ALGEBRAIC SEEDS FOR GRAPHING FUNCTIONS

Mathías A. López[1], Susanne M. Strachota[1], Bárbara M. Brizuela[1], María del Carmen Pérez-Martos[2], Angela Murphy Gardiner[3], Maria Blanton [3]

[1]Tufts University, [2]Universidad de Granada, [3]TERC

This case study of one first grade student involves the analysis of three interviews that took place before, during, and after classroom teaching experiments (CTEs). The CTEs were designed to engage children in representing algebraic concepts using graphs. Using a knowledge-in-pieces perspective, our analysis focused on identifying students' natural intuitions and ways of thinking algebraically about a functional relationship represented using graphs. Findings reveal four seeds, two of which were identified in prior studies, and how the activation and coordination of these seeds results in students' production of function graphs.

INTRODUCTION

Recent work in early algebra has shown that algebraic representations, such as variable notation (e.g., Blanton et al., 2017; Brizuela, Blanton, Gardiner et al., 2015; Brizuela, Blanton, Sawrey et al., 2015; Dougherty, 2010) and tables (Brizuela et al., 2021), are within the reach of young children and support them in engaging with algebraic reasoning. We study the natural and intuitive ways that young children's engage in interpreting and constructing of function graphs. To focus on students' intuitions we use of the knowledge-in-pieces epistemological framing (e.g., diSessa, 1993). The fundamental assumptions are that learning can leverage natural intuitions and ways of thinking. This framework has been used to research understandings of multiplication (Izsák, 2005, 2022), probability (Wagner, 2006), integrals (Jones, 2013), and early algebra (Levin & Walkoe, 2022).

Levin and Walkoe (2022) introduce the term seeds of algebraic thinking to refer to small chunks of knowledge that become available to students through interaction with their environment that help them make sense of future algebraic experiences. They describe the following features of seeds: formed in early experience, different from school algebra ideas, and neither right nor wrong as they are context-dependent. We believe this framework could provide a perspective on how students' prior experiences come into play when engaging with graphs. Understanding how students' intuitive knowledge influences their understandings of graphs could open opportunities for instruction and curriculum that build on students' prior experiences.

One seed identified by Levin and Walkoe (2022) is the covariation seed, which involves understanding how an increase in one quantity results in an increase in another. This seed helps students make sense of the effects of the dependent and independent variables in a causal relationship, such as a functional relationship (Levin & Walkoe, 2022). Levin and Walkoe (2022) presented real-life experiences that they hypothesized might be associated with the development of this seed, such as observing a bathtub fill with water. As our research focused on graphs, other intuitive knowledge

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 177-184). PME.

seeds, which have been discussed in previous literature, were likely also activated. For example, "what you see is what you get" (Elby, 2000), which captures when an individual interprets a representation or elements of a representation in a literal sense. For instance, we observed how a student interpreted the points in the graph as actual birds instead of a coordinate pair.

Following Levin and Walkoe's (2022) work, we seek to identify the seeds that are activated when graphing a functional relationship and illustrate how students coordinate these seeds to represent function graphs. We address the following research question:

Which seeds are activated when working with a function graph and how do students coordinate these seeds to construct and interpret a function graph?

METHOD

We conducted CTEs in Kindergarten and Grades 1 and 2 (ages ranged from 5-8) at an elementary school in the Northeastern United States. In Grades 1 and 2, we taught 14 lessons (see Figure 1). In Kindergarten we taught 16 lessons. We also carried out individual interviews with four students in each of the three grades. Lessons were designed by the research team and based on prior work (e.g., Blanton et al., 2015, 2017; Brizuela et al., 2015). They were taught by a teacher-researcher and were about 30-40 minutes long. All lessons and interviews were video recorded and transcribed. Here we report on three interviews from one Grade 1 student, Lucca.

