The Fokas Method for the Well-posedness of Nonlinear Dispersive Equations in Domains with a Boundary

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Dedicated to Professor Athanassios S. Fokas

Abstract The Fokas method, also known as the unified transform, is one of the most remarkable breakthroughs noted in the study of linear and integrable nonlinear partial differential equations at the turn of the new millennium. Its numerous implications, along with the elegance of the ideas forming its foundation, led the great Israel Gelfand to once describe it as one of the most exciting developments in the area of partial differential equations since the time of Fourier. In this article, we offer further evidence in support of that statement by elucidating the analogy between the Fokas method and the celebrated Fourier transform in the context of both linear and nonlinear dispersive equations. First, we review the Fokas-Gelfand derivation of the Fourier transform pair via the technique of inverse scattering but applied to linear (as opposed to nonlinear) equations—an idea that subsequently contributed to the development of the linear component of the Fokas method. Then, we discuss a novel approach for proving the well-posedness of initial-boundary value problems for general nonlinear (i.e. not necessarily integrable) dispersive equations. This approach utilizes the Fokas method in analogy with the way that the Fourier transform is used in the classical harmonic analysis-based approach for proving the well-posedness of the initial value problem for these nonlinear equations. In this regard, the new approach further establishes the Fokas method as the direct analogue of the Fourier transform in domains with a boundary.

Keywords Fokas method · Unified transform · Dispersive equations · NLS · KdV · Well-posedness · Initial-boundary value problems · Fourier transform · Inverse scattering transform

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1 Introduction

Dispersive partial differential equations describe phenomena in which waves of different wavelengths propagate at different speeds. Two prominent examples are the Korteweg-de Vries (KdV) equation[1]

$$u_t + u_{xxx} + uu_x = 0 \tag{1}$$

and the (cubic) nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} \pm |u|^2 u = 0. \tag{2}$$

In the above equations, $u = u(x,t)$ is a function of space x and time t, with the various indices denoting partial derivatives with respect to the relevant variable. Moreover, in Eq. (2), the positive sign in front of the nonlinearity corresponds to the focusing NLS and the negative sign to the defocusing NLS. Both KdV and NLS are nonlinear evolution equations that arise as approximations, under certain regimes, of the fundamental Euler equations for incompressible and inviscid flow. Furthermore, NLS has a ubiquitous presence in mathematical physics, being a central model in such diverse areas as optics, plasmas, and Bose-Einstein condensates.

When considered on the infinite line $-\infty < x < \infty$, Eqs. (1) and (2) must be supplemented with an initial condition of the form

$$u(x,0) = u_0(x) \tag{3}$$

for some given function $u_0(x)$. This is known as the initial value problem (IVP) or Cauchy problem. One can then ask whether such an IVP can be solved and, if so, how and in what sense. In particular, one can also ask whether the choice of initial data $u_0(x)$ affects the solvability of the IVP and the various properties of its solution.

A key element in regard to the above questions is the fact that both KdV and NLS have a remarkably rich structure as completely integrable systems. For KdV, this feature was discovered by Gardner et al. [18], and for NLS it was established by Zakharov and Shabat [43], while the seminal 1968 work of Lax [36] provided a solid theoretical framework for studying completely integrable equations with the help of what are nowadays known as Lax pairs. These are systems of linear equations that allow integrable nonlinear equations to be "linearized" by means of expressing them as compatibility conditions of these linear systems. For example, a Lax pair for the KdV Eq. (1) is given by the 2×2 linear system

$$\begin{aligned} &\mu_{xx} + \left(\tfrac{1}{6}u - k\right)\mu = 0, \\ &\mu_t + \left(\tfrac{1}{3}u + 4k\right)\mu_x - \tfrac{1}{6}u_x\mu = 0, \end{aligned} \qquad \mu = \mu(x,t,k), \ k \in \mathbb{C}, \tag{4}$$

[1] Although KdV also contains the linear term u_x, for our purposes it suffices to consider the simplified form (1).

