{j_{r+1}}}, ..., y_{i_k x_{j_k}}, x_{j_r}, y_{i_r}))+ \\ \sum_{1\leq p\leq m, s.t., p\neq j_1\neq ...\neq j_k} \\ LB(P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, x_p)) \end{array}\right\} \leq P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}})$$

$$P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}) \leq \min\left\{\begin{array}{c} \min_{1\leq t\leq k} P(y_{i_t x_{j_t}}), \\ \min_{1\leq t\leq k} UB(P(y_{i_1 x_{j_1}}, ..., y_{i_{t-1} x_{j_{t-1}}}, \\ y_{i_{t+1} x_{j_{t+1}}}, ..., y_{i_k x_{j_k}})), \\ \sum_{1\leq p\leq m, s.t., \exists r, 1\leq r\leq k, p=j_r} \\ UB(P(y_{i_1 x_{j_1}}, ..., y_{i_{r-1} x_{j_{r-1}}}, \\ y_{i_{r+1} x_{j_{r+1}}}, ..., y_{i_k x_{j_k}}, x_{j_r}, y_{i_r}))+ \\ \sum_{1\leq p\leq m, s.t., p\neq j_1\neq ...\neq j_k} \\ UB(P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, x_p)) \end{array}\right\}$$

where,
LB(f) denotes the lower bound of a function f and UB(f) denotes the upper bound of a function f. The bounds of $P(y_{i_1 x_{j_1}}, ..., y_{i_{r-1} x_{j_{r-1}}}, y_{i_{r+1} x_{j_{r+1}}}, ..., y_{i_k x_{j_k}}, x_{j_r}, y_{j_r})$ are given by Theorem 7 or 11, the bounds of $P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, x_p)$ are given by Theorem 6 or 9, and the bounds of $P(y_{i_1 x_{j_1}}, ..., y_{i_{t-1} x_{j_{t-1}}}, y_{i_{t+1} x_{j_{t+1}}}, ..., y_{i_k x_{j_k}})$ are given by Theorem 8 or experimental data if $k = 2$.

Theorem 9. *Suppose variable X has m values $x_1, ..., x_m$ and Y has n values $y_1, ..., y_n$, then the probability of causation $P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, x_p)$, where $1 \leq i_1, ..., i_k \leq n, 1 \leq j_1, ..., j_k, p \leq m, j_1 \neq ... \neq j_k \neq p$, is bounded as following:*

$$\max\left\{\begin{array}{c} 0, \\ \sum_{1\leq t\leq k} P(y_{i_t x_{j_t}}) + P(x_p) - k, \\ \max_{1\leq t\leq k}(LB(P(y_{i_1 x_{j_1}}, ..., y_{i_{t-1} x_{j_{t-1}}}, \\ y_{i_{t+1} x_{j_{t+1}}}, ..., y_{i_k x_{j_k}})) \\ +LB(P(y_{i_t x_{i_t}}, x_p)) - 1) \end{array}\right\} \leq P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, x_p)$$

$$P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, x_p) \leq \min\left\{\begin{array}{c} \min_{1\leq t\leq k} P(y_{i_t x_{j_t}}), \\ P(x_p), \\ \min_{1\leq t\leq k} UB(P(y_{i_1 x_{j_1}}, ..., y_{i_{t-1} x_{j_{t-1}}}, \\ y_{i_{t+1} x_{j_{t+1}}}, ..., y_{i_k x_{j_k}})), \\ \min_{1\leq t\leq k} UB(P(y_{i_t x_{i_t}}, x_p)) \end{array}\right\}$$

where,
LB(f) denotes the lower bound of a function f and UB(f) denotes the upper bound of a function f. The bounds of $P(y_{i_1 x_{j_1}}, ..., y_{i_{t-1} x_{j_{t-1}}}, y_{i_{t+1} x_{j_{t+1}}}, ..., y_{i_k x_{j_k}})$ are given by Theorem 8 or experimental data if $k = 2$ and the bounds of $P(y_{i_t x_{i_t}}, x_p)$ are given by Theorem 6.

Theorem 10. *Suppose variable X has m values $x_1, ..., x_m$ and Y has n values $y_1, ..., y_n$, then the probability of causation $P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, y_q)$, where $1 \leq i_1, ..., i_k, q \leq n, 1 \leq j_1, ..., j_k \leq m, j_1 \neq ... \neq j_k$, is bounded as follow-*

ing:

$$\max\left\{\begin{array}{c} 0, \\ \sum_{1\le t\le k} P(y_{i_t x_{j_t}}) + P(y_q) - k, \\ \max_{1\le t\le k}(LB(P(y_{i_1 x_{j_1}},...,y_{i_{t-1} x_{j_{t-1}}}, \\ y_{i_{t+1} x_{j_{t+1}}},...,y_{i_k x_{j_k}})) \\ +LB(P(y_{i_t x_{i_t}}, y_q)) - 1), \\ \sum_{1\le p\le m, \exists r, 1\le r\le k, p=j_r, q=i_r} \\ LB(P(y_{i_1 x_{j_1}},...,y_{i_{r-1} x_{j_{r-1}}}, \\ y_{i_{r+1} x_{j_{r+1}}},...,y_{i_k x_{j_k}}, x_{j_r}, y_{i_r})) + \\ \sum_{1\le p\le m, s.t., p\ne j_1\ne ...\ne j_k} \\ LB(P(y_{i_1 x_{j_1}},...,y_{i_k x_{j_k}}, x_p, y_q)) \end{array}\right\} \le P(y_{i_1 x_{j_1}},...,y_{i_k x_{j_k}}, y_q)$$

$$P(y_{i_1 x_{j_1}},...,y_{i_k x_{j_k}}, y_q) \le \min\left\{\begin{array}{c} \min_{1\le t\le k} P(y_{i_t x_{j_t}}), \\ P(y_q), \\ \min_{1\le t\le k} UB(P(y_{i_1 x_{j_1}},...,y_{i_{t-1} x_{j_{t-1}}}, \\ y_{i_{t+1} x_{j_{t+1}}},...,y_{i_k x_{j_k}})), \\ \min_{1\le t\le k} UB(P(y_{i_t x_{i_t}}, y_q)), \\ \sum_{1\le p\le m, s.t., \exists r, 1\le r\le k, p=j_r, q=i_r} \\ UB(P(y_{i_1 x_{j_1}},...,y_{i_{r-1} x_{j_{r-1}}}, \\ y_{i_{r+1} x_{j_{r+1}}},...,y_{i_k x_{j_k}}, x_{j_r}, y_{i_r})) + \\ \sum_{1\le p\le m, s.t., p\ne j_1\ne ...\ne j_k} \\ UB(P(y_{i_1 x_{j_1}},...,y_{i_k x_{j_k}}, x_p, y_q)) \end{array}\right\}$$

where,
LB(f) denotes the lower bound of a function f and UB(f) denotes the upper bound of a function f. The bounds of $P(y_{i_1 x_{j_1}},...,y_{i_{r-1} x_{j_{r-1}}}, y_{i_{r+1} x_{j_{r+1}}},...,y_{i_k x_{j_k}}, x_{j_r}, y_{j_r})$, $P(y_{i_1 x_{j_1}},...,y_{i_k x_{j_k}}, x_p, y_q)$ are given by Theorem 7 or 11, the bounds of $P(y_{i_1 x_{j_1}},...,y_{i_{t-1} x_{j_{t-1}}}, y_{i_{t+1} x_{j_{t+1}}},...,y_{i_k x_{j_k}})$ are given by Theorem 8 or experimental data if $k=2$, and the bounds of $P(y_{i_t x_{i_t}}, y_q)$ are given by Theorem 4 or 5.

Theorem 11. *Suppose variable X has m values $x_1,...,x_m$ and Y has n values $y_1,...,y_n$, then the probability of causation $P(y_{i_1 x_{j_1}},...,y_{i_k x_{j_k}}, x_p, y_q)$, where $1 \le i_1,...,i_k, q \le n, 1 \le j_1,...,j_k, p \le m, j_1 \ne ... \ne j_k \ne p$, is bounded as following:*

$$\max\left\{\begin{array}{c} 0, \\ \sum_{1\le t\le k} P(y_{i_t x_{j_t}}) + P(x_p, y_q) - k, \\ \max_{1\le t\le k}(LB(P(y_{i_1 x_{j_1}},...,y_{i_{t-1} x_{j_{t-1}}}, \\ y_{i_{t+1} x_{j_{t+1}}},...,y_{i_k x_{j_k}})) \\ +LB(P(y_{i_t x_{i_t}}, x_p, y_q)) - 1) \end{array}\right\} \le P(y_{i_1 x_{j_1}},...,y_{i_k x_{j_k}}, x_p, y_q)$$

$$P(y_{i_1 x_{j_1}},...,y_{i_k x_{j_k}}, x_p, y_q) \le \min\left\{\begin{array}{c} \min_{1\le t\le k} P(y_{i_t x_{j_t}}), \\ P(x_p, y_q), \\ \min_{1\le t\le k} UB(P(y_{i_1 x_{j_1}},...,y_{i_{t-1} x_{j_{t-1}}}, \\ y_{i_{t+1} x_{j_{t+1}}},...,y_{i_k x_{j_k}})), \\ \min_{1\le t\le k} UB(P(y_{i_t x_{i_t}}, x_p, y_q)) \end{array}\right\}$$

where,
LB(f) denotes the lower bound of a function f and UB(f) denotes the upper bound of a function f. The bounds of $P(y_{i_1 x_{j_1}},...,y_{i_{t-1} x_{j_{t-1}}}, y_{i_{t+1} x_{j_{t+1}}},...,y_{i_k x_{j_k}})$ are given by Theorem 8 or experimental data if $k=2$ and the bounds of $P(y_{i_t x_{i_t}}, x_p, y_q)$ are given by Theorem 7.