We selected Lucca's interviews for analysis because his work throughout the three interviews allowed us to construct detailed answers to our research question. All three interviews involved the same questions about the relationship between the number of birds and the number of bird wings, which can be represented as the function $y = x + x$. The students were asked to reason about the relationship and interpret tabular and graphical representations of the relationship or to construct these representations themselves.

Lucca's interview videos and transcripts were reviewed by three team members using microgenetic learning analysis (Fazio & Stiegler, 2013). We tracked instances in which students used seeds, or their own initial ways of thinking about the function graph. The team did not pre-identify the kind of thinking to track. Rather, we looked for evidence of the students' algebraic thinking (i.e., what they said or did) that indicated they were beginning to reason (conventionally or unconventionally) about the functional relationship or the representation. The team reviewed the transcripts individually and then together, until no new instances were identified.

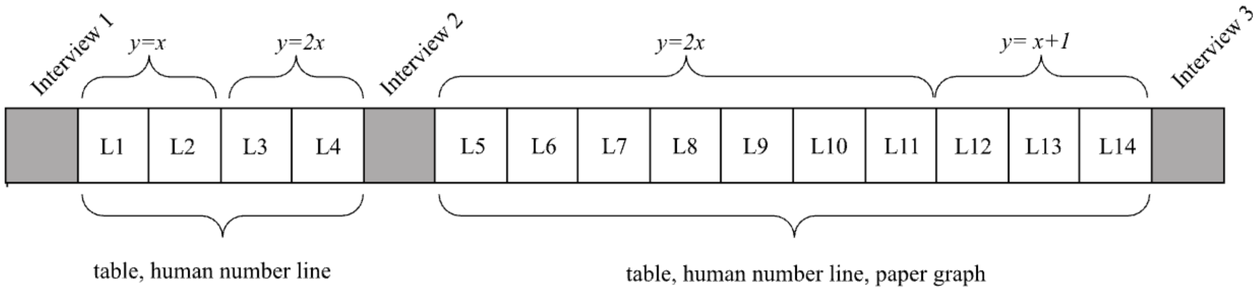

Figure 1. A representation of the Grade 1 and 2 lessons and interviews

FINDINGS

We observed Lucca use four seeds: *classifying*, *structuring*, *what you see is what you get* (Elby, 2000), and *covariation* (Levin & Walkoe, 2022). Two of those seeds, *classifying*, *structuring,* emerged from our analysis of Lucca's thinking. The other two seeds, *what you see is what you get* (Elby, 2000), and *covariation* (Levin & Walkoe, 2022), were identified previously and then observed in our data. We begin by defining *classifying* and *structuring*.

Classifying

Classifying involves sorting into, identifying, or describing a set. A set is defined by the characteristics of its elements, in this case, all the elements are quantities representing the same variable (i.e., 1 bird, 2 birds, 3 birds, and so on). Lucca likely learned to *classify* early on in real life experiences and through play.

In the interviews, we observed Lucca sort the two variables, "birds" and "bird wings," in the context of a table before activating a covariation seed. That is, before considering how these variables related, he sorted them into two sets by listing the number of birds together and the number of wings together, as depicted in Figure 2.

We also highlight that during the second interview, Lucca determined the number of bird wings for each bird. However, when asked to record the information in a table, Lucca struggled until the interviewer prompted him to add labels, or to classify the sets. It seemed that reasoning about the labels, or naming the sets that he was representing, supported Lucca in identifying and representing the two variables involved. For Lucca, classifying the numbers and naming the sets were precursors to activating a c*ovariation* or a *structuring* seed.

Even though we did not observe an externalization of the *classifying* seed when Lucca worked with the graph, we note that for *structuring* and *covariation* to be activated, *classifying* had to be activated.

