Growth of Nonsymmetric Operads

ZIHAO QI, YONGJUN XU, JAMES J. ZHANG & XIANGUI ZHAO

ABSTRACT. The paper concerns the Gelfand-Kirillov dimension and the generating series of nonsymmetric operads. Herein, an analogue of Bergman's gap theorem is proved: specifically, no finitely generated locally finite nonsymmetric operad has GK-dimension strictly between 1 and 2. For every $r \in \{0\} \cup \{1\} \cup [2, \infty)$ or $r = \infty$, we construct a single-element generated nonsymmetric operad $\mathcal{P}$ such that $\mathrm{GKdim}(\mathcal{P}) = r$. We also provide counterexamples to two expectations of Khoroshkin and Piontkovski about the generating series of operads.

1. INTRODUCTION

Let $\mathbb{F}$ be a base field. An algebra stands for a unital associative algebra over $\mathbb{F}$ unless otherwise stated. The Gelfand-Kirillov dimension (GK-dimension for short) of an algebra A is defined to be

$$\mathrm{GKdim}(A) := \sup_{V} \Big\{ \limsup_{n\to\infty} \log_n \Big[\dim_{\mathbb{F}} \Big(\sum_{i=0}^{n} V^i \Big) \Big] \Big\}$$

where the supremum is taken over all finite-dimensional subspaces V of A. Similarly, one can define the GK-dimension of nonassociative algebras. GK-dimension is a standard and powerful invariant for investigating associative and nonassociative algebras. We refer to [KL00] for more background and properties of the GK-dimension of algebras and modules.

The range of possible values for the GK-dimension of an algebra is

$$R_{\mathrm{GKdim}} := \{0\} \cup \{1\} \cup [2, \infty) \cup \{\infty\}. \tag{1.1}$$

The gap between 0 and 1 follows easily from the definition of GK-dimension. The existence problem of algebras A with $1 < \mathrm{GKdim}(A) < 2$ was open for some years until Bergman [KL00, Theorem 2.5] proved that no such algebras exist. Bergman's gap theorem is also valid for some other classes of algebras, for example, Jordan algebras [MZ96] and dialgebras [ZCY20]. However, there exist Lie algebras [Pet97] and Jordan superalgebras [PS19] with GK-dimension strictly between 1 and 2.

The notion of an operad was first introduced by Boardman-Vogt [BV73] and May [May72] in the late 1960s and early 1970s in the study of iterated loop spaces. Since 1990s, because of Ginzburg-Kapranov's Koszul duality theory of operads [GK94], Kontsevich's [Kon03] and Tamarkin's [Tam98] operadic approach to the formality theorem, as well as Getzler's study on topological field theories [Get94, Get95], operad theory has become an important tool in homological algebra, category theory, algebraic geometry, and mathematical physics.

In this paper, we investigate the GK-dimension and the generating series of non-symmetric operads. The GK-dimension of locally finite operads was defined in [KP15, p. 400] and [BYZ20, Definition 4.1]. Let $\mathcal{P}$ be a *locally finite* operad, that is, an operad $\mathcal{P}$ with each $\mathcal{P}(i)$ finite dimensional over the base field $\mathbb{F}$. The *GK-dimension* of $\mathcal{P}$ is defined to be

$$\operatorname{GKdim}(\mathcal{P}) := \limsup_{n\to\infty} \log_n \Big(\sum_{i=0}^{n} \dim_{\mathbb{F}} \mathcal{P}(i) \Big).$$

Our main result is an analogue of Bergman's gap theorem.

Theorem 1.1. *No finitely generated locally finite nonsymmetric operad has GK-dimension strictly between* 1 *and* 2.

(For the definition of a finitely generated operad, see Definition 2.16.) Bergman's gap theorem for associative algebras was proved by counting specific words that satisfy a set of conditions and form a monomial basis of the algebra under consideration, equivalently, by counting a set of *single-branched tree monomials* (defined in Section 5) where the algebra is interpreted as an operad. This method cannot be extended directly to prove the gap theorem for nonsymmetric operads since the tree monomials we want to count are not necessarily single branched. To overcome the above difficulty, we divide the underlying tree into three subtrees such that the "big" subtree is single branched and thus we can count its tree monomials similarly as for the case of associative algebras. The other two subtrees are both "small" such that their tree monomials are well controlled.

Note that if we drop the condition "finitely generated" in Theorem 1.1, then the statement does not hold. In fact, as Example 4.4 shows, any positive real number is the GK-dimension of some nonsymmetric operad.

Theorem 1.1 serves as an essential ingredient of the next result. Recall that R_{GKdim} is defined in (1.1).

Theorem 1.2.

(1) *If $\mathcal{P}$ is a finitely generated locally finite nonsymmetric operad, then we have* $\operatorname{GKdim}(\mathcal{P}) \in R_{\mathrm{GKdim}}$.

(2) *If* $r \in R_{\mathrm{GKdim}}$, *then there is a single-element generated single-branched locally finite nonsymmetric operad* $\mathcal{P}$ *such that* $\mathrm{GKdim}(\mathcal{P}) = r$.

The Gelfand-Kirillov dimension of an operad is closely related to the generating series that is defined as follows. Let $\mathcal{P}$ be a locally finite operad. The *generating series* of $\mathcal{P}$ is defined to be the formal power series [KP15, (0.1.2)]

$$G_{\mathcal{P}}(z) := \sum_{n=0}^{\infty} \dim_{\mathbb{F}} \mathcal{P}(n) z^n. \tag{1.2}$$

We recall the following definition.