Note that each theorem contains nonrecursively and recursively parts. The nonrecursively parts directly follow the Frechet inequalities. If it is realized that the nonrecursive parts are sufficient for decision-making, the recursive parts can be ignored. Furthermore, the recursive parts in theorems are guaranteed to reduce the number of hypothetical terms in the probabilities of causation by 1; therefore, the recursive parts can reach the single hypothetical term cases in Theorems 4 to 7. Moreover, Theorem 8 is the general form of PNS.

Examples

In this section, we show how the presented theorems can be used in applications. We start with our motivating example.

Choice of Treatment

An elderly patient with cancer is faced with the choice of treatment. The options from the hospital include surgery, chemotherapy, and radiation. The outcomes include ineffective, cured, and death. Given the elder patient's high risk of death from cancer surgery, the doctor of the hospital suggested radiation to the patient. So, the patient wants to know the probability that he would be cured if he chose radiation, that would die if he chose surgery, and that nothing would change if he chose chemotherapy.

Let X denotes the treatment, where x_1 denotes surgery, x_2 denotes chemotherapy, and x_3 denotes radiation. Let Y denotes the outcome, where y_1 denotes ineffective, y_2 denotes

	Surgery	Chemotherapy	Radiation
Ineffective	80 Patients	184 Patients	87 Patients
Cured	7 Patients	29 Patients	189 Patients
Death	213 Patients	87 Patients	24 Patients
Overall	300 Patients	300 Patients	300 Patients

Table 1: Experimental data collected by the hospital. Here, 300 patients were forced to receive surgery, 300 patients were forced to receive chemotherapy, and 300 patients were forced to receive radiation.

	Surgery	Chemotherapy	Radiation
Ineffective	238 Patients	10 Patients	147 Patients
Cured	20 Patients	77 Patients	72 Patients
Death	7 Patients	259 Patients	70 Patients
Overall	265 Patients	346 Patients	289 Patients

Table 2: Observational data collected by the hospital. Here, 900 patients were free to choose one of the three treatments by themselves; 265 patients chose surgery, 346 patients chose chemotherapy, and 289 patients chose radiation.

cured, and y_3 denotes death. The probability that the patient desires is the probability of causation, $P(y_{3_{x_1}}, y_{1_{x_2}}, y_{2_{x_3}})$.

The doctor provided an experimental study of 900 elderly patients where all the patients were forced to take treatment. The results are shown in Table 1.

The doctor also provided an observational study of 900 elderly patients, where all the patients were open to all treatments and chose the treatment by themselves. The results are shown in Table 2.

The experimental data provide the following estimates:

$$P(y_{1_{x_1}}) = 80/300, P(y_{2_{x_1}}) = 7/300,$$
$$P(y_{3_{x_1}}) = 213/300, P(y_{1_{x_2}}) = 184/300,$$
$$P(y_{2_{x_2}}) = 29/300, P(y_{3_{x_2}}) = 87/300,$$
$$P(y_{1_{x_3}}) = 87/300, P(y_{2_{x_3}}) = 189/300,$$
$$P(y_{3_{x_3}}) = 24/300.$$

Here, all three experimental estimates, $P(y_{3_{x_1}})$, $P(y_{1_{x_2}})$, and $P(y_{2_{x_3}})$), in the target probability of causation are higher than 0.5, which may give us the sense that the target probability of causation, $P(y_{3_{x_1}}, y_{1_{x_2}}, y_{2_{x_3}})$, would be high.

The observational data provide the following estimates:

$$P(x_1, y_1) = 238/900, P(x_1, y_2) = 20/900,$$
$$P(x_1, y_3) = 7/900, P(x_2, y_1) = 10/900,$$
$$P(x_2, y_2) = 77/900, P(x_2, y_3) = 259/900,$$

	Success	Failure	Overall
No institute	53 People	247 People	300 People
Institute A	269 People	31 People	300 People
Institute B	234 People	66 People	300 People
Institute C	151 People	149 People	300 People

Table 3: Experimental data collected by Bob. Here, 300 people were forced to take no course, 300 people were forced to take a course at institute A, 300 people were forced to take a course at institute B, and 300 people were forced to take a course at institute C.

$$P(x_3, y_1) = 147/900, P(x_3, y_2) = 72/900,$$
$$P(x_3, y_3) = 70/900.$$

We then plug the estimates into Theorem 8 (see the appendix for the detailed calculations). We obtain the bounds of the target probability of causation as follows:

$$0 \le P(y_{3_{x_1}}, y_{1_{x_2}}, y_{2_{x_3}}) \le 0.099$$

In conclusion, the probability that the patient would be cured if he chose radiation, that would die if he chose surgery, and that nothing would change if he chose chemotherapy is below 0.099, implying that the patient should not consider radiation as a treatment option.

Change of Institute

Bob is looking for a job on the job market. There are three institutes, say A, B, and C, that offer courses to help people prepare for job searches. Bob went to one of the institutes, A, and took the course, but he still failed on the job market. Thus, Bob wonders if these courses improve his chance of getting a job. What would happen if he chose the other two institutes?

Let X denotes which institute a person is chosen, where x_1 denotes that no institute is chosen, x_2 denotes institute A, x_3 denotes institute B, and x_4 denotes institute C. Let Y denotes whether a person gets a job, where y_1 denotes success in job seeking, and y_2 denotes failure in job seeking. Therefore, Bob's questions become the following two probabilities of causation, $P(y_{1_{x_3}}|x_2, y_2)$ and $P(y_{1_{x_4}}|x_2, y_2)$.

All institutes provided experimental and observational studies to illustrate their effectiveness. Bob summarized the studies in Tables 3 and 4.

The experimental data provide the estimates:

$$P(y_{1_{x_1}}) = 53/300, P(y_{2_{x_1}}) = 247/300,$$
$$P(y_{1_{x_2}}) = 269/300, P(y_{2_{x_2}}) = 31/300,$$
$$P(y_{1_{x_3}}) = 234/300, P(y_{2_{x_3}}) = 66/300,$$
$$P(y_{1_{x_4}}) = 151/300, P(y_{2_{x_4}}) = 149/300.$$

	Success	Failure	Overall
No institute	92 People	58 People	150 People
Institute A	55 People	118 People	173 People
Institute B	24 People	231 People	255 People
Institute C	599 People	23 People	622 People

Table 4: Observational data collected by Bob. Here, 1200 people were open to all institutes, 150 people chose to take no course, 173 people chose to take a course at institute A, 255 people chose to take a course at institute B, and 622 people chose to take a course at institute C.

The observational data provide the estimates:

$$P(x_1, y_1) = 92/1200, P(x_1, y_2) = 58/1200,$$
$$P(x_2, y_1) = 55/1200, P(x_2, y_2) = 118/1200,$$
$$P(x_3, y_1) = 24/1200, P(x_3, y_2) = 231/1200,$$
$$P(x_4, y_1) = 599/1200, P(x_4, y_2) = 23/1200.$$

Based on the experimental study, institute A claims that taking their course increased the success rate of finding a job from 0.177 to 0.897 and institute B claims that taking their course increased the success rate of finding a job from 0.177 to 0.780. Based on the observational study, institute C claims that taking their course increased the success rate of finding a job from 0.613 to 0.963. All of these seem useful to the job seeker, which is why Bob chose institute A previously. However, he still failed in the job market.

Now, consider the following two probabilities of causation, $P(y_{1_{x_3}}|x_2, y_2) = P(y_{1_{x_3}}, x_2, y_2)/P(x_2, y_2)$, $P(y_{1_{x_4}}|x_2, y_2) = P(y_{1_{x_4}}, x_2, y_2)/P(x_2, y_2)$, What would be the probability of success if he had chosen the other two institutes?

We plug the experimental and observational estimates into Theorem 7 to obtain the following bounds:

$$0.720 \le P(y_{1_{x_3}}|x_2, y_2) \le 1,$$
$$0 \le P(y_{1_{x_4}}|x_2, y_2) \le 0.042.$$

Now Bob can see why he should change the institute to B.

Effectiveness of Vaccine

A clinical study is conducted to test the effectiveness of the vaccine. The treatment includes vaccinated and unvaccinated. The outcomes include uninfected by the virus, asymptomatic infected, infected with mild symptoms, and infected in a severe condition.

The goal of the clinical study is to learn the probability that a patient would be infected in a severe condition if unvaccinated and would be uninfected if vaccinated, the probability that a patient would be infected in a severe condition if unvaccinated and would be asymptomatic infected if vaccinated,

	Vaccinated	Unvaccinated
Uninfected	205 People	27 People
Asymptomatic	46 People	122 People
Mild Symptoms	343 People	87 People
Severe Condition	6 People	364 People
Overall	600 People	600 People

Table 5: Experimental data of the clinical study. Here, 600 people were forced to take the vaccine and 600 people were forced to take no vaccine.

	Vaccinated	Unvaccinated
Uninfected	6 People	52 People
Asymptomatic	74 People	243 People
Mild Symptoms	632 People	147 People
Severe Condition	5 People	41 People
Overall	717 People	483 People

Table 6: Observational data of the clinical study. Here, 1200 people were free to the vaccine. 717 people chose to take the vaccine and 483 people chose to take no vaccine.

and the probability that a patient would be infected in a severe condition if unvaccinated and would be infected with mild symptoms if vaccinated.

Let X denotes vaccination with x_1 being vaccinated and x_2 being unvaccinated and Y denotes the outcome, where y_1 denotes uninfected by the virus, y_2 denotes asymptomatic infected, y_3 denotes infected with mild symptoms, and y_4 denotes infected in a severe condition. The probabilities of causation we want to evaluate are $P(y_{1_{x_1}}, y_{4_{x_2}})$, $P(y_{2_{x_1}}, y_{4_{x_2}})$, and $P(y_{3_{x_1}}, y_{4_{x_2}})$.