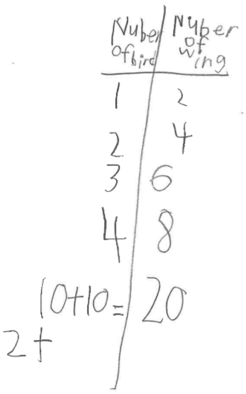

Figure 2. Lucca's self-made table during the third interview

Structuring

Structuring involves coordinating the elements of the sets in a systematic way. Throughout the three interviews, Lucca classified the numbers of birds and numbers of wings and then structured them in a table. In other words, he structured sets when he coordinated the elements in the first column (i.e., the number of birds) and the elements in the second column (i.e., the number of wings), so that he could correctly read across rows.

In the first interview, Lucca was aware of some *structure* linked to the shape of the graph. Specifically, he noticed that the points (1, 2) and (2, 4) were at the intersection of the graph grid lines. However, he did not further *structure* or specifically coordinate the corresponding elements of the sets until later interviews. For example, in the second interview Lucca constructed a self-made graph (Figure 3, left). He connected corresponding quantities with lines but did not plot points until he was prompted to by the interviewer. Once prompted Lucca drew a point on a seemingly arbitrary spot on one of those lines. The following transcript summarizes this conversation and Lucca's representation and the point are shown in the left side of Figure 3.

Interview (I): Where would you put a point to show me two birds have four wings down here?

Lucca (L): It's close to like, almost both of them.

I: Was there a math reason you put it down there?

L: Because I thought it could be anywhere on the line (as seen in Figure 3).

This example shows how Lucca activated *structuring* to coordinate bird and bird wings in a non-canonical way and highlights how seeds are neither right nor wrong since they are context dependent. Lucca's way of *structuring* was likely based on prior experience, he was aware that the point needed to be somewhere on the line, likely because of his prior experiences with the number line during the CTE.

When asked about representing the relationship in his third interview, Lucca correctly labelled the axes. He then explained that those were the correct labels because the *x*-axis corresponds to the "animal" and the *y*-axis corresponds to the "animal (body) part." Furthermore, he was able to plot and interpret the points correctly noting that each point referred to the number of birds and its corresponding number of wings. By the third interview, he could determine the number of birds corresponding to six bird wings by looking at the graph. He did this explicitly by drawing guidelines (see Figure 3, right). Another instance in which we observed Lucca activating the *structuring* seed was when he explained why he knew the location of the points:

I: Why did you put the point right there? What does it mean?

L: Because it should go on the corner.

I: What does it tell me about how many birds and bird wings there are?

L: Because the number of one bird is two bird wings. The number of two birds is four bird wings.

Next, we discuss two seeds that were identified previously in literature and observed in Lucca's interviews, *what you see is what you get* and *covariation*.

What you see is what you get

We observed the activation of this seed only during the first interview. When Lucca saw the graph, he first noticed the labels on both axes. He then attended to the numbers and when asked about the points he said, "These are probably the birds;" then after being asked about the first point, he added "This dot is probably the first bird." Here, we note that Lucca is interpreting the points as birds and, thus, is unable to see them as a coordinated pair of the number of birds and bird wings. In other words, the points were not indicating a quantity (i.e., the number of birds or the number of bird wings), but rather the birds themselves.

We attribute the activation of this seed to the fact that Lucca had never seen a graph before, therefore, he did not interpret the points as though they existed in a graph context.

Covariation

In the second interview, we observed Lucca create a table. Tables, while not the focus of this analysis, were taught in the CTEs and used throughout the interviews. When asked why he wrote the number of birds first, Lucca's response indicated that he used the same direction change *covariation* seed (Levin & Walkoe, 2022) to reason about the relationship. He explained, "You have to put the birds first to know which (one). So you know that it's the number of the birds." The interviewer probed, "So, you know the number of birds? Why did you put that one first? Why not wings first?" And Lucca further explained, "Because then you would probably get confused." Lucca's explanation suggests he activated a same direction change *covariation* seed and that his understanding of the relation between birds and bird wings at that moment was unidirectional. Lucca's unidirectional understanding surfaced again when he was unable to determine the number of birds given two bird wings. When asked the number of birds if there were two wings, Lucca said, "If there were four wings, there would be two birds."

