

SYSTOLIC INEQUALITIES FOR THE NUMBER OF VERTICES

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ABSTRACT. Inspired by the classical Riemannian systolic inequality of Gromov we present a combinatorial analogue providing a lower bound on the number of vertices of a simplicial complex in terms of its edge-path systole. Similarly to the Riemannian case, where the inequality holds under a topological assumption of “essentiality”, our proofs rely on a combinatorial analogue of that assumption. Under a stronger assumption, expressed in terms of cohomology cup-length, we improve our results quantitatively. We also illustrate our methods in the continuous setting, generalizing and improving quantitatively the Minkowski principle of Balacheff and Karam; a corollary of this result is the extension of the Guth–Nakamura cup-length systolic bound from manifolds to complexes.

1. INTRODUCTION

What is the smallest number of vertices in a simplicial complex triangulating a given topological space? Taking $\mathbb{R}P^n$ as an example, the exact minimum is known only for $n \leq 4$. Asymptotically there is a wide gap between the best known lower bound of $\frac{1}{2}(n+1)(n+2)$ from [2, §16] and the recently discovered [1] upper bound of $e^{C\sqrt{n}\log n}$, which is realized as the quotient of a centrally symmetric simplicial convex polytope.

For any simplicial complex, the length of the shortest non-contractible loop along the edges is at least 3. Hence the question of vertex minimal triangulations is the end-case of a question in systolic geometry.

In this paper we address the smallest number of vertices of triangulations such that the shortest non contractible loop along the edges has length at least s . One of our main results, Theorem 1.4 below provides a combinatorial analogue of Gromov’s systolic inequality in which the volume of a Riemannian metric is substituted by the number of vertices of a simplicial complex. This theorem answers a question from [7, Appendix 1] and is valid not only for the projective space, but much more generally, for any combinatorially n -essential complex.

Besides that, in Theorem 1.18 below we address the classical systolic inequality for Riemannian polyhedra, improving the results of [11, 18, 4, 6].

1.1. Discrete formulation of systolic inequalities. *Systolic* (or *isosystolic*) inequalities, first studied by Löwner and Pu [20] (see also [13] for a modern version of Pu’s two-dimensional result), relate the volume of a closed manifold to the *systole*, that is, the length of a shortest non-contractible curve. Gromov [10] proved *the* systolic inequality: there exists a constant c_n

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such that if M^n is an *essential* manifold, then for any Riemannian metric g on M the systole $\text{sys}(g)$ of (M, g) is bounded from above by $c_n \text{vol}(g)^{1/n}$. Here the constant c_n depends only on the dimension, and $\text{vol}(g)$ is the Riemannian volume of (M, g) . The converse—every non-essential manifold admits a metric with large systole but small volume—is also true, as first observed by Babenko [3].

Definition 1.1. The *(edge-path) systole* $\text{sys } X$ of a simplicial complex X is the smallest integer such that any closed path along the edges of X of edge-length less than $\text{sys } X$ is null-homotopic. If every component of X is simply connected, the systole is, by convention, $+\infty$.

In [7] and [15] it was observed that there exists a constant c'_n such that any simplicial complex X triangulating an essential manifold has at least $c'_n \text{sys}(X)^n$ facets (faces of dimension n). Theorem 1.4 provides a condition on a simplicial complex that implies the discrete systolic inequality in terms of the vertices (which in turn, implies the Riemannian version [7] and lower bounds for the number of faces of any given dimension which in particular improve significantly the estimate on the constant c'_n).

Unlike the estimates on the number of n -faces, lower bounds on the number of vertices (in Theorem 1.4) are not easy to derive directly from Riemannian systolic inequalities. Instead our proof adapts the approach of Larry Guth [11] and Panos Papasoglu [19], which is metaphorically referred to as “the Schoen–Yau [21] minimal hypersurface method”.

We remark that metric systolic inequalities have been successfully studied in spaces other than manifolds; see, e.g., [10, Appendix B] or [16]. Here we consider arbitrary simplicial complexes, which is not new, but allows to make definitions and present proofs in a very simple way.

Now we get down to definitions, paralleling the Riemannian definitions in the combinatorial setting. For a subset $W \subset V(X)$ of vertices of a simplicial complex X , we denote by $\langle W \rangle$ the subcomplex induced by W , that is, the subcomplex of all faces $\sigma \in X$ such that the vertices of σ are in W .

Definition 1.2. A subset $Y \subset V(X)$ is called *inessential* if the natural map $\pi_1(C) \rightarrow \pi_1(D)$ is trivial for every connected component C of $\langle Y \rangle$ and every connected component D of X .

Definition 1.3. A complex X is called *combinatorially n -essential* if its vertex set cannot be partitioned into n inessential sets or fewer.

Note that in our definition

- Combinatorially n -essential implies combinatorially m -essential whenever $m \leq n$.
- Combinatorially 1-essential is equivalent to having non-contractible loops and having finite systole.
- A space having no non-contractible loops may be considered combinatorially 0-essential, but our results do not apply to this case.

Theorem 1.4. *Let X be a combinatorially n -essential simplicial complex, $n \geq 1$. Then the number of vertices of X is at least*

$$\binom{n + \lfloor \frac{\text{sys } X}{2} \rfloor - 1}{n-1} + 2 \binom{n + \lfloor \frac{\text{sys } X}{2} \rfloor - 1}{n} - 1 \geq \binom{n + \lfloor \frac{\text{sys } X}{2} \rfloor}{n} \geq \frac{1}{n!} \left\lceil \frac{\text{sys } X}{2} \right\rceil^n.$$

A topological space is said to be *n -essential* if the classifying map $f : X \rightarrow K(\pi_1(X), 1)$ cannot be deformed to the $(n-1)$ -skeleton of $K(\pi_1(X), 1)$, see [10, Appendix B]. In an appendix to this paper we show that any triangulation of an n -essential space is combinatorially n -essential. The converse is not true, Remark 1.17 presents a combinatorially n -essential complex that is not n -essential as a topological space. See the appendix for more details on the comparison of these definitions.

1.2. Size of metric balls. The proof of theorem 1.4 will be based on considering certain metric balls and relating their properties to the systole of the simplicial complex X . For $x \in V(X)$ and an integer i , let $B(x, i) \subset V(X)$ denote the set of vertices of a simplicial complex X whose edge-distance to x is at most i . Similarly, let $S(x, i) \subset V(X)$ denote the set of vertices of X whose edge-distance to x is exactly i .

Proposition 1.5. *Let X be a simplicial complex with finite systole. Set $r := \lfloor \frac{\text{sys } X}{2} \rfloor - 1$. Then every metric ball $B(x, r) \subseteq V(X)$ is inessential, and r is the maximum integer with this property.*

Definition 1.6. We call r in the previous proposition the *homotopy triviality radius* of X .

Proof of Proposition 1.5. Consider the universal covering map $\widetilde{X} \rightarrow X$. By the definition of systole, the edge-distance between any two vertices of the same $\pi_1(X)$ -orbit is at least $\text{sys } X$. It follows that for all r such that $2r < \text{sys } X - 1$ the restriction of the covering map $\widetilde{X} \rightarrow X$ to the (preimage of the) metric ball $\langle B(x, r) \rangle$ is a trivial cover. This means that this ball is inessential.

On the other hand, if x is chosen on a non-contractible edge-loop of length $\text{sys } X$, then $\langle B(x, r) \rangle$, for $r := \lfloor \frac{\text{sys } X}{2} \rfloor$, contains this loop. $\square$

The estimate on the number of vertices of X follows from the estimate of the number of vertices in a carefully chosen ball in X of radius $r + 1$, given in the following theorem:

Theorem 1.7. *Let X be a combinatorially n -essential simplicial complex, $n \geq 1$, and r be the homotopy triviality radius of X . Then there exists a vertex $x \in X$ such that for any $i = 0, \dots, r + 1$ the number of vertices in $B(x, i)$ is at least $b_n(i)$, where positive integers $b_n(i)$ satisfy the following recursive relations:*

- $b_1(i) = 2i + 1$ for any $i = 0, \dots, r$, and $b_1(r + 1) = 2r + 2$;
- $b_n(i) = \sum_{0 \leq j \leq i} b_{n-1}(j)$ for any $i = 0, \dots, r + 1$.