Definition 1.3 ([Sta80, Zei90, BBY12, Ber14, KP15]). Let

$$F(z) := \sum_{n \geq 0} f(n) z^n$$

be a formal power series or a C^{∞}-function where $f(n) \in \mathbb{R}$ for all n.

(1) $F(z)$ is called *holonomic* (also called *D-finite* or *differentiably finite*) if it satisfies a nontrivial linear differential equation with polynomial coefficients.
(2) $F(z)$ is called *differential algebraic* if it satisfies a nontrivial algebraic differential equation with polynomial coefficients.

It is well known that

$$\text{rational} \Rightarrow \text{algebraic (over } \mathbb{R}(z)) \Rightarrow \text{holonomic} \Rightarrow \text{differential algebraic}$$

where the second implication is [Sta80, Theorem 2.1]. Several researchers have recently been studying holonomic and differential algebraic properties of $G_{\mathcal{P}}(z)$ [Ber14, KP15]. In particular, Khoroshkin-Piontkovski showed that, under moderate assumptions, operads with a finite Gröbner basis have rational, or algebraic, or differential algebraic generating series [KP15, Theorems 0.1.3, 0.1.4, and 0.1.5]. In [KP15, Section 4] Khoroshkin-Piontkovski listed several expectations and conjectures, one of which is as follows.

Expectation 1.4 ([KP15, Expectation 2]). The generating series of a generic finitely presented nonsymmetric operad is algebraic over $\mathbb{Z}[z]$.

We construct a finitely presented nonsymmetric operad such that the generating series is not holonomic (hence, not algebraic), which provides a (non-generic) counterexample to the above expectation (Example 7.4). We also prove the following result.

Proposition 1.5. *Let* $r \in R_{\mathrm{GKdim}} \setminus \{0\}$. *Then, there is a single-element generated locally finite nonsymmetric operad* $\mathcal{P}$ *with* $\mathrm{GKdim}(\mathcal{P}) = r$ *and* $G_{\mathcal{P}}(z)$ *not being holonomic. As a consequence,* $G_{\mathcal{P}}(z)$ *is neither rational nor algebraic in this case. Therefore, such an operad is a counterexample to Expectation 3 in* [KP15].

Remark 1.6. Constructions 2.3, 6.1, and 8.1 provide useful constructions of nonsymmetric operads (or symmetric ones in Construction 8.1) from graded algebras (or monomial algebras in Construction 6.1). A lot of algebraic properties of graded (or monomial) algebras can be transformed to the corresponding properties of the related operads. Using this idea, in addition to Proposition 1.5 and Example 7.4, we obtain a potential counterexample to [KP15, Expectation 1] in Example 8.5.

This paper mainly concerns nonsymmetric operads. In the final section we touch upon the symmetric ones. The following theorem is easy to prove.

Theorem 1.7.

(1) *Let $\mathcal{P}$ be a finitely generated locally finite symmetric operad. Then, we have* $\operatorname{GKdim}(\mathcal{P}) \in R_{\operatorname{GKdim}} \cup (1,2)$.
(2) *For every $r \in R_{\operatorname{GKdim}} \setminus (2,3)$, there exists a finitely generated locally finite symmetric operad $\mathcal{P}$ such that* $\operatorname{GKdim}(\mathcal{P}) = r$.
(3) *For every $r \in R_{\operatorname{GKdim}} \setminus (\{0\} \cup \{1\} \cup (2,3))$, there exists a finitely generated locally finite symmetric operad $\mathcal{P}$ such that* $\operatorname{GKdim}(\mathcal{P}) = r$ *and that $G_{\mathcal{P}}(z)$ is not holonomic.*

In light of [KP15, Ber14], it would be very interesting to determine which classes of operads have rational (respectively, algebraic, holonomic, differential algebraic) generating series. Theorem 1.7 suggests that, generically, $G_{\mathcal{P}}(z)$ is not holonomic. Theorem 1.7 (1)–(2) lead to the following question.

Question 1.8. *Let $r \in (1,2) \cup (2,3)$. Is there a finitely generated locally finite symmetric operad $\mathcal{P}$ such that* $\operatorname{GKdim}(\mathcal{P}) = r$*?*

In view of Theorem 1.7 (2)–(3) the following result of [BYZ20] is quite surprising (see [BYZ20] for details).

Theorem 1.9. *Let $\mathcal{P}$ be a 2-unitary locally finite symmetric operad. Then,* $\operatorname{GKdim}(\mathcal{P})$ *is either an integer or the infinity. If* $\operatorname{GKdim}(\mathcal{P})$ *is finite, then $\mathcal{P}$ is finitely generated and $G_{\mathcal{P}}(z)$ is rational.*

This paper is organized as follows. Section 2 recalls basic definitions and properties of nonsymmetric operads. Section 3 introduces Gröbner-Shirshov bases of nonsymmetric operads. Section 4 contains definitions, examples, and properties of the GK-dimension of nonsymmetric operads. In Section 5, we prove the main result, Theorem 1.1, and in Section 6, we prove Theorem 1.2. In Section 7 we study the generating series of operads and prove Proposition 1.5. Finally, Section 8 provides some comments and examples about symmetric operads.

2. PRELIMINARIES

Let $\mathbb{F}$ be the base field and $\mathbb{F}^* := \mathbb{F} \setminus \{0\}$. Let $\mathbb{N}$ denote the natural numbers and $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. To save space, some non-essential details are omitted here and there. But we try to provide as much detail as possible for the proof of the main result, that is, Theorem 1.1.

