

**DESCENT AND VANISHING IN CHROMATIC ALGEBRAIC  $K$ -THEORY VIA  
GROUP ACTIONS**  
**DESCENTE ET ANNULATION EN K-THÉORIE CHROMATIQUE VIA LES  
ACTIONS DES GROUPES**

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**ABSTRACT.** We prove some  $K$ -theoretic descent results for finite group actions on stable  $\infty$ -categories, including the  $p$ -group case of the Galois descent conjecture of Ausoni–Rognes. We also prove vanishing results in accordance with Ausoni–Rognes’s redshift philosophy: in particular, we show that if  $R$  is an  $\mathbb{E}_\infty$ -ring spectrum with  $L_{T(n)}R = 0$ , then  $L_{T(n+1)}K(R) = 0$ . Our key observation is that descent and vanishing are logically interrelated, permitting to establish them simultaneously by induction on the height.

**RÉSUMÉ.** Nous démontrons quelques résultats sur la descente  $K$ -théorique pour des actions de groupes finis sur des  $\infty$ -catégories stables, dont le cas des  $p$ -groupes de la conjecture de descente galoisienne d’Ausoni–Rognes. Nous obtenons aussi des résultats d’annulation en accord avec la philosophie de “décalage vers le rouge” d’Ausoni–Rognes: en particulier, nous démontrons que si  $R$  est un  $\mathbb{E}_\infty$ -anneau avec  $L_{T(n)}R = 0$ , alors on a  $L_{T(n+1)}K(R) = 0$ . Notre observation-clé est que la descente et l’annulation sont logiquement liées, ce qui permet de les établir simultanément par récurrence sur la hauteur.

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1. INTRODUCTION

In this paper, we prove some results concerning the algebraic  $K$ -theory of ring spectra and stable  $\infty$ -categories after  $T(n)$ -localization. Throughout this paper, our telescopes  $T(n)$  are taken at a fixed implicit prime  $p$  and height  $n \geq 0$ ; we adopt the convention  $T(0) = \mathbb{S}[1/p]$ . Our starting point is the following two results concerning classical commutative rings  $R$ :

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**Theorem 1.1** ([Mit90]). *For  $n \geq 2$ , we have  $L_{T(n)}K(R) = 0$ .*

**Theorem 1.2** ([Tho85], [TT90], [CMNN20]). *For  $G$  a finite group and  $R \rightarrow R'$  a  $G$ -Galois extension, the natural comparison map  $L_{T(1)}K(R) \rightarrow (L_{T(1)}K(R'))^{hG}$  is an equivalence.*

Thus, the  $K$ -theory of an ordinary commutative ring has no chromatic information beyond height one, and the localization to height one is well-behaved in its descent properties. In fact,  $T(1)$ -local  $K$ -theory is even better-behaved than suggested by Theorem 1.2: under mild finiteness hypotheses, the Galois descent can be upgraded to an étale hyperdescent result, which leads to a descent spectral sequence from étale cohomology to  $T(1)$ -local  $K$ -theory as produced by [Tho85, TT90]. Furthermore, one knows that under such conditions, the map  $K(R; \mathbb{Z}_p) \rightarrow L_{T(1)}K(R)$  from  $p$ -adic  $K$ -theory to its  $T(1)$ -localization is an equivalence in high enough degrees, i.e., one has the Lichtenbaum–Quillen conjecture, thanks to the work of Voevodsky–Rost, cf. [RØ06, CM21] for accounts. However, we will not touch on these more advanced aspects in this paper.

Moving from ordinary rings to more general ring spectra, Ausoni–Rognes suggested that the above two theorems should fit into a broader “redshift” philosophy in algebraic  $K$ -theory, [AR02, AR08]. For an  $\mathbb{E}_1$ -ring spectrum  $R$ , one expects that taking algebraic  $K$ -theory increases the “chromatic complexity” of  $R$  by one. In the setting of Theorem 1.1, the Eilenberg–MacLane spectrum  $HR$  has no chromatic information at heights  $\geq 1$ , while the result states that  $K(R) = K(HR)$  has no chromatic information at heights  $\geq 2$ ; furthermore, Theorem 1.2 and its refinement to hyperdescent control the height one information very precisely.

For  $\mathbb{E}_\infty$ -rings  $R$ , there is a particularly well-behaved notion of chromatic complexity, thanks to a theorem of Hahn [Hah16]: if  $L_{T(n)}R = 0$ , then  $L_{T(m)}R = 0$  also for all  $m > n$ . If  $R$  is an  $\mathbb{E}_\infty$ -ring, then so is  $K(R)$ , and in this setting one possible expression of the redshift philosophy would be that  $L_{T(n)}R = 0 \Leftrightarrow L_{T(n+1)}K(R) = 0$ . Here we prove half of this statement.

**Theorem A.** *Let  $R$  be an  $\mathbb{E}_\infty$ -ring and  $n \geq 0$ . If  $L_{T(n)}R = 0$ , then  $L_{T(n+1)}K(R) = 0$ .*

Recent work of Burklund–Schlank–Yuan [BSY22, Th. 9.11] and Yuan [Yua21] proves the converse of Theorem A: if  $R$  is a  $p$ -local  $\mathbb{E}_\infty$ -ring with  $L_{T(n)}R \neq 0$ , then  $L_{T(n+1)}K(R) \neq 0$ . Many special cases of Theorem A were previously known. In particular, in important specific cases, much more precise (Lichtenbaum–Quillen) statements about  $K(R)$  have been proved, as in [HW22, AKAC<sup>+</sup>22, HRW22, Aus10, AR02].

Theorem A generalizes Mitchell’s vanishing Theorem 1.1. We note that there is a more general statement which applies also to  $\mathbb{E}_1$ -rings  $A$ : if both  $L_{T(n)}A = 0$  and  $L_{T(n+1)}A = 0$ , then  $L_{T(n+1)}K(A) = 0$ ; see Corollary 4.11, which is also explored in [LMMT20].

We also have an analog of Thomason’s descent Theorem 1.2. For the statement, we need to assume  $T(n)$ -local vanishing of the  $C_p$ -Tate construction  $R^{tC_p}$  (taken with respect to the trivial action); this assumption is satisfied if  $R$  is a discrete ring and  $n = 1$ , i.e., the setting of Theorem 1.2. In addition, we need to assume the finite group  $G$  is a  $p$ -group, where  $p$  is the (throughout fixed) prime at which chromatic localizations are taken.

**Theorem B.** *Let  $R$  be an  $\mathbb{E}_\infty$ -ring and  $n \geq 0$ . Suppose  $L_{T(n)}(R^{tC_p}) = 0$  for the fixed prime  $p$ . Then for  $\mathcal{C}$  any  $R$ -linear idempotent-complete stable  $\infty$ -category equipped with an  $R$ -linear action of a finite  $p$ -group  $G$ , the homotopy fixed point comparison map for  $T(n+1)$ -local  $K$ -theory is an equivalence:*

$$(1.1) \quad L_{T(n+1)}K(\mathcal{C}^{hG}) \xrightarrow{\sim} (L_{T(n+1)}K(\mathcal{C}))^{hG}.$$

If  $R \rightarrow R'$  is a  $G$ -Galois extension of commutative rings, then by Galois descent we have  $\text{Perf}(R) \xrightarrow{\sim} \text{Perf}(R')^{hG}$ ; thus, when  $n = 0$ , Theorem B recovers the  $p$ -group case of Theorem 1.2.

But in fact the case of general  $G$  in Theorem 1.2 reduces to the  $p$ -group case by a simple transfer argument, as already pointed out and exploited by Thomason; in particular, Theorem B implies Theorem 1.2.

However, Theorem B does not hold for an arbitrary finite group  $G$ , essentially because the  $G$ -action is allowed to be arbitrary. In fact, for the trivial action of  $G$  on  $\text{Perf}(\mathbb{C})$  and  $n = 0$ , one can calculate both sides of (1.1) using Suslin's equivalence [Sus84] between topological and algebraic  $K$ -theory. One obtains that the source is the  $p$ -completed  $G$ -equivariant topological  $K$ -theory of a point while the target is  $\text{KU}_{\hat{p}}^{BG}$ . For  $G$  of prime-to- $p$  order the result is evidently false because  $\text{KU}_{\hat{p}}^{BG} = \text{KU}_{\hat{p}}$ , while for  $G$  a  $p$ -group, Theorem B amounts to the  $p$ -complete Atiyah–Segal completion theorem. Nonetheless, there is a generalization of Theorem B to arbitrary finite groups which shows that the descent question for arbitrary  $G$  reduces to that for cyclic subgroups of prime-to- $p$  order; see Theorem 6.5.

We remark that these theorems also hold with  $K$ -theory replaced by an arbitrary additive invariant, and one also has “co-descent” or “assembly map” equivalences dual to the descent statements of Theorem B; see Proposition 4.1 for more details.

Let us now give the basic example of these results. Throughout this paper, we will use the notation  $L_n^{p,f} = L_{T(0) \oplus \dots \oplus T(n)}$ , following [LMMT20]; in particular, we have the  $L_n^{p,f}$ -local sphere  $L_n^{p,f}\mathbb{S}$ . An  $L_n^{p,f}$ -local stable  $\infty$ -category is one where the mapping spectra are  $L_n^{p,f}$ -local, or equivalently one which is  $L_n^{p,f}\mathbb{S}$ -linear. By Kuhn's “blueshift” theorem [Kuh04], if a spectrum  $X$  is  $L_n^{p,f}$ -local then  $X^{tC_p}$  is  $L_{n-1}^{p,f}$ -local. Thus, from Theorem A and Theorem B we deduce the following:

**Theorem C.** *Let  $n \geq 0$ , and let  $\mathcal{C}$  be an  $L_n^{p,f}$ -local idempotent-complete stable  $\infty$ -category. Then  $L_{T(m)}K(\mathcal{C}) = 0$  for all  $m \geq n + 2$ , and for any finite  $p$ -group  $G$  acting on  $\mathcal{C}$  we have*

$$L_{T(n+1)}K(\mathcal{C}^{hG}) \xrightarrow{\sim} (L_{T(n+1)}K(\mathcal{C}))^{hG}.$$

In fact, for the proofs of Theorem A and Theorem B we proceed by first proving this special case, Theorem C. Then we combine with a recent result of Land–Mathew–Meier–Tamme [LMMT20] to the effect that  $L_{T(n)}K(R) \xrightarrow{\sim} L_{T(n)}K(L_n^{p,f}R)$  (for  $n \geq 1$ ) which lets us deduce the general case. (Actually, we also use the result of [LMMT20] in the proof of Theorem C, but in a more indirect way.)

It turns out that our arguments establish a logical connection between the vanishing and the descent theorems. This is expressed in the following result, from which we deduce all of the above theorems.

**Theorem 1.3** (Inductive vanishing, Lemma 4.9). *Let  $R$  be an  $\mathbb{E}_\infty$ -ring spectrum and  $n \geq 1$ . Then for the following conditions, we have the implications  $(A) \Rightarrow (B) \Rightarrow (C)$ :*

- (A)  $L_{T(n)}R = 0$  and  $L_{T(n)}K(R^{tC_p}) = 0$ .
- (B) For any action of a finite  $p$ -group  $G$  on an  $R$ -linear idempotent-complete stable  $\infty$ -category  $\mathcal{C}$ , the comparison map

$$L_{T(n)}K(\mathcal{C}^{hG}) \xrightarrow{\sim} (L_{T(n)}K(\mathcal{C}))^{hG}$$

is an equivalence.

- (C)  $L_{T(i)}K(R) = 0$  for  $i \geq n + 1$ .

Theorem 1.3 allows an inductive approach to simultaneously proving vanishing and descent statements. In fact, Theorem C follows immediately from it by inductively taking  $R = L_n^{p,f}\mathbb{S}$  (and replacing  $n$  with  $n + 1$ ), via Kuhn's blueshift theorem.

Concerning the general descent result Theorem B, we have already mentioned that it recovers Galois descent for  $K(1)$ -local  $K$ -theory and the  $p$ -completed  $p$ -group case of the Atiyah-Segal completion theorem. We also use it to obtain the following  $p$ -group case of a conjecture of Ausoni–Rognes [AR08, Conj. 4.2]:

**Corollary** (Corollary 4.16). *Let  $A \rightarrow B$  be a  $T(n)$ -local  $G$ -Galois extension of  $\mathbb{E}_\infty$ -rings, in the sense of Rognes [Rog08], for  $G$  a finite  $p$ -group. Then the maps  $L_{T(n+1)}K(A) \rightarrow L_{T(n+1)}(K(B)^{hG}) \rightarrow (L_{T(n+1)}K(B))^{hG}$  are equivalences.*

Besides the above thread of results, we also prove some other descent results of a slightly different nature with different techniques. Like the results of our previous paper [CMNN20], these work uniformly for all chromatic heights, including height zero, and do not assume  $G$  to be a  $p$ -group; but on the other hand they make more restrictive assumptions on the action of the group  $G$ .

**Theorem D.** *Let  $\mathcal{C}$  be a monoidal, idempotent-complete stable  $\infty$ -category with biexact tensor product equipped with a (monoidal) action of a finite group  $G$ . Let  $\text{tr}: \mathcal{C} \rightarrow \mathcal{C}^{hG}$  denote the  $G$ -equivariant biaadjoint to the forgetful functor  $\mathcal{C}^{hG} \rightarrow \mathcal{C}$ . Suppose the  $G$ -equivariant object  $\text{tr}(\mathbf{1}) \in \text{Fun}(BG, \mathcal{C}^{hG})$  has class in  $K_0(\text{Fun}(BG, \mathcal{C}^{hG}))$  equal to that of the induced  $G$ -object  $\bigoplus_G \mathbf{1}_{\mathcal{C}^{hG}} \in \text{Fun}(BG, \mathcal{C}^{hG})$ . Then the comparison map*

$$K(\mathcal{C}^{hG}) \rightarrow K(\mathcal{C})^{hG}$$

*induces an equivalence after  $T(n)$ -localization for any  $n \geq 0$  and for any prime  $p$ .*

Theorem D states that a type of normal basis property (e.g., for a Galois extension of fields, the condition on  $\text{tr}(\mathbf{1})$  follows from the normal basis theorem) for the  $G$ -equivariant object  $\text{tr}(\mathbf{1}) \in \mathcal{C}^{hG}$  implies that the homotopy fixed point comparison map for  $K$ -theory is an equivalence after  $T(n)$ -localization. The argument is based on the vanishing [Kuh04] of Tate spectra in telescopic homotopy theory. In fact, Theorem D yields another proof of Theorem 1.2 avoiding the use of  $\mathbb{E}_\infty$ -structures, see Remark 5.6.

Next, we use [MNN15] to prove a third descent result (Theorem F below), which applies in more general situations, albeit with a weaker conclusion. For this, we first formulate a generalization of the homotopy fixed point comparison maps with respect to a family of subgroups of  $G$ .

**Construction 1.4** (Comparison maps for families of subgroups). Let  $\mathcal{C}$  be an idempotent-complete stable  $\infty$ -category with an action of a finite group  $G$ . Let  $\mathcal{F}$  be a family of subgroups of  $G$ , i.e.,  $\mathcal{F}$  is nonempty and closed under subconjugation, and let  $\mathcal{O}_{\mathcal{F}}(G)$  be the category of  $G$ -sets of the form  $G/H$ ,  $H \in \mathcal{F}$ . We obtain a comparison map

$$(1.2) \quad K(\mathcal{C}^{hG}) \rightarrow \varprojlim_{G/H \in \mathcal{O}_{\mathcal{F}}(G)^{\text{op}}} K(\mathcal{C}^{hH});$$

this generalizes the homotopy fixed point comparison map, which is the case where  $\mathcal{F} = \{(1)\}$ .

The map (1.2) is dual to the type of assembly maps which (for infinite groups) are the subject of the Farrell–Jones conjecture and its variants. In the rest of this paper, we will introduce a basic condition on an  $\mathbb{E}_\infty$ -ring that guarantees the maps (1.2) are equivalences after telescopic localization, and implies a bound on the chromatic complexity of the algebraic  $K$ -theory.

**Definition 1.5** (Swan  $K$ -theory, Malkiewich [Mal17]). Let  $R$  be an  $\mathbb{E}_\infty$ -ring spectrum, and let  $G$  be a (discrete) group. We define the ring  $\text{Rep}(G, R)$ , called the *Swan  $K$ -theory of  $R$* , via

$$\text{Rep}(G, R) = K_0(\text{Fun}(BG, \text{Perf}(R))).$$

For  $G$  finite, using induction and restriction functors, one makes  $\text{Rep}(-, R)$  into a Green functor, cf. Definition 6.1.

The  $\infty$ -category  $\text{Fun}(BG, \text{Perf}(R))$  is an analog of the category of complex representations of the finite group  $G$ ; studying this in analogy with complex or modular representation theory for  $R = \text{KU}$  has been proposed by Treumann [Tre15]. In this analogy, the ring  $\text{Rep}(G, R)$  is an analog of the classical representation ring of  $G$  (to which it reduces when  $R = H\mathbb{C}$ ). In general, the calculation of the rings  $\text{Rep}(G, R)$  seems to be an interesting problem (e.g., for  $R = \text{KU}$ ), although we know very few examples.

**Definition 1.6** ( $R$ -based Swan induction). Fix a finite group  $G$  and an  $\mathbb{E}_\infty$ -ring  $R$ . If the Green functor  $\text{Rep}(-, R) \otimes \mathbb{Q}$  (for subgroups of  $G$ ) is induced from a family  $\mathcal{F}$  of subgroups of  $G$ , then we say that  *$R$ -based Swan induction* holds for the family  $\mathcal{F}$ .

In [Swa60], Swan shows that  $H\mathbb{Z}$ -based Swan induction holds for the family of cyclic groups for any finite group; see also [Swa70] for a detailed treatment of Swan  $K$ -theory for a discrete ring. For  $H\mathbb{C}$ , this is Artin's classical induction theorem for the representation ring. Via some explicit geometric arguments, we prove the following instances of Swan induction for  $\mathbb{E}_\infty$ -ring spectra.

**Theorem E.** *If  $R$  is an  $\mathbb{E}_\infty$ -ring, then for every finite group,  $R$ -based Swan induction holds for:*

- (1) *the family of abelian subgroups if  $R$  admits an  $\mathbb{E}_1$ -map from  $\text{MU}$ .*
- (2) *the family of abelian subgroups of rank  $\leq 2$  if  $R = \text{KU}$ .*
- (3) *the family of abelian subgroups of  $p$ -rank  $\leq n+1$  and  $\ell$ -rank  $\leq 1$  for primes  $\ell \neq p$  if  $R = E_n$  is Morava  $E$ -theory of height  $n$  at the prime  $p = 2$ .*

This statement subsumes Theorem 7.4, Theorem 7.13, and parts of Theorem 7.12. Informally, (2) states that while a complex representation of a finite group  $G$  is determined by its character (i.e., its restriction to cyclic subgroups), the class in  $\text{Rep}(G, \text{KU})$  of a “representation” of  $G$  with  $\text{KU}$ -coefficients should be determined by a sort of “2-character,” defined on pairs of commuting elements of  $G$ ; moreover, there should be generalizations to higher heights. We conjecture that (3) should be true for odd primes too (Conjecture 7.22).

We show that the Swan induction condition guarantees that the maps (1.2) become equivalences after telescopic localization, for every  $R$ -linear  $\infty$ -category. This relies on similar techniques as in [CMNN20].

**Theorem F** (Descent via Swan induction, see Theorem 6.4). *Let  $R$  be an  $\mathbb{E}_\infty$ -ring spectrum,  $G$  a finite group, and  $\mathcal{F}$  a family of subgroups of  $G$ . Suppose that  $R$ -based Swan induction holds for the family  $\mathcal{F}$ . Then, for every  $R$ -linear idempotent-complete stable  $\infty$ -category  $\mathcal{C}$  with an action of  $G$ , the natural map (1.2), namely*

$$K(\mathcal{C}^{hG}) \rightarrow \varprojlim_{G/H \in \mathcal{O}_{\mathcal{F}}(G)^{\text{op}}} K(\mathcal{C}^{hH});$$

*becomes an equivalence after  $T(n)$ -localization, for any  $n$  and any implicit prime  $p$ . Furthermore, the limit in (1.2) commutes with  $L_{T(n)}$ .*

In fact, Theorem F can be combined with Theorem B, yielding the following result (Theorem 6.5): if  $L_{T(n)}(R^{tC_p}) = 0$ , then the comparison map (1.2) becomes an equivalence after  $T(n+1)$ -localization, for  $\mathcal{F}$  the family of cyclic subgroups of prime-to- $p$  order.

Our final main result, which is inspired by the character theory of Hopkins–Kuhn–Ravenel [HKR00], is that a certain case of Swan induction implies the vanishing of the  $T(i)$ -localizations of algebraic  $K$ -theory for large  $i$ .

**Theorem G.** *Let  $R$  be an  $\mathbb{E}_\infty$ -ring,  $p$  a prime, and  $n > 0$ . Suppose that  $R$ -based Swan induction holds for the family of proper subgroups of  $C_p^{\times n}$ . Then  $L_{T(i)}K(R) = 0$  for  $i \geq n$  at the implicit prime  $p$ .*

As a consequence, we obtain a new proof of Mitchell's theorem (Theorem 1.1), and we recover several chromatic bounds, e.g., that if  $p = 2$  or  $n = 1$ , then we have  $L_{T(i)}K(E_n) = 0$  for  $i \geq n + 2$ . These bounds are special cases of Theorem A above, although the method is different and could be useful in other settings as well; for instance, in Theorem 7.12 we prove 2-primary Swan induction results for  $\mathrm{MO} \langle n \rangle$  which therefore implies bounds on the chromatic complexity of  $K(\mathrm{MO} \langle n \rangle)$ .

**Conventions.** We let  $\mathcal{S}$  denote the  $\infty$ -category of anima,  $\mathrm{Sp}$  denote the  $\infty$ -category of spectra,  $\mathcal{S}_G$  the  $\infty$ -category of  $G$ -anima (i.e., genuine  $G$ -spaces), and  $\mathrm{Sp}_G$  the  $\infty$ -category of (genuine)  $G$ -spectra. We denote by  $\mathbb{S}$  the unit of either of these (i.e., the sphere spectrum). We write  $\mathbb{D}X$  for the Spanier–Whitehead dual of  $X$ .

We let  $L_n^{p,f}$  denote the finite localization [Mil92] on  $\mathrm{Sp}$  away from a finite type  $n + 1$  spectrum (at the implicit prime  $p$ ). In particular, for  $n = 0$ , we have  $L_0^{p,f}(X) = X[1/p]$ . This convention follows [LMMT20]; for  $p$ -local spectra, it agrees with what is usually denoted  $L_n^f$ . Equivalently, if  $T(i)$  denotes the telescope of a  $v_i$ -self map of a finite type  $i$  complex (so by convention  $T(0)$  can be taken to be  $\mathbb{S}[1/p]$ ), then  $L_n^{p,f} = L_{T(0) \oplus \dots \oplus T(n)}$ .

We write  $K$  for connective algebraic  $K$ -theory. Most of our results hold only after telescopic localization, after which there is no difference between connective and nonconnective  $K$ -theory, and we will anyway state them in the generality of additive invariants.

We write  $\mathrm{Cat}_\infty^{\mathrm{perf}}$  for the  $\infty$ -category of small, idempotent-complete stable  $\infty$ -categories and exact functors between them, cf. [BGT13]. More generally, given an  $\mathbb{E}_\infty$ -ring  $R$ , we write  $\mathrm{Cat}_{R,\infty}^{\mathrm{perf}}$  for the  $\infty$ -category of small, idempotent-complete  $R$ -linear stable  $\infty$ -categories and  $R$ -linear functors between them, so  $\mathrm{Cat}_{R,\infty}^{\mathrm{perf}}$  is  $\mathrm{Perf}(R)$ -modules in  $\mathrm{Cat}_\infty^{\mathrm{perf}}$ . Compare [HSS17] for a treatment.

An  $\infty$ -category  $\mathcal{C}$  is called *preadditive* (also called *semiadditive* in the literature) if it is pointed, admits finite coproducts, and finite coproducts are (canonically) identified with finite products, see [GGN15, Sec. 2]. Given a preadditive  $\infty$ -category  $\mathcal{C}$ , we say that  $\mathcal{C}$  is *additive* if the  $\mathbb{E}_\infty$ -anima  $\mathrm{Hom}_{\mathcal{C}}(X, Y)$  for  $X, Y \in \mathcal{C}$  are grouplike.

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## 2. MACKEY FUNCTORS AND EQUIVARIANT ALGEBRAIC $K$ -THEORY

In this section, we review the setup of equivariant algebraic  $K$ -theory which plays an integral role in our approach to the present descent theorems. The use of equivariant algebraic  $K$ -theory refines the use of the transfer, which is central to all such descent results, going back to [Tho85].<sup>1</sup>

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<sup>1</sup>A toy example of this argument is the Galois descent for rationalized algebraic  $K$ -theory, cf. [Tho85, Th. 2.15].

In studying the comparison map  $K(\mathcal{C}^{hG}) \rightarrow K(\mathcal{C})^{hG}$ , one observes that there is also a map

$$(2.1) \quad K(\mathcal{C})_{hG} \rightarrow K(\mathcal{C}^{hG}),$$

arising from the  $G$ -equivariant functor  $\mathcal{C} \rightarrow \mathcal{C}^{hG}$  biadjoint to the forgetful functor  $\mathcal{C}^{hG} \rightarrow \mathcal{C}$ , such that the composition with the comparison map is the norm map  $K(\mathcal{C})_{hG} \rightarrow K(\mathcal{C})^{hG}$ . In the case of  $\mathcal{C} = \text{Perf}(R')$  for a  $G$ -Galois extension  $R \rightarrow R'$  of commutative rings, then Galois descent gives  $\text{Perf}(R) \simeq \text{Perf}(R')^{hG}$  and (2.1) is the map

$$K(R')_{hG} \rightarrow K(R)$$

which arises from restriction of scalars from  $R'$ -modules to  $R$ -modules. These types of transfer maps and their functorialities can be encoded using the language of (genuine)  $G$ -spectra, and some of the techniques for proving descent results can be expressed using the language of  $\mathcal{F}$ -nilpotence [MNN17, MNN19].

Several authors have considered the setup of equivariant algebraic  $K$ -theory, including Merling [Mer17], Barwick [Bar17], Malkiewich–Merling [MM19], and Barwick–Glasman–Shah [BGS20]. We will follow the setup of [Bar17, BGS20], but will try to keep the exposition mostly self-contained. In particular, we will use the theory of spectral Mackey functors, which is equivalent to the theory of  $G$ -spectra by work of Guillou–May [GM11] and Nardin [Nar16]; we will also give another proof of this equivalence in the appendix.

**Definition 2.1** (The effective Burnside  $\infty$ -category, [Bar17, Sec. 3]). For a finite group  $G$ , let  $\text{Burn}_G^{\text{eff}}$  denote the *effective Burnside  $\infty$ -category* of the category of finite (left)  $G$ -sets and  $G$ -maps; informally,  $\text{Burn}_G^{\text{eff}}$  is the nerve of the (weak)  $(2, 1)$ -category defined as follows:

- The objects of  $\text{Burn}_G^{\text{eff}}$  are finite  $G$ -sets  $S$ ,
- Given finite  $G$ -sets  $S$  and  $T$ ,  $\text{Hom}_{\text{Burn}_G^{\text{eff}}}(S, T)$  is the nerve of the groupoid of spans of finite  $G$ -sets

$$\begin{array}{ccc} & U & \\ & \searrow & \swarrow \\ S & & T \end{array}$$

and isomorphisms of spans.

- Composition is given by pullback of spans.

That is,  $\text{Burn}_G^{\text{eff}}$  is the span category of the category of finite  $G$ -sets, as in [BH21, App. C]. The  $\infty$ -category  $\text{Burn}_G^{\text{eff}}$  is preadditive, and the direct sum comes from the disjoint union of finite  $G$ -sets.

One then obtains the following definition [Bar17, Sec. 6] of a Mackey functor; this reduces to the classical notion when  $\mathcal{C}$  is the category of abelian groups.

**Definition 2.2** (Mackey functors). Given any presentable, preadditive  $\infty$ -category  $\mathcal{C}$ , we define a  $\mathcal{C}$ -valued Mackey functor (for the finite group  $G$ ) to be a  $\mathcal{C}$ -valued presheaf on  $\text{Burn}_G^{\text{eff}}$  which takes finite coproducts of  $G$ -sets to products in  $\mathcal{C}$ . We let  $\text{Mack}_G(\mathcal{C})$  denote the  $\infty$ -category of  $\mathcal{C}$ -valued Mackey functors.