The experimental and observational data of the clinical study are summarized in Tables 5 and 6, respectively.

Based on the clinical study, the researcher of the vaccine claimed that the vaccine is effective in controlling the severe condition, the number of patients with a severe condition dropped from 364 to only 6. Besides, some of the patients would be even uninfected because the number of uninfected people increased from 27 to 205.

Now, consider the probability that a patient would be in a severe condition if unvaccinated and would be uninfected by virus if vaccinated, $P(y_{1_{x_1}}, y_{4_{x_2}})$, the probability that a patient would be in a severe condition if unvaccinated and would be asymptomatic infected if vaccinated, $P(y_{2_{x_1}}, y_{4_{x_2}})$, and the probability that a patient would be in a severe condition if unvaccinated and would be infected with mild symptoms if vaccinated, $P(y_{3_{x_1}}, y_{4_{x_2}})$.

The experimental data provide the following estimates:

$$P(y_{1_{x_1}}) = 205/600, P(y_{2_{x_1}}) = 46/600,$$
$$P(y_{3_{x_1}}) = 343/600, P(y_{4_{x_1}}) = 6/600,$$
$$P(y_{1_{x_2}}) = 27/600, P(y_{2_{x_2}}) = 122/600,$$
$$P(y_{3_{x_2}}) = 87/600, P(y_{4_{x_2}}) = 364/600.$$

The observational data provide the following estimates:

$$P(x_1, y_1) = 6/1200, P(x_1, y_2) = 74/1200,$$
$$P(x_1, y_3) = 632/1200, P(x_1, y_4) = 5/1200,$$
$$P(x_2, y_1) = 52/1200, P(x_2, y_2) = 243/1200,$$
$$P(x_2, y_3) = 147/1200, P(x_2, y_4) = 41/1200.$$

We plug the estimates into Theorem 8 to obtain the bounds:

$$0 \leq P(y_{1_{x_1}}, y_{4_{x_2}}) \leq 0.039$$
$$0.037 \leq P(y_{2_{x_1}}, y_{4_{x_2}}) \leq 0.077$$
$$0.502 \leq P(y_{3_{x_1}}, y_{4_{x_2}}) \leq 0.561.$$

Thus, the probability of causation that a patient would be in a severe condition if unvaccinated and would be uninfected if vaccinated is at most 0.039, the probability that a patient would be in a severe condition if unvaccinated and would be asymptomatic infected if vaccinated is at most 0.077, and the probability that a patient would be in a severe condition if unvaccinated and would be infected with mild symptoms if vaccinated is at least 0.502.

We conclude that the vaccine is effective in controlling the severe condition, but can only make it infected with mild symptoms. The vaccine is ineffective for uninfected and asymptomatic infected if the patient would be in a severe condition if unvaccinated.

Simulated Results

In this section, we show the quality of the proposed bounds of the probabilities of causation.

We set $m = 2$ (i.e., X has two values) and $n = 3$ (i.e., Y has three values). We focus on the probability of causation, $P(y_{1_{x_1}}, y_{1_{x_2}})$. We randomly generated 1000 samples of $P(y_{1_{x_1}}, y_{1_{x_2}})$. For each sample, we then generated sample distributions (observational data and experimental data) compatible with the $P(y_{1_{x_1}}, y_{1_{x_2}})$ (see the appendix for the generating algorithm). The advantage of this generating process is that we have the real value of the probability of causation for comparison. The generating algorithm ensures that the experimental data and observational data satisfy the general relation (i.e., $P(x, y|c) \leq P(y_x|c) \leq P(x, y|c) + 1 - P(x|c)$). For a sample i, let $[a_i, b_i]$ be the bounds of the $P(y_{1_{x_1}}, y_{1_{x_2}})$ obtained from the proposed theorems. We summarized the following criteria for each sample as illustrated in Figure 1:

- lower bound : a_i;
- upper bound : b_i;
- midpoint : $(a_i + b_i)/2$;
- real value;

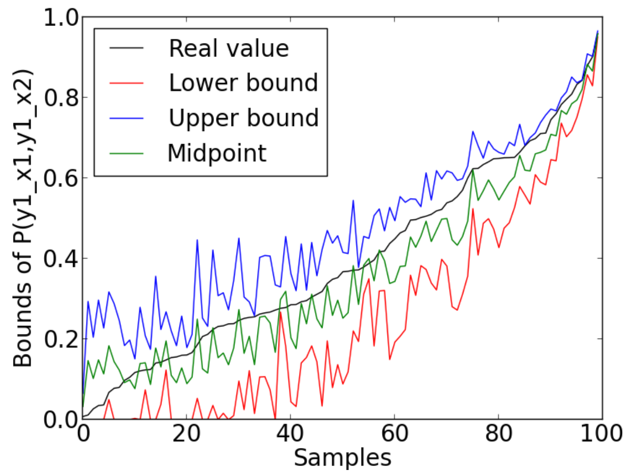

Figure 1: Bounds of the $P(y_{1_{x_1}}, y_{1_{x_2}})$ for 100 samples out of 1000.

From the Figure 1, it is clear that the proposed bounds are a good estimation of the real probability of causation. The lower and upper bounds are closely around the real value and the midpoints are almost identified with the real value. Besides, the average gap of the bounds, $\frac{\sum(b_i - a_i)}{1000}$, is 0.228, which make the bounds convincing.

Discussion

We demonstrated that nonbinary probabilities of causation help decision-maker in applications. However, we must discuss some properties of our proposed theorems further.

First, Tian-Pearl's bounds for PNS, PN, and PS are tight, implying that the bounds cannot be beaten if no additional assumption is made. However, the proposed theorems are not tight bounds, except for Theorem 7 (Theorem 7 can yield the bounds of PNS, PN, and PS, if Theorem 7 is not tight, then the bounds of PNS, PN, and PS are not tight). The main contribution of this paper is to first provide theoretical bounds for nonbinary probabilities of causation. Researchers and decision-makers require theoretical bounds. We are happy that researchers can improve or prove the tightness of our bounds in the future.

Second, Theorems 8 to 11 contain recursive bounds. One may be concerned about the computation complexity. However, suppose the number of hypothetical terms is k, then the maximum number of probabilities of causation considered in the recursion is $2^{(k+2)}$, where k is usually small.

Conclusion

We demonstrated how to obtain bounds for any probabilities of causation defined using SCM with nonbinary treatment and effect. We derived eight theorems to deliver reasonable bounds. Both examples and simulated studies are provided to support the proposed theorems.

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Appendix

Proof of Theorems

Theorem 4. *Suppose variable X has m values $x_1, ..., x_m$ and Y has n values $y_1, ..., y_n$, then the probability of causation $P(y_{i_{x_j}}, y_i)$, where $1 \le i \le n, 1 \le j \le m$, is bounded as following:*

$$\max\left\{\begin{array}{c} P(x_j, y_i), \\ P(y_{i_{x_j}}) + P(y_i) - 1 \end{array}\right\} \le P(y_{i_{x_j}}, y_i) \tag{1}$$

$$P(y_{i_{x_j}}, y_i) \le \min\left\{\begin{array}{c} P(y_{i_{x_j}}), \\ P(y_i) \end{array}\right\} \tag{2}$$

Proof. By Fréchet Inequalities, we have,

$$P(A,B) \ge \max\{0, P(A) + P(B) - 1\},$$
$$P(A,B) \le \min\{P(A), P(B)\}.$$

Thus, we have,

$$P(y_{i_{x_j}}, y_i) \ge \max\{0, P(y_{i_{x_j}}) + P(y_i) - 1\}, \tag{3}$$
$$P(y_{i_{x_j}}, y_i) \le \min\{P(y_{i_{x_j}}, P(y_i)\}.$$

Therefore, Equation 2 holds.
We also have,

$$\begin{aligned} P(y_{i_{x_j}}, y_i) &\ge P(y_{i_{x_j}}, y_i, x_j) \\ &= P(x_j, y_i) \\ &\ge 0. \end{aligned}$$

Combine with Equation 3, Equation 1 holds. □

We prove Theorems 6 and 7 first.

Theorem 6. *Suppose variable X has m values $x_1, ..., x_m$ and Y has n values $y_1, ..., y_n$, then the probability of causation $P(y_{i_{x_j}}, x_k)$, where $1 \le i \le n, 1 \le j, k \le m, j \ne k$, is bounded as following:*

$$\max\left\{\begin{array}{c} 0, \\ P(y_{i_{x_j}}) - P(x_j, y_i) \\ -1 + P(x_j) + P(x_k) \end{array}\right\} \le P(y_{i_{x_j}}, x_k) \tag{4}$$

$$P(y_{i_{x_j}}, x_k) \le \min\left\{\begin{array}{c} P(y_{i_{x_j}}) - P(x_j, y_i), \\ P(x_k) \end{array}\right\} \tag{5}$$

Proof.

$$\begin{aligned} & P(y_{i_{x_j}}, x_k) \\ =\ & P(y_{i_{x_j}}) - \sum_{t=1, t\ne k}^{m} P(y_{i_{x_j}}, x_t) \\ =\ & P(y_{i_{x_j}}) - \sum_{t=1, t\ne k,j}^{m} P(y_{i_{x_j}}, x_t) - P(y_{i_{x_j}}, x_j) \\ =\ & P(y_{i_{x_j}}) - \sum_{t=1, t\ne k,j}^{m} P(y_{i_{x_j}}, x_t) - P(x_j, y_i) \quad (6) \\ \ge\ & P(y_{i_{x_j}}) - \sum_{t=1, t\ne k,j}^{m} P(x_t) - P(x_j, y_i) \\ =\ & P(y_{i_{x_j}}) - (1 - P(x_k) - P(x_j)) - P(x_j, y_i) \\ =\ & P(y_{i_{x_j}}) - P(x_j, y_i) - 1 + P(x_j) + P(x_k). \end{aligned}$$

