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Learning as a Cornerstone of Healing, Resilience, and Community

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University at Buffalo Graduate School of Education

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Kindergarteners' Understandings and Representations of the Additive Inverse

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Abstract: This study involved a 7-lesson generalized arithmetic classroom teaching experiment (CTE) with kindergarten students. We interviewed four students individually before and after the seven weeks to explore their understandings and representations of arithmetic properties. Here, we report on students' responses to questions on the additive inverse property. Using Skemp's framework of relational and instrumental understandings (2006), our analysis revealed that most of the interviewed kindergarteners could understand the additive inverse relationally by the end of the CTE. Our interviews revealed that tables and number lines enabled students to articulate more sophisticated understandings of the additive inverse.

Introduction

In this paper we report on how kindergarten students understood the additive inverse. We approach this study through the following research question: *How do kindergartens make sense of the additive inverse using tables and number lines?* In answering this question, we leveraged prior work by focusing on children's use of visual representations to make sense of and justify algebraic relationships across the content dimension of generalized arithmetic, particularly the role of tables and number lines when working with arithmetic properties. Generalized arithmetic involves generalizing arithmetic relationships and reasoning explicitly with these generalizations (Carraher et al., 2006). These relationships include properties of numbers and operations, such as the commutative property, the additive inverse, and the additive identity.

We emphasize the significance of concentrating on generalized arithmetic because it has been recognized as a means to introduce algebraic reasoning through arithmetic activities that are familiar to young children (Carpenter et al., 2003). A recent study by Ramírez Uclés et al. (2022) reported that kindergarteners can engage in algebraic reasoning by working with arithmetic properties. Furthermore, Schifter and Russell's (2022) study noted that students can utilize representations to show relationships within a generalized arithmetic context. We build on both studies by integrating representations to explore how kindergarteners reasoning about the additive inverse using tables and number lines. Our focus on tables was motivated by Brizuela et al. (2021), who argued that tables can support young children in reasoning algebraically. We integrated number lines into our study, recognizing their essential role in representing quantities (Schliemann et al., 2013). We are aware that the structure $a - a = 0$ is recognized as the subtractive negation principle (Baroody et. al, 2009) whereas the additive inverse is $a + (-a) = 0$. However, in framing the additive inverse we opted to build on prior early algebra work in which the number a and its additive inverse $-a$, is reasoned through the relationship $a - a = 0$ (Carraher et al. 2006, Blanton et al. 2018, Ramírez-Uclés et al. 2022)

Method

Setting and participants

This paper reports on one part of a more extensive study on students' understandings and uses of representations. The CTE took place in a kindergarten (ages 5-6) class of 17 students. To develop the lessons used in the CTE, we revisited and built on previous work which have proven effect to support children's' developing understandings of variable notation in the contexts of generalized arithmetic (Brizuela et al., 2015). We modified these lessons in ways that we hypothesized could support children's understandings of visual representations, specifically tables. Since the sequence consisted of seven lessons, we opted to dismiss variable notation instruction so we could focus on representations. In the seven lesson sequence students engaged in creating, communicating, and reasoning with representations (Selling, 2016). In the generalized arithmetic lessons, we focused on three arithmetic properties of addition: three lessons on the commutative property (e.g., $a + b = b + a$); three lessons on the additive identity (e.g., $a + 0 = a$); and one lesson on the additive inverse (e.g., $a - a = 0$). In this paper, we report on three kindergartener's interviews that took place before and after this 7-week CTE.

Interviews and data Selection

Clarice, Kaitlyn, and Peter were individually interviewed before and after the 7-week generalized arithmetic intervention. We selected students based on teacher recommendations, focusing on identifying those who would be most willing to engage in conversations with a researcher. The additive inverse was contextualized by asking students about a dog, Duke, who would eat all the treats he was given. In line with the representations used during the lesson sequence, a pre-constructed table (see Figure 3), a number line with numbers zero through ten, and an open number line with only zero (see Figure 2) were given to the students during the two interviews.

Data analysis

We analyzed the video-recorded interviews using Skemp's (2006) framework of students' relational and instrumental understandings. To account for reliability, a second coder reviewed each interview to confirm the main coder's work. If there were any discrepancies, we referred to a third coder and discussed all disagreements until an agreement was reached, following an approach outlined by Syed and Nelson (2015).

Findings and discussion

As we interviewed the students, we observed different levels of sophistication in students' understandings of the additive inverse property. When asked why the total number of treats remained the same, we observed two levels of relational understanding in terms of Skemp's framework (2006): *Relational emergent* and *Relational complete*. *Relational emergent* applied to those who reasoned about the expression $a - a$, without making the result explicit (i.e. without stating "equals zero"). That is, the student does not mention that the final quantity will always be zero. For example, "He eats all of them" (Kaitlyn), "Because dogs eat all of the treats" (Clarice). These two utterances do not mention zero. *Relational complete* applied to those who reasoned about the equation $a - a = 0$, making both the expression and the result explicit. "So every time he eats treats, I know it is zero since he ate all of them" (Peter). We decided to include this distinction between both levels of relational understandings when students compared two quantities to build their arguments (Ramírez Uclés, 2022; Vergnaud, 1996) because it provided insight into the depth of students' understandings and their ability to generalize the additive inverse property.