Let  $M \in \text{Mack}_G(\mathcal{C})$ . Given a subgroup  $H \subseteq G$ , we write

$$M^H \stackrel{\text{def}}{=} M(G/H),$$

and call this the  $H$ -fixed points of  $M$ .

**Remark 2.3** (Comparison with the  $\mathcal{P}_\Sigma$ -construction). Consider the nonabelian derived  $\infty$ -category  $\mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}})$  of  $\text{Burn}_G^{\text{eff}}$ , in the sense of [Lur09, Sec. 5.5.8], i.e.,  $\mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}})$  is the  $\infty$ -category of presheaves on  $\text{Burn}_G^{\text{eff}}$  which preserve finite products, or equivalently  $\mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}})$  is obtained by freely adding sifted colimits to  $\text{Burn}_G^{\text{eff}}$ . Then  $\text{Mack}_G(\mathcal{C}) = \mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}}) \otimes \mathcal{C}$  via the Lurie tensor product, cf. [Lur17, Sec. 4.8.1].

**Construction 2.4** (The symmetric monoidal structure on Mackey functors). Suppose now  $\mathcal{C}$  is a presentably symmetric monoidal  $\infty$ -category which is preadditive. Then there is a canonical structure of a presentably symmetric monoidal structure on  $\text{Mack}_G(\mathcal{C})$ , obtained (implicitly by Day convolution) as follows. We consider the symmetric monoidal structure on  $\text{Burn}_G^{\text{eff}}$  obtained from the cartesian product on finite  $G$ -sets (and products of spans). This symmetric monoidal structure commutes with finite coproducts in each variable. Applying  $\mathcal{P}_\Sigma$ , we obtain a canonical presentably symmetric monoidal structure on  $\mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}})$  such that the Yoneda functor is symmetric monoidal, [Lur17, Prop. 4.8.1.10]. Now via the Lurie tensor product,  $\text{Mack}_G(\mathcal{C}) = \mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}}) \otimes \mathcal{C}$  then acquires the structure of a presentably symmetric monoidal  $\infty$ -category.

**Remark 2.5** (Functionality of  $\text{Mack}_G(-)$ ). Let  $\mathcal{C}, \mathcal{D}$  be presentable, preadditive  $\infty$ -categories. Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be an accessible functor which commutes with finite coproducts (but not necessarily all colimits). Then we can still define a natural functor  $\text{Mack}_G(\mathcal{C}) \rightarrow \text{Mack}_G(\mathcal{D})$  induced by  $F$  by sending a  $\mathcal{C}$ -valued Mackey functor to the corresponding  $\mathcal{D}$ -valued one (i.e., composing with  $F$ ). However, this is slightly awkward to formulate in our setup where  $\text{Mack}_G(\mathcal{C}) = \mathcal{C} \otimes \mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}})$ , since this tensor product takes place in the world of presentable  $\infty$ -categories. We can modify this by fixing a suitable cardinal  $\kappa$ , considering the  $\kappa$ -compact objects  $\mathcal{C}^\kappa \subseteq \mathcal{C}$ , then defining the cocontinuous functor  $\text{Ind}(\mathcal{C}^\kappa) \rightarrow \mathcal{D}$  and applying  $\text{Mack}_G(-) = (-) \otimes \mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}})$  to it. Varying  $\kappa$ , we obtain a functor out of  $\mathcal{C}$ . In particular, this also shows that if  $\mathcal{C}, \mathcal{D}$  are symmetric monoidal and if  $F$  has a lax symmetric monoidal structure, then  $\text{Mack}_G(\mathcal{C}) \rightarrow \text{Mack}_G(\mathcal{D})$  has a lax symmetric monoidal structure; alternatively one could see this using Day convolution, cf. [Gla16] or [Lur17, Sec. 2.2.6].

**Remark 2.6** (Spectral Mackey functors and  $G$ -spectra). Suppose  $\mathcal{C} = \text{Sp}$  is the  $\infty$ -category of spectra. Then by [GM11, Nar16], we have an equivalence between  $\text{Mack}_G(\text{Sp})$  and the  $\infty$ -category  $\text{Sp}_G$  of (genuine)  $G$ -spectra, cf. also the appendix for an independent account of this equivalence. The target of equivariant algebraic  $K$ -theory will naturally be  $\text{Mack}_G(\text{Sp})$ , and so we can equally regard equivariant algebraic  $K$ -theory as a  $G$ -spectrum.

**Example 2.7** (The case of the trivial group). Suppose  $G = (1)$  is the trivial group. Then  $\text{Burn}_{(1)}^{\text{eff}}$  is the category of finite sets and correspondences between them. This is the free preadditive category on a single generator, a result due to Cranch [Cra10], cf. [BH21, Prop. C.1] for another account. It follows that  $\text{Mack}_{(1)}(\mathcal{C}) = \mathcal{C}$ . In particular, it follows that  $\mathcal{P}_\Sigma(\text{Burn}_{(1)}^{\text{eff}})$  is the  $\infty$ -category of  $\mathbb{E}_\infty$ -anima, since this is the free presentable preadditive  $\infty$ -category on one object, cf. [GGN15].

**Construction 2.8** (Relation to the orbit category). Let  $\mathcal{O}(G)$  be the *orbit category* of  $G$ , i.e., the category of nonempty transitive  $G$ -sets. We have a natural functor  $\mathcal{O}(G) \rightarrow \text{Burn}_G^{\text{eff}}$  which sends the  $G$ -set  $S$  to  $S \in \text{Burn}_G^{\text{eff}}$  and the  $G$ -map  $f: S \rightarrow T$  to the span

$$\begin{array}{ccc} & S & \\ & \swarrow \text{id} \quad \searrow f & \\ S & & T. \end{array}$$

We also obtain a natural functor  $\mathcal{O}(G)^{\text{op}} \rightarrow \text{Burn}_G^{\text{eff}}$  in a similar (dual) manner. Suppose  $f: G/H \rightarrow G/H'$  is a morphism in  $\mathcal{O}(G)$ . Given a  $\mathcal{C}$ -valued Mackey functor  $M$ , we then obtain morphisms in  $\mathcal{C}$

$$f^*: M^{H'} \rightarrow M^H, \quad f_*: M^H \rightarrow M^{H'}.$$

Thus, given the Mackey functor  $M$ , we obtain two functors

$$\mathcal{O}(G)^{\text{op}} \rightarrow \mathcal{C}, \quad \mathcal{O}(G) \rightarrow \mathcal{C},$$

which both send  $G/H \mapsto M(G/H) = M^H$ , and such that the functoriality is via  $f^*$  in the first case and via  $f_*$  in the second case.

Next, we discuss the most basic source of Mackey functors: the Borel-equivariant ones, or those  $M$  for which  $M(G/H) \simeq M(G)^{hH}$  for all subgroups  $H \subseteq G$ . We begin with the case where  $\mathcal{C}$  is given by  $\mathbb{E}_\infty$ -monoids in anima.

**Proposition 2.9.** *There is a symmetric monoidal Bousfield localization functor  $\mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}}) \rightarrow \text{Fun}(BG, \mathcal{P}_\Sigma(\text{Burn}_{(1)}^{\text{eff}}))$  such that the essential image of its right adjoint inclusion consists of those product-preserving presheaves  $\mathcal{F}: (\text{Burn}_G^{\text{eff}})^{\text{op}} \rightarrow \mathcal{S}$  such that for each finite  $G$ -set  $S$ , the natural map  $\mathcal{F}(S) \rightarrow \mathcal{F}(G \times S)^{hG}$  is an equivalence. Here  $G$  acts on  $G$  (in the category of finite  $G$ -sets, and hence in  $\text{Burn}_G^{\text{eff}}$ ) by right multiplication.*

*Proof.* Let  $y: \text{Burn}_G^{\text{eff}} \rightarrow \mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}})$  be the Yoneda embedding, and consider the Bousfield localization functor  $L_{\mathcal{I}}$  on  $\mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}})$  with respect to the maps  $\mathcal{I} = \{y(G \times S)_{hG} \rightarrow y(S)\}$ , for each finite  $G$ -set  $S$ . Here we use the map of  $G$ -sets  $G \times S \rightarrow S$  given by projection onto the second factor, and the  $G$ -action on the source (in the category of  $G$ -sets) by right multiplication on the first factor.

Since  $y$  is symmetric monoidal and the tensor product on  $\mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}})$  commutes with colimits in each variable, the class  $\mathcal{I}$  is preserved by tensoring with objects in the image of  $y$ , and we see that this Bousfield localization  $L_{\mathcal{I}}$  respects the symmetric monoidal structure. Unwinding the definitions, we see that the image of  $L_{\mathcal{I}}$  is precisely those product-preserving presheaves  $\mathcal{F}$  as in the statement because  $\text{Hom}_{\mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}})}(y(G \times S)_{hG}, \mathcal{F}) = \mathcal{F}(G \times S)^{hG}$ . In particular, for any finite  $G$ -set  $S$  which is  $G$ -free,  $y(S)$  is  $\mathcal{I}$ -local, as one sees by unwinding the definition of mapping anima in  $\text{Burn}_G^{\text{eff}}$ .

We claim that the  $\{y(S)\}$  for  $S$  finite and  $G$ -free form a set of compact projective generators for  $L_{\mathcal{I}}\mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}})$ . Compactness and projectivity follow because for a finite free  $G$ -set  $S$ , the functor  $\mathcal{F} \mapsto \mathcal{F}(S)$  (with values in  $\mathcal{S}$ ) commutes with sifted colimits on  $\mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}})$  and carries the maps in  $\mathcal{I}$  to equivalences. Moreover, the  $y(T)$  for  $T \in \text{Burn}_G^{\text{eff}}$  can be expressed up to  $\mathcal{I}$ -equivalence as colimits of the  $y(S)$  for  $S$  finite  $G$ -free by construction; therefore, the  $\{y(S)\}$  generate. This verifies the claim about  $L_{\mathcal{I}}\mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}})$ .

The symmetric monoidal functor  $\text{Burn}_G^{\text{eff}} \rightarrow \text{Fun}(BG, \text{Burn}_{(1)}^{\text{eff}})$  which remembers an underlying set, or correspondence, with  $G$ -action extends to a cocontinuous symmetric monoidal functor  $\mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}}) \rightarrow \text{Fun}(BG, \mathcal{P}_\Sigma(\text{Burn}_{(1)}^{\text{eff}}))$ . Evidently, this functor carries the class of maps  $\mathcal{I}$  to equivalences, and factors symmetric monoidally through the Bousfield localization  $L_{\mathcal{I}}$ . It remains to show that the induced functor  $L_{\mathcal{I}}\mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}}) \rightarrow \text{Fun}(BG, \mathcal{P}_\Sigma(\text{Burn}_{(1)}^{\text{eff}}))$  is an equivalence. The compact projective generators on both sides are given by  $y(S)$  for  $S$  a finite free  $G$ -set, so it suffices to compare maps between them. Equivalently, it suffices to show that the map  $\text{Hom}_{\text{Burn}_G^{\text{eff}}}(S, T) \rightarrow \text{Hom}_{\text{Fun}(BG, \text{Burn}_{(1)}^{\text{eff}})}(S, T)$  is an equivalence for  $S, T$  finite free  $G$ -sets (in fact, it suffices for the  $G$ -action on one of them to be free). By decomposing  $S$  and  $T$  and using duality, it

suffices to prove that this map is an equivalence when  $S = *$  and  $T = G$ ; then one checks directly that both sides are the free  $\mathbb{E}_\infty$ -space on a generator and the map is an equivalence.  $\square$

**Construction 2.10** (Borel-equivariant objects). Let  $\mathcal{C}$  be a presentably symmetric monoidal, preadditive  $\infty$ -category. Tensoring the Bousfield localization of Proposition 2.9 with  $\mathcal{C}$ , we obtain a symmetric monoidal Bousfield localization functor

$$\text{Mack}_G(\mathcal{C}) \rightarrow \text{Fun}(BG, \mathcal{C}),$$

with a fully faithful lax symmetric monoidal right adjoint functor called “Borelification”,

$$(-)^{\text{Bor}}: \text{Fun}(BG, \mathcal{C}) \rightarrow \text{Mack}_G(\mathcal{C}).$$

The essential image of  $(-)^{\text{Bor}}$  (called *Borel-equivariant* objects) is given by those product-preserving presheaves  $\mathcal{F}: (\text{Burn}_G^{\text{eff}})^{\text{op}} \rightarrow \mathcal{C}$  such that for any finite  $G$ -set  $S$ , we have  $\mathcal{F}(S) \xrightarrow{\sim} \mathcal{F}(G \times S)^{hG}$ . In other words,  $\mathcal{F}$  is Borel-equivariant if and only if the restriction of  $\mathcal{F}$  to  $\mathcal{O}(G)^{\text{op}}$  is right Kan extended from the full subcategory spanned by the  $G$ -set  $G$ .

We will be interested in the above construction when  $\mathcal{C} = \text{Cat}_\infty^{\text{perf}}$ . For this, recall that  $\text{Cat}_\infty^{\text{perf}}$  is preadditive (cf. [Bar16, Prop. 4.7] for this result in the closely related context of Waldhausen  $\infty$ -categories). Moreover,  $\text{Cat}_\infty^{\text{perf}}$  is presentable [BGT13, Cor. 4.25], and symmetric monoidal under the Lurie tensor product [BGT13, Th. 3.1]. For an idempotent-complete stable  $\infty$ -category  $\mathcal{A}$  with  $G$ -action, we obtain a  $\text{Cat}_\infty^{\text{perf}}$ -valued Mackey functor  $M_{\mathcal{A}}$  such that  $M_{\mathcal{A}}(G/H) = \mathcal{A}^{hH}$ . For a map of finite  $G$ -sets  $f: G/H \rightarrow G/K$ , then  $f^*$  is the natural pullback map  $\mathcal{A}^{hK} \rightarrow \mathcal{A}^{hH}$ . We will need to know that in this case,  $f_*$  can also be described explicitly.

**Proposition 2.11.** *Let  $M \in \text{Mack}_G(\text{Cat}_\infty^{\text{perf}})$  be Borel-equivariant. Then for any map  $f: S \rightarrow T$  of finite  $G$ -sets, the functor  $f_*: M(T) \rightarrow M(S)$  (of Construction 2.8) is both left and right adjoint to  $f^*: M(S) \rightarrow M(T)$ .*

*Proof.* We will verify this by invoking from [Bar17] a construction of a  $\text{Cat}_\infty^{\text{perf}}$ -valued Mackey functor which does have the desired adjointness property, which is Borel, and whose underlying object of  $\text{Fun}(BG, \text{Cat}_\infty^{\text{perf}})$  agrees with that of  $M$ .

Let  $\mathcal{A} = M(G/\{e\}) \in \text{Fun}(BG, \text{Cat}_\infty^{\text{perf}})$ . Note first that for any map of finite sets  $f: S_0 \rightarrow T_0$ , the pullback functor  $f^*: \text{Fun}(T_0, \mathcal{A}) \rightarrow \text{Fun}(S_0, \mathcal{A})$  admits a right (and left) adjoint  $f_*: \text{Fun}(S_0, \mathcal{A}) \rightarrow \text{Fun}(T_0, \mathcal{A})$  given by summing over the fibers. Moreover, one has the base-change property: given a pullback square of finite sets

$$\begin{array}{ccc} U_0 & \longrightarrow & V_0 \\ \downarrow & & \downarrow \\ S_0 & \longrightarrow & T_0, \end{array}$$

the induced square in  $\text{Cat}_\infty^{\text{perf}}$  obtained by applying pullback everywhere is left and right adjointable [Lur09, Def. 7.3.1.2].

Now for every  $G$ -set  $S$ , we consider  $\text{Fun}_G(S, \mathcal{A}) \in \text{Cat}_\infty^{\text{perf}}$ ; this is also  $\text{Fun}(S, \mathcal{A})^{hG}$  for the diagonal  $G$ -action (with  $G$  acting on both  $S$  and  $\mathcal{A}$ ). Given a map of  $G$ -sets  $f: S \rightarrow T$ , we have a pullback functor  $f^*: \text{Fun}_G(T, \mathcal{A}) \rightarrow \text{Fun}_G(S, \mathcal{A})$ ; we obtain a  $\text{Cat}_\infty^{\text{perf}}$ -valued presheaf on the category of finite  $G$ -sets. We claim that for any map  $f: S \rightarrow T$ , the functor  $f^*: \text{Fun}_G(T, \mathcal{A}) \rightarrow \text{Fun}_G(S, \mathcal{A})$  admits an adjoint (in either direction), and furthermore that for any pullback square

of finite  $G$ -sets

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ S & \longrightarrow & T, \end{array}$$

the induced square in  $\text{Cat}_\infty^{\text{perf}}$  obtained by pullback functoriality is adjointable (in either direction). This follows from the above verification in the case of finite sets, and then taking homotopy fixed points in view of [Lur17, Cor. 4.7.4.18]. Indeed, the result of *loc. cit.* shows that given a square in  $\text{Fun}(BG, \text{Cat}_\infty)$  which is left (or right) adjointable, the square in  $\text{Cat}_\infty$  obtained by taking  $G$ -homotopy fixed points remains left (or right) adjointable.

We thus have a  $\text{Cat}_\infty^{\text{perf}}$ -valued presheaf on the category of finite  $G$ -sets,  $S \mapsto \text{Fun}_G(S, \mathcal{A})$ , and we have verified the adjointability conditions needed to apply the unfurling construction of [Bar17, Sec. 11], which produces a  $\text{Cat}_\infty^{\text{perf}}$ -valued Mackey functor  $M'$  extending the above presheaf whose restriction to  $\mathcal{O}(G)$  is given by the adjoints  $f_*$ . In particular,  $M'$  satisfies the condition of the proposition. The Mackey functor  $M'$  is Borel-complete (since this condition only depends on the restriction to the category of finite  $G$ -sets) and must therefore agree with  $M$ , since the restrictions of  $M, M'$  in  $\text{Fun}(BG, \text{Cat}_\infty^{\text{perf}})$  agree; it follows now that  $M$  has the desired property in the proposition.  $\square$

Now we describe the fundamental construction for our purposes, the equivariant algebraic  $K$ -theory of group actions, in the form constructed by Barwick–Glasman–Shah, cf. [BGS20, Sec. 8].

**Construction 2.12** (Equivariant  $K$ -theory of group actions [BGS20, Sec. 8]). It follows from Construction 2.10 that we have a lax symmetric monoidal functor of “Borelification”

$$(-)^{\text{Bor}}: \text{Fun}(BG, \text{Cat}_\infty^{\text{perf}}) \rightarrow \text{Mack}_G(\text{Cat}_\infty^{\text{perf}}),$$

and composing it with the lax symmetric monoidal algebraic  $K$ -theory functor as in Remark 2.5, we obtain a lax symmetric monoidal functor

$$(2.2) \quad K_G: \text{Fun}(BG, \text{Cat}_\infty^{\text{perf}}) \rightarrow \text{Mack}_G(\text{Sp}),$$

such that  $K_G(\mathcal{A})^H = K(\mathcal{A}^{hH})$  whenever  $\mathcal{A} \in \text{Fun}(BG, \text{Cat}_\infty^{\text{perf}})$  and  $H \subseteq G$ .

**Example 2.13** (Equivariant algebraic  $K$ -theory of  $\mathbb{E}_\infty$ -rings). Let  $R$  be an  $\mathbb{E}_\infty$ -ring with  $G$ -action. Then we write  $K_G(R)$  for  $K_G(\text{Perf}(R))$ .

We will actually need a slight generalization of the above, in order to handle invariants other than algebraic  $K$ -theory. Given a base  $\mathbb{E}_\infty$ -ring  $R$ , we consider the presentably symmetric monoidal  $\infty$ -category  $\text{Mot}_R$  of  $R$ -linear noncommutative motives, cf. [BGT13, HSS17]. We have a symmetric monoidal functor  $\mathcal{U}: \text{Cat}_{R, \infty}^{\text{perf}} \rightarrow \text{Mot}_R$  which is an additive invariant, i.e., it preserves filtered colimits and carries semiorthogonal decompositions to direct sums in  $\text{Mot}_R$ ; moreover,  $\mathcal{U}$  is initial for these data and conditions.

**Construction 2.14** ( $\text{Mot}_R$ -valued Mackey functors). Composing the functor  $(-)^{\text{Bor}}$  with  $\mathcal{U}$ , we obtain a lax symmetric monoidal functor

$$\mathcal{U}_G: \text{Fun}(BG, \text{Cat}_{R, \infty}^{\text{perf}}) \rightarrow \text{Mack}_G(\text{Mot}_R) \simeq \text{Mack}_G(\text{Sp}) \otimes \text{Mot}_R,$$

i.e.,  $\mathcal{U}_G$  takes values in Mackey functors in  $R$ -linear noncommutative motives. (Here we use Remark 2.5, since  $\mathcal{U}$  does not preserve all colimits.) Since for any  $\mathcal{A} \in \text{Cat}_{R, \infty}^{\text{perf}}$ , the algebraic  $K$ -theory  $K(\mathcal{A})$  can be recovered as  $\text{Hom}_{\text{Mot}_R}(\mathbf{1}, \mathcal{U}(\mathcal{A}))$  by [BGT13] and [HSS17], it follows that

the equivariant algebraic  $K$ -theory functor  $K_G$  is the composition of  $\mathcal{U}_G$  and the functor  $\text{id} \otimes \text{Hom}_{\text{Mot}_R}(\mathbf{1}, -) : \text{Mack}_G(\text{Sp}) \otimes \text{Mot}_R \rightarrow \text{Mack}_G(\text{Sp})$ .

Finally, in order to treat assembly-type maps for group rings, we will need to discuss the coBorel variant of the above.

**Construction 2.15** (coBorel Mackey functors). Let  $M$  be a  $\mathcal{C}$ -valued Mackey functor, for  $\mathcal{C}$  a presentable preadditive  $\infty$ -category. Note that  $\text{Mack}_G(\mathcal{C}) = \mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}}) \otimes \mathcal{C}$  is naturally tensored over  $\mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}})$ . Let  $y : \text{Burn}_G^{\text{eff}} \rightarrow \mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}})$  denote the Yoneda embedding.

We will say that  $M$  is *coBorel* if the natural map  $(M \otimes y(G))_{hG} \rightarrow M$  is an equivalence in  $\text{Mack}_G(\mathcal{C})$ . Any  $M \in \text{Mack}_G(\mathcal{C})$  admits its *coBorelification*  $M_{\text{coBor}} = (M \otimes y(G))_{hG}$ , which is the universal coBorel Mackey functor mapping to  $M$ ; this follows because the object  $y(G)_{hG}$  in  $\mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}})$  is an idempotent object for the tensor structure. (In fact, there is a natural symmetric monoidal functor from the  $\infty$ -category of  $G$ -anima to  $\mathcal{P}_\Sigma(\text{Burn}_G^{\text{eff}})$  which is the identity on  $G$ -sets; then  $y(G)_{hG}$  is the image of the  $G$ -space  $EG$ .) The coBorelification only depends on the underlying object of  $\text{Fun}(BG, \mathcal{C})$  (since  $M \otimes y(G)$  does), so we can also consider this as a functor

$$(-)_{\text{coBor}} : \text{Fun}(BG, \mathcal{C}) \rightarrow \text{Mack}_G(\mathcal{C}).$$

Dually as in Construction 2.10,  $(-)_{\text{coBor}}$  is fully faithful; the essential image consists of those  $M \in \text{Mack}_G(\mathcal{C})$  such that the dual comparison maps  $(M^{\{1\}})_{hH} \rightarrow M^H$  are equivalences for all  $H \subseteq G$ . By the universal property, we obtain a natural map  $(-)_{\text{coBor}} \rightarrow (-)^{\text{Bor}}$  which is an equivalence after forgetting to  $\text{Fun}(BG, \mathcal{C})$ . The functor  $(-)_{\text{coBor}} : \text{Fun}(BG, \mathcal{C}) \rightarrow \text{Mack}_G(\mathcal{C})$  is the left adjoint to the localization functor  $\text{Mack}_G(\mathcal{C}) \rightarrow \text{Fun}(BG, \mathcal{C})$  of Construction 2.10 (whose right adjoint was the Borelification).

We now describe the coBorelification of the  $\text{Cat}_\infty^{\text{perf}}$ -valued Mackey functors constructed above. This involves controlling homotopy orbits in  $\text{Cat}_\infty^{\text{perf}}$ . To begin with, we need some facts about limits and colimits of presentable, stable  $\infty$ -categories, cf. [Lur09, Sec. 5.5.3]. Let  $\mathcal{P}r_{\text{st}}^L$  denote the  $\infty$ -category of presentable, stable  $\infty$ -categories and left adjoint functors between them. Let  $\mathcal{P}r_{\text{st}}^R$  denote the  $\infty$ -category of presentable, stable  $\infty$ -categories and right adjoint functors between them, so we have an equivalence in  $\mathcal{P}r_{\text{st}}^L \simeq (\mathcal{P}r_{\text{st}}^R)^{\text{op}}$ . It follows that the underlying  $\infty$ -category of a colimit in  $\mathcal{P}r_{\text{st}}^L$  (of some diagram  $i \mapsto \mathcal{C}_i, i \in I$ ) can be calculated by taking the inverse limit along  $I^{\text{op}}$  of the right adjoints [Lur09, Th. 5.5.3.18]. Explicitly, via the Grothendieck construction, we can express the diagram  $I \rightarrow \mathcal{P}r_{\text{st}}^L$  in terms of a presentable fibration  $\tilde{\mathcal{C}} \rightarrow I$ , which is both a cartesian and a cocartesian fibration (cf. [Lur09, Def. 5.5.3.2]); the limit in  $\mathcal{P}r_{\text{st}}^L$  is given by the  $\infty$ -category of cocartesian sections, whereas the colimit is given by the  $\infty$ -category of cartesian sections.

We can use this to describe colimits in  $\text{Cat}_\infty^{\text{perf}}$ .

**Construction 2.16** (Colimits in  $\text{Cat}_\infty^{\text{perf}}$ ). Consider the functor  $\text{Ind} : \text{Cat}_\infty^{\text{perf}} \rightarrow \mathcal{P}r_{\text{st}}^L$  ([Lur09, Sec. 5.3.5]). This functor admits a right adjoint sending a presentable, stable  $\infty$ -category to its subcategory of compact objects; therefore,  $\text{Ind}$  commutes with all colimits. To compute a colimit in  $\text{Cat}_\infty^{\text{perf}}$  of an  $I$ -indexed diagram,  $i \mapsto \mathcal{A}_i$ , we therefore form the  $I^{\text{op}}$ -indexed diagram of  $\text{Ind}(\mathcal{A}_i)$  and the right adjoint functors, and then take the compact objects in the limit.

**Example 2.17** (Homotopy orbits in  $\text{Cat}_\infty^{\text{perf}}$ ). Let  $\mathcal{A} \in \text{Fun}(BG, \text{Cat}_\infty^{\text{perf}})$ . We claim that  $\mathcal{A}_{hG}$  is naturally described as the full subcategory of compact objects in  $\text{Ind}(\mathcal{A})^{hG}$ .

To see this, we first describe the homotopy orbits  $\text{Ind}(\mathcal{A})_{hG}$  in  $\mathcal{P}r_{\text{st}}^L$ . Form the presentable fibration over  $BG$  with fiber  $\text{Ind}(\mathcal{A})$ ; as above, the cocartesian sections give  $\text{Ind}(\mathcal{A})^{hG}$  (the homotopy limit in  $\mathcal{P}r_{\text{st}}^L$ ) while the cartesian sections give  $\text{Ind}(\mathcal{A})_{hG}$ . Since  $BG$  is an  $\infty$ -groupoid, the cartesian

and cocartesian sections are the same and we have a canonical identification  $\text{Ind}(\mathcal{A})_{hG} = \text{Ind}(\mathcal{A})^{hG}$  in  $\mathcal{P}r_{\text{st}}^L$ . The claim about  $\mathcal{A}_{hG}$  now follows by passage to compact objects.

Equivalently, we find that  $\mathcal{A}_{hG} \in \text{Cat}_\infty^{\text{perf}}$  is the full subcategory of  $\mathcal{A}^{hG}$  generated as a thick subcategory by the image of the functor  $\mathcal{A} \rightarrow \mathcal{A}^{hG}$  adjoint to the forgetful functor, since this image forms a set of compact generators of  $\text{Ind}(\mathcal{A})^{hG}$ . In particular, we have a natural fully faithful embedding  $\mathcal{A}_{hG} \subseteq \mathcal{A}^{hG}$  (which we verify below to be the norm map); it follows that for any diagram  $\mathcal{A}_j, j \in J$  in  $\text{Fun}(BG, \text{Cat}_\infty^{\text{perf}})$ , the natural map  $(\varprojlim_J \mathcal{A}_j)_{hG} \rightarrow \varprojlim_J (\mathcal{A}_j)_{hG}$  is fully faithful, since  $(-)^{hG}$  commutes with limits.