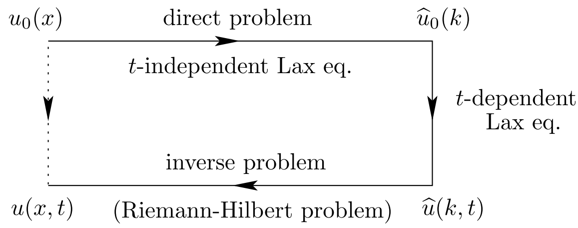

Fig. 1 Outline of the inverse scattering transform method

since KdV follows from that system under the simple symmetry condition $\mu_{xxt} = \mu_{txx}$ (which for continuous mixed derivatives follows by the Clairaut/Schwarz theorem). In [18], the authors studied the IVP (1), (3) for KdV on the infinite line by introducing the inverse scattering transform method. For the NLS equation and the IVP (2), (3), the corresponding formalism was developed in [43]. The method consists of three main steps that can be outlined as follows (see also the diagram of Fig. 1):

- the spectral analysis of the t-independent component of the Lax pair (first equation in (4)), thus mapping the initial data $u_0(x)$ to spectral data $\widehat{u}_0(k)$;
- the evolution of the spectral data $\widehat{u}_0(k)$ via the t-dependent component of the Lax pair (second equation in (4));
- the inversion of the resulting time-dependent spectral function $\widehat{u}(k,t)$ from the spectral kt-space to the physical xt-space, in order to recover the solution $u(x,t)$ of the IVP. This step typically involves the formulation and analysis of a Riemann-Hilbert problem.

The above procedure is conceptually identical to the well-known Fourier transform method for solving the IVP of linear evolution equations. In this sense, the inverse scattering transform can be regarded as a nonlinear analogue of the Fourier transform.

When available, the inverse scattering transform is a truly powerful method. Nevertheless, from the broader perspective of the analysis of nonlinear dispersive equations (and, more generally, nonlinear evolution equations), the method has some important limitations in its applicability. First and foremost, it can only be employed for integrable equations.[2] In addition, even then it comes with certain restrictions on the smoothness and decay at infinity of the initial data, e.g. on the infinite line these must belong in the class of "rapidly decaying" functions satisfying $\int_{-\infty}^{\infty} (1+|x|)\, |u_0(x)|\, dx < \infty$. These limitations rule out the vast majority of nonlinear evolution equations and, importantly, any such equation in space dimension three or higher. Moreover, even when studying integrable equations like KdV and NLS, conditions like the one above exclude large and significant classes of initial data. Indeed, as noted on page 257 of [33], the inverse scattering transform machinery seems to break down "even under very mild relaxations" of the "rapidly decaying" condition (see also [7]).

[2] There does not exist a universally accepted definition of complete integrability. Here, we identify an integrable equation by its ability to be "linearized" via a Lax pair.

Although the above limitations cannot be overcome in the context of the inverse scattering transform method, they do not pose a problem if one changes perspective and revisits the IVPs with a different goal, i.e. without the ambition of constructing an explicit solution map like the one produced via inverse scattering. In fact, the most fundamental question for the KdV and NLS IVPs is that of *well-posedness*. Originally formulated by Hadamard, this notion refers to the existence and uniqueness of solution of a given equation, as well as to the continuous dependence of that solution on the data. In the absence of well-posedness, the analysis of a model becomes pointless, regardless of the other features that this may have. For example, the "bad" Boussinesq equation, the first equation for which a soliton solution was written down, is not particularly useful otherwise since it is ill-posed. Through the years, various techniques have been developed for proving the well-posedness of IVPs that involve evolution equations. In the case of dispersive equations, a very effective such technique combines the powerful tools of harmonic analysis and the Fourier transform with the contraction mapping theorem for studying these equations in suitable Banach spaces. We hereafter refer to this technique as the Fourier transform approach.