Combine with $P(y_{i_{x_j}}, x_k) \ge 0$, Equation 4 holds.
From Equation 6, we also have,

$$\begin{aligned} & P(y_{i_{x_j}}) - \sum_{t=1, t\ne k,j}^{m} P(y_{i_{x_j}}, x_t) - P(x_j, y_i) \\ \le\ & P(y_{i_{x_j}}) - P(x_j, y_i). \end{aligned}$$

Combine with $P(y_{i_{x_j}}, x_k) \le P(x_k)$, Equation 5 holds. □

Theorem 7. *Suppose variable X has m values $x_1, ..., x_m$ and Y has n values $y_1, ..., y_n$, then the probability of causation $P(y_{i_{x_j}}, y_k, x_p)$, where $1 \le i, k \le n, 1 \le j, p \le m, j \ne p$, is bounded as following:*

$$\max\left\{\begin{array}{c} 0, \\ P(y_{i_{x_j}}) + P(x_p, y_k) \\ -1 + P(x_j) - P(x_j, y_i) \end{array}\right\} \le P(y_{i_{x_j}}, y_k, x_p) \tag{7}$$

$$P(y_{i_{x_j}}, y_k, x_p) \le \min\left\{\begin{array}{c} P(y_{i_{x_j}}) - P(x_j, y_i), \\ P(x_p, y_k) \end{array}\right\} \tag{8}$$

Proof.

$$\begin{aligned} & P(y_{i_{x_j}}, y_k, x_p) \\ =\ & P(y_{i_{x_j}}) - \sum_{(t,r)\ne p,k} P(y_{i_{x_j}}, x_t, y_r) \\ =\ & P(y_{i_{x_j}}) - \sum_{(t,r)\ne (p,k), t\ne j} P(y_{i_{x_j}}, x_t, y_r) - P(x_j, y_i) \\ \ge\ & P(y_{i_{x_j}}) - \sum_{(t,r)\ne (p,k), t\ne j} P(x_t, y_r) - P(x_j, y_i) \\ =\ & P(y_{i_{x_j}}) - (1 - P(x_j) - P(x_p, y_k)) - P(x_j, y_i) \\ =\ & P(y_{i_{x_j}}) + P(x_p, y_k) - 1 + P(x_j) - P(x_j, y_i). \end{aligned}$$

Combine with $P(y_{i_{x_j}}, y_k, x_p) \ge 0$, Equation 7 holds.
By Theorem 6, we also have,

$$\begin{aligned} & P(y_{i_{x_j}}, y_k, x_p) \\ \le\ & P(y_{i_{x_j}}, x_p) \\ \le\ & P(y_{i_{x_j}}) - P(x_j, y_i). \end{aligned}$$

Combine with $P(y_{i_{x_j}}, y_k, x_p) \le P(x_p, y_k)$, Equation 8 holds. □

Now we prove Theorem 5.

Theorem 5. *Suppose variable X has m values $x_1, ..., x_m$ and Y has n values $y_1, ..., y_n$, then the probability of causation $P(y_{i_{x_j}}, y_k)$, where $1 \le i, k \le n, 1 \le j \le m, i \ne k$, is bounded as following:*

$$\max\left\{\begin{array}{c} 0, \\ P(y_{i_{x_j}}) + P(y_k) - 1, \\ \sum_{1\le p\le m, p\ne j} \max\left\{\begin{array}{c} 0, \\ P(y_{i_{x_j}}) \\ +P(x_p, y_k) \\ -1 + P(x_j) \\ -P(x_j, y_i) \end{array}\right\} \end{array}\right\} \le P(y_{i_{x_j}}, y_k) \tag{9}$$

$$P(y_{i_{x_j}}, y_k) \leq \min\left\{\begin{array}{c} P(y_{i_{x_j}}) - P(x_j, y_i), \\ P(y_k) - P(y_k, x_j) \end{array}\right\} \quad (10)$$

Proof. By Fréchet Inequalities, we have,

$$P(A, B) \geq \max\{0, P(A) + P(B) - 1\}.$$

Thus, we have,

$$P(y_{i_{x_j}}, y_k) \geq \max\{0, P(y_{i_{x_j}}) + P(y_k) - 1\}. \quad (11)$$

By Theorem 7, we also have,

$$\begin{aligned} & P(y_{i_{x_j}}, y_k) \\ = & \sum_{p=1, p\neq j}^{m} P(y_{i_{x_j}}, y_k, x_p) \\ \geq & \sum_{p=1, p\neq j}^{m} \max\{0, P(y_{i_{x_j}}) + P(x_p, y_k) \\ & -1 + P(x_j) - P(x_j, y_i)\}. \end{aligned}$$

Combine with Equation 11, Equation 9 holds.
We also have,

$$\begin{aligned} & P(y_{i_{x_j}}, y_k) \\ = & P(y_{i_{x_j}}) - \sum_{r=1, r\neq k}^{n} P(y_{i_{x_j}}, y_r) \\ \leq & P(y_{i_{x_j}}) - P(y_{i_{x_j}}, y_i) \\ \leq & P(y_{i_{x_j}}) - P(y_{i_{x_j}}, y_i, x_j) \\ = & P(y_{i_{x_j}}) - P(x_j, y_i), \end{aligned} \quad (12)$$

and,

$$\begin{aligned} & P(y_{i_{x_j}}, y_k) \\ = & \sum_{p=1, p\neq j}^{m} P(y_{i_{x_j}}, y_k, x_p) \\ \leq & \sum_{p=1, p\neq j}^{m} P(x_p, y_k) \\ = & P(y_k) - P(y_k, x_j). \end{aligned}$$

Combine with Equation 12, Equation 10 holds. □

Theorem 8. *Suppose variable X has m values $x_1, ..., x_m$ and Y has n values $y_1, ..., y_n$, then the probability of causation $P(y_{i_{1_{x_{j_1}}}}, ..., y_{i_{k_{x_{j_k}}}})$, where $1 \leq i_1, ..., i_k \leq n, 1 \leq j_1, ..., j_k \leq m, j_1 \neq ... \neq j_k$, is bounded as following:*

$$\max\left\{\begin{array}{c} 0, \\ \sum_{1\leq t\leq k} P(y_{i_{t_{x_{j_t}}}}) - k + 1, \\ \max_{1\leq t\leq k}(LB(P(y_{i_{1_{x_{j_1}}}}, ..., y_{i_{t-1_{x_{j_{t-1}}}}}, \\ y_{i_{t+1_{x_{j_{t+1}}}}}, ..., y_{i_{k_{x_{j_k}}}})) \\ +P(y_{i_{t_{x_{i_t}}}}) - 1), \\ \sum_{1\leq p\leq m, s.t., \exists r, 1\leq r\leq k, p=j_r} \\ LB(P(y_{i_{1_{x_{j_1}}}}, ..., y_{i_{r-1_{x_{j_{r-1}}}}}, \\ y_{i_{r+1_{x_{j_{r+1}}}}}, ..., y_{i_{k_{x_{j_k}}}}, x_{j_r}, y_{i_r}))+ \\ \sum_{1\leq p\leq m, s.t., p\neq j_1\neq ...\neq j_k} \\ LB(P(y_{i_{1_{x_{j_1}}}}, ..., y_{i_{k_{x_{j_k}}}}, x_p)) \end{array}\right\} \leq P(y_{i_{1_{x_{j_1}}}}, ..., y_{i_{k_{x_{j_k}}}}) \quad (13)$$

$$P(y_{i_{1_{x_{j_1}}}}, ..., y_{i_{k_{x_{j_k}}}}) \leq \min\left\{\begin{array}{c} \min_{1\leq t\leq k} P(y_{i_{t_{x_{j_t}}}}), \\ \min_{1\leq t\leq k} UB(P(y_{i_{1_{x_{j_1}}}}, ..., y_{i_{t-1_{x_{j_{t-1}}}}}, \\ y_{i_{t+1_{x_{j_{t+1}}}}}, ..., y_{i_{k_{x_{j_k}}}})), \\ \sum_{1\leq p\leq m, s.t., \exists r, 1\leq r\leq k, p=j_r} \\ UB(P(y_{i_{1_{x_{j_1}}}}, ..., y_{i_{r-1_{x_{j_{r-1}}}}}, \\ y_{i_{r+1_{x_{j_{r+1}}}}}, ..., y_{i_{k_{x_{j_k}}}}, x_{j_r}, y_{i_r}))+ \\ \sum_{1\leq p\leq m, s.t., p\neq j_1\neq ...\neq j_k} \\ UB(P(y_{i_{1_{x_{j_1}}}}, ..., y_{i_{k_{x_{j_k}}}}, x_p)) \end{array}\right\} \quad (14)$$

where,
$LB(f)$ denotes the lower bound of a function f and $UB(f)$ denotes the upper bound of a function f. The bounds of $P(y_{i_{1_{x_{j_1}}}}, ..., y_{i_{r-1_{x_{j_{r-1}}}}}, y_{i_{r+1_{x_{j_{r+1}}}}}, ..., y_{i_{k_{x_{j_k}}}}, x_{j_r}, y_{j_r})$ are given by Theorem 7 or 11, the bounds of $P(y_{i_{1_{x_{j_1}}}}, ..., y_{i_{k_{x_{j_k}}}}, x_p)$ are given by Theorem 6 or 9, and the bounds of $P(y_{i_{1_{x_{j_1}}}}, ..., y_{i_{t-1_{x_{j_{t-1}}}}}, y_{i_{t+1_{x_{j_{t+1}}}}}, ..., y_{i_{k_{x_{j_k}}}})$ are given by Theorem 8 or experimental data if $k = 2$.

