

## ON THE CONSISTENCY STRENGTH OF $\text{MM}(\omega_1)$

NATASHA DOBRINEN, JOHN KRUEGER, PEDRO MARUN, MIGUEL ANGEL MOTA,  
AND JINDRICH ZAPLETAL

(Communicated by Vera Fischer)

ABSTRACT. We prove that the consistency of Martin's Maximum restricted to partial orders of cardinality  $\omega_1$  follows from the consistency of ZFC.

### 1. INTRODUCTION

Given the profusion of independence results which followed Cohen's discovery of the method of forcing, it has become a major objective of set theory to find natural axiomatic extensions of ZFC which decide Cantor's Continuum Problem as well as other important questions undecidable in ZFC. For example, in the last five decades forcing axioms have been widely studied and shown to have very interesting consequences regarding the continuum. Intuitively, the idea behind them is that the universe of set theory must be somehow saturated under forcing. More precisely, given a class  $\Gamma$  of partial orders and a cardinal  $\kappa$ , the *forcing axiom for  $\Gamma$  and  $\kappa$* ,  $\text{FA}(\Gamma, \kappa)$ , is the assertion that for every  $P \in \Gamma$  and every collection  $\mathcal{D}$  of size at most  $\kappa$  consisting of dense subsets of  $P$ , there is a filter  $G \subset P$  such that  $G \cap D \neq \emptyset$  for every  $D \in \mathcal{D}$ .

Some forcing axioms which are especially significant for their wide range of applications not only in set theory, but also in algebra, analysis, topology, and other fields, are Martin's Axiom for  $\omega_1$ -many dense sets ( $\text{MA}_{\omega_1}$ ) introduced by Martin, Solovay and Tennenbaum in the mid 1960's, the Proper Forcing Axiom (PFA) introduced by Baumgartner and Shelah in the early 1980's, and the Semiproper Forcing Axiom (SPFA) and Martin's Maximum (MM) introduced by Foreman, Magidor and Shelah in the mid 1980's. They are defined as  $\text{FA}(\Gamma, \omega_1)$  for  $\Gamma$  being, respectively, the class of all posets with the countable chain condition, the class of all proper posets, the class of all semiproper posets, and the class of all posets preserving stationary subsets of  $\omega_1$ , where these four classes are being presented in increasing order.

The forcing axioms  $\text{MA}_{\omega_1}$ , PFA and SPFA are known to be relatively consistent (from ZFC in the first case and modulo large cardinals in the other two) by means of forcing iterations which fall in the same class  $\Gamma$  being considered. So, this kind of construction heavily depends on certain preservation criteria. One of them is the central theorem of Shelah stating that if  $P_\alpha$  is a countable support forcing iteration

---

Received by the editors July 12, 2023, and, in revised form, October 25, 2023.

2020 *Mathematics Subject Classification*. Primary 03E35, 03E50.

This paper was originally conceived during the workshop *From  $\aleph_2$  to Infinity* sponsored by the American Institute of Mathematics and the National Science Foundation.

The first author was supported by the National Science Foundation under grant NSF DMS-1901753. The second author was supported by the Simons Foundation under Grant 631279.

of  $\{\dot{Q}_\beta: \beta < \alpha\}$  such that every  $\dot{Q}_\beta$  is a proper forcing notion in  $V^{P_\alpha \restriction \beta}$ , then  $P_\alpha$  is proper (in particular,  $P_\alpha$  does not collapse  $\omega_1$ ). Another one, also due to Shelah, holds in the context of revised countable support forcing iterations and semiproper forcings. On the other hand, there is no such preservation result for stationary set preserving posets and the classical argument for the consistency of MM (see [3] and [6]) goes in a slightly different way: it passes by showing that SPFA implies that every stationary set preserving notion of forcing is semiproper, which in turn implies the equivalence between SPFA and MM.

Let us denote by  $\text{PFA}(\omega_1)$  and  $\text{MM}(\omega_1)$  the respective restrictions of PFA and MM to posets of cardinality  $\omega_1$ . So,  $\text{PFA}(\omega_1)$  and  $\text{MM}(\omega_1)$  are defined as  $\text{FA}(\Gamma, \omega_1)$ , for  $\Gamma$  being the class of all posets of cardinality  $\omega_1$  which are proper in the first case, and the class of all posets of cardinality  $\omega_1$  which preserve stationary subsets of  $\omega_1$  in the second case. It is well-known that ZFC and  $\text{ZFC} + \text{PFA}(\omega_1)$  are equiconsistent, which follows from the fact that under CH, forcings of size  $\omega_1$  can be iterated with countable support up to length  $\omega_2$  with an  $\omega_2$ -c.c. forcing iteration (see Lemmas 2.4 and 2.5 of Chapter VIII of [5]). Shelah proved that  $\text{ZFC} +$  “there exists a strongly inaccessible cardinal” implies the consistency of  $\text{ZFC} + \text{MM}(\omega_1)$  (see Theorem 4.3 of Chapter III of [5]). The main theorem of this article states that Shelah’s inaccessible can be removed from this consistency result.