In particular, $b_n(i) \geq 2 \binom{i+n-1}{n} + \binom{i+n-1}{n-1}$ for any $i = 0, \dots, r$, and $b_n(r + 1) \geq 2 \binom{r+n}{n} + \binom{r+n}{n-1} - 1$.

These results are proved in Section 2.

1.3. Improvements under additional assumption of cohomology cup-length. We improve the bounds of Theorem 1.7 assuming additionally that the property of being essential is implied by a long nonzero product in cohomology. From here on we denote by $\mathbb{Z}_2$ the ring of residues modulo 2.

Definition 1.8. We call a complex n -cup-essential (over $\mathbb{Z}$ or $\mathbb{Z}_2$) if it admits n (not necessarily distinct) cohomology classes in degree 1 (with coefficients $\mathbb{Z}$ or $\mathbb{Z}_2$, respectively) whose cup-product is non-zero.

A cup-essential complex is essential (with the same n), as it follows from the Lusternik–Schnirelmann-type of argument (see Lemma 3.1). The torus $(S^1)^n$ is n -cup-essential both over $\mathbb{Z}$ and $\mathbb{Z}_2$. The real projective space $\mathbb{R}P^n$ is n -cup-essential over $\mathbb{Z}_2$.

Definition 1.9. We say that X has *homology triviality radius r with respect to the 1-cohomology classes ξ_i* if the restriction of every ξ_i to every metric ball $\langle B(x, r) \rangle \subseteq X$ is zero, and r is the maximum integer with this property.

Note that the existence of non-trivial degree 1 cohomology classes implies the existence of non-contractible loops and allows to define the finite systole.

Theorem 1.10. *Let X be a simplicial complex that is n -cup-essential over $\mathbb{Z}$ or $\mathbb{Z}_2$, as witnessed by the degree 1 cohomology classes $\xi_1, \dots, \xi_n$, $\xi_1 \smile \dots \smile \xi_n \neq 0$. Let r be the corresponding homology triviality radius.*

Then there is a vertex $x \in X$ such that for any $i = 0, \dots, r+1$ the number of vertices in $B(x, i)$ is at least $\check{b}_n(i)$, where positive integers $\check{b}_n(i)$ satisfy the following recursive relations:

- $\check{b}_1(i) = 2i + 1$ for any $i = 0, \dots, r$;
- $\check{b}_n(i) = \check{b}_{n-1}(i) + 2 \sum_{0 \leq j \leq i-1} \check{b}_{n-1}(j)$ for any $i = 0, \dots, r$.

In the case of $\mathbb{Z}_2$ coefficients, this theorem is basically a discrete analogue of [18, Theorem 2.3] with the improvement that X is not required to be a manifold. In the case of $\mathbb{Z}$ coefficients, to our knowledge, neither discrete nor continuous version was known before, and we discuss the corresponding continuous statement below.

Remark 1.11. Unraveling the recursion we obtain the generating function:

$$\sum_{n,i} \check{b}_n(i) u^n v^i = \sum_n \frac{u^n (1+v)^n}{(1-v)^{n+1}} = \frac{1}{1-v} \frac{1}{1-u \frac{1+v}{1-v}} = \frac{1}{1-v-u-uv}.$$

From the first identity in this formula and the expansion $\frac{1}{(1+v)^{n+1}} = \sum_i \binom{i+n}{n} v^i$ one obtains an estimate

$$2^n \binom{i}{n} \leq \check{b}_n(i) \leq 2^n \binom{i+n}{n}.$$

For example, for a fixed n and $i \rightarrow \infty$ we obtain

$$\check{b}(n, i) = 2^n \frac{i^n}{n!} (1 + o(1)).$$

For the total number of points in a cup-essential manifold we have:

Theorem 1.12. *Let X be a simplicial complex that is n -cup-essential over $\mathbb{Z}$ or $\mathbb{Z}_2$. Then the number of vertices of X is at least*

$$\sum_{k=0}^n \check{b}_k \left(\left\lfloor \frac{\text{sys } X}{2} \right\rfloor - 1 \right) \geq 2^n \binom{\left\lfloor \frac{\text{sys } X}{2} \right\rfloor}{n}.$$

If we are interested in face numbers of positive dimension, not only vertices, then Theorem 1.12 can be combined with lower bound theorems from face number theory. Recall that f_k denotes the number of k -dimensional faces. For example Kalai's lower bound theorem for simplicial PL-manifolds [14] provides a lower bound of $\binom{n-1}{k} f_0 - \binom{n}{k+1} k$ for the number of k faces, for $1 < k < n$ and roughly $n f_0$ for $k = n$ or $k = 1$. Here is another interesting application related to our motivating example:

Corollary 1.13. *If a centrally symmetric polytope in $\mathbb{R}^{n+1}$ has edge-distance between any pair of antipodal vertices at least s , then for the f -vector the following inequalities hold:*

- (1) $f_0 \geq 2^{n-1} \binom{\lfloor s/2 \rfloor}{n}$;
- (2) $f_k \geq \binom{n+1}{k} (2^n \binom{\lfloor s/2 \rfloor}{n} - 2n) + 2^{k+1} \binom{n+1}{k+1}$;
- (3) $f_{n-1} \geq 2^n n \binom{\lfloor s/2 \rfloor}{n} + 2^{n+1} - 2n^2 + 2n + 4$.

The first item follows by applying Theorem 1.12 to the quotient of the polytope by the involution $x \mapsto -x$, for $s \geq 3$, which is an n -cup-essential triangulated $\mathbb{R}P^{n-1}$. The second and third items are straightforward applications of Stanley's results [22]. In the end-case of this corollary $s = 2$ notice that any centrally symmetric polytope has at least $2n$ vertices.

The following theorem shows that when $\text{sys } X \gg n$, Theorems 1.4 and 1.12 and the previous corollary are asymptotically optimal.

Theorem 1.14. *For every n and $s \geq 3$, there exists a centrally symmetric triangulation of the n -dimensional sphere $\widetilde{X}$ such that the quotient n -projective space $X = \widetilde{X}/\mathbb{Z}_2$ has $\text{sys } X = s$ and no more than s^n vertices.*

Remark 1.15. It seems to us that the estimates on the number of vertices of Theorems 1.4 and 1.12 are valid, more generally, for Δ -complexes in place of simplicial complexes [12, Section 2.1] with the same proofs. Of course, Theorem 1.16 below does not extend since Δ -complexes may have systole equal to $1 < 3$.

The proofs of the vertex number estimates under the cup-length assumption are given in Section 3.

1.4. No assumption on the systole. The question of estimating the number of vertices in a triangulation of a manifold, mentioned in the beginning, may be viewed as a particular case of the systolic problem. Every combinatorially n -essential complex X has $\text{sys } X \geq 3$, just because any two edges with the same endpoints must be equal. In this case our theorems only provide a lower bound for $|V(X)|$ that is linear in n . In remark 1.17 below, we show that this cannot be improved.

But this linear bound may be improved under stronger assumptions. This was noticed in [2, §16], see also a series of somewhat more general estimates in [8] and [9, Section 3]. Here we provide an analogue of these results in terms of combinatorial essentiality.

Theorem 1.16. *If the barycentric subdivision of simplicial complex X is combinatorially n -essential then $|V(X)| \geq (n+1)(n+2)/2$.*

Remark 1.17. Considering a barycentric subdivision (as a step towards the topological definition of n -essential from the combinatorial one) in this theorem is important. The complete graph K_{2n+1} is combinatorially n -essential (partitioning its vertices into n or fewer parts results in a triangle) and has a linear (in terms of n) number of vertices. In particular, *one cannot even guarantee that a combinatorially n -essential complex has faces of dimension n* . See more on comparison of the notions of n -essential in the Appendix.