Proof. By Fréchet Inequalities, we have,

$$\begin{aligned} P(A_1, ..., A_n) & \geq \max\{0, \\ & \quad P(A_1) + ... + P(A_n) - n + 1\}, \\ P(A_1, ..., A_n) & \leq \min\{P(A_1), ..., P(A_n)\}. \end{aligned}$$

Thus, we have,

$$\begin{aligned} & P(y_{i_{1_{x_{j_1}}}}, ..., y_{i_{k_{x_{j_k}}}}) \\ \geq & \max\{0, \\ & P(y_{i_{1_{x_{j_1}}}}) + ... + P(y_{i_{k_{x_{j_k}}}}) - k + 1\} \\ = & \max\{0, \sum_{1\leq t\leq k} P(y_{i_{t_{x_{j_t}}}}) - k + 1\}, \end{aligned} \quad (15)$$

$$P(y_{i_{1_{x_{j_1}}}}, ..., y_{i_{k_{x_{j_k}}}})$$

$$\leq \min\{P(y_{i_1x_{j_1}}),...,P(y_{i_kx_{j_k}})\}$$
$$= \min_{1\leq t\leq k} P(y_{i_tx_{j_t}}). \quad (16)$$

Also by Fréchet Inequalities, we have,

$$P(A,B) \geq P(A)+P(B)-1.$$

Thus, we have,

$$\begin{aligned}
&P(y_{i_1x_{j_1}},...,y_{i_kx_{j_k}})\\
\geq\ &\max_{1\leq t\leq k}(P(y_{i_1x_{j_1}},...,y_{i_{t-1}x_{j_{t-1}}},\\
&y_{i_{t+1}x_{j_{t+1}}},...,y_{i_kx_{j_k}})\\
&+P(y_{i_tx_{i_t}})-1)\\
\geq\ &\max_{1\leq t\leq k}(LB(P(y_{i_1x_{j_1}},...,y_{i_{t-1}x_{j_{t-1}}},\\
&y_{i_{t+1}x_{j_{t+1}}},...,y_{i_kx_{j_k}}))\\
&+P(y_{i_tx_{i_t}})-1). \quad (17)
\end{aligned}$$

And, we have,

$$\begin{aligned}
&P(y_{i_1x_{j_1}},...,y_{i_kx_{j_k}})\\
\leq\ &\min_{1\leq t\leq k} P(y_{i_1x_{j_1}},...,y_{i_{t-1}x_{j_{t-1}}},\\
&y_{i_{t+1}x_{j_{t+1}}},...,y_{i_kx_{j_k}})\\
\leq\ &\min_{1\leq t\leq k} UB(P(y_{i_1x_{j_1}},...,y_{i_{t-1}x_{j_{t-1}}},\\
&y_{i_{t+1}x_{j_{t+1}}},...,y_{i_kx_{j_k}})). \quad (18)
\end{aligned}$$

Next, we have,

$$\begin{aligned}
&P(y_{i_1x_{j_1}},...,y_{i_kx_{j_k}})\\
=\ &\sum_{p=1}^{m} P(y_{i_1x_{j_1}},...,y_{i_kx_{j_k}},x_p)\\
=\ &\sum_{1\leq p\leq m, s.t., \exists r, 1\leq r\leq k, p=j_r} P(y_{i_1x_{j_1}},...,\\
&y_{i_{r-1}x_{j_{r-1}}},y_{i_{r+1}x_{j_{r+1}}},...,y_{i_kx_{j_k}},x_{j_r},y_{i_r})+\\
&\sum_{1\leq p\leq m, s.t., p\neq j_1\neq...\neq j_k} P(y_{i_1x_{j_1}},...,\\
&y_{i_kx_{j_k}},x_p)\\
\geq\ &\sum_{1\leq p\leq m, s.t., \exists r, 1\leq r\leq k, p=j_r} LB(P(y_{i_1x_{j_1}},...,\\
&y_{i_{r-1}x_{j_{r-1}}},y_{i_{r+1}x_{j_{r+1}}},...,y_{i_kx_{j_k}},x_{j_r},y_{i_r}))+\\
&\sum_{1\leq p\leq m, s.t., p\neq j_1\neq...\neq j_k} LB(P(y_{i_1x_{j_1}},...,\\
&y_{i_kx_{j_k}},x_p)). \quad (19)
\end{aligned}$$

And,

$$\begin{aligned}
&P(y_{i_1x_{j_1}},...,y_{i_kx_{j_k}})\\
=\ &\sum_{p=1}^{m} P(y_{i_1x_{j_1}},...,y_{i_kx_{j_k}},x_p)\\
=\ &\sum_{1\leq p\leq m, s.t., \exists r, 1\leq r\leq k, p=j_r} P(y_{i_1x_{j_1}},...,\\
&y_{i_{r-1}x_{j_{r-1}}},y_{i_{r+1}x_{j_{r+1}}},...,y_{i_kx_{j_k}},x_{j_r},y_{i_r})+\\
&\sum_{1\leq p\leq m, s.t., p\neq j_1\neq...\neq j_k} P(y_{i_1x_{j_1}},...,\\
&y_{i_kx_{j_k}},x_p)\\
\leq\ &\sum_{1\leq p\leq m, s.t., \exists r, 1\leq r\leq k, p=j_r} UB(P(y_{i_1x_{j_1}},...,\\
&y_{i_{r-1}x_{j_{r-1}}},y_{i_{r+1}x_{j_{r+1}}},...,y_{i_kx_{j_k}},x_{j_r},y_{i_r}))+\\
&\sum_{1\leq p\leq m, s.t., p\neq j_1\neq...\neq j_k} UB(P(y_{i_1x_{j_1}},...,\\
&y_{i_kx_{j_k}},x_p)). \quad (20)
\end{aligned}$$

Combine Equations 15, 17, and 19, Equation 13 holds and Combine Equations 16, 18, and 20, Equation 14 holds. □

Theorem 9. *Suppose variable X has m values $x_1,...,x_m$ and Y has n values $y_1,...,y_n$, then the probability of causation $P(y_{i_1x_{j_1}},...,y_{i_kx_{j_k}},x_p)$, where $1\leq i_1,...,i_k\leq n, 1\leq j_1,...,j_k,p\leq m, j_1\neq...\neq j_k\neq p$, is bounded as following:*

$$\max\left\{\begin{array}{c}
0,\\
\sum_{1\leq t\leq k} P(y_{i_tx_{j_t}})+P(x_p)-k,\\
\max_{1\leq t\leq k}(LB(P(y_{i_1x_{j_1}},...,y_{i_{t-1}x_{j_{t-1}}},\\
y_{i_{t+1}x_{j_{t+1}}},...,y_{i_kx_{j_k}}))\\
+LB(P(y_{i_tx_{i_t}},x_p))-1)
\end{array}\right\}$$
$$\leq P(y_{i_1x_{j_1}},...,y_{i_kx_{j_k}},x_p) \quad (21)$$

$$P(y_{i_1x_{j_1}},...,y_{i_kx_{j_k}},x_p)\leq$$
$$\min\left\{\begin{array}{c}
\min_{1\leq t\leq k} P(y_{i_tx_{j_t}}),\\
P(x_p),\\
\min_{1\leq t\leq k} UB(P(y_{i_1x_{j_1}},...,y_{i_{t-1}x_{j_{t-1}}},\\
y_{i_{t+1}x_{j_{t+1}}},...,y_{i_kx_{j_k}})),\\
\min_{1\leq t\leq k} UB(P(y_{i_tx_{i_t}},x_p))
\end{array}\right\} \quad (22)$$

where,
$LB(f)$ denotes the lower bound of a function f and $UB(f)$ denotes the upper bound of a function f. The bounds of $P(y_{i_1x_{j_1}},...,y_{i_{t-1}x_{j_{t-1}}},y_{i_{t+1}x_{j_{t+1}}},...,y_{i_kx_{j_k}})$ are given by Theorem 8 or experimental data if $k=2$ and the bounds of $P(y_{i_tx_{i_t}},x_p)$ are given by Theorem 6.

Proof. By Fréchet Inequalities, we have,

$$\begin{aligned}
P(A_1,...,A_n) &\geq \max\{0,\\
&\quad P(A_1)+...+P(A_n)-n+1\},\\
P(A_1,...,A_n) &\leq \min\{P(A_1),...,P(A_n)\}.
\end{aligned}$$

Thus, we have,

$$\begin{aligned}
& P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, x_p) \\
\geq\ & \max\{0, \\
& P(y_{i_1 x_{j_1}}) + ... + P(y_{i_k x_{j_k}}) + P(x_p) - k\} \\
=\ & \max\{0, \sum_{1 \leq t \leq k} P(y_{i_t x_{j_t}}) + P(x_p) - k\}, \quad (23) \\
& P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, x_p) \\
\leq\ & \min\{P(y_{i_1 x_{j_1}}), ..., P(y_{i_k x_{j_k}}), P(x_p)\} \\
=\ & \min\{\min_{1 \leq t \leq k} P(y_{i_t x_{j_t}}), P(x_p)\}. \quad (24)
\end{aligned}$$

Also by Fréchet Inequalities, we have,

$$\begin{aligned}
& P(A, B) \geq P(A) + P(B) - 1, \\
& P(A, B) \leq \min\{P(A), P(B)\}.
\end{aligned}$$

Thus, we have,

$$\begin{aligned}
& P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, x_p) \\
\geq\ & \max_{1 \leq t \leq k} (P(y_{i_1 x_{j_1}}, ..., y_{i_{t-1} x_{j_{t-1}}}, \\
& y_{i_{t+1} x_{j_{t+1}}}, ..., y_{i_k x_{j_k}}) \\
& + P(y_{i_t x_{i_t}}, x_p) - 1) \\
\geq\ & \max_{1 \leq t \leq k} (LB(P(y_{i_1 x_{j_1}}, ..., y_{i_{t-1} x_{j_{t-1}}}, \\
& y_{i_{t+1} x_{j_{t+1}}}, ..., y_{i_k x_{j_k}})) \\
& + LB(P(y_{i_t x_{i_t}}, x_p)) - 1). \quad (25)
\end{aligned}$$