**Proposition 2.18.** *Let  $\mathcal{A} \in \text{Fun}(BG, \text{Cat}_\infty^{\text{perf}})$ . Then the natural map of  $\text{Cat}_\infty^{\text{perf}}$ -valued Mackey functors*

$$\mathcal{A}_{\text{coBor}} \rightarrow \mathcal{A}^{\text{Bor}},$$

*is fully faithful on  $H$ -fixed points for  $H \subseteq G$ . Moreover,  $(\mathcal{A}_{\text{coBor}})^H \subseteq (\mathcal{A}^{\text{Bor}})^H = \mathcal{A}^{hH}$  is the thick subcategory generated by the image of the biadjoint  $\mathcal{A} \rightarrow \mathcal{A}^{hH}$ .*

*Proof.* Let  $H \subseteq G$ . Then we claim that the natural map (i.e., the norm map) in  $\text{Cat}_\infty^{\text{perf}}$ ,

$$(2.3) \quad \mathcal{A}_{hH} = ((\mathcal{A} \otimes G)^{hH})_{hG} \rightarrow \mathcal{A}^{hH} = ((\mathcal{A} \otimes G)_{hG})^{hH}$$

is fully faithful; this map is the  $H$ -fixed points of  $\mathcal{A}_{\text{coBor}} \rightarrow \mathcal{A}^{\text{Bor}}$ . But this follows from the observation in the previous example: we saw that if  $\mathbf{T}: \text{Fun}(BG, \text{Cat}_\infty^{\text{perf}}) \rightarrow \text{Cat}_\infty^{\text{perf}}$  is the functor  $\mathcal{B} \mapsto \mathcal{B}_{hG}$ , then if  $\mathcal{B}$  has a  $G \times H$ -action, then  $\mathbf{T}(\mathcal{B}^{hH}) \rightarrow \mathbf{T}(\mathcal{B})^{hH}$  is fully faithful. It follows that  $\mathcal{A}_{\text{coBor}} \rightarrow \mathcal{A}^{\text{Bor}}$  is fully faithful on each fixed points. To see that its essential image is the subcategory as claimed, we observe that  $\mathcal{A} \rightarrow \mathcal{A}_{hH}$  has image generating the target as a thick subcategory.  $\square$

**Example 2.19** (Assembly maps). Let  $G$  act trivially on  $\text{Perf}(R)$ . Then we find that for each subgroup  $H \subseteq G$ , one has  $\text{Perf}(R)_{hH} \simeq \text{Perf}(R[H]) \subseteq \text{Fun}(BH, \text{Perf}(R))$  is the collection of compact objects in  $\text{Fun}(BH, \text{Mod}(R))$ . In particular, the  $\text{Cat}_\infty^{\text{perf}}$ -valued Mackey functor  $(\text{Perf}(R))_{\text{coBor}}$  is precisely the one that leads to the theory of assembly maps, cf. [RV18].

**Construction 2.20** (Equivariant algebraic  $K$ -theory, coBorel version). Combining the above, we obtain a functor

$$\mathcal{U}_{G, \text{coBor}}: \text{Fun}(BG, \text{Cat}_{R, \infty}^{\text{perf}}) \xrightarrow{(-)_{\text{coBor}}} \text{Mack}_G(\text{Cat}_{R, \infty}^{\text{perf}}) \rightarrow \text{Mack}_G(\text{Mot}_R),$$

which is the coBorel version of Construction 2.14. If  $\mathcal{A}$  is an algebra object of  $\text{Fun}(BG, \text{Cat}_{R, \infty}^{\text{perf}})$ , then  $\mathcal{A}_{\text{coBor}} = \mathcal{A}^{\text{Bor}} \otimes y(G)_{hG}$  is a module over  $\mathcal{A}^{\text{Bor}}$  in  $\text{Mack}_G(\text{Cat}_{R, \infty}^{\text{perf}})$ . Therefore,  $\mathcal{U}_{G, \text{coBor}}(\mathcal{A})$  is a module over  $\mathcal{U}_G(\mathcal{A})$  (which is an algebra object of  $\text{Mack}_G(\text{Mot}_R)$ ).

### 3. REVIEW OF NILPOTENCE

To prove our descent theorems, it will be convenient to use the language of nilpotence, as in [MNN17, MNN19]. For the material in section 5 and further, we also need the variant of  $\varepsilon$ -nilpotence, as used in [CMNN20].

**Definition 3.1** (Nilpotence). Given a finite group  $G$  and a family  $\mathcal{F}$  of subgroups, a  $G$ -spectrum  $X$  is said to be  $\mathcal{F}$ -nilpotent [MNN17, Def. 6.36] if it belongs to the thick subcategory (or equivalently the thick  $\otimes$ -ideal) generated by  $G$ -spectra which are induced from subgroups in  $\mathcal{F}$ . We say that a  $G$ -spectrum  $X$  is  $(\mathcal{F}, \varepsilon)$ -nilpotent if there exists a finite set of prime numbers  $\Sigma$  such that for every finite spectrum  $F$  whose localizations at primes in  $\Sigma$  are nontrivial, then  $X$  belongs to the thick

$\otimes$ -ideal of  $G$ -spectra generated by  $F$  and the  $\mathcal{F}$ -nilpotent  $G$ -spectra. (This somewhat involved definition in particular implies that every passage to  $T(n)$ -local coefficients makes  $X$   $\mathcal{F}$ -nilpotent, and this is an if and only if for the endomorphism  $G$ -ring spectrum of  $X$ . Compare [CMNN20, Sec. 2.3] with  $A = \prod_{H \in \mathcal{F}} F(G/H_+, \mathbb{S})$ .)

**Definition 3.2** ( $\mathcal{F}$ -completeness, cf. [MNN17, Sec. 6.1]). Given a finite group  $G$  and a family  $\mathcal{F}$  of subgroups, let  $E\mathcal{F}$  be the classifying space of the family  $\mathcal{F}$  as reviewed in [MNN17, Cons. 6.3]. We say that  $X \in \mathrm{Sp}_G$  is  $\mathcal{F}$ -complete if the map  $X \rightarrow F(E\mathcal{F}_+, X)$  is an equivalence, or equivalently if  $X$  is complete with respect to the algebra object  $A = \prod_{H \in \mathcal{F}} F(G/H_+, \mathbb{S})$ . This in particular implies that  $X^G \xrightarrow{\sim} \varprojlim_{G/H \in \mathcal{O}_{\mathcal{F}}(G)^{\mathrm{op}}} X^H$ .

**Proposition 3.3.** *Given an  $(\mathcal{F}, \varepsilon)$ -nilpotent  $G$ -spectrum  $X$ , the natural comparison maps*

$$(3.1) \quad \varinjlim_{G/H \in \mathcal{O}_{\mathcal{F}}(G)} X^H \rightarrow X^G \rightarrow \varprojlim_{G/H \in \mathcal{O}_{\mathcal{F}}(G)^{\mathrm{op}}} X^H,$$

*become equivalences after applying  $L_{T(n)}$  for any height  $n$  and implicit prime  $p$ ; moreover, the functor  $L_{T(n)}$  can be applied either inside or outside the homotopy limit on the right of (3.1), i.e., the map*

$$(3.2) \quad L_{T(n)} \left( \varinjlim_{G/H \in \mathcal{O}_{\mathcal{F}}(G)^{\mathrm{op}}} X^H \right) \rightarrow \varprojlim_{G/H \in \mathcal{O}_{\mathcal{F}}(G)^{\mathrm{op}}} L_{T(n)} X^H$$

*is an equivalence.*

*Proof.* Fix  $n$  and the implicit prime  $p$ . Given an  $\mathcal{F}$ -nilpotent  $G$ -spectrum  $X$ , the maps of (3.1) are equivalences, cf. [MNN19, Prop. 2.8] (in particular,  $X$  is  $\mathcal{F}$ -complete). If  $X$  is  $\mathcal{F}$ -nilpotent then the  $T(n)$ -localization of  $X$  remains  $\mathcal{F}$ -nilpotent by a thick subcategory argument, whence (3.2) is also an equivalence. Now the collection of  $G$ -spectra for which (3.1) and (3.2) are equivalences is a thick subcategory which contains the  $\mathcal{F}$ -nilpotent  $G$ -spectra and the  $G$ -spectra of the form  $F \otimes Y$  for  $Y \in \mathrm{Sp}_G$  and  $F$  a finite torsion spectrum of type (at  $p$ )  $\geq n+1$ ; this collection therefore contains the  $(\mathcal{F}, \varepsilon)$ -nilpotent  $G$ -spectra.  $\square$

We now discuss some criteria for nilpotence, starting with the case of the family  $\mathcal{F}$  consisting only of the trivial subgroup. Let  $EG$  denote the universal free  $G$ -space (or equivalently the classifying space of the family consisting of the trivial subgroup). Let  $\widetilde{EG}$  denote the cofiber of  $EG_+ \rightarrow \mathbb{S}$  in  $\mathrm{Sp}_G$ ; it is naturally an algebra object in  $\mathrm{Sp}_G$ , as the smashing localization of  $\mathbb{S}$  in  $\mathrm{Sp}_G$  away from the localizing  $\otimes$ -ideal generated by the free  $G$ -spectra, cf. [MNN17, Prop. 6.5]. In the following, let  $R$  be an associative algebra in  $\mathrm{Sp}_G$ . Then we consider the associative algebra  $(R \otimes \widetilde{EG})^G$  in  $\mathrm{Sp}$ . Since  $EG = (G)_{hG}$  (in the  $\infty$ -category of  $G$ -anima), we have a cofiber sequence

$$R_{hG} \rightarrow R^G \rightarrow (R \otimes \widetilde{EG})^G,$$

where  $R_{hG} = R_{hG}^{\{1\}}$  and  $R_{hG} \rightarrow R^G$  is the transfer for the  $G$ -spectrum  $R$ .

**Proposition 3.4** (Criteria for  $\mathcal{F}$ -nilpotence). *An associative algebra  $R$  in  $\mathrm{Sp}_G$  is  $\mathcal{F}$ -nilpotent (for  $\mathcal{F} = \{(1)\}$ ) if and only if  $(R \otimes \widetilde{EG})^G$  is contractible.*

*Proof.* This follows from [MNN17, Th. 4.19], since  $R \otimes \widetilde{EG}$  is the localization of  $R$  (in  $\mathrm{Sp}_G$ ) away from the localizing  $\otimes$ -ideal generated by the free  $G$ -spectra.  $\square$

**Proposition 3.5** (Criterion for  $(\mathcal{T}, \varepsilon)$ -nilpotence). *Let  $R$  be an associative algebra in  $\mathrm{Sp}_G$ . Suppose that  $(R \otimes \widetilde{EG})^G$  has trivial  $T(n)$ -localization for  $n \geq 1$  and all primes  $p$  and trivial rationalization. Then  $R$  is  $(\mathcal{T}, \varepsilon)$ -nilpotent.*

*Proof.* Our assumptions imply that there exists a finite set of prime numbers  $\Sigma$  such that for every finite complex  $F$  with nontrivial localizations at primes in  $\Sigma$ , the associative algebra spectrum  $(R \otimes \widetilde{EG})^G$  belongs to the thick  $\otimes$ -ideal generated by  $F$ , cf. [CMNN20, Prop. 2.7]. Thus, the  $G$ -spectrum  $R \otimes \widetilde{EG}$  belongs to the thick  $\otimes$ -ideal generated by  $F$ ; here we use the natural adjunction  $(i_*, (-)^G) : \mathrm{Sp} \rightleftarrows \mathrm{Sp}_G$ . Therefore,  $R$  belongs to the localizing  $\otimes$ -ideal generated by  $F$  and by the  $G$ -spectrum  $G_+$  (using the fiber sequence  $R \otimes EG_+ \rightarrow R \rightarrow R \otimes \widetilde{EG}$ ), and hence it belongs to the similarly generated thick  $\otimes$ -ideal by [MNN17, Th. 4.19] (which we apply to  $\mathcal{C} = \mathrm{Sp}_G$  and the dualizable associative algebra object  $F \otimes \mathbb{D}F \times \mathbb{D}G_+ \in \mathrm{Sp}_G$ , for  $\mathbb{D}$  the categorical dual) again. The result follows.  $\square$

We next include three general results about  $\mathcal{F}$ -nilpotence for an arbitrary family. The first result states that when  $R$  is rational (i.e,  $T(0)$ -local),  $\mathcal{F}$ -nilpotence is a purely algebraic condition on  $\pi_0$ ; the second (which will only be used with  $\mathcal{F} = \mathcal{T}$ ) gives a generalization of this to  $T(n)$ -local objects. The third result allows us to transfer rational  $\mathcal{F}$ -nilpotence to  $(\mathcal{F}, \varepsilon)$ -nilpotence in the presence of an  $\mathbb{E}_\infty$ -structure, using the May nilpotence conjecture [MNN15]. For this, we let  $E\mathcal{F}$  denote the universal  $G$ -space for the family  $\mathcal{F}$  and  $\widetilde{E\mathcal{F}}$  the cofiber of the map  $E\mathcal{F}_+ \rightarrow \mathbb{S}$  in  $\mathrm{Sp}_G$ , so  $\widetilde{E\mathcal{F}}$  is the localization of  $\mathbb{S}$  away from the localizing  $\otimes$ -ideal generated by the  $\{G/H_+, H \in \mathcal{F}\}$ .

**Proposition 3.6** ([MNN19, Prop. 4.11]). *Suppose that the associative algebra  $R$  in  $\mathrm{Sp}_G$  is rational. Then  $R$  is  $\mathcal{F}$ -nilpotent if and only if the induction map  $\bigoplus_{H \in \mathcal{F}} \pi_0(R^H) \rightarrow \pi_0(R^G)$  is surjective, or equivalently has image containing the unit.*  $\square$

Let  $L_{T(i)}\mathrm{Sp}_G$  denote the full subcategory of  $\mathrm{Sp}_G$  spanned by the  $T(i)$ -local objects, i.e., those for which the  $H$ -fixed points for each subgroup  $H \subseteq G$  are  $T(i)$ -local spectra (at the implicit prime  $p$ ); this equivalence follows because the orbits form a set of compact generators for  $\mathrm{Sp}_G$ . We next give a criterion for  $\mathcal{F}$ -completeness in  $L_{T(i)}\mathrm{Sp}_G$ . This will use the vanishing of the Tate constructions in  $L_{T(i)}\mathrm{Sp}$ , due to [Kuh04], in the following equivalent form:

**Lemma 3.7.** *If  $\mathcal{C}$  is any presentable stable  $\infty$ -category and  $X \in \mathrm{Fun}(BH, \mathcal{C})$  is an  $H$ -object in  $\mathcal{C}$  for some finite group  $H$ , then  $X_{hH} \otimes T(i) \in \mathcal{C}$  belongs to the thick subcategory generated by  $X \otimes T(i)$ .*

*Proof.* An equivalent form of the telescopic Tate vanishing is that, as an object of  $\mathrm{Fun}(BH, \mathrm{Sp})$  (with trivial  $H$ -action),  $T(i)$  belongs to the thick subcategory of  $\mathrm{Fun}(BH, \mathrm{Sp})$  generated by  $T(i) \otimes H_+$ , cf. [MNN19, Prop. 5.31]. From this, the result easily follows, since  $(X \otimes T(i) \otimes H_+)_H = X \otimes T(i)$ .  $\square$

In the next result, we use the notation  $\Phi^H$  for the  $H$ -geometric fixed points functor on  $\mathrm{Sp}_G$ .

**Proposition 3.8** (Properties of  $T(i)$ -local  $G$ -spectra). *Let  $G$  be a finite group,  $\mathcal{F}$  a family of subgroups and  $i \geq 0$ . Let  $M \in L_{T(i)}\mathrm{Sp}_G$ . Then the following are equivalent:*

- (1)  $M$  is  $\mathcal{F}$ -complete.
- (2) For every finite type  $i$  complex  $F$ , the  $G$ -spectrum  $M \otimes F$  is  $\mathcal{F}$ -nilpotent.
- (3) We have  $L_{T(i)}(\Phi^H M) = 0$  for  $H \notin \mathcal{F}$ .

*Proof.* We first claim that for each family  $\mathcal{G}$  of subgroups of  $G$ , the  $G$ -spectrum  $E\mathcal{G}_+ \otimes T(i)$  is  $\mathcal{G}$ -nilpotent; we prove this by induction on  $\mathcal{G}$ . To start with, when  $\mathcal{G} = \mathcal{T}$ , then  $E\mathcal{G}_+ = (G_+)_h G$ ;

this uses the  $G$ -action on the  $G$ -space  $G$  (by right multiplication, so in the category of  $G$ -anima). It follows from Lemma 3.7 that  $E\mathcal{F}_+ \otimes T(i)$  is  $\mathcal{F}$ -nilpotent. Now we treat the inductive step. Fix a proper family  $\mathcal{G}$  such that  $E\mathcal{G}_+ \otimes T(i)$  is  $\mathcal{G}$ -nilpotent. Choose a subgroup  $H \subseteq G$  which is minimal for the property of not belonging to  $\mathcal{G}$ ; one forms a new family  $\mathcal{G}'$  obtained by adding the conjugates of  $H$  to  $\mathcal{G}$ . Then there is a cofiber sequence of pointed  $G$ -anima

$$E\mathcal{G}_+ \rightarrow E\mathcal{G}'_+ \rightarrow E\mathcal{G}'_+ \wedge \tilde{E}\mathcal{G} = (G/H_+)_{hW_H} \wedge \tilde{E}\mathcal{G},$$

where  $W_H$  is the Weyl group of  $H \subseteq G$ . Using this, the inductive assumption, and Lemma 3.7, the inductive step follows and the claim is proved.

Now we prove the result. Suppose  $M \in L_{T(i)}\mathrm{Sp}_G$  is  $\mathcal{F}$ -complete. By the thick subcategory theorem, condition (2) is independent of the choice of  $F$  and we choose a finite type  $i$  complex  $F$  such that  $F$  admits the structure of a ring spectrum; given a  $v_i$ -self map  $v$  of  $F$  which we may assume central, we can take  $T(i) = F[v^{-1}]$ . Then  $M \otimes F$  admits the structure of a  $T(i)$ -module, since  $M$  is  $T(i)$ -local. It follows that the  $\mathcal{F}$ -cellularization (cf. [MNN17, Cons. 3.2, Prop. 6.5] for an account, where cellularization is called acyclization.)  $E\mathcal{F}_+ \otimes M \otimes F$  of  $M \otimes F$  belongs to the thick  $\otimes$ -ideal of  $\mathrm{Sp}_G$  generated by  $E\mathcal{F}_+ \otimes T(i)$  and is therefore  $\mathcal{F}$ -nilpotent, by our initial claim. Consequently, the  $\mathcal{F}$ -completion  $F(E\mathcal{F}_+, E\mathcal{F}_+ \otimes M \otimes F)$  (which is  $M \otimes F$  again since this is  $\mathcal{F}$ -complete) is also  $\mathcal{F}$ -nilpotent (here we implicitly use that the  $\mathcal{F}$ -completion of a  $G$ -spectrum depends only on its  $\mathcal{F}$ -cellularization). Thus, (1) implies (2). Clearly (2) implies (3), again by smashing with  $F$ . If (3) holds, then  $M \otimes F = M \otimes T(i)$  has trivial geometric fixed points  $\Phi^H$  for  $H \notin \mathcal{F}$ , whence  $M \otimes F = E\mathcal{F}_+ \otimes M \otimes F = E\mathcal{F}_+ \otimes M \otimes T(i)$ , which we have seen is  $\mathcal{F}$ -nilpotent. Thus, (3) implies (2). Finally, (2) implies (1) by writing  $M$  (which is assumed  $T(i)$ -local) as an inverse limit of  $M \otimes F$  for suitable finite type  $i$  complexes  $F$  (e.g., generalized Moore spectra).  $\square$

**Corollary 3.9.** *Let  $R \in L_{T(i)}\mathrm{Sp}_G$  be an algebra object which is  $\mathcal{F}$ -complete. Then any  $R$ -module  $M \in L_{T(i)}\mathrm{Sp}_G$  is  $\mathcal{F}$ -complete.*

*Proof.* This follows from item (3) of Proposition 3.8, since  $\Phi^H$  is a symmetric monoidal functor.  $\square$

**Proposition 3.10.** *Let  $R \in L_{T(i)}\mathrm{Sp}_G$  be an  $\mathbb{E}_1$ -algebra, and let  $M \in L_{T(i)}\mathrm{Sp}_G$  be an  $R$ -module. Then the map  $M^G \rightarrow M^{hG}$  admits a section as  $R^G$ -modules. Similarly, the map  $L_{T(i)}(M_{hG}) \rightarrow M^G$  admits a section as  $R^G$ -modules. If  $G = C_p$ , then  $M$  is Borel-complete if and only if either of these maps is an equivalence.*

*Proof.* All of this follows because the composite map  $L_{T(i)}(M_{hG}) \rightarrow M^G \rightarrow M^{hG}$  is the norm, which is an equivalence since Tate constructions vanish in  $T(i)$ -local homotopy [Kuh04].  $\square$

**Proposition 3.11.** *Let  $R$  be an  $\mathbb{E}_\infty$ -algebra in the symmetric monoidal  $\infty$ -category  $\mathrm{Sp}_G$ . Suppose that the rationalization  $R_{\mathbb{Q}}$  is  $\mathcal{F}$ -nilpotent. Then  $R$  is  $(\mathcal{F}, \varepsilon)$ -nilpotent.*

*Proof.* By assumption, the  $\mathbb{E}_\infty$ -ring  $(R \otimes \widetilde{E\mathcal{F}})_{\mathbb{Q}}^G$  is contractible. Therefore, by the main result of [MNN15], the  $\mathbb{E}_\infty$ -ring  $(R \otimes \widetilde{E\mathcal{F}})^G$  is annihilated by  $L_{T(n)}$  for all  $n$  and implicit primes  $p$ . In particular, by [CMNN20, Prop. 2.7], there exists a finite set  $\Sigma$  of primes such that  $(R \otimes \widetilde{E\mathcal{F}})^G$  belongs to the thick  $\otimes$ -ideal of spectra generated by any finite spectrum  $F$  such that  $F_{(p)} \neq 0$  for  $p \in \Sigma$ . This implies that  $R \otimes \widetilde{E\mathcal{F}}$  belongs to the thick  $\otimes$ -ideal of  $\mathrm{Sp}_G$  generated by  $F$ , whence  $R$  belongs to the localizing  $\otimes$ -ideal generated by  $F$  and  $\{G/H_+, H \in \mathcal{F}\}$  in view of the cofiber sequence  $R \otimes E\mathcal{F}_+ \rightarrow R \rightarrow R \otimes \widetilde{E\mathcal{F}}$ . Finally, [MNN17, Th. 4.19] again implies that  $R$  belongs to the thick  $\otimes$ -ideal generated by  $F$  and  $\{G/H_+, H \in \mathcal{F}\}$  in  $\mathrm{Sp}_G$ , as desired.  $\square$

**Definition 3.12.** Let  $\mathcal{C}$  be a presentably symmetric monoidal stable  $\infty$ -category. We say that an object of  $\text{Mack}_G(\mathcal{C}) = \text{Mack}_G(\text{Sp}) \otimes \mathcal{C} \simeq \text{Sp}_G \otimes \mathcal{C}$  is  $\mathcal{F}$ -nilpotent (resp.  $(\mathcal{F}, \varepsilon)$ -nilpotent) if it belongs to the thick  $\otimes$ -ideal of  $\text{Mack}_G(\mathcal{C})$  generated by the  $\mathcal{F}$ -nilpotent (resp.  $(\mathcal{F}, \varepsilon)$ -nilpotent) objects in  $\text{Sp}_G$ .

It follows that for any cocontinuous functor  $\mathcal{C} \rightarrow \text{Sp}$ , the induced functor  $\text{Mack}_G(\mathcal{C}) \rightarrow \text{Mack}_G(\text{Sp}) \simeq \text{Sp}_G$  carries  $\mathcal{F}$ -nilpotent (resp.  $(\mathcal{F}, \varepsilon)$ -nilpotent) objects in the source to  $\mathcal{F}$ -nilpotent (resp.  $(\mathcal{F}, \varepsilon)$ -nilpotent) objects in the target. Using the adjunction  $\text{Sp} \rightleftarrows \mathcal{C}$  where the symmetric monoidal left adjoint carries  $\mathbb{S}$  to the unit, we obtain the next result.

**Proposition 3.13.** *Let  $\mathcal{C}$  be a presentably symmetric monoidal stable  $\infty$ -category, and suppose  $\mathbf{1} \in \mathcal{C}$  is compact. Let  $A$  be an object of  $\text{Mack}_G(\mathcal{C})$  which admits a unital multiplication in the homotopy category. Then  $A$  is  $\mathcal{F}$ -nilpotent (resp.  $(\mathcal{F}, \varepsilon)$ -nilpotent) in  $\text{Mack}_G(\mathcal{C})$  if and only if it is carried to an  $\mathcal{F}$ -nilpotent (resp.  $(\mathcal{F}, \varepsilon)$ -nilpotent) object of  $\text{Mack}_G(\text{Sp})$  under the functor  $\text{Hom}_{\mathcal{C}}(\mathbf{1}, -) : \text{Mack}_G(\mathcal{C}) \rightarrow \text{Mack}_G(\text{Sp})$ .*

*Proof.* Let  $i^* : \text{Mack}_G(\text{Sp}) \rightarrow \text{Mack}_G(\mathcal{C})$  denote the canonical symmetric monoidal functor (obtained from  $\text{Sp} \rightarrow \mathcal{C}$ ), and let  $i_* : \text{Mack}_G(\mathcal{C}) \rightarrow \text{Mack}_G(\text{Sp})$  denote its right adjoint (which is equally obtained by the cocontinuous functor  $\text{Hom}_{\mathcal{C}}(\mathbf{1}, -) : \mathcal{C} \rightarrow \text{Sp}$ ). By assumption,  $i_* A$  is  $\mathcal{F}$ -nilpotent (resp.  $(\mathcal{F}, \varepsilon)$ -nilpotent); thus, so is  $i^* i_* A$  and hence so is  $A$  since our assumption of a unital multiplication implies that  $A$  belongs to the thick  $\otimes$ -ideal generated by  $i^* i_* A$ .  $\square$

**Example 3.14.** Let  $\mathcal{A} \in \text{Fun}(BG, \text{Cat}_{R, \infty}^{\text{perf}})$ . Consider  $\mathcal{U}_G(\mathcal{A}) \in \text{Mack}_G(\text{Mot}_R)$ , a Mackey functor valued in  $\text{Mot}_R$ . Suppose  $\mathcal{A}$  is an algebra object of  $\text{Cat}_{R, \infty}^{\text{perf}}$  (i.e., is an  $R$ -linear monoidal stable  $\infty$ -category). Then  $\mathcal{U}_G(\mathcal{A})$  is  $\mathcal{F}$ -nilpotent (resp.  $(\mathcal{F}, \varepsilon)$ -nilpotent) if and only if the  $G$ -spectrum  $K_G(\mathcal{A})$  is  $\mathcal{F}$ -nilpotent (resp.  $(\mathcal{F}, \varepsilon)$ -nilpotent), using the representability of  $K$ -theory.

#### 4. DESCENT FOR $p$ -GROUPS; PROOF OF THEOREM A AND THEOREM B

In this section, we give the proof of Theorems A and B via Theorem 1.3. We start with the following general reduction.

**Proposition 4.1.** *Let  $R$  be an  $\mathbb{E}_2$ -ring, and let  $j \geq 0$ . Then the following are equivalent:*

- (1)  $L_{T(j)}(\Phi^{C_p} K_{C_p}(R)) = 0$ .
- (2) *The  $C_p$ -spectrum  $L_{T(j)} K_{C_p}(R)$  is Borel-complete.*
- (3) *For every  $R$ -linear idempotent-complete stable  $\infty$ -category  $\mathcal{C}$  equipped with an ( $R$ -linear) action of a finite  $p$ -group  $G$ , and every additive invariant  $E$  with values in  $T(j)$ -local spectra, we have  $E(\mathcal{C}^{hG}) \xrightarrow{\sim} E(\mathcal{C})^{hG}$ .*
- (4) *For every  $R$ -linear idempotent-complete stable  $\infty$ -category  $\mathcal{C}$  equipped with an ( $R$ -linear) action of a finite  $p$ -group  $G$ , and every additive invariant  $E$ , we have*

(4.1)  $L_{T(j)}(E(\mathcal{C})_{hG}) \xrightarrow{\sim} L_{T(j)} E(\mathcal{C}_{hG}) \xrightarrow{\sim} L_{T(j)} E(\mathcal{C}^{hG}) \xrightarrow{\sim} L_{T(j)}(E(\mathcal{C})^{hG}) \xrightarrow{\sim} (L_{T(j)} E(\mathcal{C}))^{hG}$ .