The lower bound in Theorem 1.16 should be compared with the upper bound $e^{C\sqrt{n}\log n}$ in the construction of a centrally symmetric simplicial convex polytope in [1]. It is wide open either to improve this quadratic lower bound, or to give a construction of an n -essential simplicial complex with a quadratic (or merely polynomial) in n number of vertices.

1.5. Improvements to the Riemannian inequalities. The ideas behind Theorem 1.10 can be used to improve the dimension-dependent factor in the classical systolic inequality, if one additionally assumes that the space is cup-essential (see Definition 1.8). This version of the systolic inequality for manifolds first appeared (over $\mathbb{Z}_2$) in the work of Guth [11], which pioneered the use of the minimal hypersurface method in systolic geometry. Our result applies to a wider class of complexes (Riemannian polyhedra), works over $\mathbb{Z}$ as well, and the dimensional factor is as good as the best known factor for cup-essential manifolds [18].

A *Riemannian polyhedron* is a simplicial complex whose simplices are endowed with Riemannian metrics matching on the intersection of adjacent simplices. This is a reasonable class of path metric spaces including Riemannian manifolds and admitting systolic inequalities. The systole $\text{sys } X$ of a compact non-simply connected Riemannian polyhedron X is the length of a shortest non-contractible loop in X (not necessarily along the edges of the simplicial structure).

Following the approach of Papasoglu [19], Nabutovsky [17] gave a transparent proof of the systolic inequality $\text{sys } X \leq 2(n!/2)^{1/n} \text{vol}(X)^{1/n}$ for essential Riemannian n -polyhedra. The factor $2(n!/2)^{1/n}$ is the best known. We improve Nabutovsky's estimate by a factor of $2^{1-1/n}$, assuming a cohomological cup-length condition. In fact, we prove more.

For a cohomology class $\xi \in H^1(X; \mathbb{Z})$ or $\xi \in H^1(X; \mathbb{Z}_2)$ we define $L(\xi)$ as the shortest length of a loop in X on which ξ evaluates non-trivially. The *Minkowski principle* of Balacheff–Karam [4] (see also [5]) states that if M is a closed n -manifold that is n -cup-essential over $\mathbb{Z}_2$

with $\xi_1 \smile \dots \smile \xi_n \neq 0$, then

$$\prod_{i=1}^n L(\xi_i) \leq c_n \operatorname{vol}_n X.$$

The name ‘‘Minkowski principle’’ refers to Minkowski’s second theorem on centrally symmetric convex bodies and lattices, which can be regarded as a version of this inequality for a flat Finsler torus.

The following result improves or generalizes [11, Theorem 3], [18, Theorem 2.3], [4, Theorem 2], [6, Theorem 1.1].

Theorem 1.18. *Let X be a compact Riemannian polyhedron of dimension n . Suppose it is n -cup-essential over $\mathbb{Z}$ or $\mathbb{Z}_2$, which is witnessed by the degree 1 cohomology classes $\xi_1, \dots, \xi_n \in H^1$, $\xi_1 \smile \dots \smile \xi_n \neq 0$, arranged so that $L(\xi_1) \leq \dots \leq L(\xi_n)$. Then there exists a point $x \in X$ such that the metric balls around x satisfy the following volumetric bounds:*

$$\operatorname{vol}_n B(x, r) \geq \begin{cases} \frac{(2r)^n}{n!}, & 0 < r \leq \frac{L(\xi_1)}{2}, \\ \frac{L(\xi_1)(2r)^{n-1}}{n!}, & \frac{L(\xi_1)}{2} < r \leq \frac{L(\xi_2)}{2}, \\ \dots, & \dots \\ \frac{L(\xi_1) \dots L(\xi_{n-2})(2r)^2}{n!}, & \frac{L(\xi_{n-2})}{2} < r \leq \frac{L(\xi_{n-1})}{2}, \\ \frac{L(\xi_1) \dots L(\xi_{n-1})2r}{n!}, & \frac{L(\xi_{n-1})}{2} < r \leq \frac{L(\xi_n)}{2}. \end{cases}$$

In particular, the Minkowski principle holds:

$$\prod_{i=1}^n L(\xi_i) \leq n! \operatorname{vol}_n X.$$

Even more specifically, $\operatorname{sys} X \leq (n!)^{1/n} \operatorname{vol}_n(X)^{1/n}$.

The proof is given in Section 4.

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2. USING THE COMBINATORIALLY ESSENTIAL ASSUMPTION

In order to apply induction, we generalize Theorem 1.7 and then prove the generalization. We first generalize the notion of combinatorially inessential and combinatorially essential to arbitrary covers.

Definition 2.1. Assume a simplicial complex X has a covering map $\pi : \widetilde{X} \rightarrow X$. A subset $Y \subset V(X)$ is π -inessential if the restriction $\pi^{-1}(\langle Y \rangle) \rightarrow \langle Y \rangle$ is a trivial cover. We assume that a cover over $\langle Y \rangle$ is trivial if and only if it is trivial (equal to $C \times D$ with a discrete set D) over every connected component C of $\langle Y \rangle$.

Definition 2.2. Assume a simplicial complex X has a covering map $\pi : \widetilde{X} \rightarrow X$. This covering map is called combinatorially n -essential if the vertex set of X cannot be partitioned into n or fewer π -inessential sets.

Definition 2.3. We say that a covering map $\pi : \widetilde{X} \rightarrow X$ has *homotopy triviality radius* r if every metric ball $B(x, r) \subseteq V(X)$ is π -inessential and r is the maximum integer with this property.

The definitions stated in the introduction are obtained from these definitions when $\widetilde{X} \rightarrow X$ is the universal covering map. When X has several connected components we assume that $\widetilde{X}$ is the disjoint union of the universal covers of every component.

Theorem 2.4. *Let a simplicial complex X have a covering map $\pi : \widetilde{X} \rightarrow X$ and let π be combinatorially n -essential. Let r be the homotopy triviality radius of π . Then there exists a vertex $x \in X$ such that for any $i = 0, \dots, r+1$ the number of vertices in $B(x, i)$ is at least $b_n(i)$, where positive integers $b_n(i)$ satisfy the following recursive relations:*

- $b_1(i) = 2i + 1$ for any $i = 0, \dots, r$ and $b_1(r+1) = 2r + 2$;
- $b_n(i) = \sum_{0 \leq j \leq i} b_{n-1}(j)$ for any $i = 0, \dots, r+1$.

In particular, $b_n(i) \geq 2\binom{i+n-1}{n} + \binom{i+n-1}{n-1}$ for any $i = 0, \dots, r$, and $b_n(r+1) \geq 2\binom{r+n}{n} + \binom{r+n}{n-1} - 1$.

Proof of Theorem 1.7 assuming Theorem 2.4. We may take a single component of X and consider the universal covering map $\pi : \widetilde{X} \rightarrow X$. Then π -inessential is the same as inessential for (the fundamental group of) X . Hence Theorem 2.4 applies and produces the result. $\square$

Proof of Theorem 2.4. Let us first prove the theorem for $n = 1$. In that case, consider a simple (passing once through each of its vertices) closed loop, whose lift to $\widetilde{X}$ is not closed. Such a loop exists since otherwise X would be inessential. The loop has at least $2r + 2$ vertices, because otherwise it would fit into a ball of radius r containing only loops lifting to loops. The equalities $b_1(i) = 2i + 1$ for any $i = 0, \dots, r$ and $b_1(r+1) = 2r + 2$ then easily follow.

We proceed by induction over n . Assume that $n > 1$ and that the theorem is already proven for smaller n . Find a smallest subset of vertices $Z \subset V(X)$ such that its complement $Y := V(X) \setminus Z$ is π -inessential.

By definition, the restriction of the covering map, $\pi^{-1}(\langle Z \rangle) \rightarrow \langle Z \rangle$, is then combinatorially $(n-1)$ -essential. The metric balls of Z are contained in the respective metric balls of X and the π -inessentiality property of the balls is therefore preserved. Hence the homotopy triviality radius of the restriction of the covering map is no smaller than that of π .

By the induction hypothesis, there exists a vertex $x \in Z$ such that for any $i = 0, \dots, r+1$ the number of vertices in $B(x, i) \cap Z$ is at least $b_{n-1}(i)$.