2.1. Ns operads. The following definition is copied from Definition 3.2.2.3 [BD16] (see also [LV12, Chapter 5]).

Definition 2.1. A *nonsymmetric operad* or simply *ns operad* is a collection of vector spaces $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$ equipped with an element $\mathrm{id} \in \mathcal{P}(1)$ (called the *identity element*) and maps (called *partial compositions*), for $1 \leq i \leq n$,

$$\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(n+m-1), \quad \alpha \otimes \beta \mapsto \alpha \circ_i \beta.$$

These satisfy the following properties for all $\alpha \in \mathcal{P}(n)$, $\beta \in \mathcal{P}(m)$ and $\gamma \in \mathcal{P}(r)$:

(i) (sequential axiom)

$$(\alpha \circ_i \beta) \circ_j \gamma = \alpha \circ_i (\beta \circ_{j-i+1} \gamma) \quad \text{for } i \leq j \leq i+m-1. \tag{2.1}$$

(ii) (parallel axiom)

$$(\alpha \circ_i \beta) \circ_j \gamma = \begin{cases} (\alpha \circ_{j-m+1} \gamma) \circ_i \beta, & i+m \leq j \leq n+m-1, \\ (\alpha \circ_j \gamma) \circ_{i+r-1} \beta, & 1 \leq j \leq i-1. \end{cases} \tag{2.2}$$

(iii) (unit axiom)

$$\mathrm{id} \circ_1 \alpha = \alpha, \;\; \alpha \circ_i \mathrm{id} = \alpha \quad \text{for } 1 \leq i \leq n. \tag{2.3}$$

The above definition is called the *partial definition of a ns operad*. (For the classical definition and the monoidal definition, see [LV12, Chapter 5]). Except for the final section, we only consider ns operads, and sometimes "ns" or the word "nonsymmetric" is omitted.

A collection $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$ of spaces (especially, an operad) is called *finite dimensional* if the *dimension of* $\mathcal{P}$ is finite, that is,

$$\dim \mathcal{P} := \dim \Big(\bigoplus_{n\geq 0} \mathcal{P}(n) \Big) < \infty.$$

Also, $\mathcal{P}$ is called *locally finite* (respectively, *reduced*, *connected*) if $\mathcal{P}(n)$ is finite dimensional for all $n \in \mathbb{N}$ (respectively, if $\mathcal{P}(0) = 0$, if $\mathcal{P}(1) = \mathbb{F}\,\mathrm{id} \cong \mathbb{F}$). We say a collection $\mathcal{V} = \{\mathcal{V}(n)\}_{n\geq 0}$ of spaces is a *subcollection* of $\mathcal{P}$ if $\mathcal{V}(n)$ is a subspace of $\mathcal{P}(n)$ for all $n \geq 0$. Furthermore, if the subcollection $\mathcal{V}$ is an operad with the partial compositions of $\mathcal{P}$, we call $\mathcal{V}$ a *suboperad* of $\mathcal{P}$.

To simplify notation, we also view a collection $\mathcal{G} = \{\mathcal{G}(n)\}_{n\geq 0}$ of sets (e.g., a collection of vector spaces) as a disjoint union $\mathcal{G} = \bigsqcup_{n\geq 0} \mathcal{G}(n)$ (respectively, a direct sum $\mathcal{G} = \bigoplus_{n\geq 0} \mathcal{G}(n)$), and vice versa.

As demonstrated in the following examples, an operad can be viewed as a generalization of an algebra.

Example 2.2 ([LV12, p. 137]). A unital associative algebra A can be interpreted as an operad $\mathcal{P}$ with $\mathcal{P}(1) = A$ and $\mathcal{P}(n) = 0$ for all $n \neq 1$, and the compositions in $\mathcal{P}$ are given by the multiplication of A.

The following construction is due to Dotsenko [Dot19].

Construction 2.3 *(*[Dot19, Definition 3.2 (2)]*).* Let $A := \bigoplus_{i\geq 0} A_i$ be an $\mathbb{N}$-graded algebra with unit 1_A. Suppose that A has a graded augmentation $\varepsilon : A \to \mathbb{F}$ such that $\mathfrak{m} := \ker \varepsilon$ is a maximal graded ideal of A. Let $\mathcal{P}_A(0) = 0$ and $\mathcal{P}_A(n) = A_{n-1}$ for all $n \geq 1$. Define compositions as follows:

$$\circ_i : \mathcal{P}_A(m) \otimes \mathcal{P}_A(n) \to \mathcal{P}_A(n+m-1),$$
$$a_{m-1} \otimes a_{n-1} \mapsto \begin{cases} c a_{m-1} & a_{n-1} = c 1_A,\ c \in \mathbb{F}, \\ a_{m-1} a_{n-1} & a_{n-1} \in \mathfrak{m},\ i = 1, \\ 0 & a_{n-1} \in \mathfrak{m},\ i \neq 1. \end{cases}$$

It is easy to check by definition that $\mathcal{P}_A := \{\mathcal{P}_A(n)\}_{n\geq 0}$ is an operad with id $:= 1_A$. This operad is called the *min-envelope operad* of A. Note that Piontkovski [Pio17, Theorem 3.1] used this construction to produce a counterexample to [BD16, Conjecture 10.4.1.1]. (See [Dot19, Definition 3.2 (1)] for another related construction.)

If A is generated by A_1, then $\mathcal{P}_A$ is generated by $\mathcal{P}_A(2)$. By Lemma 7.3, A is finitely generated as an algebra if and only if $\mathcal{P}_A$ is finitely generated as an operad.