It is widely known that well-posedness is affected by a number of factors, including the nature of the equation and the regularity and decay of the data. However, it is often less emphasized that it is also affected by the nature of the associated physical domain. In the case of a fully unbounded domain like the infinite line, one has an IVP; on the other hand, when the spatial domain involves a boundary (e.g. in one dimension, the half-line $0 < x < \infty$ or the finite interval $[0, 1]$), one instead has an initial-boundary value problem (IBVP). For any given equation, these two types of problems are generally very different, and this is also reflected in the analysis of their well-posedness. In fact, the well-posedness of nonlinear dispersive equations in the context of IBVPs is much less studied (and understood) than their IVP well-posedness.

Through a systematic effort that began in 2012, Alex Himonas and the author introduced a new approach for the well-posedness of IBVPs for nonlinear dispersive equations which takes advantage of the Fokas method in analogy to the way that the classical Fourier transform approach utilizes the Fourier transform. In that sense, the Fokas method can be regarded as the natural analogue of the Fourier transform in the IBVP setting. This novel well-posedness approach is reviewed in Sect. 3. Of course, the connection of the Fokas method with the Fourier transform dates back a lot further—specifically, its origins can be traced back to the 1994 paper of Fokas and Gelfand [14], where the Fourier transform pair is rediscovered through an inverse scattering analysis of the linear Schrödinger equation on the infinite line. This understanding later contributed to the realization that the Fokas method has significant implications at the level of IBVPs for *linear* equations, despite the fact that it had originally been motivated through the study of IBVPs for integrable *nonlinear* equations. For that reason, and due to the fact that the linear component of the Fokas method plays a fundamental role in the new well-posedness approach discussed in Sect. 3, in Sect. 2 we review the derivation of the Fourier transform pair in the style of [14], using some of the ideas that later led to the integrable nonlinear component of the Fokas method.

2 Inverse Scattering for Linear Equations: Rediscovering the Fourier Transform

The motivation behind the discovery of Lax pairs and the introduction and subsequent development of the inverse scattering transform method had to do with the study of integrable *nonlinear* equations; *linear* equations were not part of that motivation, since their IVP could be easily solved via the Fourier transform. Nevertheless, Fokas and Gelfand [14] came to the realization that every *linear* evolution equation can also be expressed as the compatibility condition of a Lax pair. Let us, for example, consider the Airy equation

$$u_t + u_{xxx} = 0, \tag{5}$$

which corresponds to the linear part of the KdV Eq. (1). With the help of the formal adjoint equation $-\widetilde{u}_t - \widetilde{u}_{xxx} = 0$, which is obtained by replacing ∂^j with $(-1)^j \partial^j$ in the x and t partial derivatives, we can write (5) in the divergence form $(e^{-ikx-ik^3t}u)_t + (e^{-ikx-ik^3t}[u_{xx} + iku_x - k^2u])_x = 0$, $k \in \mathbb{C}$. Seeking $M = M(x, t, k)$ such that $M_x = e^{-ikx-ik^3t}u$ and $M_t = -e^{-ikx-ik^3t}(u_{xx} + iku_x - k^2u)$, we see that the above divergence form (which is equivalent to (5)) is nothing but the symmetry requirement $M_{xt} = M_{tx}$. That is, the Airy Eq. (5) is the compatibility condition of the linear system for M, which is therefore a Lax pair for that equation. In fact, the exponential term can be absorbed by letting $M(x, t, k) = e^{-ikx-ik^3t}\mu(x, t, k)$, giving rise to the Lax pair

$$\mu_x - ik\mu = u, \quad \mu_t - ik^3\mu = -\left(u_{xx} + iku_x - k^2u\right). \tag{6}$$