Consider the following modification of Z :

$$Z' := Z \cup S(x, r+1) \setminus B(x, r).$$

Let us prove that $Y' := V(X) \setminus Z'$ is π -inessential.

Assume to the contrary, that there is a simple closed edge-path P in $\langle Y' \rangle$ whose lift to $\widetilde{X}$ is not closed. The set of vertices $S(x, r+1)$ separates the 1-skeleton of X into two disconnected components with vertex sets $B(x, r)$ and $V(X) \setminus B(x, r+1)$, respectively. Since P is simple and does not contain vertices of $S(x, r+1)$, we have that

- either $P \subset \langle B(x, r) \rangle$;
- or $P \subset \langle Y' \setminus B(x, r+1) \rangle$.

In the first case, P lifts to a closed path in $\widetilde{X}$ because π has the homotopy triviality radius r . In the second case, P lifts to a closed path in $\widetilde{X}$ because $Y' \setminus B(x, r+1) \subseteq Y$ and Y is π -inessential.

Since Y' is π -inessential, from the minimality of Z , we obtain that

$$|Z| \leq |Z'|,$$

and therefore

$$|S(x, r+1) \setminus Z| \geq |B(x, r) \cap Z|.$$

Analogously, for every $i \leq r$ we obtain that

$$|S(x, i+1) \setminus Z| \geq |B(x, i) \cap Z|.$$

Finally, for any $i = 0, \dots, r+1$ we have that

$$(2.1) \quad |B(x, i)| = |B(x, i) \cap Z| + \sum_{0 \leq j \leq i} |S(x, j) \setminus Z| \geq \sum_{0 \leq j \leq i} |B(x, j) \cap Z| \geq \sum_{0 \leq j \leq i} b_{n-1}(j) = b_n(i).$$

It remains to prove the inequality $b_n(i) \geq 2\binom{i+n-1}{n} + \binom{i+n-1}{n-1}$. The case $n = 1$ reads

$$b_1(i) \geq 2i + 1,$$

and has been already shown in the beginning of the proof.

For $n > 1$ we have:

$$\begin{aligned} b_n(i) &= \sum_{0 \leq j \leq i} b_{n-1}(j) \geq \\ &\geq \sum_{0 \leq j \leq i} \left(2\binom{j+n-2}{n-1} + \binom{j+n-2}{n-2} \right) = 2\binom{i+n-1}{n} + \binom{i+n-1}{n-1}. \end{aligned}$$

For $b_n(r+1)$, as compared to $b_n(i)$ with $i = 0, \dots, r$, the estimate is 1 less in case $n = 1$, as was described in the beginning of the proof. In the course of summation this -1 summand carries on, hence we have

$$b_n(r+1) \geq 2\binom{r+n}{n} + \binom{r+n}{n-1} - 1.$$

□

Proof of Theorem 1.4. From Proposition 1.5 we know that the restriction of the universal covering map $\pi : \widetilde{X} \rightarrow X$ to every metric ball $\langle B(x, r) \rangle$ of X is inessential, where $r = \lfloor \frac{\text{sys } X}{2} \rfloor - 1$.

By Theorem 1.7 there exists a ball of radius $r+1$ in X with at least

$$b_n(r+1) \geq 2\binom{r+n}{n} + \binom{r+n}{n-1} - 1$$

vertices, which is all we need to prove.

□

Proof of Theorem 1.16. Start with the universal covering map $\pi : \widetilde{X} \rightarrow X$ and argue, as in the proof of Theorem 2.4, considering a covering map and its restrictions.

We claim that the dimension of X must be at least n . Otherwise, the vertices of its barycentric subdivision can be properly colored with n colors, implying that π is not combinatorially n -essential over the barycentric subdivision. Hence X has a face $F \subset X$ with at least $n+1$ vertices. Remove this face from X to obtain the complex Y induced by X on $V(X) \setminus F$.

The barycentric subdivision Y' of Y is contained in the barycentric subdivision X' of X . The complex Z induced on $V(X') \setminus V(Y')$ retracts onto $\langle F \rangle$, as it is contained in the neighborhood $U(F)$, and then retracts onto any point of F . Hence Z is π -inessential. It follows that the restriction of the covering map π to the preimage of Y' is at least $(n-1)$ -essential. This allows to make a step of induction, taking Y in place of X and keeping the covering map π to measure essential and inessential. In the end we exhibit at least

$$(n+1) + n + (n-1) + \dots + 1 = \frac{(n+1)(n+2)}{2}$$

vertices of X .

□

3. USING THE CUP-ESSENTIAL ASSUMPTION

The following lemma (basically from the Lusternik–Schnirelmann theory) relates the n -essential property and the existence of a nonzero cohomology product of length n .

Lemma 3.1. *Let X be a simplicial complex with its set of vertices $V(X)$ decomposed into a disjoint union $V(X) = Y \sqcup Z$. Suppose that there are two cohomology classes ξ_1 and ξ_2 on X such that $\xi_1 \smile \xi_2 \neq 0$. Assume that ξ_1 restricted to $\langle Y \rangle$ is trivial. Then ξ_2 restricted to $\langle Z \rangle$ is not trivial.*

Proof. Let $U(Z)$ be the union of all open stars of the vertices in Z , $U(Z) \supset \langle Z \rangle$. And analogously $U(Y) \supset \langle Y \rangle$. The set $U(Z)$ retracts onto $\langle Z \rangle$ and $U(Y)$ retracts onto $\langle Y \rangle$. Since $U(Z) \cup U(Y) = X$, $\xi_1 \smile \xi_2 \neq 0$, $\xi_1|_{U(Y)} = 0$, it follows from the properties of the cohomology multiplication (the same as used in the Lusternik–Schnirelmann category estimate) that $\xi_2|_{U(Z)} \neq 0$. $\square$

Proof of Theorem 1.10 over $\mathbb{Z}$. We use induction. The base case $n = 1$ follows from Theorem 1.7 (or from the beginning of its proof).

The class ξ_1 corresponds to a classifying map

$$X \rightarrow K(\mathbb{Z}, 1) = S^1 = \mathbb{R}/\mathbb{Z}.$$

We pull back the covering map $\mathbb{R} \rightarrow S^1$ to a covering map $\pi : \widetilde{X} \rightarrow X$. This $\widetilde{X}$ has a free $\mathbb{Z}$ -action with a bounded fundamental region F , $\widetilde{X} = \bigcup_{m \in \mathbb{Z}} mF$. We may also assume that X and $\widetilde{X}$ are connected, or otherwise work in a connected component.

Let $Z \subset V(\widetilde{X})$ be a smallest set of vertices that “separates $-\infty$ from $+\infty$ ” in $\widetilde{X}$. That is, such set of vertices that for all sufficiently large m any edge-path in $\widetilde{X}$ connecting a vertex in mF to a vertex in $-mF$ contains a vertex in Z .

Note that we choose a finite Z and therefore for sufficiently large positive integer N the set Z is not connected with any of its shifts by kN , for any nonzero $k \in \mathbb{Z}$. Consider $X_N := \widetilde{X}/N\mathbb{Z}$ and the natural projections $\pi_N : X_N \rightarrow X$ and $\pi'_N : \widetilde{X} \rightarrow X_N$. Choose N sufficiently large and coprime with the order of $\xi_1 \smile \dots \smile \xi_n$. Note that π_N is an N -sheet cover and therefore $\pi_N^*(\xi_1 \smile \dots \smile \xi_n) \neq 0$, since otherwise we would have

$$N\xi_1 \smile \dots \smile \xi_n = \tau(\pi_N^*(\xi_1 \smile \dots \smile \xi_n)) = 0,$$

where $\tau : H^*(X_N; \mathbb{Z}) \rightarrow H^*(X; \mathbb{Z})$ denotes the transfer map.