For an $\mathbb{N}$-graded locally finite algebra A, its Hilbert series is defined to be $H_A(z) = \sum_{n=0}^{\infty} \dim A_n z^n$ in the same way of defining the generating series of an operad.

By the construction given in Construction 2.3, one sees that

$$G_{\mathcal{P}_A}(z) = zH_A(z). \tag{2.4}$$

A *morphism* ϕ between two operads $\mathcal{P}$ and $\mathcal{P}'$ is a collection of linear maps $\phi_n : \mathcal{P}(n) \to \mathcal{P}'(n)$, $n \geq 0$, such that $\phi_1(\mathrm{id}_{\mathcal{P}}) = \mathrm{id}_{\mathcal{P}'}$ and

$$\phi_m(u) \circ_i \phi_n(v) = \phi_{m+n-1}(u \circ_i v)$$

for all $u \in \mathcal{P}(m)$, $v \in \mathcal{P}(n)$, $1 \leq i \leq m$, $n \geq 0$. If each $\mathcal{P}(n)$ is a subspace of $\mathcal{P}'(n)$, we call $\mathcal{P}$ a *subcollection* of $\mathcal{P}'$, and write $\mathcal{P} \subseteq \mathcal{P}'$.

An *operation alphabet* (or *generating operations*) is a collection of sets $\mathcal{X}(n)$, denoted $\mathcal{X} = \{\mathcal{X}(n)\}_{n\geq 0}$. The number n is then called the *arity* of an element $x \in \mathcal{X}(n)$ and denoted by $\mathrm{Ar}(x) = n$.

Let $\mathcal{X}$ be an operation alphabet. The *free operad* over $\mathcal{X}$ is an operad $\mathcal{F}(\mathcal{X})$ that is equipped with an inclusion $\eta : \mathcal{X} \to \mathcal{F}(\mathcal{X})$ (i.e., a collection of inclusions $\eta_n : \mathcal{X}(n) \to \mathcal{F}(\mathcal{X})(n)$ for all $n \geq 0$) which satisfies the following universal property: any map $f : \mathcal{X} \to \mathcal{P}$, where $\mathcal{P}$ is an operad, extends uniquely to an operad morphism $\tilde{f} : \mathcal{F}(\mathcal{X}) \to \mathcal{P}$ [BD16, Section 3.3].

2.2. Planar rooted tree. Much as in the case of associative algebras, an operad can be presented by generators and relations, that is, as a quotient of a free operad. In this subsection, we introduce a language for working with planar rooted trees, which will be used later to construct free operads. We mainly follow the ideas in [BD16, Section 3.3] and use the language introduced in [BD16] with minor modification for the rest of the paper (see Remark 2.6).

Definition 2.4. A *rooted tree* τ consists of the following:

- a finite set $\mathrm{Vert}(\tau)$ of *vertices*, which is a disjoint union

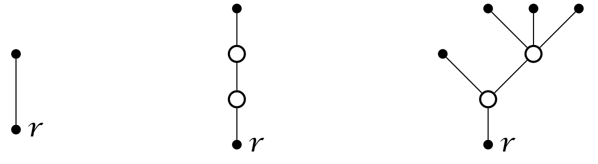
$$\mathrm{Vert}(\tau) = \mathrm{Int}(\tau) \sqcup \mathrm{Leaves}(\tau) \sqcup \{r\},$$

  where elements of the (possible empty) set $\mathrm{Int}(\tau)$ are called *internal vertices* of τ, elements of the nonempty set $\mathrm{Leaves}(\tau)$ are called *leaves* of τ, and the element r is called the *root* of τ and denoted by $\mathrm{Root}(\tau)$;
- a *parent function*

$$\mathrm{Parent} = \mathrm{Parent}_\tau : \mathrm{Vert}(\tau) \setminus \{r\} \to \mathrm{Vert}(\tau) \setminus \mathrm{Leaves}(\tau)$$

  such that $|\mathrm{Parent}^{-1}(r)| = 1$, $|\mathrm{Parent}^{-1}(v)| \geq 1$ for all $v \in \mathrm{Int}(\tau)$, and the *connectivity condition* is satisfied: for each vertex $v \neq r$, there are an $h \in \mathbb{N}^*$ and vertices $v_0 = r, v_1, \ldots, v_{h-1}, v_h = v$, such that $v_i = \mathrm{Parent}(v_{i+1})$ for $0 \leq i \leq h-1$.

When we draw a rooted tree on the plane, the root is always drawn at the bottom of the tree; a hollow circle presents an internal vertex while a black solid circle presents a leaf or the root; a solid segment between two vertices indicates that the lower vertex is the parent of the higher one. (See Figure 2.1 for examples.)



τ_0 τ_1 τ_2

FIGURE 2.1. Rooted trees

Let τ be a rooted tree. The following lemma is straightforward.

Lemma 2.5. *Suppose $v \in \mathrm{Vert}(\tau) \setminus \mathrm{Root}(\tau)$, and h and v_i for $0 \leq i \leq h$ are the same as in Definition 2.4. Then, the number h and the sequence $v_0, v_1, \ldots, v_h$ are uniquely determined by v, called, respectively the* height *of v (denoted by $\mathrm{h}(v)$) and the* path from root to v.