Following the above realization, Fokas and Gelfand applied the inverse scattering transform formalism to Lax pairs like (6) in order to solve the IVP of linear evolution equations analogously to their integrable nonlinear counterparts, i.e. as if Fourier transform were not known/available. This direction was especially motivated by a long-standing open problem, namely the advancement of the inverse scattering transform method from the IVP to the IBVP setting, e.g. for solving the KdV equation on the half-line with nonzero Dirichlet data. Indeed, as noted on page 1 of [13], when this problem was first suggested to Ablowitz and Fokas by Julian Cole in 1982, they first attempted to solve the corresponding linear problem, namely the Airy Eq. (5) on the half-line, by using an appropriate *spatial* transform. The reason for first seeking a spatial transform for the linear IBVP had to do with the observation that, in the case of the IVP, in the linear limit the inverse scattering transform reduces to the Fourier transform [2]. Thus, knowledge of the relevant spatial transform in the case of the linear IBVP could provide the basis for developing the analogue of the inverse scattering transform method for integrable nonlinear IBVPs. To their surprise, Fokas and Ablowitz could not find an appropriate spatial transform for solving the linear Airy equation on the half-line; in fact, such a transform does *not* exist for any linear evolution of spatial order higher than two[3] [13]. Taking into account that even the "simple" task of solving *linear* IBVPs via spatial transforms was an open problem, it becomes evident that any progress made in the study of linear equations via inverse scattering ideas, like the one pursued in [14] as mentioned above, could have far-reaching implications also for integrable nonlinear equations.

Let us now follow the approach of [14] in order to integrate the Lax pair (6) and hence solve the IVP for the Airy equation on the infinite line.[4] As usual in the inverse scattering transform method, we work under the assumption of existence of solution and, in particular, we assume sufficient smoothness and decay at infinity as necessary. As noted earlier (see diagram of Fig. 1), there are three main steps: the direct problem, the inverse problem, and the time evolution of the spectral data.

Direct problem. Treating t, k as parameters—and thus suppressing them from the arguments of μ, u—we integrate the t-independent part of the Lax pair (6) to obtain the following expressions for the particular solutions $\mu^{\pm}$ that correspond to zero "boundary" conditions at $\pm\infty$, i.e. $\lim_{x\to\pm\infty}\mu^{\pm}(x) = 0$:

$$\mu^{+}(x) = \int_{-\infty}^{x} e^{ik(x-y)}u(y)dy, \quad \mu^{-}(x) = -\int_{x}^{\infty} e^{ik(x-y)}u(y)dy. \tag{7}$$

Inverse problem. Changing our perspective, we use the expressions (7) in order to define μ as a piecewise function of k (this time, we suppress the dependence on x, t) by $\mu(k) = \mu^{+}(k)$ for $\text{Im}(k) > 0$ and $\mu(k) = \mu^{-}(k)$ for $\text{Im}(k) < 0$. Then, introducing the *notation*

$$\widehat{u}(k) := \int_{-\infty}^{\infty} e^{-iky}\, u(y)dy, \quad k \in \mathbb{R}, \tag{8}$$

we observe that $\mu(k)$ satisfies the following scalar *Riemann-Hilbert problem*:

- $\mu(k)$ is analytic in $\mathbb{C}\setminus\mathbb{R}$ (by the form of (7) and a Paley-Wiener theorem like Theorem 7.2.4 in [40]);
- along $\mathbb{R}$, $\mu(k)$ satisfies the jump condition $\mu^{+}(k) - \mu^{-}(k) = e^{ikx}\,\widehat{u}(k)$, $k \in \mathbb{R}$;
- integration by parts in (7) implies $\mu(k) = O(1/k)$ as $|k| \to \infty$.

The solution of this scalar Riemann-Hilbert problem is readily obtained via the Plemelj formulae (Lemma 7.2.1 in [1]) as

[3] Although a temporal Laplace transform is available, it comes with certain disadvantages, most notably its inability to generalize to the integrable nonlinear equations.