**Theorem 1.1.** *Assume CH and  $2^{\omega_1} = \omega_2$ . Then there is a countable support forcing iteration  $P_{\omega_2}$  of  $\{\dot{Q}_\beta: \beta < \omega_2\}$  with the following properties:*

- (1) *Every  $\dot{Q}_\beta$  is, in  $V^{P_{\omega_2} \restriction \beta}$ , a proper poset;*
- (2)  *$P_{\omega_2}$  is proper and has the  $\omega_2$ -chain condition;*
- (3)  *$P_{\omega_2}$  forces  $\text{MM}(\omega_1)$ .*

*Consequently, the theories ZFC and  $\text{ZFC} + \text{MM}(\omega_1)$  are equiconsistent.*

We would like to thank the referee for pointing out an error in an earlier version of this article.

## 2. STATIONARY SET PRESERVING BUT NOT PROPER

Before proving the main theorem, we give a brief sketch of the consistency that there exists a forcing poset of size  $\omega_1$  which preserves stationary subsets of  $\omega_1$  but is not proper. This fact means that the work done in the next section for destroying stationary set preserving posets which are not proper is not vacuous. Another proof was given previously by Sakai [4], who introduced a new combinatorial principle called  $\diamond^{++}$  and showed that it implies the existence of a non-proper poset of size  $\omega_1$  preserving stationary subsets of  $\omega_1$ . Since  $\diamond^{++}$  is forceable and holds in  $L$ , the existence of such a poset is consistent with ZFC. Similar to the construction in the present section, Sakai’s forcing is actually a Kurepa tree, although it is not presented as such.

It is worth pointing out in this context that a poset of size  $\omega_1$  is proper iff it is semiproper. Namely, if  $Q$  is a poset of size  $\omega_1$  and  $N$  is a countable elementary submodel, then by a straightforward argument any condition in  $Q$  is a master condition for  $N$  iff it is a semi-master condition for  $N$ . Thus, we are actually proving the consistency of the existence of a poset of size  $\omega_1$  which preserves stationary sets and is not semiproper.

The forcing for introducing such a poset is the standard forcing for adding a Kurepa tree, and the poset which is stationary set preserving but not proper in the

generic extension is the generic Kurepa tree. For any ordinal  $\alpha$ , let  $P(\alpha)$  be the poset consisting of conditions which are pairs  $(s, f)$ , where  $s$  is a countable tree of height a countable successor ordinal satisfying the usual normality properties, and  $f$  is a function whose domain is a countable subset of  $\alpha$  so that for all  $\gamma \in \text{dom}(f)$ ,  $f(\gamma)$  is an element of the top level of  $s$ . Let  $(t, g) \leq (s, f)$  if  $t$  end-extends  $s$ ,  $\text{dom}(f) \subset \text{dom}(g)$ , and for all  $\gamma \in \text{dom}(f)$ ,  $f(\gamma) \leq_t g(\gamma)$ .

The basic properties of  $P(\alpha)$  are as follows.

- (1)  $P(\alpha)$  is countably closed and, assuming  $\text{CH}$ ,  $\omega_2$ -c.c.
- (2) Assume that  $G$  is a generic filter on  $P(\alpha)$ . The union of the trees  $s$ , where  $(s, f) \in G$  for some  $f$ , is a normal  $\omega_1$ -tree which we will denote by  $T$ , and the canonical name by  $\dot{T}$ .
- (3) For each  $i < \alpha$ , the downwards closure of  $\{f(i) : \exists s (s, f) \in G, i \in \text{dom}(f)\}$  is a cofinal branch of  $T$ , which we will denote by  $b_i$ , and the canonical name by  $\dot{b}_i$ .
- (4) For all distinct  $i, j < \alpha$ , the set of  $(s, f) \in P(\alpha)$  such that  $i, j \in \text{dom}(f)$  and  $f(i) \neq f(j)$  is dense open. Hence,  $b_i \neq b_j$  in  $V[G]$ . In particular, if  $\alpha \geq \omega_2$  then  $T$  is a Kurepa tree in  $V[G]$ .

We also note that if  $\alpha < \beta$  then  $P(\alpha)$  is a regular suborder of  $P(\beta)$ , and  $P(\alpha+1)$  is forcing equivalent to  $P(\alpha) * \dot{T}$ . In fact, if  $G$  is a generic filter on  $P(\alpha)$  and  $T = \dot{T}^G$ , then a branch  $b$  through  $T$  is  $V[G]$ -generic on  $T$  iff the set  $\{(s, f) \in P(\alpha+1)/G : f(\alpha) \in b\}$  is  $V[G]$ -generic on  $P(\alpha+1)/G$ .

The next two lemmas complete the proof.