The *height* of τ is the maximal height of its internal vertices, that is,

$$\mathrm{h}(\tau) := \max\{\mathrm{h}(v) : v \in \mathrm{Int}(\tau)\}.$$

For $v \in \mathrm{Vert}(\tau) \setminus \mathrm{Leaves}(\tau)$, elements in $\mathrm{Child}(v) := \mathrm{Parent}^{-1}(v)$ are called *children* of v. More generally, we define the *descendants* of v to be

$$\mathrm{Child}^{\infty}(v) := \{v' \in \mathrm{Vert}(\tau) : \mathrm{Parent}^{i}(v') = v \text{ for some } i \in \mathbb{N}^*\}.$$

The number of children of v (respectively, the number of leaves of τ) is called the *arity* of v (respectively, of τ), denoted by $\mathrm{Ar}(v)$ (respectively, $\mathrm{Ar}(\tau)$). Define the *weight* of τ to be $\mathrm{wt}(\tau) := |\mathrm{Int}(\tau)|$. The only tree with weight 0 is called the *trivial (rooted) tree*, denoted by τ_0, the only two vertices of which are the root $\mathrm{Root}(\tau_0)$ and the leaf in $\mathrm{Child}(\mathrm{Root}(\tau_0))$ (see Figure 2.1). For a nontrivial rooted tree, the only child of the root must be an internal vertex.

When working with more than one rooted trees, we usually use subscripts (e.g., the height $\mathrm{h}_\tau(v)$) to indicate the tree under consideration. We write a map $\varphi : \mathrm{Vert}(\tau) \to \mathrm{Vert}(\tau')$ between the vertex sets of rooted trees τ and τ' as $\varphi : \tau \to \tau'$ for short. An *isomorphism* from τ to τ' is a bijective map $\varphi : \tau \to \tau'$ that respects the parent functions, that is,

$$\varphi(\mathrm{Parent}_\tau(v)) = \mathrm{Parent}_{\tau'}(\varphi(v)), \quad \forall\, v \in \mathrm{Vert}(\tau) \setminus \mathrm{Root}(\tau).$$

We say τ is *isomorphic* to τ' if there is an isomorphism from τ to τ'. It is easy to see that an isomorphism also respects the type of each vertex; more precisely, if $\varphi : \tau \to \tau'$ is an isomorphism, then $v \in \mathrm{Vert}(\tau)$ is a leaf (respectively, an internal vertex, the root) in τ if and only if so is $\varphi(v)$ in τ'.

Remark 2.6. The only difference between our rooted trees and those in Section 3.3 of [BD16] is that an internal vertex of our rooted tree always has positive arity while an internal vertex in the sense of [BD16] may have zero arity. Note that the existence of internal vertices of arity zero makes Proposition 3.4.1.6 of [BD16] false (see Example 3.3 for a counterexample).

Definition 2.7. A *planar rooted tree* (*PRT*, for short) is a rooted tree together with a planar structure, that is, a rooted tree with a total order on $\mathrm{Child}(v)$ for each $v \in \mathrm{Vert}(\tau) \setminus \mathrm{Leaves}(\tau)$.

The planar structure of a PRT τ induces a total order on $\mathrm{Vert}(\tau)$. Suppose u and u' are two different vertices and consider the paths from root to u and u':

$$v_0 = r, v_1, \dots, v_{\mathrm{h}(u)};\ v_0' = r, v_1', \dots, v_{\mathrm{h}(u')}'.$$

We say $u < u'$ if one of the following holds:

(i) Either $\mathrm{h}(u) < \mathrm{h}(u')$ and $v_0 = v_0'$, $v_1 = v_1'$, $\dots$, $v_{\mathrm{h}(u)} = v_{\mathrm{h}(u)}'$;

(ii) Or there is $1 \leq k \leq \min\{\mathrm{h}(u), \mathrm{h}(u')\} - 1$ such that $v_0 = v_0'$, $v_1 = v_1'$, $\dots$, $v_k = v_k'$ and $v_{k+1} < v_{k+1}'$.

If there is no confusion, we usually use positive integers to denote the leaves, that is, $\mathrm{Leaves}(\tau) = \{1, 2, \dots, \mathrm{Ar}(\tau)\}$, to indicate the order on $\mathrm{Leaves}(\tau)$ in the obvious way. We draw a PRT on the plane in a way that the planar order on $\mathrm{Child}(v)$

is determined by ordering the corresponding vertices left to right. A PRT τ is *isomorphic* to PRT τ' if there exists a rooted tree isomorphism $\varphi : \tau \to \tau'$ such that $\varphi(u) < \varphi(v)$ whenever $u < v$ for $u, v \in \mathrm{Vert}(\tau)$.

Example 2.8. Let τ and τ' be PRTs as drawn in Figure 2.2. It is easy to see that τ is isomorphic to τ' as rooted trees. However, they are not isomorphic as PRTs.

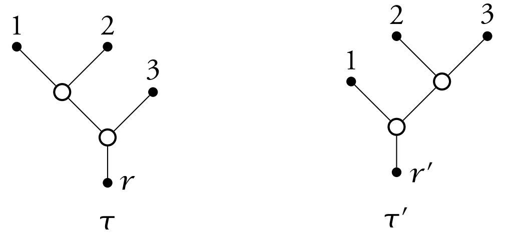


FIGURE 2.2. Non-isomorphic PRTs that are isomorphic as rooted trees

The following lemma is clear.

Lemma 2.9 *(*[BD16, Definition 3.4.2.1]*).* *Let* τ *be a* PRT, *and suppose that* $\varnothing \neq V' \subseteq \mathrm{Int}(\tau)$ *satisfies the following conditions:*

(i) *There is a unique* $v' \in V'$ *such that* $\mathrm{Parent}_\tau(v') \notin V'$.
(ii) *For each* $v'' \in V'$ *there exist* $h \in \mathbb{N}^*$ *and vertices* $v_1 = v', v_2, \dots, v_{h-1}, v_h = v''$ *such that* $v_i = \mathrm{Parent}_\tau(v_{i+1})$ *for all* $1 \le i \le h-1$.