[4] In [14], the authors illustrated their approach via the linear Schrödinger equation; the analysis is essentially the same in both cases.

$$\mu(x,k)=\frac{1}{2i\pi}\int_{-\infty}^{\infty}\frac{e^{i\lambda x}\,\widehat{u}(\lambda)}{\lambda-k}d\lambda,\quad k\notin\mathbb{R}. \tag{9}$$

Inserting this expression into the t-independent part of the Lax pair (6) and taking $|k|\to\infty$ yields the following representation for $u(x)$ in terms of the notation $\widehat{u}(k)$ introduced by (8):

$$u(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}\,\widehat{u}(k)dk,\quad x\in\mathbb{R}. \tag{10}$$

Time evolution. Observe that the expressions (7) satisfy $\lim_{x\to\pm\infty}(e^{-ikx}\mu^{\pm})=\pm\widehat{u}$. Therefore, restoring the time variable t and taking the two limits $x\to\pm\infty$ of the t-dependent part of the Lax pair (6) while assuming that $u, u_x, u_{xx}\to 0$ in those limits, we obtain the equation $\widehat{u}_t-ik^3\,\widehat{u}=0$. In view of the initial condition (3) and the notation (8), this equation implies $\widehat{u}(k,t)=e^{ik^3t}\,\widehat{u}_0(k)$, which can be combined with the representation (10) to yield the solution to the IVP (6), (3) in the explicit form

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx+ik^3t}\,\widehat{u}_0(k)dk,\quad x\in\mathbb{R},\ t\geq 0. \tag{11}$$

What is truly remarkable is the fact that the spectral transform (8), which arises *spontaneously* in the above analysis, is nothing but the celebrated *Fourier transform*! Furthermore, the solution of the relevant inverse (Riemann-Hilbert) problem readily yields the inversion of this spontaneously emerging transform, namely the *inverse Fourier transform* (10)! That is, in addition to providing the explicit solution formula to the Airy equation IVP (6), (3) (which is, of course, the well-known Fourier transform solution of this problem), the analysis of [14] leads to the *rediscovery* of the Fourier transform itself, and also to an elegant proof of its inversion!

In this regard, as noted at the beginning of this section, the contribution of [14] was of crucial importance because it suggested that devising a method for the spectral analysis of the Lax pairs of *linear* equations in the IBVP setting could provide the correct way of generalizing the inverse scattering transform method from the IVP to the IBVP setting. Soon after, this turned out to be indeed the case with the introduction and subsequent development of the Fokas method for *both* linear and integrable nonlinear equations in the IBVP setting.

3 A Novel Approach for the Well-posedness of IBVPs

The Fokas method, also known as the unified transform, was introduced by Fokas in 1997 [12] and subsequently developed by him and numerous collaborators (see [13, 17] and the references therein). The method has groundbreaking implications not only for integrable nonlinear equations, but also for linear equations. In the nonlinear case, it provides the extension of the inverse scattering transform method to the IBVP setting. In the linear case, it produces novel solution formulae for IBVPs formulated in various physical domains, with different types of nonzero boundary conditions, and in any number of spatial dimensions; as such, *the linear component of the Fokas method is the direct analogue of the Fourier transform in the IBVP setting*.

As noted in the introduction, a fundamental question for any given nonlinear dispersive equation is the one of (Hadamard) well-posedness, i.e. existence, uniqueness, and continuous dependence of the data-to-solution map. Although this topic has been studied extensively in the direction of the IVP (see, for example, the books [6, 8, 34, 37, 42] and the vast number of references therein), until recently it had remained largely unexplored in the case of IBVPs (essentially, the works [3, 9–11, 29, 30, 38]), despite the fact that this latter class of problems is very significant with regard to applications.