**Lemma 2.1.** *The poset  $P(\omega_2)$  forces that  $\dot{T}$  is not proper.*

*Proof.* Fix a large enough regular cardinal  $\lambda$ . Let  $G$  be a generic filter on  $P(\omega_2)$ . In  $V[G]$ , let  $\mathcal{X}$  denote the collection of all countable  $M \prec H(\lambda)$  satisfying: (a)  $M = N[G]$  for some countable  $N \prec H(\lambda)^V$ , (b) there exists a master condition  $q_N = (s_N, f_N) \in G$  for  $N$ , and (c) for all  $a \in T$  of height  $N \cap \omega_1$ , there exists some  $\gamma \in \text{dom}(f_N) \cap N$  such that  $f_N(\gamma) = a$ .

Working in  $V$ , for any countable elementary submodel  $N \prec H(\lambda)^V$  and  $p \in N \cap P(\omega_2)$ , it is straightforward to build a master condition  $q \leq p$  for  $N$  forcing that property (c) holds for  $\dot{T}$  in place of  $T$ . It follows that in  $V[G]$ ,  $\mathcal{X}$  is stationary in  $[H(\lambda)]^\omega$ . Let  $M = N[G] \in \mathcal{X}$  and we claim that there does not exist a master condition in  $T$  for  $M$ . Note that an element  $a$  of  $T$  is a master condition for  $N[G]$  iff for any dense open set  $D \subset T$  in  $N$ , there is some  $x \leq_T a$  such that  $x \in D \cap N$ . Suppose for a contradiction that  $a$  is such an element. By dropping  $a$  down to the height  $N \cap \omega_1$  if necessary, assume without loss of generality that  $a$  has height  $N \cap \omega_1$ . By the choice of  $q_N$ , fix  $\gamma \in \text{dom}(f_N) \cap N$  such that  $f_N(\gamma) = a$ . Since  $q_N \in G$ , it follows that  $a \in b_\gamma$ .

Let  $\dot{a}$  be a name which is forced by some  $r \leq q_N$  in  $G$  to be a master condition in  $\dot{T}$  for  $N[G]$ , has height  $N \cap \omega_1$ , and is in  $\dot{b}_\gamma$ . Then  $r * \dot{a}$  is a master condition in  $P(\omega_2) * \dot{T}$  for  $N$ . But by property (4) of  $P(\omega_2 + 1)$  the generic branches at coordinates  $\gamma$  and  $\omega_2$  must diverge. Since the dense set described in property (4) is in  $N$  and  $P(\omega_2 + 1)$  and  $P(\omega_2) * \dot{T}$  are forcing equivalent, the fact that  $r * \dot{a}$  is a master condition for  $N$  implies that the branches with coordinates  $\gamma$  and  $\omega_2$  must diverge below height  $N \cap \omega_1$ . Hence,  $r * \dot{a}$  forces that  $a$  is not in  $\dot{b}_\gamma$ , which contradicts that  $a \in b_\gamma$ .  $\square$

**Lemma 2.2.** *The poset  $P(\omega_2)$  forces that  $\dot{T}$  preserves stationary subsets of  $\omega_1$ .*

*Proof.* Suppose for a contradiction that for some generic filter  $G$  on  $P(\omega_2)$ , in  $V[G]$  there exists a stationary set  $S \subset \omega_1$ , a  $T$ -name  $\dot{C}$  for a club subset of  $\omega_1$ , and some  $x \in T$  such that  $x \Vdash_T^{V[G]} \dot{S} \cap \dot{C} = \emptyset$ . Since  $P(\omega_2)$  is  $\omega_2$ -c.c., using nice names and a density argument we can find some  $\gamma < \omega_2$  such that  $S = \dot{S}^{G \upharpoonright \gamma}$  for some  $P(\gamma)$ -name  $\dot{S}$ , the  $T$ -name  $\dot{C}$  is in  $V[G \upharpoonright \gamma]$ , and there is a condition  $(t, g) \in G$  such that  $x \leq_t g(\gamma)$ .

Now the forcing  $P(\omega_2)$  can be factored as  $P(\gamma) * \dot{T} * P(\omega_2)/\dot{G}_{\gamma+1}$ , and we can correspondingly write  $V[G] = V[G \upharpoonright \gamma][b_\gamma][H]$  for some  $H$ , where  $x \in b_\gamma$  since  $(t, g) \in G$ . By an absoluteness argument,  $x \Vdash_T^{V[G_\gamma]} \dot{S} \cap \dot{C} = \emptyset$ . Hence,  $S$  is disjoint from the club  $\dot{C}^{b_\gamma}$  in  $V[G_{\gamma+1}]$ . But then  $\dot{C}^{b_\gamma} \in V[G]$ , which contradicts that  $S$  is stationary in  $V[G]$ .  $\square$

### 3. PROVING THE MAIN THEOREM

The forcing iteration described in Theorem 1.1 will involve forcing two types of posets: (1) proper posets of size  $\omega_1$ , bookkeeping so that all such posets in the final model will have been forced with  $\omega_2$ -many times in the iteration, and (2) forcing notions which destroy posets which are stationary set preserving but not proper.