*Proof.* (1) and (2) are equivalent by Proposition 3.8; (2) is the special case of (3) where  $E = L_{T(j)} K(-)$  and  $G = C_p$  acts trivially on  $\text{Perf}(R)$ ; and (3) is a special case of (4). Thus let us show (1) implies (3) and (3) implies (4).

First, we show (1) implies (3). Since every  $p$ -group has a composition series with successive quotients cyclic of order  $p$ , we can use dévissage to reduce to the case when  $G = C_p$ . Let  $E_{C_p}(\mathcal{C}) = E(\mathcal{C}^{\text{Bor}})$  denote the  $C_p$ -spectrum obtained by applying  $E$  to the  $C_p$ -Mackey functor  $\mathcal{C}^{\text{Bor}}$  in  $\text{Cat}_{\infty}^{\text{perf}}$ . By construction,  $E_{C_p}(\mathcal{C})$  is a module in  $C_p$ -spectra over  $K_{C_p}(R)$ . In fact, this follows because

$\mathcal{C}^{\text{Bor}}$  is a module over  $\text{Perf}(R)^{\text{Bor}}$ , and  $\mathcal{U}_{C_p}(\mathcal{C}) \in \text{Mack}_{C_p}(\text{Mot})$  is therefore a module over  $K_{C_p}(R)$ . Since  $E_{C_p}(\mathcal{C})$  is  $T(j)$ -local, we find that  $E_{C_p}(\mathcal{C})$  is a module over  $L_{T(j)}K_{C_p}(R)$  and is therefore Borel-complete by Corollary 3.9.

Finally, we show (3) implies (4). To this end, we will produce a sequence of  $G$ -spectra which we will show to be Borel-complete, and which on fixed points realizes the maps in (4.1). In fact, consider the  $G$ -Mackey functors  $\mathcal{C}^{\text{Bor}}, \mathcal{C}_{\text{coBor}}$  with values in  $\text{Cat}_{\infty}^{\text{perf}}$ ; we have a natural map  $\mathcal{C}_{\text{coBor}} \rightarrow \mathcal{C}^{\text{Bor}}$ . Both are modules over  $\text{Perf}(R)^{\text{Bor}}$  in  $\text{Mack}_G(\text{Cat}_{\infty}^{\text{perf}})$ . Applying  $E$  and then  $L_{T(j)}$ , we obtain a sequence of  $G$ -spectra

$$(4.2) \quad E(\mathcal{C}_{\text{coBor}})_{\text{coBor}} \rightarrow E(\mathcal{C}_{\text{coBor}}) \rightarrow E(\mathcal{C}^{\text{Bor}}) \rightarrow E(\mathcal{C}^{\text{Bor}})^{\text{Bor}} \rightarrow (L_{T(j)}E(\mathcal{C}^{\text{Bor}}))^{\text{Bor}}.$$

Note that all of these  $G$ -spectra are modules over  $K_G(R)$ . Therefore, the  $T(j)$ -localization of the  $G$ -spectra in (4.2) are modules over the  $G$ -ring spectrum  $L_{T(j)}K_G(R)$ , which is Borel-complete by (3) (applied to the trivial  $G$ -action on  $\text{Perf}(R)$ ). Consequently, in view of Corollary 3.9, the  $T(j)$ -localizations of the  $G$ -spectra in (4.2) are all Borel-complete. Finally, all the maps of  $G$ -spectra in (4.2) induce  $T(j)$ -equivalences on underlying spectra; consequently, the  $T(j)$ -localizations induce equivalences on  $G$ -fixed points, whence the equivalences in (4).  $\square$

For future reference, we recall also the following lemma.

**Lemma 4.2.** *Let  $E_i$  denote Morava  $E$ -theory of height  $i$ . For any  $T(i)$ -local  $\mathbb{E}_{\infty}$ -algebra  $R$  over  $E_i$ , we have that  $E_i^{hC_p} \otimes_{E_i} R \xrightarrow{\sim} R^{hC_p}$ , and this is a free  $R$ -module of rank  $p^i$ . Here we always have  $C_p$  acting trivially, and the relative tensor product is algebraic, not (a priori)  $T(i)$ -localized.*

*Proof.* As  $E_i$  is complex oriented and even periodic, and the  $p$ -series  $[p](t) \in (\pi_0 E_i)[[t]]$  of its associated formal group law is a nonzerodivisor, the Gysin sequence for  $S^1 \rightarrow BC_p \rightarrow BS^1$  shows that  $E_i^{hC_p} = E_i^{BC_p}$  is also even periodic, and  $\pi_0 E_i^{hC_p} = (\pi_0 E_i)[[t]]/[p](t)$ . Since the formal group has height  $i$ , this is a free module of rank  $p^i$  over  $\pi_0 E_i$ . Since  $E_i$  is  $T(i)$ -local, Kuhn's Tate vanishing result from [Kuh04] (or the earlier [GS96]) shows that this implies that  $L_{T(i)}((E_i)_{hC_p})$  is free of rank  $p^i$ . Mapping out to an arbitrary  $T(i)$ -local  $E_i$ -module  $M$ , we deduce that

$$M^{hC_p} = \text{Hom}_{E_i}(L_{T(i)}((E_i)_{hC_p}), M) = \text{Hom}_{E_i}(L_{T(i)}((E_i)_{hC_p}), E_i) \otimes_{E_i} M = E_i^{hC_p} \otimes_{E_i} M,$$

implying all the desired claims.  $\square$

**4.1. The case of ordinary rings.** For the proof of Theorem 1.3, we will need to give an independent treatment of a special case: namely, the case where  $R$  is an ordinary ring. Note that for  $n = 1$ ,  $T(1)$  and  $K(1)$ -local homotopy coincide [Mah81, Mil81], and for all  $n$  we have  $L_{T(n)}A = 0$  if and only if  $L_{K(n)}A = 0$  whenever  $A$  is a ring spectrum, thanks to the nilpotence theorem; see [LMMT20, Lem. 2.3]. This will let us consider  $K(n)$  instead of  $T(n)$ .

What we will really need for the main proof is the following.

**Lemma 4.3.** *Let  $R$  be a commutative ring. Then the assembly map*

$$K(R)_{hC_p} \rightarrow K(R[C_p])$$

*is a  $T(n)$ -equivalence for all  $n \geq 1$ .*

The hypothesis that  $R$  should be commutative is a posteriori not necessary, by Theorem 4.12, but it will be used in the proof here.

*Proof.* In fact, we will show that the assembly map is a  $T(n)$ -local equivalence for  $n = 1$ . By Mitchell's theorem (Theorem 1.1),  $L_{T(n)}K(\mathbb{Z}) = 0$  for  $n \geq 2$ , which implies the statement also holds when  $n \geq 2$  since both sides vanish.

Thus, assume  $n = 1$ . Since  $K(A) \rightarrow K(A[1/p])$  is a  $K(1)$ -equivalence for all rings  $A$  (see [BCM20, LMMT20, Mat21] for three different proofs), we can reduce to the case where  $R$  is a  $\mathbb{Z}[1/p]$ -algebra. By transfer along the degree  $p - 1$  extension  $\mathbb{Z}[1/p] \rightarrow \mathbb{Z}[1/p, \zeta_p]$ , we can even assume  $R$  is a  $\mathbb{Z}[1/p, \zeta_p]$ -algebra. The claim is equivalent to the assertion that the  $C_p$ -spectrum  $K(\text{Perf}(R)_{\text{coBor}})$  (obtained by applying  $K(-)$  to the  $C_p$ -Mackey functor  $\text{Perf}(R)_{\text{coBor}}$ ) has the property that  $(\text{KU} \otimes K(\text{Perf}(R)_{\text{coBor}}))/p$  is Borel-complete. Equivalently, we need to show that the map

$$(4.3) \quad (\text{KU} \otimes K(R))_{hC_p} \rightarrow \text{KU} \otimes K(R[C_p])$$

induces an equivalence upon  $p$ -completion.

The  $p$ -completion of the map (4.3) admits a retraction as  $(\text{KU} \otimes K(R))_{\widehat{p}}$ -modules by Proposition 3.10. (For this argument, cf. [Mal17].) We will show that the  $p$ -completions of both sides are free  $(\text{KU} \otimes K(R))_{\widehat{p}}$ -modules of rank  $p$ , which will therefore imply the claim. Indeed, the fact that the  $p$ -completion of  $(\text{KU} \otimes K(R))_{hC_p}$  is free of rank  $p$  follows from Lemma 4.2. Moreover, the fact that  $(\text{KU} \otimes K(R[C_p]))_{\widehat{p}}$  is free of rank  $p$  follows because the standard idempotents in the group ring give  $R[C_p] \simeq R^{\times p}$  as  $R$ -algebras. This proves the claim and hence the lemma.  $\square$

**Remark 4.4** (A proof of Mitchell's theorem). The above methods also reprove the vanishing  $L_{T(n)}K(\mathbb{Z}) = 0$  for  $n \geq 2$  using similar methods; it suffices to prove  $L_{K(n)}K(\mathbb{Z}) = 0$  for such  $n$ . By Quillen's localization sequence  $K(\mathbb{F}_p) \rightarrow K(\mathbb{Z}) \rightarrow K(\mathbb{Z}[1/p])$  and Quillen's calculation  $K(\mathbb{F}_p)_{(p)} = \mathbb{Z}_{(p)}$ , it suffices to show  $L_{K(n)}K(\mathbb{Z}[1/p]) = 0$ , or again by a transfer argument  $L_{K(n)}K(\mathbb{Z}[1/p, \zeta_p]) = 0$ . We now run a similar argument as above. Let  $R = \mathbb{Z}[1/p, \zeta_p]$ . The map

$$(4.4) \quad L_{K(n)}(E_n \otimes K(R)_{hC_p}) \rightarrow L_{K(n)}(E_n \otimes K(R[C_p]))$$

admits a retraction of  $L_{K(n)}(E_n \otimes K(R))$ -modules by Proposition 3.10. We showed in the proof of Lemma 4.3 that  $K(R[C_p])$  is a free  $K(R)$ -module of rank  $p$ , whence the right-hand-side of (4.4) is a free  $L_{K(n)}(E_n \otimes K(R))$ -module of rank  $p$ . By Lemma 4.2, the left-hand-side of (4.4) is a free module of rank  $p^n$ . In particular, we obtain a split injection from a free module of rank  $p^n$  over  $L_{K(n)}(E_n \otimes K(R))$  to a free module of rank  $p$ . As  $p^n > p$ , this forces  $L_{K(n)}(E_n \otimes K(R)) = 0$ , whence the claim.

**Remark 4.5.** Suppose  $R$  is an associative  $\mathbb{Z}/p^n$ -algebra for some  $n \geq 1$ . Then the assembly map  $K(R)_{hC_p} \rightarrow K(R[C_p])$  is a  $T(0)$ -equivalence as well. In fact, after  $T(0)$ -localization this map is simply the map  $K(R)[1/p] \rightarrow K(R[C_p])[1/p]$ . This map admits a section given by the augmentation  $R[C_p] \rightarrow R$  which is surjective with nilpotent kernel, and induces an equivalence on  $K(-)[1/p]$  by [LT19, Th. 2.25]. We thank the referee for this remark.

**4.2. Extending to higher heights.** In this subsection, we prove Theorem B and Theorem A together and in full generality via an inductive argument on the height.

To obtain the desired bounds on the chromatic complexity on  $K(R)$ , we will use the following converse to chromatic blueshift.

**Theorem 4.6** (Cf. [Hah16, Prop. 4.7] and [BSY22, Th. 9.8]). *Let  $A$  be an  $\mathbb{E}_\infty$ -ring and let  $i \geq 0$ . Suppose that  $L_{T(i)}(A^{tC_p}) = 0$ . Then  $L_{T(j)}A = 0$  for  $j \geq i + 1$ .*

For the convenience of the reader, we include a deduction of Theorem 4.6 from the main theorem of [Hah16].

**Lemma 4.7.** *The  $\pi_0(E_{i+1})$ -algebra  $\pi_0(E_{i+1}^{tC_p})$  has the property that  $\pi_0(E_{i+1}^{tC_p})/(p, v_1, \dots, v_{i-1})$  is faithfully flat over the field  $\pi_0(E_{i+1})/(p, v_1, \dots, v_{i-1})[v_i^{-1}] = k((v_i))$ .*

*Proof.* Note that  $\pi_0(E_{i+1}^{tC_p})/(p, v_1, \dots, v_{i-1})$  is nonzero and has  $v_i$  invertible, since  $E_{i+1}^{tC_p}$  has trivial  $K(i+1)$ -localization but nontrivial  $K(i)$ -localization by [GS96, HS96]. Therefore, the result follows.  $\square$

*Proof.* The vanishing results for the telescopic localizations are equivalent to those for the analogous  $K(j)$ -localizations, i.e., it suffices to show that  $L_{K(j)}A = 0$  for  $j \geq i+1$  (cf. [LMMT20, Lem. 2.3]). By [Hah16, Th. 1.1], it suffices to show that  $L_{K(i+1)}A = 0$ . Therefore, without loss of generality, we may replace  $A$  with  $L_{K(i+1)}(E_{i+1} \otimes A)$  and assume that  $A$  is a  $K(i+1)$ -local  $\mathbb{E}_\infty$ - $E_{i+1}$ -algebra such that  $L_{K(i)}(A^{tC_p}) = 0$ ; we then need to show that  $A = 0$ .

Now we have

$$(4.5) \quad A^{tC_p} = A \otimes_{E_{i+1}} E_{i+1}^{tC_p},$$

by Lemma 4.2. Furthermore,  $\pi_*(E_{i+1}^{tC_p})$  is a localization of  $\pi_*(E_{i+1}^{hC_p})$  and is therefore flat over  $\pi_*(E_{i+1})$ , whence it follows from Lemma 4.7 (and the Künneth spectral sequence) that the map

$$\pi_0(A/(p, v_1, \dots, v_{i-1})[v_i^{-1}]) \rightarrow \pi_0(A^{tC_p}/(p, v_1, \dots, v_{i-1})[v_i^{-1}])$$

is faithfully flat. Our assumption is that the target vanishes since  $A^{tC_p}$  is  $L_{i-1}$ -local. Therefore, the source vanishes and we find that  $L_{K(i)}A = 0$ , whence  $L_{K(i+1)}A = 0$  by [Hah16, Th. 1.1]; thus  $A = 0$  since it is  $K(i+1)$ -local, as desired.  $\square$

We will also need to use some of the results of [LMMT20] on the chromatic behavior of algebraic  $K$ -theory.

**Theorem 4.8** ([LMMT20, Th. 3.8]). *Let  $A$  be an  $\mathbb{E}_1$ -ring and let  $n \geq 1$ . Then the map  $K(A) \rightarrow K(L_n^{p,f}A)$  is a  $T(i)$ -local equivalence for  $1 \leq i \leq n$ .*

Now we get into the proofs of Theorem B and Theorem A; the following lemma, equivalent to Theorem 1.3 from the introduction (thanks to Proposition 4.1), will be the key inductive step.

**Lemma 4.9.** *Let  $R$  be an  $\mathbb{E}_\infty$ -ring and let  $i \geq 1$ . For the following conditions, we have the implications  $(1) \Rightarrow (2) \Rightarrow (3)$ :*

- (1)  $L_{T(i)}R = 0$  and  $L_{T(i)}K(R^{tC_p}) = 0$ .
- (2)  $L_{T(i)}\Phi^{C_p}(K_{C_p}(R)) = 0$ .
- (3)  $L_{T(j)}K(R) = 0$  for all  $j \geq i+1$ .

*Proof.* In the following proof, we use the following notation: given a  $C_p$ -Mackey functor  $\mathcal{M}$  with values in  $\text{Cat}_\infty^{\text{perf}}$ , we simply write  $K(\mathcal{M})$ ,  $\text{TC}(\mathcal{M})$ , etc. for the associated  $C_p$ -spectrum obtained by applying  $K$ ,  $\text{TC}$ , etc. With this notation, we have that  $K_{C_p}(R) = K(\text{Perf}(R)^{\text{Bor}})$ ; note that our hypotheses imply that this is an  $\mathbb{E}_\infty$ -algebra in  $C_p$ -spectra.

We start by showing (1) implies (2). First, we reduce to the case where  $R$  is connective. Indeed, given a coconnective spectrum  $X$ , the  $C_p$ -Tate construction  $X^{tC_p}$  is annihilated by  $L_n^{p,f}$  for any  $n \geq 0$ ; this follows by dévissage and filtered colimits (note that  $(-)^{tC_p}$  commutes with filtered colimits on coconnective spectra) from the case where  $X$  is an Eilenberg–MacLane spectrum in a single degree. Therefore, the map  $(\tau_{\geq 0}R)^{tC_p} \rightarrow R^{tC_p}$  induces an equivalence on  $L_n^{p,f}$ -localizations for any  $n \geq 0$ . It follows that  $L_{T(i)}K((\tau_{\geq 0}R)^{tC_p}) \xrightarrow{\sim} L_{T(i)}K(R^{tC_p})$  by Theorem 4.8. Therefore, the

hypotheses of (1) hold for  $R$  if and only if they hold for  $\tau_{\geq 0}R$ , so we may assume  $R$  is connective; note also that the conclusion of (2) for  $\tau_{\geq 0}R$  implies it for  $R$ .

Now we have the categorical Mackey subfunctor  $\text{Perf}(R)_{\text{coBor}} \subseteq \text{Perf}(R)^{\text{Bor}}$ . By definition, this map of categorical  $C_p$ -Mackey functors is an equivalence on underlying objects (both have underlying object of  $\text{Cat}_{\infty}^{\text{perf}}$  given by  $\text{Perf}(R)$ ), and on  $C_p$ -fixed points it is given by the inclusion  $\text{Perf}(R[C_p]) \subseteq \text{Fun}(BC_p, \text{Perf}(R))$ , see Proposition 2.18 and Example 2.19. The Verdier quotient of categorical Mackey functors  $\text{Perf}(R)^{\text{Bor}} / \text{Perf}(R)_{\text{coBor}}$  is therefore a  $\text{Cat}_{\infty}^{\text{perf}}$ -valued  $C_p$ -Mackey functor with trivial underlying object and  $C_p$ -fixed points given by the Verdier quotient  $\text{Fun}(BC_p, \text{Perf}(R)) / \text{Perf}(R[C_p])$ , which is linear over the  $\mathbb{E}_{\infty}$ -ring  $R^{tC_p}$  (cf. [NS18, Sec. I.3]). Applying  $K$ -theory, we obtain a cofiber sequence of  $C_p$ -spectra

$$K(\text{Perf}(R)_{\text{coBor}}) \rightarrow K_{C_p}(R) \rightarrow K(\text{Perf}(R)^{\text{Bor}} / \text{Perf}(R)_{\text{coBor}}).$$

If we suppose that  $L_{T(i)}K(R^{tC_p}) = 0$ , then by the above, it follows that  $L_{T(i)}\Phi^{C_p}(-)$  annihilates the last term. Therefore, in order to prove (2), it suffices to show that  $L_{T(i)}\Phi^{C_p}K(\text{Perf}(R)_{\text{coBor}}) = 0$ .

Now  $K(\text{Perf}(R)_{\text{coBor}})$  of any  $\mathbb{E}_1$ -algebra is group-ring  $K$ -theory, cf. Example 2.19. Thus, since  $R$  is now assumed connective, we may apply the Dundas–Goodwillie–McCarthy theorem [DGM13] to obtain a pullback square of  $C_p$ -spectra,

$$\begin{array}{ccc} K(\text{Perf}(R)_{\text{coBor}}) & \longrightarrow & \text{TC}(\text{Perf}(R)_{\text{coBor}}) \\ \downarrow & & \downarrow \\ K(\text{Perf}(\pi_0 R)_{\text{coBor}}) & \longrightarrow & \text{TC}(\text{Perf}(\pi_0 R)_{\text{coBor}}). \end{array}$$

Here  $K(\text{Perf}(\pi_0 R)_{\text{coBor}})$  has trivial  $T(i)$ -localized geometric fixed points by Lemma 4.3. Thus, to prove (2), it suffices to prove that  $L_{T(i)}\Phi^{C_p} \text{TC}(\text{Perf}(A)_{\text{coBor}}) = 0$  whenever  $A$  is a connective  $\mathbb{E}_1$ -ring with  $L_{T(i)}A = 0$  (as this holds for both  $A = R$  by hypothesis and  $A = \pi_0 R$  trivially).

Now we use an expression for the  $p$ -completion of  $\Phi^{C_p} \text{TC}(\text{Perf}(A)_{\text{coBor}})$  given in the work of Hesselholt–Nikolaus, [HN19, Th. 1.4.1]. Indeed,  $\Phi^{C_p} \text{TC}(\text{Perf}(A)_{\text{coBor}})$  is the cofiber of the assembly map  $\text{TC}(A) \otimes BC_{p+} \rightarrow \text{TC}(A[C_p])$  and *loc. cit.* shows that after  $p$ -completion, this cofiber becomes  $\text{THH}(A; \mathbb{Z}_p)_{h\mathbb{T}_p}[1] \otimes C_p$  for  $\mathbb{T}_p$  the  $p$ -fold cover of the circle  $\mathbb{T}$ . In particular, our assumption that  $L_{T(i)}A = 0$  thus implies that  $L_{T(i)}(\Phi^{C_p} \text{TC}(\text{Perf}(A)_{\text{coBor}})) = 0$  as desired. This shows (1) implies (2).

Finally, we show (2) implies (3). The Borel-completion of the  $C_p$ -spectrum  $K_{C_p}(R)$  is the Borel-complete  $C_p$ -spectrum associated to the trivial  $C_p$ -action on  $K(R)$ ; in particular, we have a map of  $\mathbb{E}_{\infty}$ -rings  $\Phi^{C_p}K_{C_p}(R) \rightarrow K(R)^{tC_p}$ . It follows from (2) that  $L_{T(i)}(K(R)^{tC_p}) = 0$ , whence (3) by Theorem 4.6.  $\square$

We now prove Theorem A and Theorem B from the introduction, by starting with their special case Theorem C, which we restate here:

**Theorem 4.10.** *Let  $n \geq 0$ , and let  $\mathcal{C}$  be an  $L_n^{p,f}$ -local stable  $\infty$ -category. Then  $L_{T(m)}K(\mathcal{C}) = 0$  for all  $m \geq n+2$ , and for any finite  $p$ -group  $G$  acting on  $\mathcal{C}$  we have*

$$L_{T(n+1)}K(\mathcal{C}^{hG}) \xrightarrow{\sim} (L_{T(n+1)}K(\mathcal{C}))^{hG}.$$

*Proof.* Taking  $R = L_n^{p,f}\mathbb{S}$ , by applying Lemma 4.9 and Proposition 4.1 it suffices to show that  $L_{T(n+1)}R = 0$  and  $L_{T(n+1)}K(R^{tC_p}) = 0$ . The first vanishing follows from the definition. As for the second vanishing, we use induction on  $n$ . When  $n = 0$  we have  $R = \mathbb{S}[1/p]$  so  $R^{tC_p} = 0$ . When

$n > 0$ , Kuhn's blueshift theorem [Kuh04] shows that  $R^{tC_p}$  is  $L_{n-1}^{p,f}$ -local, whence  $K(R^{tC_p})$  is a module over  $K(L_{n-1}^{p,f} \mathbb{S})$  and we conclude by induction.  $\square$

As a corollary of combining this theorem with the results of [LMMT20] (in particular, Theorem 4.8), one obtains the following purity result in  $T(n)$ -local  $K$ -theory; this also appears in [LMMT20] and is explored further there.

**Corollary 4.11.** *Let  $A$  be an  $\mathbb{E}_1$ -ring, and let  $n \geq 1$ . Then the map  $A \rightarrow L_{T(n-1) \oplus T(n)} A$  induces an equivalence on  $L_{T(n)} K(-)$ .*

*Proof.* By Theorem 4.8, we may assume that  $A$  is already  $L_n^{p,f}$ -local. We have a pullback square

$$\begin{array}{ccc} \mathbb{K}(A) & \longrightarrow & \mathbb{K}(L_{T(n-1) \oplus T(n)} A) \\ \downarrow & & \downarrow \\ \mathbb{K}(L_{n-2}^{p,f} A) & \longrightarrow & \mathbb{K}(L_{n-2}^{p,f} (L_{T(n-1) \oplus T(n)} A)), \end{array}$$

since both vertical homotopy fibers are given by the (non-connective)  $K$ -theory of the thick subcategory of  $\text{Perf}(A)$  generated by  $A \otimes F$ , for  $F$  a finite type  $n-1$  complex. The result now follows from Theorem C, which shows that the spectra on the bottom row are  $T(n)$ -acyclic.  $\square$

Now we can input this back in to our arguments and obtain Theorem A and Theorem B, which we combine and restate here.

**Theorem 4.12.** *Let  $R$  be an  $\mathbb{E}_\infty$ -ring.*

(1) *Suppose  $L_{T(n)}(R^{tC_p}) = 0$  for some  $n \geq 0$ . Let  $\mathcal{C}$  be an  $R$ -linear idempotent-complete stable  $\infty$ -category equipped with an  $R$ -linear action of a finite  $p$ -group  $G$ . Let  $E$  be an additive invariant of  $R$ -linear idempotent-complete stable  $\infty$ -categories. Then the natural maps induce equivalences*

$$L_{T(n+1)}(E(\mathcal{C})_{hG}) \xrightarrow{\sim} L_{T(n+1)}E(\mathcal{C}_{hG}) \xrightarrow{\sim} L_{T(n+1)}E(\mathcal{C}^{hG}) \xrightarrow{\sim} L_{T(n+1)}(E(\mathcal{C})^{hG}) \xrightarrow{\sim} (L_{T(n+1)}E(\mathcal{C}))^{hG}.$$

(2) *Suppose  $L_{T(n+1)}R = 0$  for some  $n \geq -1$ . Then  $L_{T(j)}K(R) = 0$  for  $j \geq n+2$ .*

*Proof.* For (1), by Lemma 4.9 and Proposition 4.1 it suffices to show that  $L_{T(n+1)}R = 0$  and  $L_{T(n+1)}K(R^{tC_p}) = 0$ . The first follows from Theorem 4.6; for the second, we also get the weaker vanishing  $L_{T(n+1)}(R^{tC_p}) = 0$  (Hahn's theorem, [Hah16]) so this follows from the purity result Corollary 4.11. For (2), Hahn's theorem shows  $L_{T(n+2)}R = 0$  as well, so this follows from Corollary 4.11.  $\square$

**Remark 4.13.** The converse of part (2) of Theorem 4.12 is proved (for  $p$ -local  $\mathbb{E}_\infty$ -rings) in [BSY22, Th. 9.11] using the nonvanishing of  $L_{T(n+1)}K(E_n)$  proved in [Yua21].

**4.3. Comparison with the redshift conjectures.** Finally, we discuss the relationship of our results to redshift. Conjecture 4.2 of [AR08] predicts that if  $A \rightarrow B$  is a  $K(n)$ -local  $G$ -Galois extension of  $\mathbb{E}_\infty$ -rings in the sense of [Rog08], then  $L_{T(n+1)}K(A) \simeq L_{T(n+1)}(K(B)^{hG})$ . Here we will prove this conjecture in the case where  $G$  is a  $p$ -group. In fact, we will allow the (a priori more general) case of a  $T(n)$ -local  $G$ -Galois extension.

We recall that the condition of being a  $T(n)$ -local  $G$ -Galois extension, in which the map  $B \otimes_A B \rightarrow \prod_G B$  need only be a  $T(n)$ -equivalence, is much weaker than being a  $G$ -Galois extension of underlying  $\mathbb{E}_\infty$ -ring spectra (also known as a “global Galois extension”), and fundamental examples

such as the Galois extensions of the  $K(n)$ -local sphere produced by Devinatz–Hopkins are only  $T(n)$ -locally (or  $K(n)$ -locally) Galois. Thus the descent results in our previous paper [CMNN20] do not apply to them. Moreover, even in the case of underlying  $G$ -Galois extensions our previous results required being able to verify an extra condition: the rational surjectivity of the transfer map.

In the global case, we directly obtain from Theorem 4.12 and Galois descent the following.