The main reason for this disproportion is the absence of the Fourier transform from the IBVP setting. Indeed, in the case of dispersive equations, the proof of well-posedness for the IVP relies heavily on the rich and powerful collection of harmonic analysis techniques that surround the Fourier transform. Importantly, the solution via the Fourier transform of the associated forced linear IBVP provides the starting point for defining the iteration map used for proving existence and uniqueness of solution via a fixed point argument (contraction mapping approach). Hence, in the case of IBVPs, without even a way of solving the linearized equations (recall discussion in Sect. 2), it is not surprising that very little progress had been made towards a *general approach* for establishing well-posedness of these problems in the case of (dispersive) nonlinear equations.

A systematic effort towards this goal began in 2012, when the author arrived at the University of Notre Dame to work under the mentorship of Professor Alex Himonas. The main idea had been proposed to Himonas by Fokas a few years earlier, in 2008, and consisted in employing the explicit solution formulae produced by the Fokas method in the case of (forced) linear IBVPs in order to set up the iterations for proving the well-posedness of the corresponding nonlinear problems via contraction mapping. The main source of optimism in regard to this suggestion was that, as mentioned earlier, for linear equations, the Fokas method is the analogue of the Fourier transform in the IBVP setting. Hence, it seemed reasonable to expect that the Fokas solution formulae could fulfill the role of generating iteration maps for nonlinear IBVPs in the same way that the Fourier transform formulae do in the case of nonlinear IVPs.

Regardless of how natural this idea may at first seem, however, when attempting to implement it one is quickly met with important challenges. For example, one must figure out how to obtain estimates in those function spaces that are natural to dispersive equations—such as Sobolev spaces or Bourgain spaces, which are typically studied (and even defined) with the help of the Fourier transform—when the Fokas solution formulae involve integrals along *complex* contours of the spectral k-plane (as opposed to the Fourier transform (8), which is defined only for $k \in \mathbb{R}$).

Another challenge has to do with the correct function space for the boundary data. For example, in the case of the IVP (2), (3) for the NLS equation, the initial datum $u_0(x)$ is typically placed in Sobolev spaces H^s and the solution is obtained in the associated Hadamard-type spaces $C_t H_x^s$ (at least for smooth enough data, i.e.

high enough s). However, on the half-line, one must *additionally* prescribe data at the boundary $x = 0$, e.g. via the Dirichlet boundary condition $u(0, t) = g_0(t)$ and so one must determine a suitable function space also for $g_0(t)$. Whether or not this space depends on the space H^s for $u_0(x)$ and if so, the precise relationship between the two spaces, is a question that adds to the complexity of IBVPs when compared to the IVP.[5]

A combination of ideas inspired by aspects of the Fokas method, together with suitably adapted results from the classical harmonic analysis toolbox used for the IVP, made it possible to pursue Fokas's suggestion and introduce an approach for establishing the well-posedness of IBVPs for nonlinear dispersive equations in a way conceptually analogous to the Fourier transform approach used for the IVP. This new approach has been employed for various problems involving the NLS, KdV, "good" Boussinesq, and biharmonic Schrödinger equations [15, 16, 20, 22, 24, 25, 35, 39], while it has also proved effective outside the dispersive class, for a nonlinear reaction-diffusion model [26]. In the new approach, the key to overcoming the challenges described above was the study of what we refer to as the *pure linear IBVP*. This problem consists of the homogeneous linearized version of the equation under study, supplemented with zero initial data and *nonzero but compactly supported* boundary data. The pure linear IBVP can be thought of as the simplest *genuine* IBVP, since it incorporates the challenges of an IBVP without the "distractions" caused by the initial data and the nonlinearity/forcing.