And, we have,

$$\begin{aligned}
& P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, x_p) \\
\leq\ & \min_{1 \leq t \leq k} \min\{P(y_{i_1 x_{j_1}}, ..., y_{i_{t-1} x_{j_{t-1}}}, \\
& y_{i_{t+1} x_{j_{t+1}}}, ..., y_{i_k x_{j_k}}), P(y_{i_t x_{j_t}}, x_p)\} \\
\leq\ & \min_{1 \leq t \leq k} \min\{UB(P(y_{i_1 x_{j_1}}, ..., y_{i_{t-1} x_{j_{t-1}}}, \\
& y_{i_{t+1} x_{j_{t+1}}}, ..., y_{i_k x_{j_k}})), UB(P(y_{i_t x_{j_t}}, x_p))\}. (26)
\end{aligned}$$

Combine Equations 23 and 25, Equation 21 holds and Combine Equations 24 and 26, Equation 22 holds. □

Theorem 10. *Suppose variable X has m values $x_1, ..., x_m$ and Y has n values $y_1, ..., y_n$, then the probability of causation $P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, y_q)$, where $1 \leq i_1, ..., i_k, q \leq n, 1 \leq j_1, ..., j_k \leq m, j_1 \neq ... \neq j_k$, is bounded as following:*

$$\max\left\{\begin{array}{c}
0, \\
\sum_{1 \leq t \leq k} P(y_{i_t x_{j_t}}) + P(y_q) - k, \\
\max_{1 \leq t \leq k}(LB(P(y_{i_1 x_{j_1}}, ..., y_{i_{t-1} x_{j_{t-1}}}, \\
y_{i_{t+1} x_{j_{t+1}}}, ..., y_{i_k x_{j_k}})) \\
+LB(P(y_{i_t x_{i_t}}, y_q)) - 1), \\
\sum_{1 \leq p \leq m, \exists r, 1 \leq r \leq k, p = j_r, q = i_r} \\
LB(P(y_{i_1 x_{j_1}}, ..., y_{i_{r-1} x_{j_{r-1}}}, \\
y_{i_{r+1} x_{j_{r+1}}}, ..., y_{i_k x_{j_k}}, x_{j_r}, y_{i_r})) + \\
\sum_{1 \leq p \leq m, s.t., p \neq j_1 \neq ... \neq j_k} \\
LB(P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, x_p, y_q))
\end{array}\right\} \leq P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, y_q) \quad (27)$$

$$P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, y_q) \leq$$

$$\min\left\{\begin{array}{c}
\min_{1 \leq t \leq k} P(y_{i_t x_{j_t}}), \\
P(y_q), \\
\min_{1 \leq t \leq k} UB(P(y_{i_1 x_{j_1}}, ..., y_{i_{t-1} x_{j_{t-1}}}, \\
y_{i_{t+1} x_{j_{t+1}}}, ..., y_{i_k x_{j_k}})), \\
\min_{1 \leq t \leq k} UB(P(y_{i_t x_{i_t}}, y_q)), \\
\sum_{1 \leq p \leq m, s.t., \exists r, 1 \leq r \leq k, p = j_r, q = i_r} \\
UB(P(y_{i_1 x_{j_1}}, ..., y_{i_{r-1} x_{j_{r-1}}}, \\
y_{i_{r+1} x_{j_{r+1}}}, ..., y_{i_k x_{j_k}}, x_{j_r}, y_{i_r})) + \\
\sum_{1 \leq p \leq m, s.t., p \neq j_1 \neq ... \neq j_k} \\
UB(P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, x_p, y_q))
\end{array}\right\} \quad (28)$$

where,
LB(f) denotes the lower bound of a function f and UB(f) denotes the upper bound of a function f. The bounds of $P(y_{i_1 x_{j_1}}, ..., y_{i_{r-1} x_{j_{r-1}}}, y_{i_{r+1} x_{j_{r+1}}}, ..., y_{i_k x_{j_k}}, x_{j_r}, y_{j_r})$, $P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, x_p, y_q)$ are given by Theorem 7 or 11, the bounds of $P(y_{i_1 x_{j_1}}, ..., y_{i_{t-1} x_{j_{t-1}}}, y_{i_{t+1} x_{j_{t+1}}}, ..., y_{i_k x_{j_k}})$ are given by Theorem 8 or experimental data if $k = 2$, and the bounds of $P(y_{i_t x_{i_t}}, y_q)$ are given by Theorem 4 or 5.

Proof. By Fréchet Inequalities, we have,

$$\begin{aligned}
P(A_1, ..., A_n) &\geq \max\{0, \\
& \quad P(A_1) + ... + P(A_n) - n + 1\}, \\
P(A_1, ..., A_n) &\leq \min\{P(A_1), ..., P(A_n)\}.
\end{aligned}$$

Thus, we have,

$$\begin{aligned}
& P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, y_q) \\
\geq\ & \max\{0, \\
& P(y_{i_1 x_{j_1}}) + ... + P(y_{i_k x_{j_k}}) + P(y_q) - k\}
\end{aligned}$$

$$= \max\{0, \sum_{1\le t\le k} P(y_{i_t x_{j_t}}) + P(y_q) - k\}, \quad (29)$$

$$P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, y_q) \le \min\{P(y_{i_1 x_{j_1}}), ..., P(y_{i_k x_{j_k}}), P(y_q)\} = \min\{\min_{1\le t\le k} P(y_{i_t x_{j_t}}), P(y_q)\}. \quad (30)$$

Also by Fréchet Inequalities, we have,

$$P(A,B) \ge P(A) + P(B) - 1,$$
$$P(A,B) \le \min\{P(A), P(B)\}.$$

Thus, we have,

$$\begin{aligned} & P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, y_q) \\ \ge\ & \max_{1\le t\le k}(P(y_{i_1 x_{j_1}}, ..., y_{i_{t-1} x_{j_{t-1}}}, y_{i_{t+1} x_{j_{t+1}}}, ..., y_{i_k x_{j_k}}) + P(y_{i_t x_{i_t}}, y_q) - 1) \\ \ge\ & \max_{1\le t\le k}(LB(P(y_{i_1 x_{j_1}}, ..., y_{i_{t-1} x_{j_{t-1}}}, y_{i_{t+1} x_{j_{t+1}}}, ..., y_{i_k x_{j_k}})) + LB(P(y_{i_t x_{i_t}}, y_q)) - 1). \end{aligned} \quad (31)$$

And, we have,

$$\begin{aligned} & P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, y_q) \\ \le\ & \min_{1\le t\le k} \min\{P(y_{i_1 x_{j_1}}, ..., y_{i_{t-1} x_{j_{t-1}}}, y_{i_{t+1} x_{j_{t+1}}}, ..., y_{i_k x_{j_k}}), P(y_{i_t x_{j_t}}, y_q)\} \\ \le\ & \min_{1\le t\le k} \min\{UB(P(y_{i_1 x_{j_1}}, ..., y_{i_{t-1} x_{j_{t-1}}}, y_{i_{t+1} x_{j_{t+1}}}, ..., y_{i_k x_{j_k}})), UB(P(y_{i_t x_{j_t}}, y_q))\}. \end{aligned} \quad (32)$$

Next, we have,

$$\begin{aligned} & P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, y_q) \\ =\ & \sum_{p=1}^{m} P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, y_q, x_p) \\ =\ & \sum_{1\le p\le m, s.t., \exists r, 1\le r\le k, p=j_r, q=i_r} P(y_{i_1 x_{j_1}}, ..., y_{i_{r-1} x_{j_{r-1}}}, y_{i_{r+1} x_{j_{r+1}}}, ..., y_{i_k x_{j_k}}, x_{j_r}, y_{i_r}) + \\ & \sum_{1\le p\le m, s.t., p\ne j_1\ne ...\ne j_k} P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, x_p, y_q) \\ \ge\ & \sum_{1\le p\le m, s.t., \exists r, 1\le r\le k, p=j_r, q=i_r} LB(P(y_{i_1 x_{j_1}}, ..., y_{i_{r-1} x_{j_{r-1}}}, y_{i_{r+1} x_{j_{r+1}}}, ..., y_{i_k x_{j_k}}, x_{j_r}, y_{i_r})) + \\ & \sum_{1\le p\le m, s.t., p\ne j_1\ne ...\ne j_k} LB(P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, x_p, y_q)). \end{aligned} \quad (33)$$

And,

$$\begin{aligned} & P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, y_q) \\ =\ & \sum_{p=1}^{m} P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, y_q, x_p) \\ =\ & \sum_{1\le p\le m, s.t., \exists r, 1\le r\le k, p=j_r, q=i_r} P(y_{i_1 x_{j_1}}, ..., y_{i_{r-1} x_{j_{r-1}}}, y_{i_{r+1} x_{j_{r+1}}}, ..., y_{i_k x_{j_k}}, x_{j_r}, y_{i_r}) + \\ & \sum_{1\le p\le m, s.t., p\ne j_1\ne ...\ne j_k} P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, x_p, y_q) \\ \le\ & \sum_{1\le p\le m, s.t., \exists r, 1\le r\le k, p=j_r, q=i_r} UB(P(y_{i_1 x_{j_1}}, ..., y_{i_{r-1} x_{j_{r-1}}}, y_{i_{r+1} x_{j_{r+1}}}, ..., y_{i_k x_{j_k}}, x_{j_r}, y_{i_r})) + \\ & \sum_{1\le p\le m, s.t., p\ne j_1\ne ...\ne j_k} UB(P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, x_p, y_q)). \end{aligned} \quad (34)$$