The projection π'_N induces an isomorphism of simplicial complexes $\langle Z \rangle \rightarrow \langle Z_N \rangle$ by the choice of N , where $Z_N := \pi'_N(Z)$. The subset Z_N blocks any cycle in X_N which could lift to a cycle connecting $-\infty$ and $+\infty$, as it follows from the separation property of Z . Therefore $\pi_N^*(\xi_1)$ is zero on $\langle V(X_N) \setminus Z_N \rangle$. By Lemma 3.1, $\pi_N^*(\xi_2) \smile \dots \smile \pi_N^*(\xi_n)$ is nonzero on $\langle Z_N \rangle$. From the isomorphism of simplicial complexes $\langle Z \rangle \cong \langle Z_N \rangle$, the product

$$\pi^*(\xi_2) \smile \dots \smile \pi^*(\xi_n) \in H^{n-1}(Z; \mathbb{Z})$$

is also nonzero.

The projection of $B(x, r) \cap Z$ is contained in the corresponding r -ball $B(\pi(x), r)$ of X . By the choice of r , the classes $\pi^*(\xi_2), \dots, \pi^*(\xi_n)$ vanish on $\langle B(x, r) \cap Z \rangle$. Hence the inductive assumption with n replaced by $n - 1$ applies to Z . Let $x \in Z$ be the point obtained from the inductive assumption.

Similar to the proof of Theorem 1.7, we have that for every $i + 1 \leq r$

$$|S(x, i + 1) \setminus Z| \geq |B(x, i) \cap Z|.$$

In the current setting this inequality can be improved.

Let $S^+(x, i+1) \subset S(x, i+1) \setminus Z$ be the subset of those vertices in $S(x, i+1)$ which are connected to $+\infty$, i.e., for which there exists an edge path in $\widetilde{X}$ not using any vertex of Z and connecting them to a vertex in mF for any sufficiently large m . Likewise, let $S^-(x, i+1) \subset S(x, i+1) \setminus Z$ be the subset of those vertices in $S(x, i+1)$ which are edge-connected to $-\infty$ in $\widetilde{X} \setminus Z$.

Remove $B(x, i)$ from Z and add $S^+(x, i+1)$ instead. Denote the obtained vertex set by Z' . For $i+1 \leq r$, we have that Z' still “separates $-\infty$ from $+\infty$ ” in $\widetilde{X}$. Indeed, any path from $-\infty$ to $+\infty$ in $\widetilde{X}$ not passing through Z' has to pass through $Z \setminus Z' = Z \cap B(x, i)$. Then this path has to reach $+\infty$ from $B(x, i)$. When this path passes through $S(x, i+1)$ for the last time, it does in fact pass through $S^+(x, i+1)$ by the definition of $S^+(x, i+1)$.

From the minimality of our choice of Z , we get that for any $i+1 \leq r$

$$|S^+(x, i+1)| \geq |B(x, i) \cap Z|.$$

Likewise,

$$|S^-(x, i+1)| \geq |B(x, i) \cap Z|.$$

We now apply the inductive assumption that bounds $|B(x, i) \cap Z|$ from below for all $i \leq r$ noting that each of these $B(x, i)$ projects bijectively onto $B(\pi(x), i)$ by the definition of the cohomology triviality radius. So, for all $i \leq r$ we have:

$$\begin{aligned} |B(\pi(x), i)| &= |B(x, i)| = |B(x, i) \cap Z| + \sum_{0 \leq j \leq i} |S(x, j) \setminus Z| \geq \\ &\geq |B(x, i) \cap Z| + \sum_{0 \leq j \leq i} |S^+(x, j)| + \sum_{0 \leq j \leq i} |S^-(x, j)| \geq \\ &\geq \check{b}_{n-1}(i) + 2 \sum_{0 \leq j \leq i-1} \check{b}_{n-1}(j) = \check{b}_n(i). \end{aligned}$$

□

Proof of Theorem 1.10 over $\mathbb{Z}_2$. We use induction. The base case $n = 1$ follows from Theorem 1.7 (or from the beginning of its proof).

The class ξ_1 corresponds to a classifying map

$$X \rightarrow K(\mathbb{Z}_2, 1) = \mathbb{R}P^\infty.$$

We pull back the double covering map $S^\infty \rightarrow \mathbb{R}P^\infty$ to a double covering map $\pi : \widetilde{X} \rightarrow X$. This $\widetilde{X}$ has a free involution $\tau : \widetilde{X} \rightarrow \widetilde{X}$. We may also assume that X and $\widetilde{X}$ are connected, or otherwise work in a connected component.

Let $Z \subset V(X)$ be a smallest set of vertices such that the covering map π is trivial on $\langle V(X) \setminus Z \rangle$. Then ξ_1 is zero over $\langle V(X) \setminus Z \rangle$ and the restriction of the product $\xi_2 \smile \cdots \smile \xi_n$ is nonzero over $\langle Z \rangle$ by Lemma 3.1.

Any metric r -ball of $\langle Z \rangle$ is contained in the corresponding r -ball of X . By the cohomology triviality radius assumption on X , we have that the classes $\xi_2, \dots, \xi_n$ vanish on any r -ball of $\langle Z \rangle$. Hence the inductive assumption with n replaced by $n-1$ applies to $\langle Z \rangle$. Let $x \in Z$ be the point obtained from the inductive assumption and put $Y = V(X) \setminus Z$.

Similar to the proof of Theorem 1.7, we have that for every $i+1 \leq r$

$$|S(x, i+1) \setminus Z| \geq |B(x, i) \cap Z|,$$

but we are going to improve this bound.

Let $\tilde{Z}$ be the lift of Z to $\tilde{X}$. Let the lift of $\langle S(x, i+1) \rangle$ to $\tilde{X}$ consists of two disconnected copies of $\langle S(x, i+1) \rangle$ from the cohomology triviality assumption, let them be $\tilde{S}$ and $\tau\tilde{S}$. The same applies to the lift of $\langle Y \rangle$, it consists of two disconnected copies $\langle \tilde{Y} \rangle, \langle \tau\tilde{Y} \rangle \subset V(\tilde{X})$. Put

$$\tilde{S}^+ = (\tilde{S} \cap \tilde{Y}) \cup (\tau\tilde{S} \cap \tau\tilde{Y}).$$

The components of $\langle \tilde{S}^+ \rangle$ in the above union formula are disconnected and therefore $\langle \tilde{S}^+ \rangle$ is a trivial double cover over $\langle S^+ \rangle$, for the corresponding subset $S^+ \subset S(x, i+1) \cap Y$. Analogously,

$$\tilde{S}^- = (\tilde{S} \cap \tau\tilde{Y}) \cup (\tau\tilde{S} \cap \tilde{Y})$$

generates $\langle S^- \rangle$, which is a trivial double cover over $\langle S^- \rangle$ for the corresponding $S^- \subset S(x, i+1) \cap Y$. Note that $S^+ \cap S^- = \emptyset$.

Remove $B(x, i)$ from Z and add S^+ instead. Denote the obtained vertex set by Z' . The new complement Y' then equals $Y \cup B(x, i) \setminus S^+$. We need to show that Z' and Y' may serve as Z and Y . For this, it is sufficient to assume that ξ_1 evaluates to 1 on a loop γ passing through the vertices of Y' and obtain a contradiction.

If γ does not touch $B(x, i)$ then it is fully contained in Y and the contradiction is obtained by the choice of Y . Otherwise let γ start and end in $B(x, i)$. Then its lift $\tilde{\gamma}$ starts in $\tilde{B}$ and ends in $\tau\tilde{B}$, where $\tilde{B}$ is a component of the lift of $B(x, i)$ to $\tilde{X}$ corresponding to $\tilde{S}$.

The path $\tilde{\gamma}$ will need to leave $\tilde{B} \cup \tilde{S}$ through $\tilde{S}^-$ (since γ did not touch S^+) and hence it gets into $\tau\tilde{Y}$ after this by the definition of $\tilde{S}^-$. At some point it has to touch $\tau\tilde{S}$ and, since it passes in $\tau\tilde{Y}$, this will happen in $\tilde{S}^+$. Hence γ touches S^+ , a contradiction.