**Corollary 4.14.** *Let  $A \rightarrow B$  be a faithful  $G$ -Galois extension of  $\mathbb{E}_\infty$ -rings (the faithfulness assumption is imposed to ensure Galois descent, in the form of [Mat16, Th. 9.4] or [Ban17]) with  $G$  a finite  $p$ -group, and suppose that  $L_{T(n)}(A^{tC_p}) = 0$ . Then the maps  $L_{T(n+1)}K(A) \rightarrow L_{T(n+1)}(K(B)^{hG}) \rightarrow (L_{T(n+1)}K(B))^{hG}$  are equivalences.  $\square$*

Now we consider the  $T(n)$ -local case, where we can also obtain results, but with an additional argument. Given a  $T(n)$ -local  $\mathbb{E}_\infty$ -ring  $A$ , we write  $K'(A)$  for the  $K$ -theory of the small, symmetric monoidal, stable  $\infty$ -category  $\mathcal{D}(A)$  of dualizable objects in  $T(n)$ -local  $A$ -modules. We have a natural inclusion  $\text{Perf}(A) \subseteq \mathcal{D}(A)$ , whence a map of  $\mathbb{E}_\infty$ -rings  $K(A) \rightarrow K'(A)$ . The next result (together with Theorem A) shows that this map is a  $T(n+1)$ -equivalence and implies that  $K$  or  $K'$  can be used equivalently in the Ausoni–Rognes conjecture.

**Proposition 4.15.** *Let  $A$  be a  $T(n)$ -local  $\mathbb{E}_\infty$ -ring. Then the homotopy fiber of  $K(A) \rightarrow K'(A)$  is naturally a module over  $K(L_{n-1}^{p,f}\mathbb{S})$ .*

*Proof.* Indeed, consider the Verdier quotient  $\mathcal{D}(A)/\text{Perf}(A)$ . We claim that this stable  $\infty$ -category is naturally  $L_{n-1}^{p,f}\mathbb{S}$ -linear. To this end, we need to show that if  $F$  is a finite type  $n$  complex, then for any  $M \in \mathcal{D}(A)$ , we have  $M \otimes F \in \text{Perf}(A)$  (so that this vanishes in the Verdier quotient). To this end, we observe that dualizability implies that the functor

$$\text{Hom}_{L_{T(n)}\text{Mod}(A)}(M, -) : L_{T(n)}\text{Mod}(A) \rightarrow L_{T(n)}\text{Mod}(A)$$

commutes with all colimits. Tensoring with  $F$ , we find that  $M \otimes F$  is a compact object of  $L_{T(n)}\text{Mod}(A)$ ; this uses that tensoring with  $F$  yields a colimit-preserving functor  $L_{T(n)}\text{Sp} \rightarrow \text{Sp}$ . Since  $L_{T(n)}\text{Mod}(A)$  is compactly generated by  $A \otimes F$ , it follows that  $M \otimes F$  belongs to the thick subcategory generated by  $A \otimes F$  and is therefore a perfect  $A$ -module, whence the result.  $\square$

Given a  $T(n)$ -local  $G$ -Galois extension  $A \rightarrow B$  with  $G$  a finite group, we have  $\mathcal{D}(A) \simeq \mathcal{D}(B)^{hG}$ ; this follows because  $L_{T(n)}\text{Mod}(A) \simeq (L_{T(n)}\text{Mod}(B))^{hG}$  by Galois descent,<sup>2</sup> and using [Lur17, Prop. 4.6.1.11] to commute the formation of dualizable objects over homotopy fixed points. Therefore, the next result follows in a similar manner; this proves [AR08, Conj. 4.2] in the case of a  $p$ -group.

**Corollary 4.16.** *Let  $A \rightarrow B$  be a  $T(n)$ -local  $G$ -Galois extension, with  $G$  a finite  $p$ -group. Then  $L_{T(n+1)}K(A) \xrightarrow{\sim} L_{T(n+1)}(K(B)^{hG}) \xrightarrow{\sim} (L_{T(n+1)}K(B))^{hG}$ .*

Let  $G$  be a finite  $p$ -group acting on a  $T(n)$ -local  $\mathbb{E}_\infty$ -ring  $B'$ ; then the map  $A' \stackrel{\text{def}}{=} B'^{hG} \rightarrow B'$  is  $T(n)$ -locally  $G$ -Galois, cf. [BCSY22, Cor. 7.31].

*Proof.* In fact, Theorem B yields the analog of this result with  $K'(-)$  replacing  $K$ , since  $\mathcal{D}(-)$  satisfies  $T(n)$ -local Galois descent. Using Proposition 4.15, we find that the difference between the statements for  $K'(-)$  and  $K(-)$  is controlled by modules over  $K(L_{n-1}^{p,f}\mathbb{S})$ , which have trivial  $T(n+1)$ -localizations by Theorem A.  $\square$

<sup>2</sup>Note that  $A \rightarrow B$  is automatically  $T(n)$ -locally faithful. In fact,  $A \simeq B^{hG} \simeq L_{T(n)}B_{hG}$ , so tensoring with  $B$  is conservative on  $L_{T(n)}\text{Mod}(A)$ .

**Example 4.17.** An important example of a  $T(n)$ -local (pro-)Galois extension is the map  $L_{K(n)}\mathbb{S} \rightarrow E_n$ , where the profinite group in question is the extended Morava stabilizer group  $\mathbb{G}_n$ . We have a short exact sequence

$$1 \rightarrow \mathbb{S}_n \rightarrow \mathbb{G}_n \rightarrow \text{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \rightarrow 1,$$

where  $\mathbb{S}_n$  has an open subgroup which is pro- $p$ . For each open subgroup  $H \subseteq \mathbb{G}_n$ , we write  $E_n^{hH}$  for the (Devinatz–Hopkins) continuous homotopy fixed points, [DH04]. Now, we choose an open subgroup  $U$  of  $\mathbb{G}_n$  such that  $U \cap \mathbb{S}_n$  is a pro- $p$ -group. Then for any normal inclusion  $V' \trianglelefteq V \subseteq U$  of open subgroups, we have  $L_{T(n+1)}K(E_n^{hV}) \xrightarrow{\sim} (L_{T(n+1)}K(E_n^{hV'}))^{h(V/V')}$ , i.e., we obtain Galois descent for the  $K(n)$ -local finite Galois extensions of  $E_n^{hU}$ , and we therefore obtain a sheaf of  $T(n+1)$ -local spectra on finite continuous  $U$ -sets. This follows from the descent for  $p$ -groups proved above as well as the descent for finite étale extensions, proved as in [CMNN20]. Our methods do not (to the best of our knowledge) yield hyperdescent for this sheaf, which would closely relate  $L_{T(n+1)}K(E_n)$  and  $L_{T(n+1)}K(L_{K(n)}\mathbb{S})$ .

**Remark 4.18.** Finally, [AR08, Conjecture 4.3] predicts that for appropriate  $K(n)$ -local  $\mathbb{E}_\infty$ -ring spectra  $B$  (e.g.,  $L_{K(n)}\mathbb{S}$ ), and for a finite type  $n+1$ -complex  $V$ , the map  $V \otimes K(B) \rightarrow L_{T(n+1)}(V \otimes K(B))$  is an equivalence in high enough degrees; this is a higher chromatic analog of the Lichtenbaum–Quillen conjecture, cf. [AR02, Aus10, HW22, AKAC<sup>+</sup>22, HRW22] for instances where such statements are proved. Our methods are certainly not strong enough to prove such statements; however, this conjecture would imply the weaker assertion  $L_{T(n+i)}K(B) = 0$  for  $i \geq 2$ , which we have proved above as Theorem A.

## 5. DESCENT BY NORMAL BASES; PROOF OF THEOREM D

In this section, we will give another condition that guarantees  $T(n)$ -local descent, which will work uniformly for all  $n$  (including  $n = 0$ ).

**Construction 5.1** (The transfer). We use the transfer map of the finite group  $G$ , which is a map of spectra

$$\text{tr}_{BG}: BG_+ \rightarrow \mathbb{S}.$$

The adjoint map of anima  $BG \rightarrow \Omega^\infty \mathbb{S}$  arises from interpreting the target as the  $K$ -theory of the category  $\text{Fin}$  of finite sets (the Barratt–Priddy–Quillen theorem), and considering the  $G$ -action on the  $G$ -set  $G$  (by right multiplication), so we take the composite map  $BG \rightarrow \text{Fin}^\simeq \rightarrow \Omega^\infty K(\text{Fin}) = \Omega^\infty \mathbb{S}$ . Our basic tool will be the following observation:

**Theorem 5.2.** *For any  $n \geq 0$  and implicit prime  $p$ , the map  $L_{T(n)}(\text{tr}_{BG}): L_{T(n)}BG_+ \rightarrow L_{T(n)}\mathbb{S}$  admits a section.*

*Proof.* This follows from (and is equivalent to, as explained in [CM17]) the vanishing of Tate spectra in the  $T(n)$ -local category, due to Kuhn [Kuh04]. In fact, this vanishing yields that  $L_{T(n)}BG_+ \xrightarrow{\sim} C^*(BG, L_{T(n)}\mathbb{S})$  via the norm map, and the transfer is the composite of the norm with the projection  $C^*(BG, L_{T(n)}\mathbb{S}) \rightarrow L_{T(n)}\mathbb{S}$ , which clearly admits a section.<sup>3</sup>  $\square$

**Proposition 5.3.** *Let  $R$  be an associative algebra in  $\text{Sp}_G$ . Suppose that there is a factorization of  $BG_+ \xrightarrow{\text{tr}_{BG}} \mathbb{S} \rightarrow R^G$  through the  $R$ -transfer  $R_{hG} \rightarrow R^G$ . Then  $R$  is  $(\mathcal{T}, \varepsilon)$ -nilpotent.*

<sup>3</sup>Note also that as explained in [CM17], the existence of the section in the essential case  $G = C_p$  follows via the Bousfield–Kuhn functor [Bou01, Kuh89] from the Kahn–Priddy theorem [KP78], which states that  $\Omega^{\infty+1}(\text{tr}_{BG}): \Omega^{\infty+1}BG_+ \rightarrow \Omega^{\infty+1}\mathbb{S}$  has a section.

*Proof.* By Proposition 3.5, it suffices to show that  $L_{T(n)}(R^G/R_{hG}) = L_{T(n)}(R \otimes \widetilde{EG})^G = 0$  for any  $p$  and  $n$  (including  $p = 0$ ). For this, it suffices to show that the map

$$L_{T(n)}(R_{hG}) \rightarrow L_{T(n)}(R^G)$$

has image on  $\pi_0$  containing the unit. But this follows because we have seen above that the map  $L_{T(n)}(\text{tr}_{BG}): L_{T(n)}(BG_+) \rightarrow L_{T(n)}\mathbb{S}$  has image (on  $\pi_0$ ) containing the unit.  $\square$

We will apply this below to associative  $G$ -ring spectra of a particularly special kind, where one has a homotopy commutative diagram

$$(5.1) \quad \begin{array}{ccc} R_{hG} & \longrightarrow & R^G \\ \uparrow & & \uparrow \\ BG_+ & \xrightarrow{\text{tr}_{BG}} & \mathbb{S}, \end{array}$$

in which the factorization  $BG_+ \rightarrow R_{hG}$  required in Proposition 5.3 is obtained as the  $G$ -homotopy orbits of the map  $\mathbb{S} \rightarrow R$ ; in particular, these satisfy the conditions of Proposition 5.3. Note that this now is merely a condition on the algebra  $R$  in  $\text{Sp}_G$ , namely that the diagram

$$(5.2) \quad \begin{array}{ccc} R_{hG} & \longrightarrow & R^G \\ (\eta)_{hG} \uparrow & & \uparrow \eta \\ BG_+ & \xrightarrow{\text{tr}_{BG}} & \mathbb{S}, \end{array}$$

should commute up to homotopy, where  $\eta$  denotes the unit map. Such  $R$  arise via the following categorical construction, namely by taking  $R = K_G(\mathcal{C})$  below.

Let  $(\mathcal{C}, \otimes, \mathbf{1})$  be a monoidal, stable  $\infty$ -category equipped with a  $G$ -action. Let  $f: G \rightarrow *$  be the map of  $G$ -sets. We use the induction functor  $f_*: \mathcal{C} \rightarrow \mathcal{C}^{hG}$  (biadjoint to the forgetful functor  $\mathcal{C}^{hG} \rightarrow \mathcal{C}$ ); we note that this is  $G$ -equivariant with respect to the trivial action on the target. Since  $\mathbf{1} \in \mathcal{C}$  is  $G$ -invariant, we obtain a  $G$ -action on  $f_*(\mathbf{1}) \in \mathcal{C}^{hG}$ .

**Definition 5.4** (The normal basis condition). We say that the  $G$ -action on  $\mathcal{C}$  as above satisfies the *normal basis property* if the object  $f_*(\mathbf{1}) \in \text{Fun}(BG, \mathcal{C}^{hG})$  defines the same  $K_0$ -class as the object  $\mathbf{1}_{\mathcal{C}^{hG}} \otimes G_+ \in \text{Fun}(BG, \mathcal{C}^{hG})$ .

In other words, the normal basis condition implies that the following diagram, which is *not* commutative,

$$(5.3) \quad \begin{array}{ccc} \mathcal{C}_{hG}^{\simeq} & \xrightarrow{f_*} & (\mathcal{C}^{hG})^{\simeq} \\ \uparrow [1]_{hG} & & \uparrow \mathbf{1}_{\mathcal{C}^{hG}} \\ BG & \xrightarrow{\text{tr}_{BG}} & \Omega^\infty \mathbb{S}, \end{array}$$

gives rise to two objects in  $\text{Fun}(BG, \mathcal{C}^{hG})$  with the same  $K_0$ -class; the class obtained by going right and up is the normal basis class  $\mathbf{1}_{\mathcal{C}^{hG}} \otimes G_+$ , while the class obtained by going up and right is  $f_*(\mathbf{1})$ . It follows that we do have a homotopy commutative diagram if we replace the top right in (5.3) with  $\Omega^\infty K(\mathcal{C}^{hG})$ .

We now prove the following result, which is a slight refinement of Theorem D.

**Theorem 5.5.** Suppose  $R$  is an  $\mathbb{E}_\infty$ -ring,  $\mathcal{C}$  is an algebra object of  $\text{Cat}_{R,\infty}^{\text{perf}}$  equipped with an action of a finite group  $G$ , and the  $G$ -action on  $\mathcal{C}$  satisfies the normal basis property. Then  $\mathcal{U}_G(\mathcal{C}) \in \text{Mack}_G(\text{Mot}_R)$  is  $(\mathcal{T}, \varepsilon)$ -nilpotent. In particular, for any additive invariant  $E$  on  $\text{Cat}_{R,\infty}^{\text{perf}}$ , the map  $E(\mathcal{C}^{hG}) \rightarrow E(\mathcal{C})^{hG}$  induces an equivalence after  $T(i)$ -localization for any  $i$  and any implicit prime  $p$ , including  $p = 0$ .

*Proof.* We show that  $\mathcal{U}_G(\mathcal{C}) \in \text{Mack}_G(\text{Mot}_R)$  is  $(\mathcal{T}, \varepsilon)$ -nilpotent, which also implies the other claims. By Example 3.14, it suffices to show that  $K_G(\mathcal{C})$  is  $(\mathcal{T}, \varepsilon)$ -nilpotent. By Proposition 5.3, it suffices to show that we have a factorization of the  $BG_+ \rightarrow \mathbb{S}$  through  $K(\mathcal{C})_{hG} \rightarrow K(\mathcal{C}^{hG})$ . However, this follows from the diagram (5.3) (which is not commutative, but which becomes homotopy commutative when we replace the upper right by  $\Omega^\infty K(\mathcal{C}^{hG})$  by our hypotheses).  $\square$

**Remark 5.6** (Alternative proof of Theorem 1.2). Let  $R \rightarrow R'$  be a  $G$ -Galois extension of commutative rings. Then Zariski locally on  $R$ , one has the normal basis property (even before passage to  $K_0$ ): the  $R[G]$ -module  $R'$  is locally isomorphic to  $R[G]$ ; indeed, this follows because of the usual normal basis theorem when  $R$  is a field, and hence more generally a local ring. Using Zariski descent for  $K$ -theory [TT90], one reduces to this case, whence the result via Theorem 5.5.

**Remark 5.7.** In fact, the above argument for Theorem 1.2 is valid more generally (with  $L_{T(n)}$ -localization for any  $n$ ) if  $R \rightarrow R'$  is a  $G$ -Galois extension of  $\mathbb{E}_\infty$ -ring spectra in the sense of [Rog08] in the case where  $\pi_0(R) \rightarrow \pi_0(R')$  is additionally  $G$ -Galois, i.e.,  $R \rightarrow R'$  is étale in the sense of [Lur17, Sec. 7.5]. This is a special case of the results of [CMNN20], which assume a weaker condition on  $R \rightarrow R'$ , but which essentially use the  $\mathbb{E}_\infty$ -structures on the algebras in question.

## 6. SWAN INDUCTION AND APPLICATIONS; PROOFS OF THEOREM F AND G

In this section, we recall the notion of Swan  $K$ -theory, and prove Theorems F and G from the introduction. In the final section, we will give a number of examples of Swan induction.

The Swan  $K$ -theory of a ring spectrum with respect to a finite group was introduced by Malkiewich in [Mal17], following ideas of Swan [Swa60, Swa70] who defined it for discrete rings. Throughout the subsection, let  $R$  be an  $\mathbb{E}_\infty$ -ring spectrum.

**Definition 6.1** ([Mal17, Def. 4.11]). Given a finite group  $G$ , we let  $\text{Rep}(G, R)$  denote the Grothendieck ring of the stable  $\infty$ -category  $\text{Fun}(BG, \text{Perf}(R))$ . We will call this the *Swan  $K$ -theory of  $R$  with respect to  $G$* . The groups  $\{\text{Rep}(H, R)\}_{H \subseteq G}$  form a Green functor, as the  $\pi_0$  of the  $\mathbb{E}_\infty$ -algebra  $K_G(R)$  in  $\text{Sp}_G$ .

For a family  $\mathcal{F}$  of subgroups of  $G$ , we will say that  $\mathcal{F}$ -based Swan induction holds for  $R$  if there exist classes  $x_H \in \text{Rep}(H, R) \otimes \mathbb{Q}$  for  $H \in \mathcal{F}$  such that

$$(6.1) \quad 1 = \sum_{H \in \mathcal{F}} \text{Ind}_H^G(x_H) \in \text{Rep}(G, R) \otimes \mathbb{Q},$$

for  $\text{Ind}_H^G: \text{Rep}(H, R) \rightarrow \text{Rep}(G, R)$  the map obtained by induction of representations on  $R$ -modules.

**Remark 6.2.** The condition that  $\mathcal{F}$ -based Swan induction holds for  $R$  is precisely the condition that  $K_G(R) \otimes \mathbb{Q} \in \text{Sp}_G$  is  $\mathcal{F}$ -nilpotent, in light of Proposition 3.4.

**Example 6.3** (Classes in  $\text{Rep}(G, R)$ ). Let  $M$  be a finite  $G$ -CW complex. Then  $R \otimes M_+$  defines an object of  $\text{Fun}(BG, \text{Perf}(R))$  and consequently an element  $[R \otimes M_+] \in \text{Rep}(G, R)$ . If  $M$  has the homotopy type of a  $G$ -CW complex, then a cell decomposition shows that the class  $[R \otimes M_+]$  actually belongs to the image of the map  $A(G) \rightarrow \text{Rep}(G, R)$ , for  $A(G)$  the Burnside ring.

We now prove the descent statement in the  $K$ -theory of  $R$ -linear  $\infty$ -categories that Swan induction implies (this is Theorem F); the use of a rational statement to deduce telescopic ones follows [CMNN20].

**Theorem 6.4** (Descent via Swan induction). *Let  $R$  be an  $\mathbb{E}_\infty$ -ring and let  $G$  be a finite group. Suppose that  $R$ -based Swan induction holds for the family  $\mathcal{F}$ . Then for any  $R$ -linear  $\infty$ -category  $\mathcal{C}$  equipped with a  $G$ -action, and for any additive invariant  $E$  on  $\text{Cat}_{R,\infty}^{\text{perf}}$ , the maps*

$$(6.2) \quad E(\mathcal{C}^{hG}) \rightarrow \varprojlim_{G/H \in \mathcal{O}_{\mathcal{F}}(G)^{\text{op}}} E(\mathcal{C}^{hH})$$

and

$$(6.3) \quad \varinjlim_{G/H \in \mathcal{O}_{\mathcal{F}}(G)} E(\mathcal{C}_{hH}) \rightarrow E(\mathcal{C}_{hG})$$

become an equivalence after  $T(n)$ -localization, for any  $n$  and any implicit prime  $p$ .

*Proof.* For the first claim, it suffices to show that  $\mathcal{U}_G(\mathcal{C}) \in \text{Mack}_G(\text{Mot}_R)$  is  $(\mathcal{F}, \varepsilon)$ -nilpotent. By multiplicativity, this reduces to showing that  $\mathcal{U}_G(\text{Perf}(R))$  is  $(\mathcal{F}, \varepsilon)$ -nilpotent, where the  $G$ -action on  $\text{Perf}(R)$  is trivial; for this in turn, it suffices to show that  $K_G(R) \in \text{Sp}_G$  is  $(\mathcal{F}, \varepsilon)$ -nilpotent as in Example 3.14. By Proposition 3.11, it suffices to show that  $K_G(R)_{\mathbb{Q}}$  is  $\mathcal{F}$ -nilpotent, which is precisely the condition of  $R$ -based Swan induction for  $\mathcal{F}$  (Remark 6.2).

For the second claim, we use the coBorel construction of Construction 2.20. We claim that  $\mathcal{U}_{G,\text{coBor}}(\mathcal{C}) \in \text{Mack}_G(\text{Mot}_R)$  is  $(\mathcal{F}, \varepsilon)$ -nilpotent. But this follows because it is a module over  $\mathcal{U}_G(\text{Perf}(R))$ , which we have just seen is  $(\mathcal{F}, \varepsilon)$ -nilpotent; thus  $\mathcal{U}_{G,\text{coBor}}(\mathcal{C})$  is  $(\mathcal{F}, \varepsilon)$ -nilpotent. This implies that (6.3) becomes an equivalence after  $T(n)$ -localization, cf. Proposition 3.3.  $\square$

Now we record a variant of Theorem 6.4 specifically in the context where  $R = L_n^{p,f} \mathbb{S}$ , and where the localization is precisely at height  $n+1$ ; this relies on similar techniques as in section 4, and follows from combining them with the above. In fact, this yields a slight refinement of the results of section 4 to non- $p$ -groups.

**Theorem 6.5.** *Fix  $n \geq 0$ . Let  $R$  be an  $\mathbb{E}_\infty$ -ring such that  $L_{T(n)}(R^{tC_p}) = 0$ . Let  $\mathcal{C}$  be an  $R$ -linear idempotent-complete stable  $\infty$ -category equipped with an  $R$ -linear action of a finite group  $G$ . Then for any additive invariant  $E$ , the maps (6.2) and (6.3) become equivalences after  $T(n+1)$ -localization, for  $\mathcal{F}$  the family of cyclic subgroups of  $G$  of prime-to- $p$ -power order.*

*Proof.* Let us first observe that we may replace  $R$  by its connective cover  $\tau_{\geq 0}R$ . Indeed, via the Postnikov tower, we find that  $(\tau_{\leq -1}R)^{tC_p}$  is annihilated by  $T(n)$ . Thus, the vanishing assumption for  $R$  is equivalent to the same assumption for  $\tau_{\geq 0}R$ , whence we may assume for the rest of the argument that  $R$  is connective.

Now observe that for any finite group  $H$ , we have  $L_{T(n)}(R^{tH}) = 0$ . Indeed, we reduce to the case where  $H$  is a  $p$ -group by restricting to a  $p$ -Sylow. Inductively, we have a normal subgroup  $C_p \trianglelefteq H$ . The norm map  $R_{hH} \rightarrow R^{hH}$  factors as

$$R_{hH} = (R_{hC_p})_{h(H/C_p)} \rightarrow (R^{hC_p})_{h(H/C_p)} \rightarrow (R^{hC_p})^{h(H/C_p)} = R^{hH}.$$

By induction on  $H$  and the assumption  $L_{T(n)}(R^{tC_p}) = 0$ , we see that each map above has  $T(n)$ -acyclic cofiber, whence  $R^{tH}$  is  $T(n)$ -acyclic.

The rest of the argument will closely follow that of Lemma 4.9. We will show that  $L_{T(n+1)}K_G(R)$  is  $\mathcal{F}$ -complete, for  $\mathcal{F}$  as in the statement. This will imply the result. Indeed, for any additive invariant  $E$ , the  $G$ -spectra  $E(\mathcal{C}^{\text{Bor}}), E(\mathcal{C}_{\text{coBor}}), F(E\mathcal{F}_+, E(\mathcal{C}^{\text{Bor}}))$  are modules over  $K_G(R)$ . If

$L_{T(n+1)}K_G(R)$  is shown to be  $\mathcal{F}$ -complete, then the  $G$ -spectra

$$L_{T(n+1)}E(\mathcal{C}^{\text{Bor}}), L_{T(n+1)}E(\mathcal{C}_{\text{coBor}}), L_{T(n+1)}F(E\mathcal{F}_+, E(\mathcal{C}^{\text{Bor}}))$$

will be  $\mathcal{F}$ -complete by Corollary 3.9 and thus become  $\mathcal{F}$ -nilpotent after smashing with a finite type  $(n+1)$ -spectrum by Proposition 3.8. The result then follows in light of [MNN19, Prop. 2.8].

To see that  $L_{T(n+1)}K_G(R)$  is  $\mathcal{F}$ -complete, we let  $\mathcal{D} = \text{Perf}(R)$  with trivial  $G$ -action. We have the fully faithful inclusion of  $G$ -Mackey functors  $\mathcal{D}_{\text{coBor}} \rightarrow \mathcal{D}^{\text{Bor}}$  (Proposition 2.18); the cofiber takes values in  $R^{tH}$ -linear  $\infty$ -categories for various subgroups  $H \subseteq G$ ; indeed, this follows because the Verdier quotient  $\text{Fun}(BH, \text{Perf}(R)) / \text{Perf}(R[H])$  is linear over  $R^{tH}$  as in [NS18, Sec. I.3]. Therefore, using Theorem A, we find an equivalence of  $G$ -Mackey functors  $L_{T(n+1)}K(\mathcal{D}_{\text{coBor}}) \simeq L_{T(n+1)}K(\mathcal{D}^{\text{Bor}}) = L_{T(n+1)}K_G(R)$ . Thus, it suffices to show that  $L_{T(n+1)}K(\mathcal{D}_{\text{coBor}})$  is  $\mathcal{F}$ -complete, or equivalently that its  $T(n+1)$ -local geometric fixed points vanish at all subgroups except possibly those which are cyclic of prime-to- $p$  order (Proposition 3.8).

Now  $K(\mathcal{D}_{\text{coBor}})$  is group-ring  $K$ -theory (Example 2.19). Since  $R$  is connective, we obtain from the Dundas–Goodwillie–McCarthy theorem [DGM13] a pullback square of  $G$ -spectra,

$$\begin{array}{ccc} L_{T(n+1)}K(\mathcal{D}_{\text{coBor}}) & \longrightarrow & L_{T(n+1)}\text{TC}(\text{Perf}(R)_{\text{coBor}}) \\ \downarrow & & \downarrow \\ L_{T(n+1)}K(\text{Perf}(\pi_0R)_{\text{coBor}}) & \longrightarrow & L_{T(n+1)}\text{TC}(\text{Perf}(\pi_0R)_{\text{coBor}}). \end{array}$$

Now we obtain from [LRRV19, Th. 1.2] that the  $G$ -spectra  $\text{TC}(\text{Perf}(R)_{\text{coBor}}), \text{TC}(\text{Perf}(\pi_0R)_{\text{coBor}})$  are modulo  $p$  induced from the family of cyclic subgroups of  $G$ ; in particular, their geometric fixed points at non-cyclic subgroups vanish modulo  $p$ . Finally, the term  $L_{T(n+1)}K(\text{Perf}(\pi_0R)_{\text{coBor}})$  vanishes for  $n \geq 1$  by Mitchell’s theorem; if  $n = 0$ , it follows from Theorem 6.4 and Swan’s induction theorem from [Swa60] (reproved below as Theorem 7.5) that  $L_{T(n+1)}K(\text{Perf}(\pi_0R)_{\text{coBor}})$  is induced from the family of cyclic subgroups.

Thus, we find that  $L_{T(n+1)}K(\mathcal{D}_{\text{coBor}}) = L_{T(n+1)}K(\mathcal{D}^{\text{Bor}}) = L_{T(n+1)}K_G(R)$  is complete for the family of cyclic subgroups. In particular, the  $T(n+1)$ -local geometric fixed points vanish for non-cyclic subgroups. Suppose then that  $G$  is cyclic and has order divisible by  $p$ ; we must show that  $L_{T(n+1)}\Phi^G K(\mathcal{D}^{\text{Bor}}) = 0$ .