In the following sections, we report Kaitlyn and Clarice's second interview, and Peter's first interview during these three instances, we observed how they used different representations to make sense of the additive inverse.

Kaitlyn's second interview

In the second interview, the open number line helped Kaitlyn to think about the general case. When asked how she would represent "many treats," Kaitlyn pointed to an arbitrary place on the right side of the number line, indicating some indeterminate quantity of treats (see Figure 1, left).

Figure 1
Kaitlyn Points to an Arbitrary Place to Show many Treats and then Points to Zero on an Open Number line.

Kaitlyn was working with an open number line, which had only zero. When asked to think about any number of treats, she would point to the right side of the number line, indicating she was thinking of a large number. She did not need to write an arbitrary number to refer to this quantity. We take this as evidence Kaitlyn was comfortable working with any number of treats. Then, when the interviewer told her that Duke had eaten all his treats and asked her to show how many treats were left, "And then he eats all of them, what does it look like?", she moved her hand along the number line to the left side and pointed to zero (see Figure 1, right). We emphasize that the distance covered between the two movements were the same. Kaitlyn represented any number a by moving to the right, and then represented its additive inverse $-a$ by moving to the left the same distance. Kaitlyn's utterance regarding how the dog eats all his treats, "Because he eats all of them," combined with her gestures along the number line to show $a - a = 0$, are evidence of a *relational understanding* and her beginning to generalize the inverse property.

Clarice's second interview

In the second interview, Clarice's *relational understanding* can be associated with her tabular representation. First, Clarice was able to record the information correctly as seen in Figure 6. Then, when asked about the table, she noticed that the first and second columns were the same, and the third would always be zero:

> Interviewer: What is happening in the chart?
> Clarice: There is only zeros here (points to the last column in Figure 6)
> Interviewer: What else do you notice?
> Clarice: There is one two and there is another two. There is another three too here. And a four right here. And ones right here.

In addition, Clarice knew that the dog would eat all the treats he was given initially as she would always respond "zero" when asked about two, three, and four threats. Additionally, when asked for any number of treats she used the open number line to indicate that the answer would be zero if the dog had an unknown number of treats (see Figure 2). The interview explained, "I gave him a lot of treats. I have no idea how many because I didn't count them. Can you show me on that number line how many he had left?" And Clarice drew a vertical line below zero.

Figure 2
Clarice's Work on the Open Number Line in the Second Interview

We infer that the use of the number line allowed her to think about and represent a general case, even though she was not able to verbalize it she wrote two parenthesis "( )" when the interviewed asked about a lot of treats. Moreover, her comparison of equal quantities is represented in the lines she drew, indicating that the numbers connected are equal (see Figure 3). Therefore, we argue that Clarice used the table to understand the additive inverse relationally: she understood that subtracting the quantity from an initial amount would consistently yield zero, as both the initial quantity and the subtracted quantity were equal.

Figure 3
Clarice's Table in the Second Interview

Number of Treats Duke has	Number of treats Duke eats	How many treats are left
1	1	0
2	2	0
3	3	0
4	4	0

Peter's first interview

Starting with the first interview, Peter demonstrated a relational understanding of the additive inverse property. For him, the transformation of the initial quantity was the same for each case, "So every time he eats the treats, I know it is zero since he ate all of them." Despite this, Peter had difficulties working with the table. He would point at the wrong column or misplace numbers. Like Kaitlyn, Peter provided evidence of a similar strategy using the number line to show $a - a = 0$ in the first interview. Peter used gestures to support this verbal explanation. He pointed to the right of the number line when asked by the interviewer to show "a bunch of treats" and then pointed to zero when asked to show how many were left. That is, while he was not able to use the table to represent his understanding of the additive inverse, he was able to use the number line to show his relational understanding and to think about the general case.

Conclusions

We agree with Ramírez Uclés et al. (2022) that given that these kindergarten students had not been formally taught subtraction, the table and number line might be useful tools because they do not require experience with the minus symbol in an expression, yet they help students communicate their thinking. Furthermore, the open number line allowed Kaitlyn and Peter to think about indeterminate quantities. We therefore argue that these might be useful representations to use in a generalized arithmetic context.

We underscore the significance of representations to support young learners in articulating mathematical concepts, particularly when confronted with topics such as arithmetic properties that are intrinsically challenging to verbally express (Ramírez-Ucles et al., 2022). Discussing the abstract notions of zero or indeterminate quantities can be daunting, so visual representations such as number lines and tables provide an alternative through which students can communicate their ideas.

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