Interestingly, we observed a shift in the direction of his thinking about the relationship when he interpreted this relationship in a graph context. When Lucca constructed his graph, he actually connected the numbers of bird wings (the *y*-axis) with the numbers of birds (the *x*-axis), which can be seen in his self-made graph (see Figure 3, left). In addition, when given a premade graph and asked to show (i.e., point to) the number of bird wings for three birds he answered, "There's no point. That's not possible" and gestured up the *y*-axis. Based on Lucca's response we assume he had shifted the focus of the directionality of his *covariation* seed, and therefore was unable to reason about three birds. Instead, Lucca seemed to be thinking about three wings and responded that it is "not possible" because he knew no number of birds would have three wings.

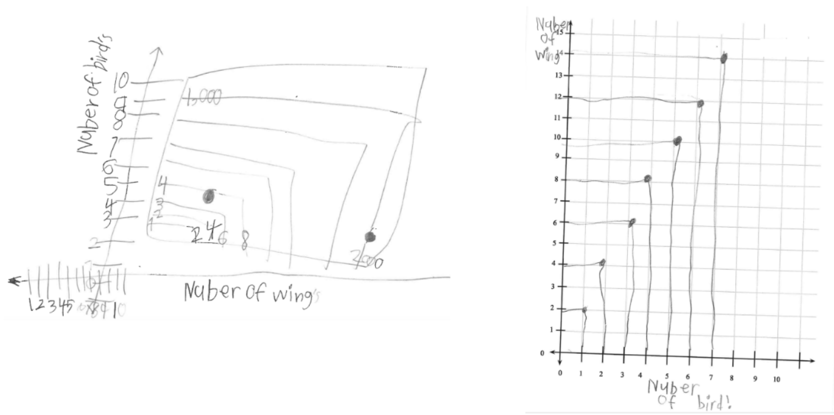

Figure 3. Lucca's self-made graph from his second interview (Left) and Lucca's graph from his third interview (Right)

DISCUSSION

By Lucca's third interview, we observed him plot a function graph, and we argue that at this point, he was able to do so because he coordinated *classifying*, *structuring* and *covariation seeds*. In other words, we believe that Lucca was able to plot the points because he activated *classifying*, *structuring,* and *covariation seeds* in concert. Even though we did not observe an externalization of the *classifying* seed in the graph context, we note that for *structuring* and *covariation* to be activated, Lucca had to activate and previously engage in *classifying.* Moreoever, we observed several instances in which Lucca engaged in *classifying* in the context of. table, but we do not report those instances here, because there are outside of our focus on graphs.

Note that the *what you see is what you get* seed was likely not activated in these later moments when Lucca graphed because at this point in his development he was no longer relying on that seed. We assume that through his experiences in the CTEs, Lucca became familiar with the function graph, recognizing the relationship being represented rather than just its compelling visual attributes, which are likely to cue the activation of this seed (Elby, 2000).

In the following we briefly summarize our observations of Lucca coordinating the activation of the *classifying*, *structuring*, and *covariation seeds*. We hypothesize that the coordination of the three seeds was a developing ability to activate a *coordination* *class* for representing a functional relationship using a graph. From a knowledge-in-piece framework, a *coordination class* is a task-specific collection of resources students use to engage with the task (Izsák et al., 2022). Here we briefly describe our observations of Lucca beginning to coordinate seeds for graphing functional relationships.

First, Lucca was able to *classify* the number of birds and the number of bird wings in two different sets. He did this by constructing a table listing the number of birds and the number of bird wings together (Figure 2).