In fact, since there is an inclusion  $H \trianglelefteq G$  with  $G/H \simeq C_p$ , we have the transfer map

$$(K(\mathcal{D}^{\text{Bor}})^H)_{hC_p} \rightarrow K(\mathcal{D}^{\text{Bor}})^G.$$

This map, or equivalently  $K(\mathcal{D}^{hH})_{hC_p} \rightarrow K(\mathcal{D}^{hG})$ , is  $T(n+1)$ -locally an equivalence thanks to Theorem 4.12 (applied to the residual  $C_p$ -action on  $\mathcal{D}^{hH}$ ). Since it factors through the map  $(E\mathcal{P} \otimes K(\mathcal{D}^{\text{Bor}}))^G \rightarrow K(\mathcal{D}^{\text{Bor}})^G$  for  $\mathcal{P}$  the family of proper subgroups, it follows that this last map has  $T(n+1)$ -local image (on  $\pi_0$ ) containing the unit, whence  $L_{T(n+1)}\Phi^G K(\mathcal{D}^{\text{Bor}}) = 0$  as desired.  $\square$

Next, we prove Theorem G, which was inspired by the generalized character theory of Hopkins, Kuhn, and Ravenel [HKR00] as well as the results of [MNN19]. We will apply this below to recover some cases of the chromatic bounds on  $K$ -theory spectra.

**Proposition 6.6.** *Fix a prime  $p$  and a non-negative integer  $n$ . Let  $R$  be an  $\mathbb{E}_\infty$ -ring spectrum and  $G = C_p^{\times n}$ . Suppose that the sum of the rationalized transfer maps*

$$(6.4) \quad \bigoplus_{H \subsetneq G} R^0(BH) \otimes \mathbb{Q} \rightarrow R^0(BG) \otimes \mathbb{Q}$$

is a surjection (or equivalently has image containing the unit). Then  $L_{T(n+i)}R \simeq 0$  for all  $i \geq 0$  (at the prime  $p$ ).

*Proof.* A  $T(n)$ -local ring spectrum is contractible if and only if its  $K(n)$ -localization is contractible, cf. [LMMT20, Lem. 2.3]. Therefore, it suffices to prove that  $L_{K(n+i)}R = 0$ . To verify the desired vanishing, we can replace  $R$  by the  $\mathbb{E}_\infty$ - $R$ -algebra  $L_{K(n+i)}(E_{n+i} \otimes R)$ ; by naturality of the transfer map, the hypotheses of the result are preserved by this replacement. Thus, we may assume throughout that  $R = L_{K(n+i)}R$  receives an  $\mathbb{E}_\infty$ -map from  $E_{n+i}$ . By the main result of [MNN15] then, it suffices to show  $\pi_0 R \otimes \mathbb{Q} = 0$ .

When  $n = 0$ , the left hand side of (6.4) is 0, so  $\pi_0 R \otimes \mathbb{Q} = 0$ . So it suffices to consider the case  $n > 0$ . Since each transfer map factors through a maximal proper subgroup, the surjectivity of (6.4) is equivalent to the surjectivity of

$$(6.5) \quad \bigoplus_{C_p^{\times(n-1)} \cong H \subsetneq G} R^0(BH) \otimes \mathbb{Q} \rightarrow R^0(BG) \otimes \mathbb{Q}.$$

The left hand side of this equation contains  $\frac{p^n - 1}{p - 1}$ -copies of  $R^0(BC_p^{\times(n-1)})$ . Since  $R$  is a  $K(n+i)$ -local  $E_{n+i}$ -module, it follows that for any  $k$ , one has that  $C^*(BC_p^{\times k}, R) = C^*(BC_p^{\times k}, E_{n+i}) \otimes_{E_{n+i}} R$  is a free  $R$ -module of rank  $p^{(n+i)k}$  (cf. Lemma 4.2). In particular, the right-hand-side of (6.5) is free over  $\pi_0(R) \otimes \mathbb{Q}$  of rank  $p^{(n+i)n}$ , while each summand on the left-hand-side has rank  $p^{(n+i)(n-1)}$ . Using the surjectivity of (6.5), we find that if  $\pi_0(R) \otimes \mathbb{Q} \neq 0$ , then we would conclude the inequality of ranks,

$$p^{(n+i)(n-1)} \frac{p^n - 1}{p - 1} \geq p^{(n+i)n}.$$

However, we see easily that this inequality cannot hold if  $i \geq 0$  and  $n > 0$ . This contradiction proves the result.  $\square$

**Theorem 6.7.** *Let  $p$  be a prime,  $n \geq 0$  and  $R$  an  $\mathbb{E}_\infty$ -ring spectrum. Suppose that  $R$ -based Swan induction holds for the family of proper subgroups of  $C_p^{\times n}$ . Then  $L_{T(i)}K(R) = 0$  for  $i \geq n$  at the prime  $p$ .*

*Proof.* We write  $G = C_p^{\times n}$  and consider the  $\mathbb{E}_\infty$ -algebra in  $G$ -spectra  $K_G(R)$ . By assumption,  $K_G(R)_{\mathbb{Q}}$  is nilpotent for the family of proper subgroups, cf. Remark 6.2. There is a natural map of  $\mathbb{E}_\infty$ -algebras in  $\mathrm{Sp}_G$  of the form  $K_G(R)_{\mathbb{Q}} \rightarrow (K_G(R)^{\mathrm{Bor}})_{\mathbb{Q}}$ , so the target is also nilpotent for the family of proper subgroups. But  $K_G(R)^{\mathrm{Bor}}$  is simply the Borel-equivariant  $G$ -spectrum associated to the trivial  $G$ -action on  $K(R)$ , so the condition that  $(K_G(R)^{\mathrm{Bor}})_{\mathbb{Q}}$  should be nilpotent for the family of proper subgroups is exactly that the map (6.4) (with  $K(R)$  replacing  $R$ ) should be a surjection. Thus, the result follows from Proposition 6.6.  $\square$

## 7. SWAN INDUCTION THEOREMS; PROOF OF THEOREM E

In this section, we establish several examples of Swan induction theorems for structured ring spectra, and prove Theorem E. In particular, we show that one always has Swan induction for the family of abelian subgroups for MU (Theorem 7.4), for the cyclic groups for  $\mathbb{Z}$  (Theorem 7.5, recovering results of [Swa60]) or for  $\mathbb{S}[1/|G|]$  (Theorem 7.6), for the rank  $\leq 2$  abelian subgroups for KU (Theorem 7.13), and for the rank  $\leq n+1$  abelian subgroups for  $E_n$  at  $p = 2$  (Theorem 7.12).

**7.1. Geometric arguments.** Throughout, let  $R$  be an  $\mathbb{E}_\infty$ -ring spectrum. We first observe the following basic features of the Swan induction property.

**Remark 7.1.** (1) If  $R$  is an  $\mathbb{E}_\infty$ -ring such that one has  $R$ -based Swan induction with respect to a family of subgroups  $\mathcal{F}$  of some group  $G$ , and  $R'$  is an  $\mathbb{E}_\infty$ -ring admitting a map from  $R$  (even an  $\mathbb{E}_1$ -map suffices), then  $R'$ -based Swan induction for  $\mathcal{F}$  and  $G$  holds as well (this was used in the proof of Proposition 6.6).  
(2) In order to prove that  $R$ -based Swan induction holds with respect to a family  $\mathcal{F}$  of subgroups of  $G$ , it suffices to show that for every subgroup  $H \subseteq G$  which is not in  $\mathcal{F}$ , then one has Swan induction with respect to the family of proper subgroups of  $H$ . This is an elementary observation about Green functors, cf. [MNN17, Prop. 6.40].  
(3) Suppose  $G \twoheadrightarrow G'$  is a surjection, and  $R$ -based Swan induction holds for the family of proper subgroups of  $G'$ . Then  $R$ -based Swan induction holds for the family of proper subgroups of  $G$ .

In this subsection, we give geometric proofs of Swan induction in several cases. Our basic tool is the following.

**Proposition 7.2.** *Let  $M$  be a  $G$ -space such that:*

- (1)  *$M$  admits a finite  $G$ -CW structure.*
- (2)  *$M$  has isotropy in the family  $\mathcal{F}$  of subgroups of  $G$ .*
- (3) *There is an equivalence*

$$(7.1) \quad R \otimes M_+ \simeq \bigoplus_{k=1}^n \Sigma^{2i_k} R \in \text{Fun}(BG, \text{Perf}(R))$$

for some integers  $i_1, \dots, i_n$ . Here we equip the  $\Sigma^{2i_k} R$  with the trivial  $G$ -action.<sup>4</sup>

Then  $R$ -based Swan induction holds for the family  $\mathcal{F}$ .

*Proof.* We consider the object  $X = R \otimes M_+ \in \text{Fun}(BG, \text{Perf}(R))$  and calculate its  $K_0$ -class  $[X]$  in two different ways.

- (1) By assumption,  $M$  has a finite  $G$ -CW decomposition with equivariant cells of the form  $G/H \times D^n$ . The  $G$ -cells necessarily satisfy  $H \in \mathcal{F}$  by hypothesis on the isotropy of  $M$ . It follows that there exist integers  $n_H, H \in \mathcal{F}$  such that

$$(7.2) \quad [X] = \sum_{H \in \mathcal{F}} n_H [R \otimes G/H_+] = \sum_{H \in \mathcal{F}} \text{Ind}_H^G(n_H) \in \text{Rep}(G, R).$$

- (2) The assumption (3) gives an equivalence in  $\text{Fun}(BG, \text{Perf}(R))$  between  $X$  and a direct sum of  $n > 0$  even shifts of the unit. It follows that

$$(7.3) \quad [X] = n \in \text{Rep}(G, R).$$

Equating (7.2) and (7.3), we obtain the result.  $\square$

The first condition in Proposition 7.2 will be satisfied if, for example,  $M$  is a compact smooth manifold with  $G$ -action, by the equivariant triangulation theorem [Ill78]. We can check the condition (3) of Proposition 7.2 via the following result.

<sup>4</sup>In fact, for the argument, it suffices that the class in  $\text{Rep}(G, R)$  of  $R \otimes M_+$  is a nonzero integer. This would be satisfied, for example, if there are odd suspensions of  $R$  that appear in (7.1), as long as the Euler characteristic is nonzero.

**Proposition 7.3.** *Let  $M$  be a  $G$ -space with the homotopy type of a finite  $G$ -CW complex. Then  $M$  satisfies condition (3) of Proposition 7.2 if and only if:*

- (1) *The  $R^*$ -cohomology  $R^*(M_{hG})$  is a free module on generators in even degrees over  $R^*(BG)$ .*
- (2) *The natural map*

$$R^*(*) \otimes_{R^*(BG)} R^*(M_{hG}) \rightarrow R^*(M)$$

*is an isomorphism.*

*Proof.* In fact, using (1), we can produce a  $G$ -equivariant map from a sum of shifts of the unit into  $R \otimes \mathbb{D}M_+ \in \text{Fun}(BG, \text{Perf}(R))$ , by choosing a basis of  $R^*(M_{hG}) = \pi_{-*} \text{Hom}_{\text{Fun}(BG, \text{Perf}(R))}(R, R \otimes \mathbb{D}M_+)$ ; the induced map is an equivalence in  $\text{Fun}(BG, \text{Perf}(R))$  by the second condition.  $\square$

**Theorem 7.4.** *Suppose there exists an  $\mathbb{E}_1$ -map  $\text{MU} \rightarrow R$ . Then  $R$ -based Swan induction holds for the family of abelian subgroups (for any finite group  $G$ ).*

*Proof.* We fix an embedding  $G \subseteq \text{U}(n)$  and consider the action on the flag variety  $M = F = \text{U}(n)/T$  for  $T \subseteq \text{U}(n)$  a maximal torus. As a smooth  $G$ -manifold,  $M$  admits a finite  $G$ -CW structure. The stabilizers of the  $G$ -action are abelian (as they are contained in conjugates of  $T$ ). By [MNN17, Prop. 7.49], we obtain an equivalence of the form (7.1). Alternatively, we can use Proposition 7.3 and the projective or flag bundle formula to see this. Therefore, we can apply Proposition 7.2 to conclude.  $\square$

When  $R$  is a discrete commutative ring, a classical theorem of Swan [Swa60] states that one has Swan induction for the family of *cyclic* subgroups. We give a geometric proof of Swan's theorem in the spirit of some of our other results.

**Theorem 7.5** (Swan [Swa60]). *Let  $R$  be an  $\mathbb{E}_\infty$ -ring which admits an  $\mathbb{E}_1$ -map from  $H\mathbb{Z}$ . Then  $R$ -based Swan induction holds for the family of cyclic groups (for any finite group  $G$ ).*

*Proof.* Without loss of generality, we may take  $R = H\mathbb{Z}$ . By Theorem 7.4 and downward induction based on Remark 7.1, we see that it suffices to consider  $G = C_p^{\times 2}$  for some prime  $p$ . We consider the  $p$ -dimensional projective Heisenberg representation of  $G$  on  $\mathbb{C}^p$ , given by the matrices

$$(7.4) \quad A = \begin{bmatrix} 1 & & & \\ & \zeta_p & & \\ & & \zeta_p^2 & \\ & & & \ddots \\ & & & & \zeta_p^{p-1} \end{bmatrix}, \quad B = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ 1 & & & \end{bmatrix}.$$

Here  $A$  is a diagonal matrix whose eigenvalues are the powers of a primitive  $p$ th root  $\zeta_p$  of unity, and  $B$  is the permutation matrix for a cyclic permutation. Since the matrices  $A$  and  $B$  commute up to scalars, they define a projective representation of  $G$ , yielding an embedding  $G \subseteq \text{PGL}_p(\mathbb{C})$ .

The group  $\text{PGL}_p(\mathbb{C})$  acts naturally on  $\mathbb{CP}^{p-1}$ , and the action of the subgroup  $G \subseteq \text{PGL}_p(\mathbb{C})$  has no fixed points. It follows that the class  $[H\mathbb{Z} \otimes \mathbb{CP}_+^{p-1}]$  in  $\text{Rep}(G, \mathbb{Z})$  is a sum of classes induced from proper subgroups. To calculate the class  $[H\mathbb{Z} \otimes \mathbb{CP}_+^{p-1}]$  in another manner, we can also consider the finite Postnikov filtration  $\{\tau_{\leq 2i}(H\mathbb{Z} \otimes \mathbb{CP}_+^{p-1})\}$  whose successive subquotients are even suspensions  $\Sigma^{2i}\mathbb{Z}$ . The  $G$ -action on each of the (shifted discrete) associated graded terms is trivial because the  $G$ -action extends to an action of the connected group  $\text{PGL}_p(\mathbb{C})$ . Therefore, we find that

$[H\mathbb{Z} \otimes \mathbb{CP}_+^{p-1}] = p \in \text{Rep}(G, \mathbb{Z})$ . It follows that we have integers  $n_H$  for each  $H \subsetneq G$  such that

$$(7.5) \quad p = \sum_{H \subsetneq G} n_H \text{Ind}_H^G(1) \in \text{Rep}(G, \mathbb{Z}). \quad \square$$

**Theorem 7.6.** *Let  $G$  be a finite group. If  $R$  is any  $\mathbb{E}_\infty$ -ring with  $|G| \in \pi_0(R)^\times$ , then  $R$ -based Swan induction holds for the family of cyclic subgroups of  $G$ .*

*Proof.* Without loss of generality, we may take  $R = \mathbb{S}[1/|G|]$ . We have  $\text{Fun}(BG, \text{Perf}(R)) \simeq \text{Perf}(R[G])$ ; equivalently, an  $R[G]$ -module is perfect if and only if its underlying  $R$ -module is perfect, and similarly for every subgroup of  $G$ . This follows from the fact that taking homotopy  $G$ -fixed points commutes with arbitrary colimits in the  $\infty$ -category of  $R$ -modules. Recall that if  $A$  is a connective  $\mathbb{E}_1$ -ring, then the natural map  $K_0(A) \rightarrow K_0(\pi_0 A)$  is an isomorphism. We thus find a chain of isomorphisms

$$\begin{aligned} \text{Rep}(G, R) &= K_0(\text{Fun}(BG, \text{Perf}(R))) \simeq K_0(R[G]) \\ &\simeq K_0(\pi_0 R[G]) \simeq K_0(\text{Fun}(BG, \text{Perf}(\mathbb{Z}[1/|G|]))). \end{aligned}$$

Applying Swan's theorem (Theorem 7.5 above), we conclude the result.  $\square$

Next, we prove a Swan induction result for KU. We give a geometric argument here that only works at small primes; we will prove the result in full generality later in Theorem 7.13.

**Theorem 7.7.** *For any finite group  $G$ , KU-based Swan induction holds for the family of abelian subgroups of  $G$  whose  $p$ -part for  $p \in \{2, 3, 5\}$  has rank  $\leq 2$ .*

*Proof.* In view of Theorem 7.4, we may assume that  $G$  is abelian. We then reduce to proving that one has Swan induction for  $G = C_p^{\times 3}$  and  $p \in \{2, 3, 5\}$  for the family of proper subgroups (cf. Remark 7.1). Since  $p \leq 5$ , we have an embedding  $G \hookrightarrow \Gamma$  for  $\Gamma$  a suitable simply connected compact Lie group whose image is not contained in any maximal torus: by the results of [Bor61], it suffices to choose  $\Gamma$  such that  $H^*(\Gamma; \mathbb{Z})$  has  $p$ -torsion in its cohomology, e.g., we can take  $\Gamma = E_8$ . Let  $T \subseteq \Gamma$  be a maximal torus and consider the  $\Gamma$ -action on the flag variety  $\Gamma/T$ , as well as its restricted  $G$ -action. We will show that the  $G$ -space  $\Gamma/T$  satisfies the hypotheses of Proposition 7.2, for  $\mathcal{F}$  the family of proper subgroups. First of all,  $G$  acts without fixed points since it is not contained in any maximal torus of  $\Gamma$ . By [MNN17, Cor. 8.17], we have an equivalence

$$\Gamma/T_+ \otimes \text{KU} \simeq \bigoplus_1^{|W|} \text{KU} \in \text{Fun}(B\Gamma, \text{Perf}(\text{KU})),$$

where  $W$  is the Weyl group of  $\Gamma$ . Restricting to  $G$ , this proves hypothesis (3) of Proposition 7.2 and thus our result.  $\square$

Next, we include some results which are specific to the prime 2, based on the use of representation spheres; they have the advantage of applying at arbitrary chromatic heights. We first need two lemmas that will enable us to recognize the triviality of group actions (for which we fix an arbitrary prime  $p$ ).

**Lemma 7.8.** *Let  $E$  be an even  $\mathbb{E}_1$ -ring spectrum such that  $\pi_* E$  is  $p$ -torsion-free. Let  $M \in \text{Fun}(BC_p, \text{Perf}(E))$  be such that:*

- (1) *The underlying  $E$ -module of  $M$  is equivalent to a direct sum of copies of  $E$ .*
- (2) *The  $C_p$ -action on  $\pi_* M$  is trivial.*

Then  $M$  is equivalent to a direct sum of copies of the unit in  $\text{Fun}(BC_p, \text{Perf}(E))$ .

*Proof.* Let  $\{x_i\} \subseteq \pi_0 M$  be a basis of the free  $\pi_* E$ -module  $\pi_* M$ . For each  $i$ , we want to produce a  $C_p$ -equivariant map of  $E$ -modules

$$(7.6) \quad E \rightarrow M$$

which carries  $1 \mapsto x_i$  in homotopy. Taking the direct sum of these maps, we will have the desired equivalence. Equivalently, to produce (7.6), we need to show that the image of  $\pi_*(M^{hC_p}) \rightarrow \pi_* M$  contains each  $x_i$ . However, the  $E_2$ -term of the homotopy fixed point spectral sequence for  $\pi_*(M^{hC_p})$  is concentrated in even total degree by our assumptions, and thus collapses. This shows that there are no obstructions to producing the maps (7.6) and thus to providing the equivalence of the lemma.  $\square$

**Lemma 7.9.** *Fix a group  $G$ . Let  $E$  be an  $\mathbb{E}_1$ -ring spectrum such that  $\pi_*(C^*(BG; E))$  is even and  $p$ -torsion-free. Let  $M \in \text{Fun}(B(C_p \times G), \text{Perf}(E))$  be such that:*

- (1) *The underlying object of  $M$  in  $\text{Fun}(BG, \text{Perf}(E))$  is equivalent to a direct sum of copies of the unit.*
- (2) *The  $C_p$ -action on  $\pi_*(M^{hG})$  is trivial.*

*Then  $M$  is equivalent to a direct sum of copies of the unit in  $\text{Fun}(B(C_p \times G), \text{Perf}(E))$ : in particular, the  $C_p \times G$ -action is trivial.*

*Proof.* We use that  $\text{Fun}(B(C_p \times G), \text{Perf}(E)) = \text{Fun}(BC_p, \text{Fun}(BG, \text{Perf}(E)))$ . The thick subcategory of  $\text{Fun}(BG, \text{Perf}(E))$  generated by the unit is equivalent to  $\text{Perf}(C^*(BG; E))$ , via the functor  $(-)^{hG}$ . Thus, the result follows from Lemma 7.8 applied to the object  $M^{hG} \in \text{Fun}(BC_p, \text{Perf}(C^*(BG; E)))$ .  $\square$

**Proposition 7.10.** *Let  $p$  be a prime,  $n \geq 1$  and  $G$  be an elementary abelian  $p$ -group of rank  $n+2$ . Let  $R$  be a  $T(n)$ -local, even  $\mathbb{E}_\infty$ -ring under  $E_n$  such that  $\pi_* R$  is torsion-free. Let  $M \in \text{Fun}(BG, \text{Perf}(R))$ . Suppose that for each  $H \subsetneq G$ , the object  $\text{Res}_H^G M \in \text{Fun}(BH, \text{Perf}(R))$  is equivalent to a direct sum of copies of the unit. Then  $M$  is equivalent to a direct sum of copies of the unit in  $\text{Fun}(BG, \text{Perf}(R))$ .*

The above proposition establishes a very weak result towards the general expectation that the complexity of the representation theory over  $\mathbb{E}_\infty$ -rings should stabilize once the rank of the group is a bit larger than the chromatic complexity of the coefficients. A more subtle such result would be our Conjecture 7.22.

*Proof of Proposition 7.10.* Let  $G' \subsetneq G$  be a maximal proper subgroup, and fix a complement  $C_p \simeq H \subseteq G$ , so that  $G = G' \times H$ . By our assumptions and Lemma 7.9, it suffices to show that the  $H$ -action on  $\pi_*(M^{hG'})$  is trivial (cf. Lemma 4.2, which shows that  $C^*(BG'; R)$  is even and torsion-free).

To see this, we claim that the map of  $C^*(BG', R)$ -modules

$$(7.7) \quad M^{hG'} \rightarrow \prod_{G'' \simeq C_p^{\times n} \subsetneq G'} M^{hG''}$$

is injective on homotopy. Since  $M$  is  $G'$ -equivariantly isomorphic to a sum of copies of the unit, it suffices to verify the injectivity of  $\pi_*(7.7)$  with  $M$  replaced by  $R$ ; this case follows because

$$(7.8) \quad C^*(BG', R) \rightarrow \prod_{G'' \simeq C_p^{\times n} \subsetneq G'} C^*(BG'', R)$$

is injective on  $\pi_*(-) \otimes \mathbb{Q}$  in light of [MNN19, Th. 3.18 and Prop. 5.36] (this is essentially a consequence of the character theory of [HKR00]) since both sides are torsion-free.

For any  $G'' \subsetneq G'$ , we have an induced  $H$ -action on the  $C^*(BG'', R)$ -module  $M^{hG''}$ , and the map in (7.7) is  $H$ -equivariant. Since  $M$  restricts to a direct sum of copies of the unit for every proper subgroup of  $G$ , the  $H$ -action on  $M^{hG''}$  is trivial; indeed, this follows because the action of the *proper* subgroup generated by  $G''$  and  $H$  on  $M$  is trivial. It follows from the injectivity on homotopy of (7.7) and this observation that the  $H$ -action on  $M^{hG'}$  is trivial on homotopy groups. This completes the proof.  $\square$

We now start considering representation spheres. For a based space  $B$ , let  $B\langle n \rangle$  be the  $(n-1)$ st connective cover of  $B$ , so the first potentially non-trivial homotopy group is in degree  $n$ . Let  $\text{MO}\langle n \rangle$  be the Thom spectrum associated to the  $J$ -homomorphism  $\text{BO}\langle n \rangle \rightarrow \text{BO} \rightarrow B\text{GL}_1 \mathbb{S}$ . We define the function  $\phi$  for all integers  $n \geq 1$  via  $\phi(n) = 8a + 2^b$  when  $n = 4a + b + 1$  with  $0 \leq b \leq 3$ .

**Lemma 7.11.** *Let  $n \geq 1$ ,  $G = C_2^{\times n}$  and consider the characters  $\{\eta_i\}_{1 \leq i \leq n}$  of  $G$  obtained by pulling back the sign character along the  $n$  projection maps. Define  $\alpha = \prod_{i=1}^n (1 - \eta_i) \in \text{RO}(G)$ . Then for any  $\text{MO}\langle \phi(n) \rangle$ -oriented  $\mathbb{E}_1$ -ring spectrum  $R$ , there is an equivalence  $S^\alpha \otimes R \simeq R$  in  $\text{Fun}(BG, \text{Perf}(R))$ .*

*Proof.* We claim that the map

$$(7.9) \quad BC_2^{\times n} \xrightarrow{\prod_{i=1}^n (1 - \eta_i)} \text{BO}$$

lifts to  $\text{BO}\langle \phi(n) \rangle$ . This implies that for any  $\text{MO}\langle \phi(n) \rangle$ -oriented  $R$ , there is an equivalence  $S^\alpha \otimes R \simeq R$  in  $\text{Fun}(BG, \text{Perf}(R))$  as desired.

By Bott periodicity,  $\phi(n)$  is the degree of the  $n$ th nonzero homotopy group of  $\text{BO}$ , starting with  $\phi(1) = 1$ . We argue inductively on  $n \geq 1$  to construct the lifting. If  $\eta$  is the sign representation of  $C_2$ , then the virtual representation sphere  $S^{1-\eta}$  is classified by a map

$$BC_2 \simeq \text{BO}(1) \rightarrow \text{BO}$$

which lifts to  $\text{BO}\langle \phi(1) \rangle = \text{BO}\langle 1 \rangle$  because  $BC_2$  is connected. This settles the base case  $n = 1$ .

Suppose now we have a lifting of (7.9) to  $\text{BO}\langle \phi(n) \rangle$  for some  $n \geq 1$ . The next classifying map is obtained as follows:

$$BC_2^{\times n+1} \simeq BC_2 \times BC_2^{\times n} \xrightarrow{(1 - \eta_{n+1}) \otimes \prod_{i=1}^n (1 - \eta_i)} \text{BO}(1) \wedge \text{BO}\langle \phi(n) \rangle \rightarrow \text{BO} \wedge \text{BO} \xrightarrow{\otimes} \text{BO}.$$

Since  $\text{BO}(1) \wedge \text{BO}\langle \phi(n) \rangle$  is  $\phi(n)$ -connected, the composite of the last two maps lifts to  $\text{BO}\langle \phi(n+1) \rangle$  as desired.  $\square$

**Theorem 7.12.** *Let  $G$  be an abelian 2-group. Let  $R$  be an  $\mathbb{E}_\infty$ -ring spectrum. Suppose that for some  $n \geq 1$ , we have either:*

- (1)  $R = E_n$ , a Lubin-Tate theory of height  $n$  at the prime 2.
- (2)  $R$  is  $\text{MO}\langle \phi(n+2) \rangle$ -orientable (as an  $\mathbb{E}_1$ -ring and with  $\phi$  as defined before Lemma 7.11).

*Then  $R$ -based Swan induction holds for the family of subgroups of  $G$  of rank at most  $n+1$ .*

*Proof.* By pulling back along the map to a maximal elementary abelian quotient of  $G$ , it suffices to treat the case where  $G = C_2^{\times n+2}$  and prove that  $R$ -based Swan induction holds for the family of proper subgroups, cf. Remark 7.1. Let  $\varepsilon_1, \dots, \varepsilon_{n+2}$  be independent sign characters  $G \rightarrow \{\pm 1\}$ ,

i.e.,  $\{\varepsilon_i\}$  is a basis for the  $\mathbb{F}_2$ -vector space  $\text{Hom}(G, \{\pm 1\})$ . Let  $\eta_i \in \text{RO}(G)$  ( $1 \leq i \leq n+2$ ) be the class of the associated one-dimensional representation (as in Lemma 7.11). We consider the class

$$\alpha = \prod_{i=1}^{n+2} (1 - \eta_i) \in \text{RO}(G),$$

and the associated representation  $G$ -sphere  $S^\alpha \in \text{Sp}_G$ . Note that for any proper subgroup  $H \subsetneq G$ ,  $\alpha$  restricts to a class in  $\text{RO}(H)$  which is divisible by 2 and therefore comes from the complex representation ring. This follows because  $\alpha$  belongs to the  $(n+2)$ -th power of the augmentation ideal, and for any  $m$ , the augmentation ideal in  $\text{RO}(C_2^{\times m})/2 = \mathbb{F}_2[C_2^{\times m}]$  has its  $(m+1)$ th power equal to zero.