From the minimality of our choice of Z , we now get that for any $i+1 \leq r$

$$|S^+| \geq |B(x, i) \cap Z|.$$

Similarly,

$$|S^-| \geq |B(x, i) \cap Z|,$$

and in total

$$|S(x, i+1) \setminus Z| \geq 2|B(x, i) \cap Z|,$$

We now apply the inductive assumption that bounds $|B(x, i) \cap Z|$ from below for all $i \leq r$ and obtain for any $i \leq r$:

$$\begin{aligned} |B(x, i)| &= |B(x, i) \cap Z| + \sum_{0 \leq j \leq i} |S(x, j) \setminus Z| \geq \\ &\geq |B(x, i) \cap Z| + 2 \sum_{0 \leq j \leq i-1} |B(x, j) \cap Z| \geq \check{b}_{n-1}(i) + 2 \sum_{0 \leq j \leq i-1} \check{b}_{n-1}(j) = \check{b}_n(i). \end{aligned}$$

□

Proof of Theorem 1.12. Put $r = \lfloor \frac{\text{sys } X}{2} \rfloor - 1$. The restriction of any ξ_i to a metric ball of X of radius r is inessential, since such balls only contain contractible loops.

By Theorem 1.10 we find one such ball B_n with at least $\check{b}_n(r)$ vertices. The restriction of $\xi_1 \smile \dots \smile \xi_{n-1}$ to the complex $\langle V(X) \setminus B_n \rangle$ is nonzero by Lemma 3.1. By Theorem 2.4, we find a ball of this subcomplex with $\check{b}_{n-1}(r)$ vertices, and so on. Altogether, we exhibit at least

$$\sum_{k=0}^n \check{b}_k(r) \geq \sum_{k=0}^n 2^k \binom{r}{k} \geq 2^n \binom{r}{n} = 2^n \binom{\lfloor \frac{\text{sys } X}{2} \rfloor - 1}{n}.$$

vertices of X .

□

Proof of Theorem 1.14. For every n , we construct a $\mathbb{Z}_2$ -symmetric triangulation X_n of the sphere S^n with no more than $2s^n$ vertices and with the edge-distance between the opposite vertices at least s . The $\mathbb{Z}_2$ -quotient of this complex has the desired properties.

We proceed by induction over n . For $n = 1$, let X_1 be the polygon with $2s$ vertices, it evidently does the job.

Suppose we have already constructed X_n . Consider $(s - 1)$ copies of X_n which we call *layers*. For each i add a cylinder $X_n \times [0, 1]$ with $X_n \times \{0\}$ and $X_n \times \{1\}$ identified with the i^{th} and the $(i + 1)^{\text{th}}$ layers respectively. Add cones over the first and the last layer with apexes S and N respectively. The resulting space is homeomorphic to S^{n+1} . The $\mathbb{Z}_2$ symmetry sends a vertex v in the i^{th} layer to the vertex $-v$ in the $(s - i)^{\text{th}}$ layer, where $-v$ is the vertex symmetric to v in X_n . The north pole N is sent to the south pole S and vice versa.

It remains to triangulate the cylinders $X_n \times [0, 1]$ between the layers. Assign a unique integer label to each vertex of X_n so that the integers assigned to opposite vertices have the same absolute value but different sign. This way the vertices of X_n are ordered according to their labels. For every top-dimensional simplex $\Delta \subset X_n$, triangulate $\Delta \times [0, 1]$ in each cylinder according to the order of vertices of Δ : the i^{th} top-dimensional simplex of $\Delta \times [0, 1]$ has the first i vertices of $\Delta \times \{0\}$ and the last $(n + 2 - i)$ vertices of $\Delta \times \{1\}$ as its vertices. Denote the resulting triangulation by X_{n+1} .

It is easy to see that X_{n+1} is symmetric and has not more than $(s - 1)2s^n + 2 \leq 2s^{n+1}$ vertices. Its vertices are only connected by an edge to the vertices in the adjacent layers, and if a vertex v in the i^{th} layer is connected by an edge with a vertex u in the $(i + 1)^{\text{th}}$ layer then v and u were connected by an edge in X_n as well. We obtain that

- an edge-path between a vertex in some i^{th} layer and its opposite in the $(s - i)^{\text{th}}$ layer, not passing through the poles, has length at least s ;
- an edge-path between the poles has length at least s , because there are $(s - 1)$ layers between the poles;
- an edge path between arbitrary vertices passing through a pole has length at least s , since the union of this path and its antipodal image is a closed path passing through both poles, whose length must be at least $2s$.

□

4. MINKOWSKI PRINCIPLE FOR RIEMANNIAN POLYHEDRA

We prove inductively a certain technical version of Theorem 1.18. To state it, it is convenient to introduce the function showing up in the volume bound. For fixed numbers $0 \leq L_1 \leq \dots \leq L_n$, define the following monotone continuous function:

$$V_n(r; L_1, \dots, L_n) := \begin{cases} \frac{(2r)^n}{n!} & \text{if } 0 < r \leq L_1/2, \\ \frac{L_1(2r)^{n-1}}{n!} & \text{if } L_1/2 < r \leq L_2/2, \\ \dots & \\ \frac{L_1 \dots L_{n-2}(2r)^2}{n!} & \text{if } L_{n-2}/2 < r \leq L_{n-1}/2, \\ \frac{L_1 \dots L_{n-1}2r}{n!} & \text{if } L_{n-1}/2 < r \leq L_n/2, \\ \frac{L_1 \dots L_n}{n!} & \text{if } r > L_n/2. \end{cases}$$

Lemma 4.1. *For any $t \geq 0$ and any $0 \leq L_1 \leq L_2 \leq \dots \leq L_n$ the following inequality holds*

$$2 \int_0^r V_{n-1}(t; L_1, \dots, L_{n-1}) dt \geq V_n(r; L_1, \dots, L_n).$$

Proof. If $r \leq L_1/2$,

$$2 \int_0^r V_{n-1}(t; L_1, \dots, L_{n-1}) dt = 2 \int_0^r \frac{(2t)^{n-1}}{(n-1)!} dt = \frac{(2r)^n}{n!} = V_n(r; L_1, \dots, L_n).$$

If $r \in (L_i/2, L_{i+1}/2]$, then inducting in i we can assume that

$$2 \int_0^{L_i/2} V_{n-1}(t; L_1, \dots, L_{n-1}) dt \geq V_n(L_i/2; L_1, \dots, L_n) = \frac{L_1 \dots L_{i-1} L_i^{n-i+1}}{n!}.$$

In order to get $2 \int_0^r V_{n-1}(t; L_1, \dots, L_{n-1}) dt \geq V_n(r; L_1, \dots, L_n)$, we need the following inequality:

$$2 \int_{L_i/2}^r V_{n-1}(t; L_1, \dots, L_{n-1}) dt \geq \frac{L_1 \dots L_i ((2r)^{n-i} - L_i^{n-i})}{n!}.$$

The left-hand side evaluates to

$$2 \int_{L_i/2}^r \frac{L_1 \dots L_i (2t)^{n-i-1}}{(n-1)!} dt = \frac{L_1 \dots L_i ((2r)^{n-i} - L_i^{n-i})}{(n-1)!(n-i)},$$

thus proving the bound.

For $r > L_n/2$, the inequality follows from the fact that $V_n(r; L_1, \dots, L_n)$ remains constant. $\square$

Theorem 4.2. *Let X be a compact Riemannian polyhedron of dimension n . Suppose it is n -cup-essential over $\mathbb{Z}$ or over $\mathbb{Z}_2$, which is witnessed by the degree 1 cohomology classes $\xi_1, \dots, \xi_n$, $\xi_1 \smile \dots \smile \xi_n \neq 0$. Suppose that there are real numbers $0 < L_1 \leq \dots \leq L_n$ such that the class ξ_i vanishes when restricted to a ball of radius less than $L_i/2$, for all i . Let $0 < \rho < L_1/2$, $0 < \varepsilon < 1$ be any real numbers. Then there exists a point $x \in X$ such that $\text{vol}_n B(x, r) \geq (1 - \varepsilon) V_n(r; L_1, \dots, L_n)$ for all $r \geq \rho$.*

Proof in the integral case. If $n = 1$, take a point x on a shortest loop on which ξ_1 evaluates non-trivially. Then $B(x, r)$ intersects this loop along a curve of length at least $2r$ for $r \leq L(\xi_1)/2$, and the statement follows.