In order to prove that the forcing iteration is  $\omega_2$ -c.c., we will use a property introduced by Shelah (see Chapter VIII Section 2 of [5]).

**Definition 3.1.** A poset  $R$  satisfies the  $\omega_2$ -properness isomorphism condition ( $\omega_2$ -p.i.c. for short) if and only if for every large enough regular cardinal  $\theta$ , for every well-ordering  $<$  of  $H_\theta$  and for all ordinals  $\alpha < \beta < \omega_2$  the following holds: if  $N_\alpha$  and  $N_\beta$  are countable elementary submodels of  $(H_\theta, \in, <, R)$  such that  $\alpha \in N_\alpha$ ,  $\beta \in N_\beta$ ,  $N_\alpha \cap \omega_2 \subset \beta$ ,  $N_\alpha \cap \alpha = N_\beta \cap \beta$ ,  $p \in N_\alpha \cap R$  and  $\pi : N_\alpha \rightarrow N_\beta$  is an isomorphism satisfying  $\pi(\alpha) = \beta$  and  $\pi \upharpoonright (N_\alpha \cap N_\beta) = id$ , then there exists a master condition  $q$  for  $N_\alpha$ , extending  $p$  and  $\pi(p)$ , such that

$$q \Vdash_R \pi''(\dot{G} \cap \check{N}_\alpha) = \dot{G} \cap \check{N}_\beta.$$

Every proper poset of size  $\omega_1$  has the  $\omega_2$ -p.i.c., and if CH holds, then every  $\omega_2$ -p.i.c. poset satisfies the  $\omega_2$ -chain condition. Moreover, by Lemma 2.4 of Chapter VIII of [5], under the assumption of CH, if  $P_{\omega_2}$  is a countable support forcing iteration of  $\{\dot{Q}_\beta : \beta < \omega_2\}$  such that every  $\dot{Q}_\beta$  has the  $\omega_2$ -p.i.c. in  $V^{P_{\omega_2} \upharpoonright \beta}$ , then  $P_{\omega_2}$  has the  $\omega_2$ -chain condition. Therefore, CH implies that  $P_{\omega_2}$  does not collapse cardinals. In the context of our specific iteration, we will apply this result by taking each  $\dot{Q}_\beta$  to be either a name for a proper poset of size  $\omega_1$  or a name for a poset  $Q$  as described in the next theorem.

**Theorem 3.2.** *There exists a proper countably distributive poset  $Q$  of cardinality  $2^{\omega_1}$  with the  $\omega_2$ -p.i.c. satisfying that for every poset  $P$  of cardinality  $\omega_1$ , if  $P$  is not proper, then*

$$\Vdash_Q \check{P} \text{ does not preserve stationary subsets of } \omega_1.$$

With this new ingredient, and assuming CH together with  $2^{\omega_1} = \omega_2$ , the construction of a countable support forcing iteration  $P_{\omega_2}$  witnessing Theorem 1.1 is very natural. Since  $2^{\omega_1} = \omega_2$ , we can fix a function  $\Phi : \omega_2 \rightarrow H_{\omega_2}$  with the property that  $\{\beta \in \omega_2 : \Phi(\beta) = x\}$  is unbounded in  $\omega_2$  for each  $x \in H_{\omega_2}$ . At stage

$\beta < \omega_2$ , if  $\Phi(\beta)$  is a  $P_{\omega_2} \restriction \beta$ -name for a proper poset of cardinality  $\omega_1$ , then let  $\dot{Q}_\beta = \Phi(\beta)$ . Otherwise, let  $\dot{Q}_\beta$  be a  $P_{\omega_2} \restriction \beta$ -name for a poset  $Q$  as in Theorem 3.2.

We claim that  $P_{\omega_2}$  forces  $\text{MM}(\omega_1)$ . Namely, for every  $P_{\omega_2}$ -name  $\dot{P}$  for a poset of cardinality  $\omega_1$  and for every sequence  $(\dot{D}_i)_{i < \omega_1}$  of  $P_{\omega_2}$ -names for dense subsets of  $\dot{P}$ , there is a high enough  $\beta < \omega_2$  such that  $\dot{P}$  and all members of  $(\dot{D}_i)_{i < \omega_1}$  are  $P_{\omega_2} \restriction \beta$ -names and  $\Phi(\beta) = \dot{P}$ . So,  $\dot{Q}_\beta$  is either  $\dot{P}$  or a  $P_{\omega_2} \restriction \beta$ -name for a poset  $Q$  as in Theorem 3.2 depending on whether or not  $\dot{P}$  is a  $P_{\omega_2} \restriction \beta$ -name for a proper poset. This is possible thanks to the  $\omega_2$ -chain condition of  $P_{\omega_2}$ , the fact that  $P_{\omega_2}$  has cardinality  $2^{\omega_1} = \omega_2$ , and the unboundedness assumption on the bookkeeping function  $\Phi$ .

The rest of this section is devoted to proving Theorem 3.2.