In the case of the Dirichlet half-line problem for the NLS Eq. (2), the pure linear IBVP is given by

$$\begin{aligned} &iu_t + u_{xx} = 0, \quad 0 < x < \infty, \ 0 < t < T, \\ &u(x, 0) = 0, \quad u(0, t) = g(t), \quad \text{supp}(g) \subset (0, T), \end{aligned} \tag{12}$$

where $T > 0$ is fixed (since we are interested in local well-posedness). Using the Fokas method, the solution of problem (12) is found to be

$$u(x, t) = \frac{1}{\pi} \int_{\mathcal{C}} e^{ikx - ik^2 t} k \widehat{g}(-k^2) dk, \tag{13}$$

where $\widehat{g}(-k^2)$ is the Fourier transform (8) of $g(t)$ evaluated at $-k^2$ and the complex contour $\mathcal{C}$ is the positively oriented boundary of the first quadrant of the complex k-plane. Below, we illustrate how the Fokas formula (13) can be used in order to estimate the solution of (12) for each $t \in [0, T]$ as a function in the Sobolev space $H^s(0, \infty)$, $s \geq 0$, on the half-line. Note that this space can be defined either as a restriction of the infinite-line space $H^s(\mathbb{R})$ or, directly, via the norm equal to the sum of the $L^2(0, \infty)$-norms of the derivatives up to order s (using the Slobodeckij

[5] In some cases, there exist results on the time regularity of the IVP solution that can provide helpful insights about the regularity of the boundary data [32]. In general, however, such results may not be available.

seminorm if s is fractional). We shall only provide the details for the case $s = 0$, which corresponds to $L^2(0,\infty)$; the full estimation can be found in [15].

The contour $\mathcal{C}$ comprises the positive halves of the real and imaginary axes. Denoting the respective parts of the solution by u_{re} and u_{im}, we have $u = u_{\text{re}} + u_{\text{im}}$ with

$$u_{\text{re}}(x,t) = \frac{1}{\pi}\int_0^\infty e^{ikx}\cdot e^{-ik^2t}\, k\,\widehat{g}(-k^2)dk,\ u_{\text{im}}(x,t) = \frac{1}{\pi}\int_0^\infty e^{-kx}\cdot e^{ik^2t}\, k\,\widehat{g}(k^2)dk.$$

Since the expression for u_{re} also makes sense for $x < 0$, it can be regarded as a function on the infinite line. Thus, by the Plancherel theorem,

$$\sup_{t\in[0,T]} \|u_{\text{re}}(t)\|_{L_x^2(0,\infty)} \lesssim \left\|e^{-ik^2t}\, k\,\widehat{g}(-k^2)\right\|_{L_k^2(0,\infty)} \simeq \|g\|_{H_t^{\frac{1}{4}}(\mathbb{R})}. \tag{14}$$

On the other hand, the expression for u_{im} does not make sense for $x < 0$, thus a different idea is needed. In particular, observe that, up to a constant, the $L_x^2(0,\infty)$-norm of u_{im} is just the $L_x^2(0,\infty)$-norm of the Laplace transform with respect to k of the quantity $e^{ik^2t}\, k\,\widehat{g}(k^2)$. Hence, by the boundedness of the Laplace transform in $L^2(0,\infty)$ [19],

$$\sup_{t\in[0,T]} \|u_{\text{im}}(t)\|_{L_x^2(0,\infty)} \lesssim \left\|e^{ik^2t}\, k\,\widehat{g}(k^2)\right\|_{L_k^2(0,\infty)} \simeq \|g\|_{H_t^{\frac{1}{4}}(\mathbb{R})}. \tag{15}$$

Together, estimates (14) and (15) imply that if the boundary datum of the pure linear IBVP (12) belongs to $H_t^{1/4}$ then the solution of this problem belongs to $C_t L_x^2(0,\infty)$. Furthermore, through the generalizations of these estimates for $s \geq 0$, the Sobolev space $H_t^{(2s+1)/4}$ spontaneously emerges as the correct space for the Dirichlet boundary datum $g_0(t)$. This fact is corroborated via a separate analysis of the time regularity of the homogeneous and forced linear Schrödinger IVPs, which actually shows that the above choice of space for the boundary datum is sharp. Eventually, via a contraction mapping argument, the various linear estimates derived with the help of the Fokas method solution formula (12) imply the Hadamard well-posedness of the Dirichlet problem for NLS on the half-line. More precisely:

Theorem *([15]). Suppose $1/2 < s \leqslant 3/2$. Then, the IBVP for the cubic NLS Eq.* (2) *on the half-line with initial data $u_0 \in H^s(0,\infty)$ and Dirichlet boundary data $g_0 \in H^{(2s+1)/4}(0,T)$ is well-posed in the sense of Hadamard. In particular, there exists a unique solution $u \in C\left([0,T^*];H^s(0,\infty)\right)$, which satisfies*

$$\sup_{t\in[0,T^*]} \|u(t)\|_{H^s(0,\infty)} \leqslant c_s\Big(\|u_0\|_{H^s(0,\infty)} + \|g_0\|_{H^{\frac{2s+1}{4}}(0,T)}\Big)$$

with $c_s = c(s) > 0$ and $0 < T^* \leqslant \min\Big\{T, c_s\Big(\|u_0\|_{H^s(0,\infty)} + \|g_0\|_{H^{\frac{2s+1}{4}}(0,T)}\Big)^{-4}\Big\}$, and the data-to-solution map $\{u_0, g_0\} \mapsto u$ is locally Lipschitz continuous.

The above result can also be established for the general semilinear Schrödinger equation of nonlinearity $\alpha > 1$. Moreover, by adapting the proof of the famous Strichartz estimates [41] that are used for sharp well-posedness of the NLS IVP, it is possible to extend the above result to the interval $0 \le s < 1/2$ and hence obtain *sharp* well-posedness on the half-line (like for the IVP, the solution will now belong in a finer space motivated by the Strichartz estimates). Indeed, a sharp result of this kind was proved in [21], where the approach introduced in [15] was advanced *for the first time to higher than one spatial dimensions* for the NLS equation on the half-plane $\mathbb{R} \times \mathbb{R}^+$.

In fact, the analysis carried out in [21] and, more recently, in [23], led to a remarkable and perhaps unexpected discovery, namely that the celebrated $X^{s,b}$ spaces, which were introduced by Bourgain [4, 5] as *solution* spaces for proving the sharp well-posedness of the periodic and non-periodic NLS and KdV IVPs, now arise spontaneously as *boundary data* spaces in the estimation of the Fokas method solution for the pure linear IBVP associated with NLS on the half-plane. More precisely, for initial data $u_0 \in H^s(\mathbb{R} \times \mathbb{R}^+)$, it is shown in [21] that the Dirichlet boundary data must belong to a certain restriction of the space $X^{s,1/4} \cap X^{0,(2s+1)/4}$. In the case of the Neumann and Robin problems studied in [23], the corresponding space is a restriction of $X^{s,-1/4} \cap X^{0,(2s-1)/4}$.

In lieu of an epilogue, we emphasize that, despite the substantial progress made during the last decade on the well-posedness of nonlinear IBVPs via the novel Fokas-method-inspired approach outlined above, a plethora of important problems remain open. For example, recently the new approach was further extended in the direction of Bourgain spaces [27, 28], improving the result of [16] for the KdV equation on the half-line from H^s with $3/4 < s < 1$ (which is consistent with the IVP result of [31]) down to $s > -3/4$, matching the IVP result of [32]. Nevertheless, although the results of [27, 28, 32] are optimal with respect to contraction mapping techniques, they are not sharp in general, since it was recently shown in [33] without using a contraction mapping technique that the KdV IVP is well-posed in H^{-1}. Whether or not this result also holds on the half-line is currently unknown. The adaptation of the new approach to other higher-dimensional equations and/or domains such as the quarter-plane is another interesting direction that should be explored.

In conclusion, the Fokas method has provided the key to developing an effective, universal approach for the rigorous well-posedness of IBVPs that involve nonlinear dispersive (and non-dispersive) equations. This is yet another aspect of the remarkable impact that the method has had on the analysis of linear and nonlinear IBVPs since its introduction in 1997. Furthermore, it is also indicative of the influence that the method will continue to have on the field for the years to come.

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