Combine Equations 29, 31, and 33, Equation 27 holds and Combine Equations 30, 32, and 34, Equation 28 holds. □

Theorem 11. *Suppose variable X has m values $x_1, ..., x_m$ and Y has n values $y_1, ..., y_n$, then the probability of causation $P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, x_p, y_q)$, where $1 \le i_1, ..., i_k, q \le n, 1 \le j_1, ..., j_k, p \le m, j_1 \ne ... \ne j_k \ne p$, is bounded as following:*

$$\max\left\{\begin{array}{c} 0, \\ \sum_{1\le t\le k} P(y_{i_t x_{j_t}}) + P(x_p, y_q) - k, \\ \max_{1\le t\le k}(LB(P(y_{i_1 x_{j_1}}, ..., y_{i_{t-1} x_{j_{t-1}}}, \\ y_{i_{t+1} x_{j_{t+1}}}, ..., y_{i_k x_{j_k}})) \\ + LB(P(y_{i_t x_{i_t}}, x_p, y_q)) - 1) \end{array}\right\} \le P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, x_p, y_q) \quad (35)$$

$$P(y_{i_1 x_{j_1}}, ..., y_{i_k x_{j_k}}, x_p, y_q) \le \min\left\{\begin{array}{c} \min_{1\le t\le k} P(y_{i_t x_{j_t}}), \\ P(x_p, y_q), \\ \min_{1\le t\le k} UB(P(y_{i_1 x_{j_1}}, ..., y_{i_{t-1} x_{j_{t-1}}}, \\ y_{i_{t+1} x_{j_{t+1}}}, ..., y_{i_k x_{j_k}})), \\ \min_{1\le t\le k} UB(P(y_{i_t x_{i_t}}, x_p, y_q)) \end{array}\right\} \quad (36)$$

where,
LB(f) denotes the lower bound of a function f and UB(f) denotes the upper bound of a function f. The bounds of $P(y_{i_1 x_{j_1}}, ..., y_{i_{t-1} x_{j_{t-1}}}, y_{i_{t+1} x_{j_{t+1}}}, ..., y_{i_k x_{j_k}})$ are given by Theorem 8 or experimental data if $k = 2$ and the bounds of $P(y_{i_t x_{i_t}}, x_p, y_q)$ are given by Theorem 7.

Proof. By Fréchet Inequalities, we have,

$$\begin{aligned}P(A_1,...,A_n) &\geq \max\{0,\\ &\quad P(A_1)+...+P(A_n)-n+1\},\\ P(A_1,...,A_n) &\leq \min\{P(A_1),...,P(A_n)\}.\end{aligned}$$

Thus, we have,

$$\begin{aligned}&P(y_{i_1 x_{j_1}},...,y_{i_k x_{j_k}},x_p,y_q)\\ \geq\ &\max\{0,\\ &P(y_{i_1 x_{j_1}})+...+P(y_{i_k x_{j_k}})+P(x_p,y_q)-k\}\\ =\ &\max\{0,\sum_{1\leq t\leq k}P(y_{i_t x_{j_t}})+P(x_p,y_q)-k\}, \quad (37)\\ &P(y_{i_1 x_{j_1}},...,y_{i_k x_{j_k}},x_p,y_q)\\ \leq\ &\min\{P(y_{i_1 x_{j_1}}),...,P(y_{i_k x_{j_k}}),P(x_p)\}\\ =\ &\min\{\min_{1\leq t\leq k}P(y_{i_t x_{j_t}}),P(x_p,y_q)\}. \quad (38)\end{aligned}$$

Also by Fréchet Inequalities, we have,

$$\begin{aligned}P(A,B)\geq P(A)+P(B)-1,\\ P(A,B)\leq \min\{P(A),P(B)\}.\end{aligned}$$

Thus, we have,

$$\begin{aligned}&P(y_{i_1 x_{j_1}},...,y_{i_k x_{j_k}},x_p,y_q)\\ \geq\ &\max_{1\leq t\leq k}(P(y_{i_1 x_{j_1}},...,y_{i_{t-1} x_{j_{t-1}}},\\ &y_{i_{t+1} x_{j_{t+1}}},...,y_{i_k x_{j_k}})\\ &+P(y_{i_t x_{i_t}},x_p,y_q)-1)\\ \geq\ &\max_{1\leq t\leq k}(LB(P(y_{i_1 x_{j_1}},...,y_{i_{t-1} x_{j_{t-1}}},\\ &y_{i_{t+1} x_{j_{t+1}}},...,y_{i_k x_{j_k}}))\\ &+LB(P(y_{i_t x_{i_t}},x_p,y_q))-1). \quad (39)\end{aligned}$$

And, we have,

$$\begin{aligned}&P(y_{i_1 x_{j_1}},...,y_{i_k x_{j_k}},x_p,y_q)\\ \leq\ &\min_{1\leq t\leq k}\min\{P(y_{i_1 x_{j_1}},...,y_{i_{t-1} x_{j_{t-1}}},\\ &y_{i_{t+1} x_{j_{t+1}}},...,y_{i_k x_{j_k}}),P(y_{i_t x_{j_t}},x_p,y_q)\}\\ \leq\ &\min_{1\leq t\leq k}\min\{UB(P(y_{i_1 x_{j_1}},...,y_{i_{t-1} x_{j_{t-1}}},\\ &y_{i_{t+1} x_{j_{t+1}}},...,y_{i_k x_{j_k}})),\\ &UB(P(y_{i_t x_{j_t}},x_p,y_q))\}. \quad (40)\end{aligned}$$

Combine Equations 37 and 39, Equation 35 holds and Combine Equations 38 and 40, Equation 36 holds. □

Calculation in the Examples

Choice of Treatment First, by Theorem 8,

$$\begin{aligned}&P(y_{3_{x_1}},y_{1_{x_2}},y_{2_{x_3}})\\ \geq\ &\max\{P(y_{3_{x_1}})+P(y_{1_{x_2}})+P(y_{2_{x_3}})-2,0\}\\ =\ &\max\{213/300+184/300+189/300-2,0\}\\ =\ &0.\end{aligned}$$

and,

$$\begin{aligned}&P(y_{3_{x_1}},y_{1_{x_2}},y_{2_{x_3}})\\ \leq\ &\min\{P(y_{3_{x_1}}),P(y_{1_{x_2}}),P(y_{2_{x_3}})\}\\ =\ &\min\{213/300,184/300,189/300\}\\ =\ &0.613.\end{aligned}$$

Second, by Theorem 8, we have,

$$\begin{aligned}0.323\leq P(y_{3_{x_1}},y_{1_{x_2}})\leq 0.340,\\ 0.243\leq P(y_{1_{x_2}},y_{2_{x_3}})\leq 0.386,\\ 0.340\leq P(y_{3_{x_1}},y_{2_{x_3}})\leq 0.472.\end{aligned}$$

Then, by Theorem 8 again,

$$\begin{aligned}&P(y_{3_{x_1}},y_{1_{x_2}},y_{2_{x_3}})\\ \geq\ &\max\{LB(P(y_{3_{x_1}},y_{1_{x_2}}))+P(y_{2_{x_3}})-1,\\ &LB(P(y_{1_{x_2}},y_{2_{x_3}}))+P(y_{3_{x_1}})-1,\\ &LB(P(y_{3_{x_1}},y_{2_{x_3}}))+P(y_{1_{x_2}})-1\}\\ =\ &\max\{0.323+189/300-1,\\ &0.243+213/300-1,\\ &0.340+184/300-1\}\\ =\ &-0.047.\end{aligned}$$

and,

$$\begin{aligned}&P(y_{3_{x_1}},y_{1_{x_2}},y_{2_{x_3}})\\ \leq\ &\min\{LB(P(y_{3_{x_1}},y_{1_{x_2}})),\\ &LB(P(y_{1_{x_2}},y_{2_{x_3}})),\\ &LB(P(y_{3_{x_1}},y_{2_{x_3}}))\}\\ =\ &\min\{0.340,0.386,0.472\}\\ =\ &0.340.\end{aligned}$$

Third, by Theorem 11, we have,

$$\begin{aligned}0\leq P(x_1,y_3,y_{1_{x_2}},y_{2_{x_3}})\leq 0.008,\\ 0\leq P(x_2,y_1,y_{3_{x_1}},y_{2_{x_3}})\leq 0.011,\\ 0\leq P(x_3,y_2,y_{3_{x_1}},y_{1_{x_2}})\leq 0.080.\end{aligned}$$

Then, by Theorem 8,

$$\begin{aligned}&P(y_{3_{x_1}},y_{1_{x_2}},y_{2_{x_3}})\\ \geq\ &LB(P(x_1,y_3,y_{1_{x_2}},y_{2_{x_3}}))+\\ &LB(P(x_2,y_1,y_{3_{x_1}},y_{2_{x_3}}))+\\ &LB(P(x_3,y_2,y_{3_{x_1}},y_{1_{x_2}}))\\ =\ &0.\end{aligned}$$

and,

$$\begin{aligned}&P(y_{3_{x_1}},y_{1_{x_2}},y_{2_{x_3}})\\ \leq\ &UB(P(x_1,y_3,y_{1_{x_2}},y_{2_{x_3}}))+\\ &UB(P(x_2,y_1,y_{3_{x_1}},y_{2_{x_3}}))+\\ &UB(P(x_3,y_2,y_{3_{x_1}},y_{1_{x_2}}))\\ =\ &0.099.\end{aligned}$$

Combine all the bounds of $P(y_{3_{x_1}}, y_{1_{x_2}}, y_{2_{x_3}})$, we then have,

$$\max\{0, -0.047, 0\} \leq P(y_{3_{x_1}}, y_{1_{x_2}}, y_{2_{x_3}}),$$
$$P(y_{3_{x_1}}, y_{1_{x_2}}, y_{2_{x_3}}) \leq \min\{0.613, 0.340, 0.099\}$$