**Definition 3.3.** In an  $\omega_1$ -preserving forcing extension  $V[G]$ , a *continuous  $V$ -reflection sequence* is a sequence  $\langle \bar{M}_\alpha : \alpha \in C \rangle$  such that:

- (1)  $C \subset \omega_1$  is a closed unbounded set;
- (2) for each  $\alpha \in C$ ,  $\bar{M}_\alpha$  is the transitive collapse of some elementary submodel of  $(H_{\omega_2}^V, \in)$  such that  $\alpha = \omega_1^{\bar{M}_\alpha}$ ;
- (3) (*continuity*) for every  $\alpha \in C$  and every function  $x: \alpha^{<\omega} \rightarrow \alpha$  in the model  $\bar{M}_\alpha$  there is  $\gamma \in \alpha$  such that for every ordinal  $\delta \in C$  between  $\gamma$  and  $\alpha$ ,  $x \restriction \delta^{<\omega} \in \bar{M}_\delta$  (which implies by (2) above that  $\delta$  is closed under  $x$ );
- (4) (*reflection*) for every stationary set  $S \subset [H_{\omega_2}^V]^\omega$  in  $V$ , the set  $\{\alpha \in C : \bar{M}_\alpha$  is the transitive collapse of some element of  $S\} \subset \omega_1$  is stationary.

**Proposition 3.4.** Let  $V[G]$  be an  $\omega_1$ -preserving forcing extension in which there exists a continuous  $V$ -reflection sequence. In  $V$ , let  $P$  be a forcing of cardinality  $\omega_1$  which is not proper. Then  $V[G] \models P$  does not preserve stationary subsets of  $\omega_1$ .

*Proof.* First, move to  $V[G]$  and fix a continuous  $V$ -reflection sequence  $\langle \bar{M}_\alpha : \alpha \in C \rangle$ .

*Claim 3.5.* For every function  $x: \omega_1^{<\omega} \rightarrow \omega_1$  in  $V$ , for all but countably many  $\alpha \in C$ ,  $x \restriction \alpha^{<\omega} \in \bar{M}_\alpha$  holds.

*Proof.* By the reflection property of the sequence, the set  $S = \{\alpha \in C : x \restriction \alpha^{<\omega} \in \bar{M}_\alpha\} \subset \omega_1$  is stationary. Use the continuity property of the sequence to find a regressive function  $f: S \rightarrow \omega_1$  such that for every ordinal  $\alpha \in S$  and every ordinal  $\delta \in C$  between  $f(\alpha)$  and  $\alpha$ ,  $x \restriction \delta^{<\omega} \in \bar{M}_\delta$ . Use Fodor's lemma to find an ordinal  $\gamma \in \omega_1$  such that the set  $\{\alpha \in S : f(\alpha) = \gamma\}$  is stationary. It is immediate from the definitions that for every ordinal  $\delta \in C$  greater than  $\gamma$ ,  $x \restriction \delta^{<\omega} \in \bar{M}_\delta$ .  $\square$

Now, let  $P$  be a poset of cardinality  $\omega_1$  which is not proper, which we may assume without loss of generality has underlying set  $\omega_1$ . If  $P$  collapses  $\omega_1$ , then it also collapses  $\omega_1$  in  $V[G]$ , and hence is not stationary set preserving. So assume that  $P$  preserves  $\omega_1$ . Note that  $P$  can be coded in  $(H_{\omega_2}^V, \in)$  by a function  $x: \omega_1^{<\omega} \rightarrow \omega_1$  in  $V$  (for example, by the characteristic function of its partial ordering). By the claim, thinning out  $C$  to a closed unbounded subset if necessary, we may assume that for every  $\alpha \in C$ ,  $P \restriction \alpha \in \bar{M}_\alpha$ .

Now, return to  $V$  and observe that since  $P$  is not proper, by a pigeonhole argument there must be a condition  $p \in P$  and a stationary set  $S \subset [H_{\omega_2}^V]^\omega$  such that no model in  $S$  has a master condition below  $p$ . Move to  $V[G]$  and use the reflection property of the sequence to conclude that the set  $T = \{\alpha \in C : \bar{M}_\alpha$  is the transitive

collapse of some model in  $S\}$  is stationary. It will be enough to show that in  $V[G]$ ,  $p \Vdash_P \check{T}$  is nonstationary.

To this end, let  $\dot{E}$  be the  $P$ -name for the set  $\{\alpha \in C : \text{the } P\text{-generic filter has nonempty intersection with every maximal antichain of } P \restriction \alpha \text{ in the model } \bar{M}_\alpha\}$ .

*Claim 3.6.*  $\Vdash_P \dot{E} \subset \omega_1^V$  is a closed unbounded set.

*Proof.* First, argue for the unboundedness. Let  $q \in P$  be a condition and  $\gamma \in \omega_1$  be an ordinal. Back in  $V$ , consider the set  $U \subset [H_{\omega_2}]^\omega$  of all models which contain  $\gamma$  and have a master condition below  $q$ . Let us prove that the set  $U$  is stationary.