Then, V' *defines a* PRT τ' *as follows:*

$$\mathrm{Root}(\tau') = \mathrm{Parent}_\tau(v'),\ \mathrm{Int}(\tau') = V',\ \mathrm{Leaves}(\tau') = \Big(\bigcup_{v \in V'} \mathrm{Child}(v)\Big) \setminus V',$$

and the parent function and the planar structure of τ' *are the restrictions of the parent function and the planar structure of* τ. *We call* τ' *a* subtree *of* τ, *denoted* $\tau' \subseteq \tau$.

Note that, in Lemma 2.9, condition (i) implies that each v_i in condition (ii) belongs to V'.

Let τ be a PRT, $v \in \mathrm{Int}(\tau)$, and $V' = (\mathrm{Int}(\tau) \cap \mathrm{Child}^\infty(v)) \cup \{v\}$. Then, V' satisfies the conditions in Lemma 2.9 and thus defines a subtree τ' of τ. We call this subtree the *maximal subtree* of τ rooted at $\mathrm{Parent}(v)$ and containing internal vertex v. Given $r' \in \mathrm{Vert}(\tau)$ and $v' \in \mathrm{Int}(\tau) \cap \mathrm{Child}(r')$, it is easy to see that there is a unique maximal subtree of τ that is rooted at r' and contains v'. We denote it by $\mathrm{MaxSub}_\tau(r', v')$.

2.3. Free ns operads. Let $\mathcal{X}$ be an operation alphabet. A *labelling* of a nontrivial PRT τ is a map $\mathsf{x} : \mathrm{Int}(\tau) \to \mathcal{X}$ such that $\mathrm{Ar}(\mathsf{x}(v)) = \mathrm{Ar}(v) = |\mathrm{Child}(v)|$ for all $v \in \mathrm{Int}(\tau)$. A *nontrivial tree monomial* in $\mathcal{X}$ is a pair $T = (\tau, \mathsf{x})$ of a nontrivial PRT τ and a labelling x of τ. Define the tree monomial for the trivial tree to be the *trivial tree monomial*, denoted by $1 := (\tau_0, \emptyset)$. The PRT τ is called the *underlying tree* of tree monomial T, denoted by $\mathrm{Tr}(T) := \tau$. More generally, if W is a set of tree monomials, denote $\mathrm{Tr}(W) := \{\tau : \tau = \mathrm{Tr}(T),\ T \in W\}$. The *arity* and *weight* of a tree monomial $T = (\tau, \mathsf{x})$ are defined as $\mathrm{Ar}(T) := \mathrm{Ar}(\tau)$ and $\mathrm{wt}(T) := \mathrm{wt}(\tau)$ respectively. Let $\mathrm{TM}(\mathcal{X})$ denote the set of tree monomials in $\mathcal{X}$ together with the trivial tree monomial.

Example 2.10. Suppose that $\mathcal{X}(1) = \{a\}$ and $\mathcal{X}(2) = \{b, c\}$. In Figure 2.3 are examples of tree monomials in $\mathcal{X}$.

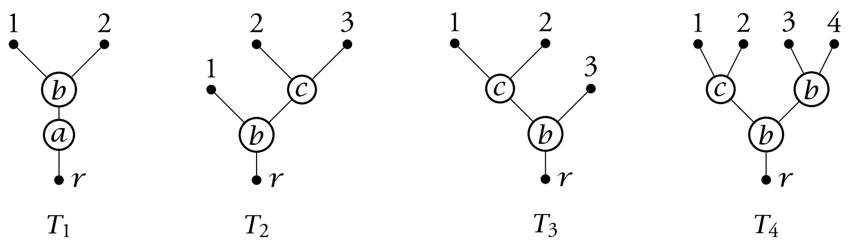


FIGURE 2.3. Tree monomials

Definition 2.11. Suppose $T = (\tau, \mathsf{x}) \in \mathrm{TM}(\mathcal{X})$ and τ' is a subtree of τ. Then, the tree monomial $T' := (\tau', \mathsf{x}|_{\mathrm{Int}(\tau')})$ is called a *submonomial* of T. The tree monomial T is said *divisible* by $T_1 = (\tau_1, \mathsf{x}_1) \in \mathrm{TM}(\mathcal{X})$ if τ contains a subtree τ_1' isomorphic to τ_1 via $\phi : \tau_1' \to \tau_1$ and $\mathsf{x}(v) = \mathsf{x}_1(\phi(v))$ for all $v \in \mathrm{Int}(\tau_1')$.

A *tree polynomial* in $\mathcal{X}$ with coefficients in $\mathbb{F}$ is an $\mathbb{F}$-linear combination of tree monomials of the same arity. The *support* of a tree polynomial f, denoted by $\mathrm{Supp}(f)$, is the set of all tree monomials that appear in f with nonzero coefficients. The arity of f is defined to be the arity of a tree monomial in $\mathrm{Supp}(f)$. Denote the vector space of all tree polynomials of arity $n \geq 1$ by $\mathcal{T}(\mathcal{X})(n)$ and $\mathcal{T}(\mathcal{X})(0) = 0$. (Note that we only consider $\mathcal{X}$ with $\mathcal{X}(0) = \emptyset$ in this paper.) Let $\mathcal{T}(\mathcal{X}) := \{\mathcal{T}(\mathcal{X})(n)\}_{n \geq 0}$. In order to make $\mathcal{T}(\mathcal{X})$ an operad, we need to define compositions of tree polynomials. We first define the graftings of PRTs.