Now assume $n > 1$. For brevity, we assume fixed values of $L_1, \dots, L_n$ and write

$$V_n(r) = V_n(r; L_1, \dots, L_n) \quad \text{and} \quad V_{n-1}(r) = V_{n-1}(r; L_1, \dots, L_{n-1}).$$

The class ξ_n is classified by a map $X \rightarrow K(\mathbb{Z}, 1) = \mathbb{R}/\mathbb{Z}$, which gives rise to a covering map $\pi : \widetilde{X} \rightarrow X$. The action of $\mathbb{Z}$ on $\widetilde{X}$ allows us to speak about “separating $-\infty$ from $+\infty$ ” in $\widetilde{X}$, as in the proof of Theorem 1.10.

Consider all bounded $(n-1)$ -dimensional subpolyhedra¹ in $\widetilde{X}$, separating $-\infty$ from $+\infty$. For example, the boundary of any reasonable fundamental domain of the $\mathbb{Z}$ -action is such a subpolyhedron. Let v be the infimum of their $(n-1)$ -volumes. Pick such a subpolyhedron Z with $\text{vol}_{n-1} < v + \delta$, where $\delta > 0$ is a small number to be specified later. Consider $X_N := \widetilde{X}/N\mathbb{Z}$ where N is a positive integer satisfying two properties:

- N is sufficiently large, so that the projection $\pi'_N : \widetilde{X} \rightarrow X_N$ induces a homeomorphism between Z and $Z_N := \pi'_N(Z)$;
- N is coprime with the order of $\xi_1 \smile \dots \smile \xi_n$, so that its pullback under the projection $\pi_N : X_N \rightarrow X$ is non-zero.

Just like in the proof of Theorem 1.10, $\pi_N^*(\xi_n)$ vanishes on $X_N \setminus Z_N$. By the topological version of Lemma 3.1, $\pi_N^*(\xi_1) \smile \dots \smile \pi_N^*(\xi_{n-1})$ restricts non-trivially to an arbitrarily small neighborhood of Z_N in X_N ; taking this neighborhood sufficiently small so that it retracts to

¹A *subpolyhedron* is a subspace admitting the structure of a simplicial complex whose cells are embedded smoothly in the ambient polyhedron. The Riemannian metric is inherited from the ambient polyhedron, allowing one to measure intrinsic distances and volumes.

Z_N , we see that $\pi_N^*(\xi_1) \smile \cdots \smile \pi_N^*(\xi_{n-1})$ restricts non-trivially to Z_N . Since $Z \simeq Z_N$, we have a long non-zero cohomology product in $H^*(Z; \mathbb{Z})$, and we are in position to apply the induction hypothesis for Z .

Apply the induction hypothesis to Z with modified $\hat{\rho} = \rho(\varepsilon/2)^{1/n}$ and $\hat{\varepsilon} = \varepsilon/4$. It outputs a point $x \in Z$ such that $\text{vol}_{n-1} B_Z(x, r) \geq (1 - \hat{\varepsilon})V_{n-1}(r)$ for all $r \geq \hat{\rho}$. Here $B_Z(\cdot)$ denotes a metric ball in the intrinsic metric of Z , whereas the notation $B(\cdot)$ and $S(\cdot)$ will refer to $\widetilde{X}$. The intrinsic distances dominate the extrinsic ones, so we have

$$\text{vol}_{n-1}(B(x, r) \cap Z) \geq (1 - \hat{\varepsilon})V_{n-1}(r) \quad \text{for all } r \geq \hat{\rho}.$$

We now show that

$$(\star) \quad \text{vol}_{n-1} S(x, r) \geq 2 \left(1 - \frac{\varepsilon}{2}\right) V_{n-1}(r) \quad \text{for almost all } r \geq \hat{\rho}.$$

We only consider those r for which $S(x, r)$ is a subpolyhedron. We can assume this is true for almost all r , perturbing the distance function slightly. Fix any such $r \geq \hat{\rho}$. Introduce $S^+ \subset S(x, r)$ (respectively, S^-) as the subset of points connected to $+\infty$ (respectively, $-\infty$) in $\widetilde{X} \setminus Z$. The set $Z \setminus (\text{int } B(x, r) \cap Z) \cup S^+$ still separates $-\infty$ from $+\infty$, hence we have a volume bound:

$$v \leq \text{vol}_{n-1} (Z \setminus (\text{int } B(x, r) \cap Z) \cup S^+) < v + \delta - (1 - \hat{\varepsilon})V_{n-1}(r) + \text{vol}_{n-1} S^+.$$

Therefore, taking $\delta = \frac{\varepsilon}{4}V_{n-1}(\rho)$, we obtain

$$\text{vol}_{n-1} S^+ > (1 - \hat{\varepsilon})V_{n-1}(r) - \delta \geq \left(1 - \frac{\varepsilon}{2}\right) V_{n-1}(r).$$

A similar bound can be established for $\text{vol}_{n-1} S^-$, and adding those up we obtain $(\star)$.

Now that we have $(\star)$, we can integrate it over $[\hat{\rho}, r]$ and use the coarea inequality:

$$\begin{aligned} \text{vol}_{n-1} B(x, r) &\geq \int_{\hat{\rho}}^r \text{vol}_{n-1} S(x, t) dt \geq 2 \left(1 - \frac{\varepsilon}{2}\right) \int_{\hat{\rho}}^r V_{n-1}(t) dt > \\ &> 2 \left(1 - \frac{\varepsilon}{2}\right) \int_0^r V_{n-1}(t) dt - 2 \int_0^{\hat{\rho}} V_{n-1}(t) dt \geq \\ &\stackrel{\text{Lem. 4.1}}{\geq} \left(1 - \frac{\varepsilon}{2}\right) V_n(r) - \frac{(2\hat{\rho})^n}{n!} \geq (1 - \varepsilon)V_n(r), \end{aligned}$$

where the last inequality holds for all $r \geq \rho$ by the choice of $\hat{\rho} = \rho(\varepsilon/2)^{1/n}$. For $r < L_n/2$, the projection π sends $B(x, r)$ to X isometrically, so we obtain the desired volume bound in X for the radii in the range $[\rho, L_n/2]$. For $r \geq L_n/2$ the bound holds vacuously since $V_n(r)$ becomes constant. $\square$

Proof in the mod 2 case. The strategy is the same as in the discrete case and the volumetric computations are the same as in the proof over $\mathbb{Z}$, so we only briefly sketch the argument.

The base case $n = 1$ is done as before. Suppose that $n > 1$ and that X is connected. The class ξ_n corresponds to a classifying map $X \rightarrow \mathbb{R}P^\infty$, along which we pull back the double covering map $S^\infty \rightarrow \mathbb{R}P^\infty$. This way we obtain a covering map $\pi : \widetilde{X} \rightarrow X$, with a free involution $\tau : \widetilde{X} \rightarrow \widetilde{X}$.

Consider all bounded $(n-1)$ -dimensional subpolyhedra of X such that π restricts to a trivial cover over the complement $X \setminus Z$. Let v be the infimal volume of such subpolyhedra. Pick a subpolyhedron Z among those with the volume less than $v + \delta$, where δ is chosen as in the integral case. Just like in the discrete case, the product $\xi_1 \smile \cdots \smile \xi_{n-1}$ restricts non-trivially to Z , and one can apply the inductive assumption for Z , with parameters $\hat{\rho}$ and $\hat{\varepsilon}$ chosen as in the integral case.