To this end, let  $f: H_{\omega_2}^{<\omega} \rightarrow H_{\omega_2}$  be a function. To find a model  $M \in U$  closed under the function  $f$ , let  $\theta$  be a large enough regular cardinal and let  $X = \langle N_\alpha : \alpha \in \omega_1 \rangle$  be a continuous increasing tower of countable elementary submodels of  $H_\theta$  containing  $f$  as an element, and let  $N = \bigcup_\alpha N_\alpha$ . Let  $H \subset P$  be a generic filter containing  $q$ , and consider the models  $N_\alpha[H]$  for  $\alpha \in \omega_1$  and  $N[H]$ . Since the poset  $P$  is a subset of  $N$ , it is clear that  $N[H] \cap V = N$ . The models  $\langle N_\alpha[H] : \alpha \in \omega_1 \rangle$  form a continuous increasing sequence of countable subsets of  $N[H]$ , so  $Y = \langle N_\alpha[H] \cap V : \alpha \in \omega_1 \rangle$  is a continuous increasing sequence of countable subsets of  $N[H] \cap V = N$ . Since  $\omega_1$  is preserved passing to  $V[H]$ , the sequences  $X$  and  $Y$  must intersect at some point, i.e. there must be an ordinal  $\alpha \in \omega_1$  such that  $N_\alpha[H] \cap V = N_\alpha$ . Fix a condition  $r \leq q$  in  $H$  such that  $r \Vdash_P^V N_\alpha[\dot{H}] \cap V = N_\alpha$ . Then  $r$  is a master condition for  $N_\alpha \cap H_{\omega_2}$ , and  $N_\alpha \cap H_{\omega_2}$  is a model in the set  $U$  closed under the function  $f$ .

In  $V[G]$  again, use the reflection property to find an ordinal  $\alpha \in C$  which is greater than  $\gamma$  and such that  $\bar{M}_\alpha$  is the transitive collapse of some model  $M$  in  $U$ . By the definition of  $U$ , fix a master condition  $r \leq q$  for  $M$ . It easily follows that  $r$  forces that  $\check{\alpha} \in \dot{E}$ . This completes the proof of the unboundedness of  $\dot{E}$ .

For the closure, suppose that some condition  $q \in P$  forces an ordinal  $\alpha \in C$  to be a limit point of  $\dot{E}$ . To show that  $q \Vdash \check{\alpha} \in \dot{E}$ , let  $A \in \bar{M}_\alpha$  be a maximal antichain of  $P \restriction \alpha$  in the model  $\bar{M}_\alpha$  and let  $r \leq q$ ; we must find a condition in  $A \cap \alpha$  compatible with  $r$ . To do this, apply the continuity property to a suitable function to find an ordinal  $\gamma \in \alpha$  such that for every ordinal  $\delta \in C$  between  $\gamma$  and  $\alpha$ ,  $A \cap \delta$  is a maximal antichain of  $P \restriction \delta$  in the model  $\bar{M}_\delta$ . Since  $r$  forces that  $\alpha$  is a limit point of  $\dot{E}$ , find a condition  $s \leq r$  and an ordinal  $\delta \in C$  between  $\gamma$  and  $\alpha$  such that  $s \Vdash \check{\delta} \in \dot{E}$ . By the definition of the name  $\dot{E}$ , there must be an element of  $A \cap \delta$  compatible with the condition  $s$ , and hence with  $r$ . This completes the proof.  $\square$

It is now clear from the definitions that in  $V[G]$ ,  $p \Vdash_P \dot{E} \cap \check{T} = \emptyset$ . The proof of the proposition is complete.  $\square$

We now define the poset for adding a continuous  $V$ -reflection sequence.

**Definition 3.7.**  $Q$  is the set of all pairs  $q = \langle a_q, b_q \rangle$  where

- (1)  $a_q$  is a function whose domain is a closed countable subset of  $\omega_1$  called the *support of  $q$* ,  $\text{supp}(q)$ ;
- (2) for every ordinal  $\alpha \in \text{supp}(q)$ , writing  $M = a_q(\alpha)$ , we have that  $M$  is the transitive collapse of a countable elementary submodel of  $(H_{\omega_2}^V, \in)$  such that  $\omega_1^M = \alpha$ ;
- (3) (continuity) for every  $\alpha \in \text{supp}(q)$  and every function  $x: \alpha^{<\omega} \rightarrow \alpha$  in the model  $a_q(\alpha)$ , there is  $\gamma \in \alpha$  such that for every ordinal  $\delta \in \text{supp}(q)$  between  $\gamma$  and  $\alpha$ ,  $x \restriction \delta^{<\omega} \in a_q(\delta)$ ;
- (4)  $b_q$  is a countable set of functions from  $\omega_1^{<\omega}$  to  $\omega_1$ .