Definition 2.12 ***(*****[BD16, Definition 3.3.3.2]*****)*****.** Suppose that τ_1 and τ_2 are PRTs. Let $\ell \in \mathrm{Leaves}(\tau_1) = \{1, 2, \ldots, \mathrm{Ar}(\tau_1)\}$. We define a PRT $\tau_1 \circ_\ell \tau_2$, called the result of *partial grafting* of τ_2 to τ_1 at ℓ, as follows:

$$\begin{aligned}
\mathrm{Root}(\tau_1 \circ_\ell \tau_2) &:= \mathrm{Root}(\tau_1),\\
\mathrm{Int}(\tau_1 \circ_\ell \tau_2) &:= \mathrm{Int}(\tau_1) \sqcup \mathrm{Int}(\tau_2),\\
\mathrm{Leaves}(\tau_1 \circ_\ell \tau_2) &:= \mathrm{Leaves}(\tau_1) \sqcup \mathrm{Leaves}(\tau_2) \setminus \{\ell\};
\end{aligned}$$

the parent function and the planar structure on $\tau_1 \circ_\ell \tau_2$ are induced respectively by the parent functions and planar structures of τ_1 and τ_2 with two exceptions: for the only vertex $v \in \mathrm{Child}_{\tau_2}(\mathrm{Root}(\tau_2))$, define $\mathrm{Parent}_{\tau_1 \circ_\ell \tau_2}(v) := \mathrm{Parent}_{\tau_1}(\ell)$; the total order needed by the planar structure puts v in the place of ℓ.

Partial graftings of PRTs induce partial compositions of tree monomials.

Definition 2.13. Given two tree monomials $T_1 = (\tau_1, \mathsf{x}_1)$ and $T_2 = (\tau_2, \mathsf{x}_2)$, we define the *partial composition* $T_1 \circ_\ell T_2$ for $1 \leq \ell \leq \mathrm{Ar}(T_1)$ to be the tree monomial $T = (\tau, \mathsf{x})$, where $\tau = \tau_1 \circ_\ell \tau_2$ and

$$\mathsf{x}(v) := \begin{cases} \mathsf{x}_1(v) & v \in \mathrm{Int}(\tau_1), \\ \mathsf{x}_2(v) & v \in \mathrm{Int}(\tau_2). \end{cases}$$

Example 2.14. Let T_1 and T_2 be the same as in Example 2.10 (see Figure 2.3). All possible compositions $T_1 \circ_\ell T_2$ are demonstrated in Figure 2.4.

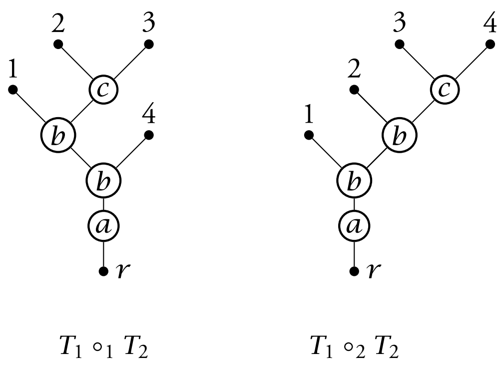


FIGURE 2.4. Partial compositions of tree monomials

Extending compositions of tree monomials by multilinearity to the collection $\mathcal{T}(\mathcal{X})$ gives *partial compositions* of tree polynomials:

$$\circ_i : \mathcal{T}(\mathcal{X})(n) \otimes \mathcal{T}(\mathcal{X})(m) \to \mathcal{T}(\mathcal{X})(n+m-1),$$
$$\alpha \otimes \beta \mapsto \alpha \circ_i \beta, \quad 1 \leq i \leq n.$$

The following lemma is easy to prove.

Lemma 2.15. *Equipped with the partial compositions defined above, $\mathcal{T}(\mathcal{X})$ is the* free reduced ns operad generated by $\mathcal{X}$.

An *ideal* $\mathcal{I}$ of an operad $\mathcal{P}$ is a subcollection of $\mathcal{P}$ such that each composition $f \circ_i g$ belongs to $\mathcal{I}$ whenever f or g belongs to $\mathcal{I}$. Suppose S is a subcollection of $\mathcal{P}$. The *ideal of $\mathcal{P}$ generated by* S, denoted by (S), is the smallest (by inclusion) ideal of $\mathcal{P}$ containing S.

We are ready now to define a presentation of an operad by generators and relations.

Definition 2.16. Suppose that an operad $\mathcal{P}$ is the quotient of the free operad $\mathcal{T}(\mathcal{X})$ by some ideal $\mathcal{I}$, and that $\mathcal{I}$ is generated by a subcollection $\mathcal{R} \subset \mathcal{I}$. We say that the operad $\mathcal{P}$ is *presented by generators $\mathcal{X}$ and relations $\mathcal{R}$*. We call $\mathcal{P}$ *finitely generated* (respectively, *finitely presented*) if $\mathcal{P}$ can be presented by a finite set $\mathcal{X}$ of generators; that is, $\bigsqcup_{n\geq 0} \mathcal{X}(n)$ is a finite set (respectively, by a finite set $\mathcal{X}$ of generators and a finite-dimensional subcollection $\mathcal{R}$ of relations).