Finally, we obtain,

$$0 \leq P(y_{3_{x_1}}, y_{1_{x_2}}, y_{2_{x_3}}) \leq 0.099.$$

Change of Institute First, by Theorem 7,

$$\begin{aligned} & P(y_{1_{x_3}}, x_2, y_2) \\ \geq\ & \max\{0, \\ & P(y_{1_{x_3}}) + P(x_2, y_2) - 1 + P(x_3) - P(x_3, y_1)\} \\ =\ & \max\{0, \\ & 234/300 + 118/1200 - 1 + 255/1200 - 24/1200\} \\ =\ & 85/1200. \end{aligned}$$

and,

$$\begin{aligned} & P(y_{1_{x_3}}, x_2, y_2) \\ \leq\ & \min\{P(y_{1_{x_3}}) - P(x_3, y_1), P(x_2, y_2)\} \\ =\ & \min\{234/300 - 24/1200, 118/1200\} \\ =\ & 118/1200. \end{aligned}$$

Also,

$$P(y_{1_{x_3}}|x_2, y_2) = P(y_{1_{x_3}}, x_2, y_2)/P(x_2, y_2).$$

Thus,

$$\frac{85/1200}{118/1200} \leq P(y_{1_{x_3}}|x_2, y_2) \leq \frac{118/1200}{118/1200},$$
$$0.720 \leq P(y_{1_{x_3}}|x_2, y_2) \leq 1.$$

Again, by Theorem 7,

$$\begin{aligned} & P(y_{1_{x_4}}, x_2, y_2) \\ \geq\ & \max\{0, \\ & P(y_{1_{x_4}}) + P(x_2, y_2) - 1 + P(x_4) - P(x_4, y_1)\} \\ =\ & \max\{0, \\ & 151/300 + 118/1200 - 1 + 622/1200 - 599/1200\} \\ =\ & 0. \end{aligned}$$

and,

$$\begin{aligned} & P(y_{1_{x_4}}, x_2, y_2) \\ \leq\ & \min\{P(y_{1_{x_4}}) - P(x_4, y_1), P(x_2, y_2)\} \\ =\ & \min\{151/300 - 599/1200, 118/1200\} \\ =\ & 5/1200. \end{aligned}$$

Also,

$$P(y_{1_{x_4}}|x_2, y_2) = P(y_{1_{x_4}}, x_2, y_2)/P(x_2, y_2).$$

Thus,

$$\frac{0}{118/1200} \leq P(y_{1_{x_4}}|x_2, y_2) \leq \frac{5/1200}{118/1200},$$
$$0 \leq P(y_{1_{x_4}}|x_2, y_2) \leq 0.042.$$

Effectiveness of Vaccine First, by Theorem 7, we have the following,

$$\begin{aligned} 0 \leq P(y_{4_{x_2}}, x_1, y_1) &\leq 0.005, \\ 0 \leq P(y_{1_{x_1}}, x_2, y_4) &\leq 0.034, \\ 0.037 \leq P(y_{4_{x_2}}, x_1, y_2) &\leq 0.062, \\ 0 \leq P(y_{2_{x_1}}, x_2, y_4) &\leq 0.015, \\ 0.502 \leq P(y_{4_{x_2}}, x_1, y_3) &\leq 0.527, \\ 0 \leq P(y_{3_{x_1}}, x_2, y_4) &\leq 0.034. \end{aligned}$$

Then by Theorem 8,

$$\begin{aligned} & P(y_{1_{x_1}}, y_{4_{x_2}}) \\ \geq\ & \max\{0, P(y_{1_{x_1}}) + P(y_{4_{x_2}}) - 1, \\ & LB(P(y_{4_{x_2}}, x_1, y_1)) + LB(P(y_{1_{x_1}}, x_2, y_4))\} \\ =\ & \max\{0, 205/600 + 364/600 - 1, 0 + 0\} \\ =\ & 0. \end{aligned}$$

and,

$$\begin{aligned} & P(y_{1_{x_1}}, y_{4_{x_2}}) \\ \leq\ & \min\{P(y_{1_{x_1}}), P(y_{4_{x_2}}), \\ & UB(P(y_{4_{x_2}}, x_1, y_1)) + UB(P(y_{1_{x_1}}, x_2, y_4))\} \\ =\ & \min\{205/600, 364/600, 0.005 + 0.034\} \\ =\ & 0.039. \end{aligned}$$

Thus,

$$0 \leq P(y_{1_{x_1}}, y_{4_{x_2}}) \leq 0.039.$$

Also by Theorem 8,

$$\begin{aligned} & P(y_{2_{x_1}}, y_{4_{x_2}}) \\ \geq\ & \max\{0, P(y_{2_{x_1}}) + P(y_{4_{x_2}}) - 1, \\ & LB(P(y_{4_{x_2}}, x_1, y_2)) + LB(P(y_{2_{x_1}}, x_2, y_4))\} \\ =\ & \max\{0, 46/600 + 364/600 - 1, 0.037 + 0\} \\ =\ & 0.037. \end{aligned}$$

and,

$$\begin{aligned} & P(y_{2_{x_1}}, y_{4_{x_2}}) \\ \leq\ & \min\{P(y_{2_{x_1}}), P(y_{4_{x_2}}), \\ & UB(P(y_{4_{x_2}}, x_1, y_2)) + UB(P(y_{2_{x_1}}, x_2, y_4))\} \\ =\ & \min\{46/600, 364/600, 0.062 + 0.015\} \\ =\ & 0.077. \end{aligned}$$

Thus,

$$0.037 \leq P(y_{2_{x_1}}, y_{4_{x_2}}) \leq 0.077.$$

Again by Theorem 8,

$$\begin{aligned} & P(y_{3_{x_1}}, y_{4_{x_2}}) \\ \geq\ & \max\{0, P(y_{3_{x_1}}) + P(y_{4_{x_2}}) - 1, \\ & LB(P(y_{4_{x_2}}, x_1, y_3)) + LB(P(y_{3_{x_1}}, x_2, y_4))\} \\ =\ & \max\{0, 343/600 + 364/600 - 1, 0.502 + 0\} \\ =\ & 0.502. \end{aligned}$$

and,

$$
\begin{aligned}
& P(y_{3_{x_1}}, y_{4_{x_2}}) \\
\leq\ & \min\{P(y_{3_{x_1}}), P(y_{4_{x_2}}), \\
& UB(P(y_{4_{x_2}}, x_1, y_3)) + UB(P(y_{3_{x_1}}, x_2, y_4))\} \\
=\ & \min\{343/600, 364/600, 0.527 + 0.034\} \\
=\ & 0.561.
\end{aligned}
$$

Thus,

$$0.502 \leq P(y_{3_{x_1}}, y_{4_{x_2}}) \leq 0.561.$$

Distribution Generating Algorithm

Here, the sample distribution generating algorithm in the simulated study is presented. It generated both experimental and observational data compatible with the fractions of response types of individuals. The data satisfy the general relation between experimental and observational data.

Algorithm 1: Generate samples for simulated study

Input: num, number of samples needed.
Output: num sample distributions (observational data and experimental data).

$count = 0$;
while $count < num$ **do**
  //$rand(0,1)$ is the function that random uniformly generate a number from 0 to 1.
  $a = []$;
  for $i = 1$ to 8 **do**
    $a.append(rand(0,1))$;
  end for
  $a.append(1.0)$;
  $a.sort()$;
  //f is the fractions of response types of individuals, $f[0] = P(y_{1_{x_1}}, y_{1_{x_2}}), ..., f[8] = P(y_{3_{x_1}}, y_{3_{x_2}})$.
  $f = []$;
  $f[0] = a[0]$;
  for $i = 1$ to 8 **do**
    $f[i] = a[i] - a[i-1]$;
  end for
  // Generate experimental data.
  $P(y_{1_{x_1}}) = f[0] + f[1] + f[2]$;
  $P(y_{2_{x_1}}) = f[3] + f[4] + f[5]$;
  $P(y_{3_{x_1}}) = f[6] + f[7] + f[8]$;
  $P(y_{1_{x_2}}) = f[0] + f[3] + f[6]$;
  $P(y_{2_{x_2}}) = f[1] + f[4] + f[7]$;
  $P(y_{3_{x_2}}) = f[2] + f[5] + f[8]$;
  // Generate observational data.
  $P(x_1, y_1) = rand(0, P(y_{1_{x_1}}))$;
  $P(x_1, y_2) = rand(0, P(y_{2_{x_1}}))$;
  $P(x_1) = rand(P(x_1, y_1) + P(x_1, y_2), \min\{P(x_1, y_1) + 1 - P(y_{1_{x_1}}), P(x_1, y_2) + 1 - P(y_{1_{x_2}})\})$;
  $P(x_1, y_3) = P(x_1) - P(x_1, y_1) - P(x_1, y_2)$;
  $P(x_2) = 1 - P(x_1)$
  $P(x_2, y_1) = rand(0, \min\{P(y_{1_{x_2}}), P(x_2)\})$;
  $P(x_2, y_2) = rand(0, \min\{P(y_{2_{x_2}}), P(x_2) - P(x_2, y_1)\})$;
  $P(x_2, y_3) = P(x_2) - P(x_2, y_1) - P(x_2, y_2)$;
  //Validate the data, the experimental data and observational data should satisfies the following: $P(x,y) \leq P(y_x) \leq P(x,y) + 1 - P(x)$.
  $mark = True$
  for $i = 1$ to 3 **do**
    for $j = 1$ to 2 **do**
      if $P(y_{i_{x_j}}) < P(x_j, y_i)$ or $P(y_{i_{x_j}}) > P(x_j, y_i) + 1 - P(x_j)$ **then**
        $mark = False$;
      end if
    end for
  end for
  if $mark == False$ **then**
    $continue$;
  end if
  $count = count + 1$;
end while