The ordering is given by  $r \leq q$  if  $\text{supp}(r)$  is an end-extension of  $\text{supp}(q)$ ,  $a_q \subset a_r$ ,  $b_q \subset b_r$ , and for every ordinal  $\alpha \in \text{supp}(r) \setminus \text{supp}(q)$  and every function  $x \in b_q$ ,  $x \restriction \alpha^{<\omega} \in a_r(\alpha)$ .

It is not difficult to see that the forcing  $Q$  has cardinality  $2^{\omega_1}$ , and the relation  $\leq$  is transitive and reflexive. The relevant forcing properties of  $Q$  are all derived from the following consideration:

**Definition 3.8.** Let  $M$  be a countable elementary submodel of  $H(\kappa)$ , for a large enough regular cardinal  $\kappa$ . Let  $g \subset M$  be a filter meeting all open dense subsets of  $Q$  in  $M$ .

- (1) Let  $a = \bigcup_{s \in g} a_s \cup \{\langle M \cap \omega_1, \bar{M} \rangle\}$ , where  $\bar{M}$  is the transitive collapse of the model  $M \cap H_{\omega_2}$ ;
- (2) let  $b$  be the set of all functions from  $\omega_1^{<\omega}$  to  $\omega_1$  belonging to the model  $M$ ;
- (3) let  $r(M, g) = \langle a, b \rangle$ .

**Proposition 3.9.**  $r(M, g)$  is a condition in  $Q$  which is a common lower bound of all conditions in  $g$ .

*Proof.* Write  $\alpha = M \cap \omega_1$ . First of all, the fact that  $g$  is a filter shows that  $\bigcup_{s \in g} a_s$  is a function, and its domain  $c$  is a subset of  $M \cap \omega_1$  which is closed except perhaps at its supremum. A simple density argument shows that in fact  $\sup(c) = \alpha$ . Thus, to verify that  $r(M, g)$  is a condition, it is only necessary to check the continuity of  $a$  at  $\alpha$ . Let  $y: \alpha^{<\omega} \rightarrow \alpha$  be any function in the model  $\bar{M}$ , and let  $x \in M$  be the function whose collapse is  $y$ . By a density argument, there must be a condition  $s \in g$  such that  $x \in b_s$ . The definition of the ordering on  $Q$  then shows that the ordinal  $\gamma = \max \text{supp}(s)$  witnesses the continuity condition for  $\alpha$  and  $x$ .

To verify that for every condition  $s \in g$ ,  $r(M, g) \leq s$  holds, it is enough to verify that for every ordinal  $\delta \in \text{dom}(a) \setminus \text{supp}(s)$  and every  $x \in b_s$ , it is the case that  $x \restriction \delta^{<\omega} \in a(\delta)$ . For  $\delta \in \alpha$  this is immediately clear from the assumption that  $g$  is a filter. If  $\delta = \alpha$ , then  $x \in M$  since  $x \in b_s$  and  $s \in M$ ; by the elementarity of  $M$  we conclude again that  $x \restriction \delta^{<\omega}$  belongs to  $a(\alpha)$ , since it is the transitive collapse image of the function  $x$ .  $\square$

**Corollary 3.10.** The poset  $Q$  satisfies the following properties:

- (1) proper;
- (2) countably distributive;
- (3)  $\omega_2$ -p.i.c.

*Proof.* For (1), let  $q \in Q$  be a condition and let  $M$  be a countable elementary submodel of  $H(\kappa)$ , for a large enough regular cardinal  $\kappa$ , such that  $q$  and  $Q$  are in  $M$ . Construct a filter  $g \subset M \cap Q$  containing the condition  $q$  and meeting all dense open subsets of  $Q$  which belong to the model  $M$ . It is immediate that  $r(M, g)$  is a master condition for the model  $M$  below  $q$ . For (2), if in addition  $\{D_n: n \in \omega\}$  is a countable collection of open dense subsets of  $Q$  and  $M$  is selected in such a way that each  $D_n$  is in  $M$ , then  $r(M, g)$  is a condition below  $q$  in the intersection  $\bigcap_n D_n$ .

For (3), suppose that  $M, N$  are two isomorphic countable elementary submodels. By the Mostowski collapse lemma, the isomorphism is unique, and we denote it by  $\pi: M \rightarrow N$ . Let  $q \in M \cap Q$  be an arbitrary condition and let  $g \subset M \cap Q$  be a filter having  $q$  as an element and meeting all open dense subsets of  $Q$  which belong

to the model  $M$ . It will be enough to show that there is a condition  $r$  extending all the elements of the set  $g \cup \pi''g$ . To find  $r$ , write  $r(M, g) = \langle a_M, b_M \rangle$  and  $r(N, \pi''g) = \langle a_N, b_N \rangle$ , and observe that  $a_M = a_N$  since the isomorphism  $\pi$  fixes  $M \cap \omega_1 = N \cap \omega_1$  and because the two models  $M$  and  $N$  have the same transitive collapse. So,  $r(M, g)$  and  $r(N, \pi''g)$  are compatible as witnessed by the common extension  $r = \langle a_M, b_M \cup b_N \rangle$  and  $r$  works as desired.  $\square$