Given a nontrivial character  $\mu$  of  $G$ , considered as a 1-dimensional real representation, the cofiber sequence  $S(\mu)_+ \rightarrow \mathbb{S} \rightarrow S^\mu$  shows that the class  $[S^\mu] \in \text{Rep}(G, \mathbb{S})$  has the property that  $[S^\mu] - 1 = [S(\mu)_+]$  is induced from a proper subgroup, namely the kernel of  $\mu$ . Consequently, if  $V$  is a sum of nontrivial characters in  $\text{RO}(G)$ , then  $[S^V] - 1 \in \text{Rep}(G, \mathbb{S})$  is a sum of classes induced from proper subgroups in  $\text{Rep}(G, \mathbb{S})$ . Expanding out the product defining  $\alpha$  into a sum of characters, we find only a single trivial representation (since the  $\varepsilon_i$  are linearly independent). It follows that in  $\text{Rep}(G, \mathbb{S})$ , one has

$$(7.10) \quad [S^\alpha] = -[S^{\alpha-1}] = -1 + C,$$

for  $C$  a sum of classes induced from proper subgroups.

We also claim that

$$S^\alpha \otimes R \simeq R \text{ in } \text{Fun}(BG, \text{Perf}(R)).$$

Under the first hypothesis, this follows from Proposition 7.10 since the restriction of  $\alpha$  to a proper subgroup is a complex representation sphere, and thus trivializable. Under the second hypothesis, this follows from Lemma 7.11. Consequently,  $[S^\alpha \otimes R] = 1 \in \text{Rep}(G, R)$ . By (7.10), it follows that  $1 = -1 + C$ , so  $2 \in \text{Rep}(G, R)$  is a sum of classes induced from proper subgroups, as desired.  $\square$

Note that this argument cannot work at odd primes, since all representations of  $C_p$  are complex for  $p > 2$ .

*Proof of Theorem E, (3).* This follows from Theorem 7.6 to handle the prime-to-2 case combined with Theorem 7.12, which handles the prime 2.  $\square$

**7.2. Swan induction for KU.** In this subsection, we prove the Swan induction theorem for KU. Note that we have already given (geometric) proofs earlier for the  $p$ -part with  $p \leq 5$ , see Theorem 7.7.

**Theorem 7.13.** *Let  $G$  be any finite group. Then KU-based Swan induction holds for the family of abelian subgroups of rank  $\leq 2$ .*

To prove Theorem 7.13, it suffices (cf. Remark 7.1) to treat the case of  $G = C_p^{\times 3}$  for an arbitrary prime  $p$ . Our proof will rely essentially on twisted  $K$ -theory. We will first need various preliminaries.

**Construction 7.14** (Twists of  $K$ -theory). There is a natural map of anima

$$K(\mathbb{Z}, 3) \rightarrow B\text{GL}_1(\text{KU}),$$

where  $B\text{GL}_1(\text{KU})$  is the classifying space of trivial invertible KU-modules, cf. [ABG10, Sec. 7] for an account, which induces the identity on  $\pi_3$ . Consequently, for any finite group  $G$ , we have a

natural map

$$(7.11) \quad H^3(G; \mathbb{Z}) \rightarrow \text{Pic}(\text{Fun}(BG, \text{Perf}(\text{KU}))),$$

where the right-hand-side is the Picard group of the symmetric monoidal  $\infty$ -category  $\text{Fun}(BG, \text{Perf}(\text{KU}))$ . Given a class  $\tau \in H^3(G; \mathbb{Z})$ , we let  $\text{KU}_\tau$  be the associated object of  $\text{Fun}(BG, \text{Perf}(\text{KU}))$ .

We will be especially interested in the case  $G = C_p^{\times 2}$ . Choosing a nonzero class  $\tau \in H^3(G; \mathbb{Z}) = \mathbb{F}_p$ , we obtain an invertible object  $\text{KU}_\tau \in \text{Fun}(B(C_p^{\times 2}), \text{Perf}(\text{KU}))$ . The induced map  $B(C_p^{\times 2}) \rightarrow B\text{GL}_1(\text{KU})$  is nontrivial, as it lifts uniquely to the 3-connective cover  $\tau_{\geq 3} B\text{GL}_1(\text{KU})$ , and  $K(\mathbb{Z}, 3)$  splits off as a direct factor from here; this means that  $\text{KU}_\tau$  is not equivalent to the unit in  $\text{Fun}(BG, \text{Perf}(\text{KU}))$ .

In the next lemma, to distinguish the factors, we write  $C_p^a \subseteq C_p^{\times 2}$  for the first factor and  $C_p^b \subseteq C_p^{\times 2}$  for the second. Note that  $\text{KU}^0(BC_p^b)$  is isomorphic to the completion of the representation ring  $R(C_p^b)$  at the augmentation ideal by the Atiyah–Segal completion theorem [AS69, Ati61]. If  $\zeta$  is a nontrivial character of  $C_p^b$ , then  $R(C_p^b)$  is free on the powers of  $[\zeta]$ .

**Lemma 7.15.** *Let  $\tau \in H^3(C_p^{\times 2}; \mathbb{Z})$  be a nontrivial element.*

- (1) *The underlying object  $\text{KU}_\tau|_{C_p^b}$  in  $\text{Fun}(BC_p^b, \text{Perf}(\text{KU}))$  is isomorphic to the unit.*
- (2) *The residual  $C_p^a$ -action on  $(\text{KU}_\tau)^{hC_p^b} \simeq C^*(BC_p^b, \text{KU})$  has the property that the action by a generator in  $C_p^a$  acts by multiplication by  $[\zeta]^i$ , for some  $0 < i < p$ .*

*Proof.* The first assertion follows because  $\tau$  restricts to zero in  $H^3(C_p; \mathbb{Z}) = 0$ . The second assertion follows because the action of a generator is necessarily given by multiplication by an element of  $\text{KU}^0(BC_p)$  whose  $p$ th power is the identity. Moreover, this generator is necessarily nontrivial or the entire  $C_p^{\times 2}$ -action on  $\text{KU}_\tau$  would be trivial by Lemma 7.9.  $\square$

Our key tool is the following result. We consider the  $C_p^{\times 2}$ -action on  $\mathbb{CP}^{p-1}$  arising from the projective representation on  $\mathbb{C}^p$  as in the proof of Theorem 7.5. We identify the KU-linearization of this action.

**Proposition 7.16.** *We have a decomposition in  $\text{Fun}(B(C_p^{\times 2}), \text{Perf}(\text{KU}))$*

$$(7.12) \quad \text{KU} \otimes \mathbb{D}\mathbb{CP}_+^{p-1} \simeq \bigoplus_{\tau \in H^3(C_p^{\times 2}; \mathbb{Z})} \text{KU}_\tau.$$

*Proof.* Again, we label the first and second factors of  $C_p^{\times 2}$  by  $C_p^a, C_p^b$ . We first calculate  $\text{KU}_{C_p^b}^*(\mathbb{CP}^{p-1})$ , i.e., the  $C_p^b$ -equivariant KU-theory of  $\mathbb{CP}_+^{p-1}$ . Fix a nontrivial character  $\zeta$  of  $C_p^b$ . The underlying  $C_p^b$ -space of  $\mathbb{CP}_+^{p-1}$  is the projectivization of the representation  $1 \oplus \zeta \oplus \cdots \oplus \zeta^{\otimes(p-1)}$  of  $C_p^b$ .

By the projective bundle theorem, it follows that there is an isomorphism of  $R(C_p^b)$ -algebras,

$$\text{KU}_{C_p^b}^*(\mathbb{CP}^{p-1}) \simeq R(C_p^b)[x] / \prod_{i=0}^{p-1} (x - [\zeta^i]),$$

cf. [Seg68, Prop. 3.9]. Here  $x$  is the class of the tautological line bundle on  $\mathbb{CP}^{p-1}$ , which is (canonically)  $C_p^b$ -equivariant. We have a residual  $C_p^a$ -action on this  $R(C_p^b)$ -algebra. Using the definition of  $x$  as the class of a tautological bundle, one checks that a generator of  $C_p^a$  carries  $x$  to  $x[\zeta^i]$  for an appropriate  $i \neq 0$ . Explicitly, we consider the  $C_p^b$ -equivariant line bundle on  $\mathbb{CP}^{p-1}$  given by the set of pairs  $(x, v)$  for  $x \in \mathbb{CP}^{p-1}$  and  $v \in x$ ; the  $C_p^b$ -equivariant structure is by action

on the pair. The claim follows by noting that  $C_p^a, C_p^b$  act on  $\mathbb{C}^p$ , but their actions fail to commute by a  $p$ th root of unity.

From this, it follows that  $(\mathrm{KU} \otimes \mathbb{D}\mathbb{C}\mathbb{P}_+^{p-1})|_{C_p^b}$  is a direct sum of  $p$  copies of the unit in  $\mathrm{Fun}(BC_p^b, \mathrm{Perf}(\mathrm{KU}))$ . Therefore, we have  $(\mathrm{KU} \otimes \mathbb{D}\mathbb{C}\mathbb{P}_+^{p-1})^{hC_p^b} \simeq \bigoplus_{i=0}^{p-1} C^*(BC_p^b, \mathrm{KU})$ . By the comparison between equivariant and Borel-equivariant  $K$ -theory, and the above calculation, we see that the residual  $C_p^a$  acts on the  $i$ th factor by multiplication by  $[\zeta^i] \in R(C_p^b) \rightarrow \mathrm{KU}^0(BC_p^b)$  (up to renumbering factors).

Now we prove the desired equivalence. It suffices to compare the  $C_p^b$ -homotopy fixed points of both sides of (7.12),  $C_p^a$ -equivariantly as free modules over the even, torsion-free  $\mathbb{E}_\infty$ -ring spectrum  $C^*(BC_p^b; \mathrm{KU})$ . We will do this by a homotopy fixed-point spectral sequence argument. On  $\pi_0$ , we have seen from the previous paragraph and Lemma 7.15 that  $\pi_0((\mathrm{KU} \otimes \mathbb{D}\mathbb{C}\mathbb{P}_+^{p-1})^{hC_p^b})$  and  $\pi_0(\bigoplus_\tau \mathrm{KU}_\tau^{hC_p^b})$  are isomorphic as  $\pi_0(C^*(BC_p^b, \mathrm{KU}))$ -modules equipped with a  $C_p^a$ -action. Using the homotopy fixed point spectral sequence (and observing that there is no room for obstructions since all the summands  $\pi_0(\mathrm{KU}_\tau^{hC_p^b})$  for  $\tau \neq 0$  have trivial higher  $C_p^a$ -cohomology), we can produce  $C_p^a$ -equivariant maps  $\mathrm{KU}_\tau^{hC_p^b} \rightarrow (\mathrm{KU} \otimes \mathbb{D}\mathbb{C}\mathbb{P}_+^{p-1})^{hC_p^b}$  for each  $\tau$ , and the direct sum of these is an equivalence.  $\square$

*Proof of Theorem 7.13.* It suffices to show that KU-based Swan induction holds for the family of proper subgroups of  $C_p^{\times 3}$ . We first observe that  $\mathrm{Rep}(-, \mathrm{KU}) \otimes \mathbb{Q}$  is a Green functor and thus  $\mathrm{Rep}(G, \mathrm{KU}) \otimes \mathbb{Q}$  receives a map from the rationalized Burnside ring  $A(G) \otimes \mathbb{Q}$ . For any finite group  $G$ , we have complementary idempotents  $e_G, \tilde{e}_G$  in  $A(G) \otimes \mathbb{Q}$  (which is isomorphic to a product of copies of  $\mathbb{Q}$  over conjugacy classes of subgroups  $H \subseteq G$ ) such that:

- (1)  $e_G$  is a  $\mathbb{Q}$ -linear combination of the classes of the  $G$ -sets  $G/H, H \subsetneq G$ .
- (2) For each  $H \subsetneq G$ , the restriction of  $e_G$  to  $A(H) \otimes \mathbb{Q}$  is equal to 1. Equivalently, for each  $H \subsetneq G$ , the homomorphism  $A(G) \otimes \mathbb{Q} \rightarrow \mathbb{Q}$  which sends a  $G$ -set  $T$  to  $|T^H|$  carries  $e_G$  to 1.
- (3)  $e_G + \tilde{e}_G = 1$ .

Let  $M$  be a rational Green functor for the group  $G$ , so that we have a ring map  $A(G) \otimes \mathbb{Q} \rightarrow M(G)$ . Then  $M$  is induced from the family of proper subgroups (equivalently,  $1 \in M(G)$  is a sum of classes induced from proper subgroups) if and only if this map carries  $e_G$  to 1 (or, equivalently, sends  $\tilde{e}_G$  to zero); indeed, this follows because multiplication by  $e_G$  acts as the identity on classes induced from a proper subgroup.

Our strategy of proof is to verify this identity in  $\mathrm{Rep}(C_p^{\times 3}, \mathrm{KU}) \otimes \mathbb{Q}$  directly using (7.12), using a relation (proved in the next paragraph) between the idempotent for  $C_p^{\times 2}$  and the class of  $\mathbb{C}\mathbb{P}^{p-1}$ .

Consider  $\mathrm{Rep}(C_p^{\times 2}, R)$  for any  $\mathbb{E}_\infty$ -ring  $R$ . In this case, we have another expression for the image of  $e_{C_p^{\times 2}}$  under  $A(C_p^{\times 2}) \otimes \mathbb{Q} \rightarrow \mathrm{Rep}(C_p^{\times 2}, R) \otimes \mathbb{Q}$  (for which we will simply write  $e_{C_p^{\times 2}}$ ). In fact, we claim that

$$(7.13) \quad [R \otimes \mathbb{C}\mathbb{P}_+^{p-1}]_p = e_{C_p^{\times 2}} \in \mathrm{Rep}(C_p^{\times 2}, R) \otimes \mathbb{Q},$$

for the  $C_p^{\times 2}$ -action on  $\mathbb{C}\mathbb{P}^{p-1}$  arising from the  $p$ -dimensional projective representation as above. In other words,  $p e_{C_p^{\times 2}}$  is the class of  $R \otimes \mathbb{C}\mathbb{P}_+^{p-1} \in \mathrm{Fun}(BC_p^{\times 2}, \mathrm{Perf}(R))$  in rationalized  $K_0$ . To see this, we first observe that any finite  $G$ -CW complex has a well-defined Euler characteristic taking values in  $A(G)$  which can be calculated by taking the Euler characteristic of the cellular chains. Now  $\mathbb{C}\mathbb{P}^{p-1}$  is a finite fixed-point-free  $C_p^{\times 2}$ -complex such that the fixed points under any proper subgroup have Euler characteristic  $p$  (since these fixed points will either be  $p$  distinct points or  $\mathbb{C}\mathbb{P}^{p-1}$ ). This

implies the associated class in  $A(C_p^{\times 2})$  is  $pe_{C_p^{\times 2}}$  as desired, by the above characterization of the idempotent  $e_G$ , whence the claim.

We specialize now to the case where  $R = \text{KU}$ . Let  $\tau$  be a generator of  $H^3(C_p^{\times 2}; \mathbb{Z}) = \mathbb{F}_p$  and let  $x = [\text{KU}\tau]$ . Then, combining (7.13) and the decomposition of Proposition 7.16, we conclude

$$\frac{1+x+\cdots+x^{p-1}}{p} = e_{C_p^{\times 2}} \in \text{Rep}(C_p^{\times 2}, \text{KU}) \otimes \mathbb{Q}.$$

Note that  $x^p = 1$ , so the left hand side is clearly idempotent. This also determines the complementary idempotent; we therefore have

$$(7.14) \quad \frac{1}{p} \prod_{j=1}^{p-1} (1-x^j) = \tilde{e}_{C_p^{\times 2}},$$

since  $\frac{1}{p} \prod_{j=1}^{p-1} (1-x^j)$  is the complementary idempotent to  $\frac{1+x+\cdots+x^{p-1}}{p}$  in the group ring  $\mathbb{Z}[1/p, x]/(x^p - 1) = \mathbb{Z}[1/p, \zeta_p] \times \mathbb{Z}[1/p]$ .

Our goal is to show that  $\tilde{e}_{C_p^{\times 3}} = 0$  in  $\text{Rep}(C_p^{\times 3}, \text{KU}) \otimes \mathbb{Q}$ , which is equivalent to the Swan induction claim. Given an elementary abelian  $p$ -group  $G$  of rank  $\text{rk}(G) \geq 2$ , one has  $\tilde{e}_G = \prod_{\phi: G \twoheadrightarrow G'} \phi^* e_{\tilde{G}'}$ , where the product ranges over all surjections  $G \twoheadrightarrow G'$  with  $\text{rk}(G') = \text{rk}(G) - 1$ . This follows since the given product over all  $\phi$  is an idempotent in  $A(G) \otimes \mathbb{Q}$  with trivial restriction to proper subgroups and with image under the  $G$ -fixed point map  $A(G) \rightarrow \mathbb{Q}$  equal to 1. Therefore, we can express  $\tilde{e}_{C_p^{\times 3}}$  as the product

$$(7.15) \quad \prod_{\phi: C_p^{\times 3} \twoheadrightarrow C_p^{\times 2}} \phi^* \tilde{e}_{C_p^{\times 2}},$$

using the pullback in the representation ring. We will now analyze this using group rings.

For any finite group  $G$ , we have a natural map  $H^3(BG; \mathbb{Z}) \rightarrow \text{Pic}(\text{Fun}(BG, \text{Perf}(\text{KU})))$ , which defines a map of commutative rings

$$(7.16) \quad \varphi_G: \mathbb{Q}[H^3(BG; \mathbb{Z})] \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \text{Rep}(G, \text{KU}),$$

which is compatible with pullback in  $G$ . By (7.14), there exists a class in the group ring  $\mathbb{Q}[H^3(BC_p^{\times 2}; \mathbb{Z})]$  whose image under  $\varphi_{C_p^{\times 2}}$  is precisely the idempotent  $\tilde{e}_{C_p^{\times 2}}$ . Using the expression (7.15), we see that there is a class  $u \in \mathbb{Q}[H^3(BC_p^{\times 3}; \mathbb{Z})]$  whose image under  $\varphi_{C_p^{\times 3}}$  is  $\tilde{e}_{C_p^{\times 3}}$ . Moreover,  $u$  restricts to zero in  $\mathbb{Q}[H^3(BH; \mathbb{Z})]$  for all proper subgroups  $H \subsetneq C_p^{\times 3}$ . By the next two lemmas, this is enough to force  $u = 0$ , which proves the theorem.  $\square$

**Lemma 7.17.** *Let  $X \simeq C_p^{\times 3}$  be a rank 3 elementary abelian  $p$ -group, so  $H^3(X; \mathbb{Z})$  is also a rank 3 elementary abelian  $p$ -group by the Künneth formula. As  $Z \subseteq X$  ranges over the rank 2 subgroups of  $X$ , the maps  $H^3(X; \mathbb{Z}) \rightarrow H^3(Z; \mathbb{Z}) \simeq \mathbb{F}_p$  range over the nonzero maps  $H^3(X; \mathbb{Z}) \rightarrow \mathbb{F}_p$ , up to scalars.*

*Proof.* The construction which sends  $Z \subseteq X$  to the kernel of the surjection  $H^3(X; \mathbb{Z}) \rightarrow H^3(Z; \mathbb{Z})$  establishes a map

$$(7.17) \quad \Psi: \{\text{2-dimensional subspaces } Z \subseteq X\} \rightarrow \{\text{hyperplanes in } H^3(X; \mathbb{Z})\}.$$

We need to show that (7.17) is a bijection. Note that both sides are finite sets of the same cardinality, and that the map is  $\text{Aut}(X) = \text{GL}_3(\mathbb{F}_p)$ -equivariant (using the induced action on  $H^3(X; \mathbb{Z})$ ).

Choose a decomposition  $X = V \oplus W$  where  $V$  has rank 2 and  $W$  has rank 1. By the universal coefficient theorem, we have a natural short exact sequence

$$(7.18) \quad 0 \rightarrow H^3(V; \mathbb{Z}) \rightarrow H^3(X; \mathbb{Z}) \rightarrow \text{Tor}_1(H^2(V; \mathbb{Z}), H^2(W; \mathbb{Z})) \rightarrow 0,$$

where  $\text{Tor}_1(H^2(V; \mathbb{Z}), H^2(W; \mathbb{Z}))$  has rank two. On the left-hand-side of (7.17), we consider the collection  $C$  of subspaces  $Z \subseteq X$  such that the composite  $Z \subseteq X \rightarrow V$  is not surjective; equivalently,  $Z = L \oplus W$  for some  $L \subseteq V$  a 1-dimensional subspace. Note that  $|C| = p+1$ . On the right-hand-side, consider the collection  $D$  of hyperplanes in  $H^3(X; \mathbb{Z})$  which contain  $H^3(V; \mathbb{Z})$ ; the exact sequence (7.18) also easily shows  $|D| = p+1$ .

We claim that  $\Psi^{-1}(D) = C$ . In fact, given a two-dimensional subspace  $Z \subseteq X$  such that  $H^3(V; \mathbb{Z}) \rightarrow H^3(X; \mathbb{Z}) \rightarrow H^3(Z; \mathbb{Z})$  is zero, it follows easily that the map  $Z \rightarrow X \rightarrow V$  fails to be surjective, and conversely. The group  $\text{Aut}(V) \subseteq \text{Aut}(X)$  (via the diagonal embedding, fixing  $W$ ) preserves and acts transitively on  $C$ . Moreover,  $\text{Aut}(V) \subseteq \text{Aut}(X)$  preserves and acts transitively on  $D$ , because we have an  $\text{Aut}(V)$ -equivariant identification  $H^2(V; \mathbb{Z}) \simeq H^1(V; \mathbb{Q}/\mathbb{Z}) = \text{Hom}(V, \mathbb{F}_p)$ , and  $D$  is identified with the set of lines in  $H^2(V; \mathbb{Z})$ . Therefore, for  $c \in C$ , we necessarily have that  $\Psi^{-1}(\Psi(c))$  consists of a single point since the fibers of  $\Psi$  at points of  $D$  must all have the same cardinality. Since  $\text{Aut}(X)$  acts transitively on the set of 2-dimensional subspaces of  $X$ , it follows easily that (7.17) is an isomorphism as desired.  $\square$

**Lemma 7.18.** *Let  $A$  be a finite abelian group. Let  $x \in \mathbb{Q}[A]$  be an element such that for every map  $A \rightarrow C$ , for  $C$  a cyclic group, the image of  $x$  under  $\mathbb{Q}[A] \rightarrow \mathbb{Q}[C]$  is zero. Then  $x = 0$ .*

*Proof.* We can extend scalars to  $\mathbb{C}$ . Then we have a natural isomorphism  $\mathbb{C}[A] \simeq \prod_{A^\vee} \mathbb{C}$ , for  $A^\vee$  the group of characters of  $A$ . Our assumption is that for any map  $C' \rightarrow A^\vee$  with  $C'$  cyclic, the restriction  $\prod_{A^\vee} \mathbb{C} \rightarrow \prod_{C'} \mathbb{C}$  annihilates  $x$ ; this clearly forces  $x = 0$ .  $\square$

**7.3. Applications to chromatic complexity.** In this subsection, we record the applications of the above Swan induction theorems to chromatic bounds for the  $K$ -theory of certain ring spectra. We recover another new proof of Mitchell's theorem and are able to treat some special cases of Theorem A.

**Corollary 7.19** (Mitchell [Mit90]). *For  $i \geq 2$ , we have  $L_{T(i)} K(\mathbb{Z}) \simeq 0$  (for any prime  $p$ ).*

*Proof.* Combine Theorem 6.7 and Theorem 7.5 (Swan induction for  $H\mathbb{Z}$ ).  $\square$

**Corollary 7.20.** *Let  $E_n$  be a height  $n \geq 1$  Lubin-Tate theory at the prime 2 and  $G \subseteq \mathbb{G}_n$  a finite subgroup of the extended Morava stabilizer group. Then  $L_{T(n+m)} K(E_n^{hG}) = 0$  for all  $m \geq 2$ .*

*Proof.* When  $G$  is the trivial subgroup, this follows directly from Theorem 7.12 and Theorem 6.7. The general case then follows from the Galois descent theorem [CMNN20, Thm. 1.10], which gives  $L_{T(n+m)} K(E_n^{hG}) \simeq (L_{T(n+m)} K(E_n))^{hG} \simeq 0$ .  $\square$

We next recover the following result. At  $p \geq 5$ , the result is a consequence of the calculations of Ausoni–Rognes [AR02] and Ausoni [Aus10], which determine the mod  $(p, v_1)$  homotopy groups of  $K(\ell)$  (resp.  $K(\text{ku})$ ) at such primes (and in particular yield the stronger Lichtenbaum–Quillen style claim that  $K(\text{ku})/(p, v_1)$  agrees with its  $T(2)$ -localization in high degrees). For all primes, this result has been recently proved by Angelini-Knoll–Salch [AKS20] and Hahn–Raksit–Wilson [HRW22]. The result is also a special case of Theorem A.

**Corollary 7.21.** *For  $i \geq 3$ , we have  $L_{T(i)} K(\text{KU}) = L_{T(i)} K(\text{KO}) \simeq 0$  (at any prime  $p$ ).*

*Proof.* By Galois descent [CMNN20], it suffices to handle the case of  $\mathrm{KU}$ . In this case, Theorem 6.7 together with Theorem 7.13 imply the result.  $\square$

Motivated by the above results, we conjecture the following Swan induction result for  $E_n$ ; we have proved it at  $p = 2$  in Theorem 7.12, or for  $n = 1$  as a consequence of Theorem 7.13.

**Conjecture 7.22.** Let  $p$  be a prime,  $n \geq 1$ ,  $E_n$  a Lubin-Tate theory of height  $n$  at the prime  $p$  and  $G$  finite group. Then  $E_n$ -based Swan induction holds for the family of those abelian subgroups of  $G$  for which:

- (1) The prime-to- $p$  part is cyclic.
- (2) The  $p$ -part has rank  $\leq n + 1$ .

**Remark 7.23** (A purely algebraic question). Finally, Theorem 6.7 can be used to prove that  $L_{K(1)}K(\mathbb{F}_p) = 0$ , which is a consequence of the stronger result  $K(\mathbb{F}_p; \mathbb{Z}_p) = H\mathbb{Z}_p$  proved by Quillen; indeed, one sees that  $H\mathbb{F}_p$ -based Swan induction holds for the trivial family in  $C_p$  using the filtration of the regular representation  $\mathbb{F}_p[C_p]$  by trivial representations. One also knows that  $L_{K(1)}K(\mathbb{Z}/p^n) = 0$  for any  $n \geq 1$ , cf. [LMMT20, BCM20, Mat21] for three proofs. Can this result also be proved using Theorem 6.7, i.e., does  $H\mathbb{Z}/p^n$ -based Swan induction hold for the trivial family in  $C_p$ ?

#### APPENDIX A. MACKEY FUNCTORS AND ORTHOGONAL $G$ -SPECTRA

This appendix provides a fairly self-contained proof of the fact that, for a finite group  $G$ , the symmetric monoidal  $\infty$ -categories afforded by orthogonal  $G$ -spectra and by spectral Mackey functors are equivalent. This result is due originally to Guillou and May [GM11] (ignoring the monoidal structure), and was revisited by Barwick and Barwick-Glasman-Shah [Bar17, BGS20] in the context of more general parametrized homotopy theory, see specifically [Nar16, Thm. A.4]. Compared to their work, our approach is streamlined by ignoring all models (as used by [GM11]), and by not addressing any universal properties of Mackey functors (as in [Bar17, BGS20]).

The motivation for giving our proof of their result is the immediate need of the present paper: We use categorical methods to construct Mackey functors, and then apply descent results proven for the homotopy theory of orthogonal  $G$ -spectra to them. Our work also yields a new proof of the equivariant Barratt-Priddy-Quillen theorem (which however uses the non-equivariant one).

Throughout, let  $G$  denote a finite group. We refer the reader to [MNN17, Sec. 5] for a quick account of the symmetric monoidal  $\infty$ -category  $\mathrm{Sp}_G$  extracted from the model category of orthogonal  $G$ -spectra. We denote by  $\mathcal{O}(G)$  the orbit category of  $G$ , by  $\mathcal{S}$  the  $\infty$ -category of anima, by  $\mathcal{S}_G := \mathrm{Fun}(\mathcal{O}(G)^{\mathrm{op}}, \mathcal{S})$  the presentable, cartesian closed  $\infty$ -category of  $G$ -anima (see [BH21, Lem. 2.1]), by  $\mathcal{S}_{G,\bullet} \simeq \mathcal{S}_{G,*/}$  the presentable, closed symmetric monoidal  $\infty$ -category of based  $G$ -anima, and by  $\Sigma_G^\infty: \mathcal{S}_{G,\bullet} \rightarrow \mathrm{Sp}_G$  the suspension spectrum functor. We consider  $\mathcal{S}_G$  with its cartesian monoidal structure. In the appendix, we will write the units of  $G$ -spectra and spectral Mackey functors by  $1$  rather than  $\mathbb{S}$ .