He then identified these two different sets in the graph by referring to the labels, and then *structured the sets*, when he described the number of wings as twice the number of birds. The external representation of his *structuring* was evident when Lucca drew guidelines to plot the points, coordinating the increment in x with the increment in y. This moment is also evidence that he activated the *covariation* seed because he describes a "resulting change in output" given information about the input (Levin & Walkoe, 2022, p. 1306). Moreover, Lucca plotted points, indicating that he understood how to *structure* the two sets (i.e., he understood how to represent that specific set elements were coordinated).

CONCLUSION

Different students may activate different seeds in order to graph a function; we do not suggest that Lucca's coordination class will generalize to all students. However, there is significance in analyzing moment-to-moment reasoning and attention to interactions between different seeds to understand students' mental activities. As seen in this work, we identified two elements of Lucca's knowledge which we conceptualized as *classifying* and *structuring*. We believe that these seeds, in coordination with a *covariation* seed supported Lucca in graphing a function.

Future research could focus on exploring how the two seeds we report activate in different problem contexts and representations as well as the possibilities of these seeds refining over time. Additionally, exploring how different students coordinate their seeds to engage in graphing could allow for a better understanding of what experiences incite their activation. Finally, we highlight the potential in using a seeds framework because it supports us in moving "away from the predominant preoccupation with numerical calculations" and placing the "focal emphasis on typical and important ways of mathematical thinking" (Dörfler, 2008, p. 159) many of which are intuitive and natural, based on prior experiences, and captured with the seeds approach to mathematics learning.

Acknowledgement

This research is supported by the National Science Foundation's DRK-12 Award #1154355. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

Blanton, M., Brizuela, B. M., Gardiner, A. M., Sawrey, K., & Newman-Owens, A. (2017). A progression in first-grade children's thinking about variable and variable notation in functional relationships. *Educational Studies in Mathematics, 95*(2), 181–202. http://www.jstor.org/stable/45184560

Brizuela, B. M., Blanton, M., Gardiner, A. M., Newman-Owens, A., & Sawrey, K. (2015). A first grade student's exploration of variable and variable notation / Una alumna de primer grado explora las variables y su notación. *Estudios de Psicología: Studies in Psychology, 36*(1), 138-165.

Brizuela, B. M., Blanton, M., Sawrey, K., Newman-Owens, A., & Gardiner, A. M. (2015). Children's Use Of Variables and Variable Notation To Represent Their Algebraic Ideas. *Mathematical Thinking and Learning, 17*, 1-30.

diSessa, A. A. (1993). Toward an epistemology of physics. Cognition and instruction, 105-225.

Dougherty, B. (2008). Measure up: A quantitative view of early algebra. In J.J. Kaput, D.W. Carraher, & M.L. Blanton (Eds.), Algebra in the early grades (pp. 389-412). Lawrence Erlbaum Associates.

Dörfler, W. (2008). En route from patterns to algebra: Comments and reflections. ZDM Mathematics Education, 40, 143–160.

Elby, A. (2000). What students' learning of representations tells us about constructivism. *Journal of Mathematical Behavior, 19*, 481–502.

Izsák, A. (2005). "You Have to Count the Squares": Applying Knowledge in Pieces to Learning Rectangular Area. *The Journal of the Learning Sciences, 14*(3), 361–403.

Izsák, A., Beckmann, S., & Stark, J. (2022) Seeking Coherence in the Multiplicative Conceptual Field: A Knowledge-in-Pieces Account. *Cognition and Instruction, 40*(3), 305-350.

Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. *The Journal of Mathematical Behavior, 32*, 122-141.

Fazio, L. K., Siegler, R. S. (2013) Microgenetic Learning Analysis: A Distinction without a Difference: Commentary on Parnafes and diSessa. *Human Development, 56*(1): 52–58.

Levin, M., Walkoe, J. (2022) Seeds of algebraic thinking: a Knowledge in Pieces perspective on the development of algebraic thinking. *ZDM Mathematics Education 54*, 1303–1314.

Wagner, P. A. (2006). Probability, Decision Theory, and a Curricular Approach to Developing Good Thinking. *Journal of Thought, 41*, 23-38.