The induction hypothesis outputs a point $x \in Z$ such that $\text{vol}_{n-1}(B(x, r) \cap Z) \geq (1 - \varepsilon)V_{n-1}(r)$ for all $r \geq \hat{\rho}$. The surgery on Z that replaces $B(x, r) \cap Z$ by $S(x, r)$ produces a subpolyhedron of volume at least v , and this gives a lower bound for $\text{vol}_{n-1} S(x, r)$. To obtain a bound that is twice as good, we do the surgery more carefully. Similarly to the discrete case, one introduces non-overlapping subsets S^+ and S^- of $S(x, r)$ in a way that makes ξ_n vanish on the complement of $Z \setminus (\text{int } B(x, r) \cap Z) \cup S^+$ and on the complement of $Z \setminus (\text{int } B(x, r) \cap Z) \cup S^-$. The surgery on Z that replaces $B(x, r) \cap Z$ by S^+ or S^- produces a subpolyhedron of volume at least v , this gives lower bounds on $\text{vol}_{n-1} S^+$ and $\text{vol}_{n-1} S^-$, which we integrate and get the desired bound on $\text{vol}_n B(x, r)$. The computations carry over from the integral case *verbatim*. $\square$

Lemma 4.3. *Let $\xi \in H^1(X; \mathbb{Z})$ or $\xi \in H^1(X; \mathbb{Z}_2)$, and let $r < L(\xi)/2$. Then ξ restricts to any ball of radius r trivially.*

Proof. Take an arbitrary loop γ inside a ball of radius r . Introduce many points p_i along the loop, and consider connect x to p_i by a geodesic segment g_i of length at most r . For each i , consider the triangular curve γ_i glued of g_i , g_{i+1} , and the short path between p_i and p_{i+1} along γ . If the p_i are scattered densely enough, the length of each γ_i is less than $L(\xi)$, hence $\xi(\gamma_i) = 0$. But when oriented appropriately, the γ_i add up to γ ; therefore, $\xi(\gamma) = 0$. $\square$

Proof of Theorem 1.18. It follows from Lemma 4.3 that one can apply Theorem 4.2 with $L_1 = L(\xi_1), \dots, L_n = L(\xi_n)$ (rearranging the ξ_i if necessary).

We would like to set $\varepsilon = 0$ and $\rho = 0$ in the conclusion of Theorem 4.2. This will be done using the continuity of the function $v(x, r) := \frac{\text{vol}_n B(x, r)}{V_n(r)}$.

First, we fix ρ and get rid of ε . Consider the function $v(x) = \min_{r \in [\rho, R]} v(x, r)$. We know that for any ε there is x such that $v(x) \geq (1 - \varepsilon)$. Since $v(\cdot)$ is upper semi-continuous and X is compact, we conclude that there is x such that $v(x) \geq 1$.

Now we get rid of ρ . Assume that for any x there exists $0 < r \leq R$ such that $v(x, r) < 1$. Consider the function $t(x) = \sup \{r \mid v(x, r) < 1\}$. By our assumption, $t(\cdot)$ is defined everywhere and positive. It can be verified that $t(\cdot)$ is lower semi-continuous, so it attains a positive minimum on X . Taking ρ less than this minimum, we get a contradiction with the result of the previous paragraph. Therefore, there is x such that $v(x, r) \geq 1$ for all $r \in (0, R]$. $\square$

APPENDIX: DEFINITIONS OF ESSENTIALITY

The statements of this section are not used in the proofs of the results listed in the introduction. But they may be useful to put the results in a wider context of systolic inequalities.

Lemma A.1. *The two definitions of n -essential for a topological space X and a positive integer n are equivalent:*

- (i) *The classifying map $f : X \rightarrow K(\pi_1(X), 1)$ cannot be deformed to the $(n - 1)$ -skeleton of $K(\pi_1(X), 1)$.*
- (ii) *X cannot be covered by n or fewer open sets $U_i \subseteq X$ so that for every connected component C of every U_i and every connected component D of X the map $\pi_1(C) \rightarrow \pi_1(D)$ is trivial (in this case we call U_i inessential).*

Proof. Set $G := \pi_1(X)$. If (i) fails for a certain deformed map f then one covers the $(n - 1)$ -skeleton $K^{(n-1)}$ of $K(G, 1)$ with n inessential open sets $K^{(n-1)} = V_1 \cup \dots \cup V_n$. This may be done, for example, using the barycentric subdivision of $K^{(n-1)}$ and the proper coloring of its vertices in n colors, then V_i is a union of stars of vertices of color i in the barycentric subdivision. Then the open sets $U_i = f^{-1}(V_i) \subseteq X$ show that (ii) fails as well.

In the opposite direction, we may work with every component of X separately and assume X is connected and has a universal cover. If $X = U_1 \cup \dots \cup U_n$ with inessential U_i , then the

universal covering space $\widetilde{X}$ equals $\widetilde{U}_1 \cup \cdots \cup \widetilde{U}_n$, so that G acts freely on the components of every $\widetilde{U}_i$ (this is the meaning of “inessential” in terms of the universal cover). Hence there exist G -equivariant maps $f_i : \widetilde{U}_i \rightarrow G$ and using a G -invariant partition of unity ρ_i subordinate to the open cover $\{\widetilde{U}_i\}$ we obtain

$$\widetilde{f}(x) = \rho_1(x)f_1(x) * \cdots * \rho_n(x)f_n(x).$$

This $\widetilde{f}$ a G -equivariant map from $\widetilde{X}$ to the join $\underbrace{G * \cdots * G}_n$. Passing to quotients, one obtains $f : X \rightarrow \underbrace{G * \cdots * G}_n / G$, which may be viewed as a map from X to the $(n-1)$ -skeleton of $K(G, 1) = \underbrace{G * \cdots * G}_\infty / G$, inducing an isomorphisms of the fundamental groups. This shows that X is not n -essential in terms of maps as well. $\square$

Lemma A.2. *If a connected n -dimensional complex is n -essential as a topological space then it is combinatorially n -essential in the sense of Definition 1.3.*

Note that example in Remark 1.17 shows that the opposite direction of the lemma is false.

Proof of Lemma A.2. A partition of a vertex set $V(X) = Y_1 \sqcup \cdots \sqcup Y_n$ of a triangulation produces an open cover $X = U(Y_1) \cup \cdots \cup U(Y_n)$, where $U(Y_i)$ is the union of open stars of the vertices of Y_i , which retracts onto $\langle Y_i \rangle$ and is therefore inessential whenever $\langle Y_i \rangle$ is such. Now we apply the equivalence from Lemma A.1. $\square$

Remark A.3. Lemma A.2 cannot be extended to more general cell complexes from simplicial complexes. Take a cube $Q_n = [-1, 1]^n$. Its vertices are split into two subsets O and E depending on whether the number of -1 ’s is odd or even. No vertex from O is connected by an edge of the cube to another vertex of O , and the same for E . When n is even, the antipodal involution $\tau(x) = -x$ takes E to E and O to O . The quotient $X = \partial Q_n / \tau$ is then a cell complex homeomorphic to $(n-1)$ -essential $\mathbb{R}P^n$. The vertices of X split into two types, E/τ and O/τ , both inducing no edge and therefore being inessential.

A weaker version in the opposite direction does hold.

Lemma A.4. *If all sufficiently fine (iterated) barycentric subdivisions of a complex X are combinatorially n -essential then the complex is n -essential as a topological space.*

Proof. We use Lemma A.1 and assume that X is not topologically n -essential. Then there is a cover $X = \bigcup_i U_i$ by n or fewer inessential open subsets. It remains to show that some barycentric subdivision of X allows splitting of vertices into n or fewer non-essential vertex sets.

We consider a partition of unity $\sum_i \rho_i \equiv 1$ subordinate to $\{U_i\}$. The compact sets

$$F_i = \{x \in X \mid \rho_i(x) \geq 1/n\}$$

are contained in their respective open sets U_i , and $X = \bigcup_i F_i$. Let $\delta > 0$ be the smallest over i distance between the disjoint compact sets F_i and $X \setminus U_i$ (assuming some metric on X).

Now we take an iterated barycentric subdivision T of X whose all faces have diameter strictly less than δ . Partition its vertex set $V(T)$ into a disjoint union $\sqcup V_i = V(T)$ such that $V_i \subset F_i$ for all i . For any V_i , the induced subcomplex $\langle V_i \rangle$ is then fully contained in U_i by the choice of δ and T . The restriction of the universal covering map to $\langle V_i \rangle$ is then trivial, since it is trivial over U_i . Hence every V_i is an inessential set of vertices. $\square$

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