**Corollary 3.11.** *Let  $G \subset Q$  be a generic filter. In the model  $V[G]$ , let  $F = \bigcup \{a : \exists r \in G \ a = a_r\}$ . Then  $F$  is a continuous  $V$ -reflection sequence.*

*Proof.* It is immediately clear that  $\text{dom}(F)$  is a closed unbounded subset of  $\omega_1$  and that  $F$  satisfies the continuity property. Thus, it will be enough to verify the reflection property. For this, return to the ground model, let  $q \in Q$  and let  $S \subset [H_{\omega_2}]^\omega$  be a stationary set. Let also  $\dot{E}$  be a  $Q$ -name for a closed unbounded subset of  $\omega_1$ . It will be enough to find a condition  $r \leq q$  and an ordinal  $\alpha \in \text{supp}(r)$  such that  $a_r(\alpha)$  is the transitive collapse of a model in  $S$  and  $r \Vdash \check{\alpha} \in \dot{E}$ .

To this end, use the stationarity of the set  $S$  to find a countable elementary submodel  $M$  of  $H(\kappa)$  for some large enough regular cardinal  $\kappa$ , containing both  $q$  and  $\dot{E}$  such that  $M \cap H_{\omega_2} \in S$ . Find a filter  $g \subset Q \cap M$  generic over  $M$  containing the condition  $q$ , and let  $r = r(M, g)$  and  $\alpha = M \cap \omega_1$ . It is clear that  $r \leq q$ ,  $r \Vdash \check{\alpha} \in \dot{E}$  since  $r$  is a master condition for  $M$ , and  $a_r(\alpha)$  is a model isomorphic to  $M \cap H_{\omega_2} \in S$ .  $\square$

The proofs of Theorems 3.2 and 1.1 are complete.

#### 4. FINAL REMARKS

We finish this article with a few open questions. The first of them is motivated by the size of the poset  $Q$  used in Theorem 3.2 and the second one comes from the search for fragments of MM implying (as PFA does) that the continuum is equal to  $\omega_2$ .

**Question 4.1.** Does  $\text{PFA}(\omega_1)$  imply  $\text{MM}(\omega_1)$ ?

**Question 4.2.** Does  $\text{MM}(\omega_1)$  imply  $2^\omega = \omega_2$ ?

**Question 4.3.** Does  $\text{MM}(\omega_1)$  imply that every stationary set preserving forcing of size  $\omega_1$  is proper? Does this conclusion hold in the model constructed above?

In [2], Asperó and Mota proved that the forcing axiom  $\text{FA}(\Gamma, \omega_1)$ , for  $\Gamma$  being the class of all finitely proper posets of cardinality  $\omega_1$ , is consistent with the continuum being arbitrarily large. Very recently, Asperó and Golshani have improved that result by showing that  $\text{PFA}(\omega_1)$  is also compatible with  $2^\omega > \omega_2$  (see [1]). Therefore, a positive answer for our first question would imply a negative answer for the second one.

#### ACKNOWLEDGMENT

We thank James Cummings, Itay Neeman, and Dima Sinapova for having organized that meeting which took place on May 15-19, 2023 at the AIM Institute in San Jose, California.



## REFERENCES

- [1] D. Asperó and M. Golshani, *The proper forcing axiom for  $\aleph_1$ -sized posets, Prikry-type proper forcing, and the size of the continuum*, Preprint, [arXiv:2209.01395](#), 2023.
- [2] David Asperó and Miguel Angel Mota, *Forcing consequences of PFA together with the continuum large*, Trans. Amer. Math. Soc. **367** (2015), no. 9, 6103–6129, DOI 10.1090/S0002-9947-2015-06205-9. MR3356931
- [3] M. Foreman, M. Magidor, and S. Shelah, *Martin's maximum, saturated ideals, and nonregular ultrafilters. I*, Ann. of Math. (2) **127** (1988), no. 1, 1–47, DOI 10.2307/1971415. MR924672
- [4] H. Sakai,  *$\omega_1$ -stationary preserving poset of size  $\omega_1$  which is not semi-proper*, available at <http://www2.kobe-u.ac.jp/~hsakai/>, unpublished.
- [5] Saharon Shelah, *Proper and improper forcing*, 2nd ed., Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998, DOI 10.1007/978-3-662-12831-2. MR1623206
- [6] Saharon Shelah, *Semiproper forcing axiom implies Martin maximum but not  $\text{PFA}^+$* , J. Symbolic Logic **52** (1987), no. 2, 360–367, DOI 10.2307/2274385. MR890443

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, INDIANA 46556  
*Email address:* ndobrine@nd.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, DENTON, TEXAS 76203  
*Email address:* jkrueger@unt.edu

DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PENNSYLVANIA 15213  
*Email address:* pmarun@andrew.cmu.edu

DEPARTAMENTO DE MATEMÁTICAS, ITAM, 01080, MEXICO CITY, MEXICO  
*Email address:* motagaytan@gmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32611  
*Email address:* zapletal@ufl.edu