To set the notation for Mackey functors, we denote by  $\mathrm{Fin}_G$  the category of finite  $G$ -sets, by  $\mathrm{Span}(\mathrm{Fin}_G)$  the  $(2, 1)$ -category of spans on  $\mathrm{Fin}_G$  (cf. [BH21, App. C]) and set

$$\mathrm{Mack}_G := \mathrm{Mack}_G(\mathrm{Sp}) = \mathrm{Fun}^\times(\mathrm{Span}(\mathrm{Fin}_G)^{\mathrm{op}}, \mathrm{Sp}),$$

the category of finite product-preserving presheaves on  $\mathrm{Span}(\mathrm{Fin}_G)$  with values in the  $\infty$ -category  $\mathrm{Sp}$  of spectra. Note that  $\mathrm{Span}(\mathrm{Fin}_G) = \mathrm{Burn}_G^{\mathrm{eff}}$  as recalled in Definition 2.1, but we stick to the former notation in the Appendix, in order to be compatible with our main references. We recall

that  $\text{Mack}_G \simeq \mathcal{P}_\Sigma(\text{Span}(\text{Fin}_G)) \otimes \text{Sp}$ , cf. Remark 2.3. Below, we will recall the suspension functor in the Mackey context, to be denoted

$$\Sigma_{\mathcal{M}}^\infty : \mathcal{S}_{G,\bullet} \longrightarrow \text{Mack}_G.$$

The cartesian product on  $\text{Fin}_G$  induces a symmetric monoidal structure on  $\text{Span}(\text{Fin}_G)$ , we endow  $\text{Mack}_G$  with the symmetric monoidal structure given by Day-convolution and denote it by  $\otimes$ . This is the unique symmetric monoidal structure which is bicocontinuous and is such that  $\Sigma_{\mathcal{M}}^\infty$  is symmetric monoidal.

Our main result is the following.

**Theorem A.1.** *There is a unique symmetric monoidal left-adjoint  $L : \text{Sp}_G \rightarrow \text{Mack}_G$  such that  $L \circ \Sigma_G^\infty \simeq \Sigma_{\mathcal{M}}^\infty$ , and  $L$  is an equivalence.*

The rest of this section will provide a proof of this result.

The construction of  $L$  rests on the following result, and we thank Markus Hausmann for providing a key reference in its proof. Compare also [GM20, App. C] for a treatment.

**Theorem A.2.** *The suspension  $\Sigma_G^\infty : \mathcal{S}_{G,\bullet} \longrightarrow \text{Sp}_G$  is the initial example of a presentably symmetric monoidal functor (in other words, a map in  $\text{CAlg}(\text{Pr}^L)$ ) which inverts the functor  $S^V \otimes -$  for all finite-dimensional, orthogonal representations  $V$  of  $G$ .*

*Proof.* We have the symmetric monoidal suspension functor  $\Sigma_G^\infty : \mathcal{S}_{G,\bullet} \longrightarrow \text{Sp}_G$ . By construction of  $\text{Sp}_G$ , the representation spheres map to invertible objects in  $\text{Sp}_G$ . Now the initial presentably symmetric monoidal  $\infty$ -category  $\mathcal{C}$  equipped with a cocontinuous, symmetric monoidal functor from  $\mathcal{S}_{G,\bullet}$  inverting the representation spheres is discussed (in a more general context) in [Rob15, Sec. 2.1], cf. also [BH21, Lem. 4.1]; note that the representation spheres are symmetric objects by [GM20, Lem. C.5] and that the required cyclic invariance condition is easily checked. Equivalently, we can also perform this construction at the level of small finitely cocomplete symmetric monoidal  $\infty$ -categories by restricting to the compact objects. By [Rob15, Prop. 2.19, Cor. 2.22], we find that this formal inversion (in  $\text{Pr}_{\text{st}}^L$ ) is given by the colimit of smashing with  $S^V$  on  $\mathcal{S}_{G,\bullet}$ , as  $V$  ranges over  $G$ -representations. By construction, we obtain a canonical, cocontinuous symmetric monoidal functor  $\mathcal{C} \rightarrow \text{Sp}_G$ . It follows from the above that the mapping anima between the finite  $G$ -sets  $T, T'$  are computed in the same way (namely, as  $\varinjlim_V \text{Hom}_{\mathcal{S}_{G,\bullet}}(S^V \wedge T_+, S^V \wedge T'_+)$ ), so  $\mathcal{C} \rightarrow \text{Sp}_G$  is fully faithful on compact generators, whence the result.  $\square$

To construct  $\Sigma_{\mathcal{M}}^\infty$ , recall from [BH21, §9.1 before Lem. 9.4] the canonical cartesian monoidal functor  $\iota : \text{Fin}_{G,+} \rightarrow \text{Span}(\text{Fin}_G)$  and the symmetric monoidal equivalence  $\mathcal{P}_\Sigma(\text{Fin}_{G,+}) \simeq \mathcal{S}_{G,\bullet}$  ([BH21, Lem. 2.1]). This induces  $\Sigma_{\mathcal{M}}^\infty$ , to be defined as the composition

$$\Sigma_{\mathcal{M}}^\infty := \left( \mathcal{S}_{G,\bullet} \simeq \mathcal{P}_\Sigma(\text{Fin}_{G,+}) \xrightarrow{\mathcal{P}_\Sigma(\iota)} \mathcal{P}_\Sigma(\text{Span}(\text{Fin}_G)) = \text{Fun}^\times(\text{Span}(\text{Fin}_G)^{\text{op}}, \mathcal{S}) \rightarrow \text{Mack}_G \right),$$

where the final map is the stabilization. By construction,  $\Sigma_{\mathcal{M}}^\infty$  is a map in  $\text{CAlg}(\text{Pr}^L)$ .

Theorem A.2 tells us that to construct the functor  $L$  in Theorem A.1, we need to see that  $\Sigma_{\mathcal{M}}^\infty$  inverts all representation spheres. We will do this by constructing from scratch on  $\text{Mack}_G$  what will a posteriori turn out to be geometric fixed point functors, and by establishing some of their basic properties. Denote by  $\mathcal{P}$  the family of proper subgroups of  $G$  and recall the cofiber sequence in  $\mathcal{S}_{G,\bullet}$ , defining  $\widetilde{E\mathcal{P}}$ :

$$\varinjlim_{G/H \in \mathcal{O}(G)_\mathcal{P}} G/H_+ \simeq E\mathcal{P}_+ \longrightarrow *_+ = S^0 \longrightarrow \widetilde{E\mathcal{P}},$$

(cf. [MNN19, Appendix A.1]). The least formal part of our argument is the following.

**Lemma A.3.** *We have an equivalence  $(\Sigma_{\mathcal{M}}^{\infty}(\widetilde{EP}))_{(G/G)} \simeq \mathbb{S}$  in  $\text{Sp}$  (since the source is an  $\mathbb{E}_{\infty}$ -ring, the equivalence is uniquely specified).*

To see this, we will need the following result on manipulating colimits. For an  $\infty$ -category  $\mathcal{C}$ , we denote by  $\mathcal{C}^{\triangleright}$  the result of freely adjoining a final object to  $\mathcal{C}$ , and by  $\mathcal{C}^{\simeq}$  the maximal underlying subgroupoid of  $\mathcal{C}$ . The construction  $\mathcal{C} \mapsto \mathcal{C}^{\simeq}$  is right adjoint to the inclusion  $\mathcal{S} \simeq \mathcal{G}rp_{\infty} \subseteq \mathcal{C}at_{\infty}$  of  $\infty$ -groupoids into all  $\infty$ -categories.

**Proposition A.4.** *Let  $\mathcal{C}$  be an  $\infty$ -category and  $F: \mathcal{C}^{\triangleright} \rightarrow \mathcal{S}$  the functor defined by  $F(c) = ((\mathcal{C}^{\triangleright})_{/c})^{\simeq}$ . Then the canonical map of anima  $\varinjlim_{\mathcal{C}} F \rightarrow \varinjlim_{\mathcal{C}^{\triangleright}} F$  is equivalent to the inclusion  $\mathcal{C}^{\simeq} \subseteq (\mathcal{C}^{\triangleright})^{\simeq}$ .*

*Proof.* The closely related functor  $F': \mathcal{C}^{\triangleright} \rightarrow \mathcal{C}at_{\infty}$  defined by  $F'(c) := (\mathcal{C}^{\triangleright})_{/c}$  classifies the cocartesian codomain fibration  $cd: \text{Fun}(\Delta^1, \mathcal{C}^{\triangleright}) \rightarrow \mathcal{C}^{\triangleright}$  given by evaluation on 1 [Lur09, Cor. 2.4.7.12]. It follows that  $F = (-)^{\simeq} \circ F'$  classifies the left fibration  $cd': \text{Fun}(\Delta^1, \mathcal{C}^{\triangleright})^{\text{left}} \rightarrow \mathcal{C}^{\triangleright}$  obtain by passing from  $\text{Fun}(\Delta^1, \mathcal{C}^{\triangleright})$  to the sub-simplicial set  $\text{Fun}(\Delta^1, \mathcal{C}^{\triangleright})^{\text{left}} \subseteq \text{Fun}(\Delta^1, \mathcal{C}^{\triangleright})$  consisting of all simplices all of whose edges are  $cd$ -cocartesian. Informally then, the objects of  $\text{Fun}(\Delta^1, \mathcal{C}^{\triangleright})^{\text{left}}$  are the morphisms in  $\mathcal{C}^{\triangleright}$ , and the morphisms are the commuting squares in which the map between sources is an equivalence.

We now observe that evaluation at zero,  $ez: \text{Fun}(\Delta^1, \mathcal{C})^{\text{left}} \rightarrow \mathcal{C}^{\simeq}$ , is a Cartesian fibration which has all fibers contractible (because each of them has an initial object). In particular,  $ez$  is a weak equivalence, and an inverse equivalence is provided by sending objects to identity morphisms. We have thus seen that  $\varinjlim_{\mathcal{C}} F \simeq \mathcal{C}^{\simeq}$ .

Furthermore, the canonical map  $\varinjlim_{\mathcal{C}} F \rightarrow \varinjlim_{\mathcal{C}^{\triangleright}} F$  is equivalent to the obvious map  $\text{Fun}(\Delta^1, \mathcal{C})^{\text{left}} \rightarrow \text{Fun}(\Delta^1, \mathcal{C}^{\triangleright})^{\text{left}}$ , the target of which is equivalent to the fiber over the cone point, namely  $(\mathcal{C}^{\triangleright})^{\simeq}$ . One checks that this identifies the canonical map with the inclusion  $\mathcal{C}^{\simeq} \subseteq (\mathcal{C}^{\triangleright})^{\simeq}$ , as claimed.  $\square$

*Proof of Lemma A.3.* Applying Proposition A.4 with  $\mathcal{C} = \mathcal{O}(G)_{\mathcal{P}}$  the category of orbits with proper isotropy (hence  $\mathcal{C}^{\triangleright} = \mathcal{O}(G)$ ), we obtain a cofiber sequence in anima

$$\varinjlim_{G/H \in \mathcal{O}(G)_{\mathcal{P}}} (\mathcal{O}(G)/(G/H))^{\simeq} \simeq \mathcal{O}(G)_{\mathcal{P}}^{\simeq} \hookrightarrow \mathcal{O}(G)^{\simeq} \longrightarrow * \sqcup +$$

where the final map sends all orbits with proper isotropy to  $+$ , and sends  $G/G$  to  $*$ . We can consider this as a cofiber sequence in pointed anima  $\mathcal{S}_{\bullet}$  of the form

$$\varinjlim_{G/H \in \mathcal{O}(G)_{\mathcal{P}}} (\mathcal{O}(G)/(G/H))_{+}^{\simeq} \simeq \mathcal{O}(G)_{\mathcal{P},+}^{\simeq} \longrightarrow \mathcal{O}(G)_{+}^{\simeq} \longrightarrow S^0 = *_{+}$$

where the final map sends all orbits with proper isotropy to the base-point  $+$ , and sends  $G/G$  to  $*$ . Applying the free commutative monoid functor  $\mathbb{P}: \mathcal{S}_{\bullet} \rightarrow \text{CMon}(\mathcal{S})$  yields a cofiber sequence in  $\text{CMon}(\mathcal{S})$ :

$$(A.1) \quad \varinjlim_{G/H \in \mathcal{O}(G)_{\mathcal{P}}} (\text{Fin}_G/(G/H))^{\simeq} \longrightarrow \text{Fin}_{\widetilde{G}}^{\simeq} \longrightarrow \text{Fin}^{\simeq}$$

in which the final map is identified with taking  $G$ -fixed points. To see this, observe that  $\mathcal{O}(G)_{/(G/H)} \simeq \mathcal{O}(H)$  and that  $\mathbb{P}(\mathcal{O}(H)_{+}^{\simeq}) \simeq \text{Fin}_{\widetilde{H}}^{\simeq}$ , as can be checked most easily using the general formula  $\mathbb{P}(Z) = \bigvee_{n \geq 0} (Z^{\times n} \times_{\Sigma_n} E\Sigma_{n+})$ .

We denote by  $(-)^+$  the group completion on  $\text{CMon}(\mathcal{S})$ , and observe that

$$\Omega^\infty(\Sigma_{\mathcal{M}}^\infty(G/H_+)(G/G)) = \text{Hom}_{\text{Span}(\text{Fin}_G)}(G/G, G/H)^+ \simeq (\text{Fin}_G/(G/H))^{\simeq, +}.$$

We thus see that the delooping of the group completion of the cofiber sequence (A.1) is a cofiber sequence in  $\text{Sp}$  of the form

$$(\Sigma_{\mathcal{M}}^\infty(E\mathcal{P}))(G/G) \longrightarrow \Sigma_{\mathcal{M}}^\infty(S^0)(G/G) \longrightarrow \Sigma_{\mathcal{M}}^\infty(\widetilde{E\mathcal{P}})(G/G) \simeq \mathbb{S},$$

using the Barratt–Priddy–Quillen theorem that  $\Omega^\infty(\mathbb{S}) \simeq \text{Fin}^{\simeq, +}$ .  $\square$

Next, we will need to discuss restriction for Mackey functors.

**Construction A.5** (Restriction for Mackey functors). Let  $H \subseteq G$  be a subgroup.

- (1) We have a symmetric monoidal and coproduct preserving functor

$$\text{Res}_H^G: \text{Span}(\text{Fin}_G) \longrightarrow \text{Span}(\text{Fin}_H).$$

This sends a  $G$ -set  $U$  to the underlying  $H$ -set of  $U$ , and behaves accordingly on correspondences (cf. [BH21, App. C.3]).

- (2) We also have a functor

$$G \times_H (-): \text{Span}(\text{Fin}_H) \longrightarrow \text{Span}(\text{Fin}_G),$$

which takes a  $H$ -set  $T$  to the  $G$ -set  $G \times_H T$ , and behaves analogously on correspondences. Note that the construction  $T \mapsto G \times_H T$  on finite  $H$ -sets preserves fiber products.

**Proposition A.6.** *Both the functors  $(\text{Res}_H^G(-), G \times_H (-)): \text{Span}(\text{Fin}_G) \rightleftarrows \text{Span}(\text{Fin}_H)$  and the functors  $(G \times_H (-), \text{Res}_H^G(-)): \text{Span}(\text{Fin}_H) \rightleftarrows \text{Span}(\text{Fin}_G)$  are biadjoint.*

*Proof.* If  $S$  is a finite  $H$ -set and  $T$  is a finite  $G$ -set, then we have an equivalence of categories

$$(\text{Fin}_G)_{/(G \times_H S) \times T} \xrightarrow{\sim} (\text{Fin}_H)_{/S \times \text{Res}_H^G(T)},$$

given by pulling back along the  $H$ -map  $S \times T \rightarrow (G \times_H S) \times T$ . Using that  $\text{Hom}_{\text{Span}(\text{Fin}_G)}(X, Y) \simeq (\text{Fin}_G)_{/X \times Y}^{\sim}$ , the result follows. See also [BH21, App. C.3] for a more general treatment of  $\text{Span}(-)$  as an  $(\infty, 2)$ -functor.  $\square$

We now define restriction for Mackey functors, essentially by left Kan extension. Namely, we define the symmetric monoidal, cocontinuous functor  $\text{Res}_{H, \mathcal{M}}^G: \text{Mack}_G \rightarrow \text{Mack}_H$  to be  $\text{Res}_{H, \mathcal{M}}^G := \mathcal{P}_\Sigma(\text{Res}_H^G) \otimes \text{id}_{\text{Sp}_G}$ .

As an example, note that for  $F \in \text{Mack}_G$  and subgroups  $H' \subseteq H \subseteq G$  we have

$$\text{Res}_{H, \mathcal{M}}^G(F)(H/H') \simeq F(G/H').$$

To see this, since both sides are colimit preserving functors of  $F$ , it suffices to check the case when  $F$  is the suspension of an orbit, and then the claim is immediate from the second adjunction in Proposition A.6.

**Proposition A.7.** *Assume  $F \in \text{Mack}_G$  is such that for all proper subgroups  $H \subseteq G$  we have  $\text{Res}_{H, \mathcal{M}}^G(F) \simeq *$ . Then the canonical map*

$$F(G/G) \longrightarrow \left( \Sigma_{\mathcal{M}}^\infty(\widetilde{E\mathcal{P}}) \otimes F \right) (G/G)$$

*is an equivalence.*

*Proof.* First observe that for all subgroups  $H \subseteq G$ , the suspension  $\Sigma_{\mathcal{M}}^{\infty}(G/H_+) \in \text{Mack}_G$  is self-dual. It then follows that for all proper subgroups  $H \subseteq G$ , the spectrum

$$(F \otimes \Sigma_{\mathcal{M}}^{\infty}(G/H_+))(G/G) \simeq F(G/H) \simeq \text{Res}_{H,\mathcal{M}}^G(F)(H/H) \simeq *$$

is contractible, and hence that  $(F \otimes \Sigma_{\mathcal{M}}^{\infty}(E\mathcal{P}_+))(G/G) \simeq *$ . The result follows.  $\square$

We next introduce geometric fixed points in the Mackey context. The fixed point functor  $(-)^G : \text{Fin}_G \rightarrow \text{Fin}$  commutes with pullbacks and hence induces a functor on span categories. This functor preserves finite coproducts and the cartesian product, hence the functor

$$\Phi_{\mathcal{M}}^G := \mathcal{P}_{\Sigma}(\text{Span}((-)^G)) \otimes \text{id}_{\text{Sp}} : \text{Mack}_G \rightarrow \mathcal{P}_{\Sigma}(\text{Span}(\text{Fin})) \otimes \text{Sp} \simeq \text{Sp}$$

commutes with all colimits and is symmetric monoidal. By construction, it takes the expected values on orbits, namely  $\Phi_{\mathcal{M}}^G(\Sigma_{\mathcal{M}}^{\infty}(G/H_+))$  is contractible for a proper subgroup  $H \subseteq G$ , and equivalent to  $\mathbb{S}$  for  $H = G$ . In fact, more generally, for each  $X \in \mathcal{S}_{G,\bullet}$  we have

$$\Phi_{\mathcal{M}}^G(\Sigma_{\mathcal{M}}^{\infty}(X)) \simeq \Sigma^{\infty}(X(G/G)).$$

For a subgroup  $H \subseteq G$ , we denote  $\Phi_{\mathcal{M}}^{G,H} := \Phi_{\mathcal{M}}^H \circ \text{Res}_{H,\mathcal{M}}^G$ .

We will need to know that our geometric fixed points are given by the familiar construction:

**Proposition A.8.** *There is an equivalence<sup>5</sup> of functors  $\Phi_{\mathcal{M}}^G(-) \simeq (\Sigma_{\mathcal{M}}^{\infty}(\widetilde{E\mathcal{P}}) \otimes (-))(G/G)$ .*

*Proof.* First, we have a natural transformation for  $F \in \text{Mack}_G$  given by  $F(G/G) \rightarrow \Phi_{\mathcal{M}}^G(F)$ , since  $F(G/G)$  is corepresented by the unit and  $\Phi_{\mathcal{M}}^G(-)$  is symmetric monoidal by construction. Since the target is unaffected by replacing  $F$  by  $\Sigma_{\mathcal{M}}^{\infty}(\widetilde{E\mathcal{P}} \otimes F)$ , we obtain a map  $(\Sigma_{\mathcal{M}}^{\infty}(\widetilde{E\mathcal{P}} \otimes (-)))(G/G) \rightarrow \Phi_{\mathcal{M}}^G(-)$ . This map is an equivalence on all orbits, because for a subgroup  $H \subseteq G$  we can compute that

$$(\Sigma_{\mathcal{M}}^{\infty}(\widetilde{E\mathcal{P}}) \otimes \Sigma_{\mathcal{M}}^{\infty}(G/H_+))(G/G) \simeq (\Sigma_{\mathcal{M}}^{\infty}(\widetilde{E\mathcal{P}} \otimes (G/H_+)))(G/G)$$

is contractible if  $H$  is proper, and is  $\mathbb{S}$  if  $H = G$  by Lemma A.3. Therefore, the result follows since both functors preserve colimits.  $\square$

This allows to easily establish the basic properties of geometric fixed points in the Mackey context:

**Proposition A.9.** *The family  $\{\Phi_{\mathcal{M}}^{G,H}\}_{H \subseteq G}$  of symmetric monoidal left adjoints is jointly conservative.*

*Proof.* Assume  $\Phi_{\mathcal{M}}^{G,H}(F) \simeq *$  for all  $H \subseteq G$ , and we need to see that  $F \simeq *$ .

This is clear for trivial  $G$ , and we argue by induction on the group order in general. We can thus assume that  $\text{Res}_{H,\mathcal{M}}^G(F) \simeq *$  for all proper subgroups  $H \subseteq G$ . In particular then, for all proper subgroups  $H \subseteq G$  we know that

$$F(G/H) = \text{Res}_{H,\mathcal{M}}^G(F)(H/H) = *$$

is contractible, and need to see that  $F(G/G)$  is as well. But combining Proposition A.7 and Proposition A.8, we see that  $F(G/G) \simeq \Phi_{\mathcal{M}}^G(F) = \Phi_{\mathcal{M}}^{G,G}(F)$ , and this is contractible by assumption.  $\square$

This finally lets us check that suspension for Mackey functors inverts all representation spheres.

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<sup>5</sup>The equivalence is constructed in the proof, we will only need an abstract equivalence.

**Proposition A.10.** *For every representation  $V$  of  $G$ ,  $\Sigma_{\mathcal{M}}^{\infty}(S^V) \in \text{Mack}_G$  is invertible.*

*Proof.* We first note that  $\Sigma_{\mathcal{M}}^{\infty}(S^V) \in \text{Mack}_G$  is at least dualizable. Since  $\text{Mack}_G$  is stable, the dualizable objects are stable under finite colimits, and  $S^V$  is a finite colimits of orbits. It thus suffices to remark that the orbits are dualizable (in fact, self-dual) already in  $\text{Span}(\text{Fin}_G)$ . Once we know  $\Sigma_{\mathcal{M}}^{\infty}(S^V) \in \text{Mack}_G$  is dualizable, it will be invertible if and only if it becomes so after applying any family of jointly conservative symmetric monoidal functors. By Proposition A.9 it will thus suffice to see that for every subgroup  $H \subseteq G$ , the spectrum  $\Phi_{\mathcal{M}}^{G,H}(\Sigma_{\mathcal{M}}^{\infty}(S^V))$  is invertible, but this follows from a direct computation:

$$\Phi_{\mathcal{M}}^{G,H}(\Sigma_{\mathcal{M}}^{\infty}(S^V)) = \Phi_{\mathcal{M}}^H(\text{Res}_{H,\mathcal{M}}^G(\Sigma_{\mathcal{M}}^{\infty}(S^V))) \simeq \Sigma^{\infty}((S^V)^H) \simeq S^{\dim(V^H)}. \quad \square$$

We can now complete the proof of our main result.

*Proof of Theorem A.1.* Theorem A.2 and Proposition A.10 show that there is a unique symmetric monoidal left adjoint  $L : \text{Sp}_G \longrightarrow \text{Mack}_G$  such that  $L \circ \Sigma_G^{\infty} \simeq \Sigma_{\mathcal{M}}^{\infty}$ . It remains to see that  $L$  is an equivalence. Denote by  $R$  the right adjoint of  $L$ . Since both  $\text{Sp}_G$  and  $\text{Mack}_G$  are generated under colimits by dualizable objects (namely the suspensions of orbits), it follows from [BDS16, Thm. 1.3] that  $R$  admits itself a right adjoint, hence preserves colimits, and that the adjunction  $(L, R)$  satisfies a projection formula. Furthermore,  $R$  is conservative because the image of its left adjoint  $L$  contains a set of generators. We can thus apply [MNN17, Prop. 5.29] to conclude that the adjunction  $(L, R)$  induces an adjoint equivalence

$$\text{Mod}_{\text{Sp}_G}(R(1_{\text{Mack}_G})) \simeq \text{Mack}_G,$$

and it remains to see that the counit of the adjunction

$$(A.2) \quad 1_{\text{Sp}_G} \longrightarrow R(L(1_{\text{Sp}_G})) \simeq R(1_{\text{Mack}_G})$$

is an equivalence.

Now we use induction on the group order. Given a proper subgroup  $H \subsetneq G$ , we have a commutative diagram in  $\text{CAlg}(\text{Pr}^L)$ ,

$$\begin{array}{ccc} \text{Sp}_G & \longrightarrow & \text{Mack}_G \\ \downarrow & & \downarrow \\ \text{Sp}_H & \longrightarrow & \text{Mack}_H, \end{array}$$

by the universal property of  $\text{Sp}_G$ . The inductive hypothesis gives that the bottom horizontal arrow is an equivalence. This implies that if  $X \in \text{Sp}_G$ , then  $\text{Hom}_{\text{Sp}_G}(G/H_+, X) = \text{Hom}_{\text{Mack}_G}(G/H_+, X)$  since both sides are calculated as maps out of the unit in  $\text{Sp}_H$  (resp.  $\text{Mack}_H$ ). In particular, this implies that (A.2) restricts to an equivalence after restriction to proper subgroups; therefore, it suffices to see that  $\Phi^G$  (i.e., geometric fixed points for orthogonal spectra) turns this map into an equivalence. Since  $\Phi^G(1_{\text{Sp}_G}) = \mathbb{S}$  and we are looking at a map of commutative algebras, it suffices in fact to see that there is an equivalence of spectra  $\Phi^G(R(1_{\text{Mack}_G})) \simeq \mathbb{S}$ . This follows from the following computation:

$$\begin{aligned} \Phi^G(R(1_{\text{Mack}_G})) &\simeq \left( \Sigma_G^{\infty}(\widetilde{E\mathcal{P}}) \otimes R(1_{\text{Mack}_G}) \right)^G \simeq \left( R \left[ L(\Sigma_G^{\infty}(\widetilde{E\mathcal{P}})) \otimes 1_{\text{Mack}_G} \right] \right)^G \simeq \\ &\simeq L(\Sigma_G^{\infty}(\widetilde{E\mathcal{P}}))(G/G) \simeq (\Sigma_{\mathcal{M}}^{\infty}(\widetilde{E\mathcal{P}}))(G/G) \simeq \mathbb{S}. \end{aligned}$$

This computation used in turn: The definition of  $\Phi^G$ , the projection formula for  $(L, R)$ , the fact that  $(R(-))^G \simeq (-)(G/G)$  (by adjointness of  $L$  and  $R$ ), the fact that  $L \circ \Sigma_G^\infty \simeq \Sigma_M^\infty$ , and finally Lemma A.3.  $\square$

As promised earlier, our account yields the following proof of the equivariant Barratt–Priddy–Quillen theorem (originally due to [GM11]), which by-passes any loop-space theory (but uses the non-equivariant version).

**Corollary A.11.** *For a finite group  $G$ , there is an equivalence in  $\text{CMon}(\mathcal{S})$*

$$\varinjlim_V \Omega^V S^V \simeq (\text{Fin}_G)^{\simeq, +},$$

where the colimit is taken along any cofinal system of representations of  $G$ , and  $(-)^+$  denotes group completion.

For the proof, one simply computes the endomorphism anima of the unit of both  $\text{Sp}_G$  and  $\text{Mack}_G$  from the definition, and compares the result.

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