3. GRÖBNER-SHIRSHOV BASES OF NS OPERADS

In this section, we follow the ideas in [BD16, Chapter 3] to introduce Gröbner-Shirshov bases (also known as Gröbner bases) theory for ns operads. Much as in the case of associative algebras, the Gröbner-Shirshov basis method is useful for the computation of GK-dimension of an operad.

Let $\mathcal{X}$ be an operation alphabet and $\mathcal{X}^*$ be the free monoid generated by $\mathcal{X}$. Recall that a total order $>$ on $\mathcal{X}^*$ is called a *monomial order* on $\mathcal{X}^*$ if $>$ is a well-order and $u_1 > u_2$ implies $u_1u_3 > u_2u_3$ and $u_3u_1 > u_3u_2$ for all $u_1, u_2, u_3 \in \mathcal{X}^*$.

A collection of total orders $\succ_n$ of $\mathrm{TM}(\mathcal{X})(n)$, $n \geq 0$, is called a *monomial order* on $\mathrm{TM}(\mathcal{X})$ if the following conditions are satisfied:

(i) Each $\succ_n$ is a well-order.
(ii) Each partial composition is a strictly increasing function in each of its arguments, that is if $T_0, T_0' \in \mathrm{TM}(\mathcal{X})(m)$, $T_1, T_1' \in \mathrm{TM}(\mathcal{X})(n)$, $1 \leq i \leq m$, then

$$\begin{aligned} T_0 \circ_i T_1 &\succ_{m+n-1} T_0' \circ_i T_1 \quad \text{if } T_0 \succ_m T_0', \\ T_0 \circ_i T_1 &\succ_{m+n-1} T_0 \circ_i T_1' \quad \text{if } T_1 \succ_n T_1'. \end{aligned}$$

Now, we introduce a monomial order on $\mathrm{TM}(\mathcal{X})$. Let $T = (\tau, x)$ be a tree monomial. For each leaf ℓ of τ, we record the labels of internal vertices of the path from the root to ℓ, forming a word in alphabet $\mathcal{X}$. The sequence of these words, ordered by the planar structure on $\mathrm{Leaves}(\tau)$, is called the *path sequence* of the tree monomial T, denoted by $\mathrm{Path}(T)$.

Example 3.1. The path sequences of the tree monomials in Figure 2.3 are

$$\begin{aligned} \mathrm{Path}(T_1) &= (ab, ab), & \mathrm{Path}(T_2) &= (b, bc, bc), \\ \mathrm{Path}(T_3) &= (bc, bc, b), & \mathrm{Path}(T_4) &= (bc, bc, bb, bb). \end{aligned}$$

It is not difficult to prove that a tree monomial is uniquely determined by its path sequence (see [BD16, Lemma 3.4.1.4]). Given a monomial order $>$ on $\mathcal{X}^*$, we define an order (still denoted by $>$, called the *path extension* of the monomial order on $\mathcal{X}^*$) on $\mathrm{TM}(\mathcal{X})$ by using path sequences of tree monomials. Suppose $T, T' \in \mathrm{TM}(\mathcal{X})$, $\mathrm{Path}(T) = (u_1, u_2, \ldots, u_m)$ and $\mathrm{Path}(T') = (u_1', u_2', \ldots, u_n')$. We say $T > T'$ if either $m > n$, or $m = n$ and there exists $1 \leq i \leq m$ such that $u_1 = u_1'$, $u_2 = u_2'$, $\ldots$, $u_{i-1} = u_{i-1}'$, $u_i > u_i'$.

The following lemma is copied from [BD16, Proposition 3.4.1.6].

Lemma 3.2. *Suppose $\mathcal{X} = \bigsqcup_{n\geq 1} \mathcal{X}(n)$; namely, $\mathcal{X}(0) = \varnothing$. The path extension of a monomial order on $\mathcal{X}^*$ is a monomial order on* $\mathrm{TM}(\mathcal{X})$.

Note that the above statement does not hold if we allow internal vertices of arity zero in a PRT. See the following example.

Example 3.3. Let $\mathcal{X} = \{a, b\}$ with $a \in \mathcal{X}(2)$ and $b \in \mathcal{X}(0)$. Consider the

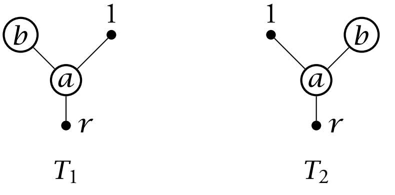


FIGURE 3.1. Tree monomials with $\mathcal{X}(0) \neq \varnothing$

tree monomials T_1 and T_2 in Figure 3.1. The path sequences of the tree monomials are

$$\mathrm{Path}(T_1) = (ab, a), \quad \mathrm{Path}(T_2) = (a, ab).$$

Under the path extension of a degree-lexicographic order on $\mathcal{X}^*$, we have $T_1 > T_2$. However, if we denote T_3 the tree monomial with only one internal vertex labeled by b, then the compositions $T_1 \circ_1 T_3$ and $T_2 \circ_1 T_3$ are equal, which shows that the path extension is not a monomial order.

Fix a monomial order on $\mathrm{TM}(\mathcal{X})$. Suppose

$$g = a_1U_1 + a_2U_2 + \cdots + a_nU_n \in \mathcal{T}(\mathcal{X})$$

where $n \geq 1$, each $a_i \in \mathbb{F}^*$, $U_i \in \mathrm{TM}(\mathcal{X})$, and $U_1 > U_2 > \cdots > U_n$. Then, U_1 (respectively, a_1, a_1U_1) is called the *leading monomial* (respectively, *leading coefficient*, *leading term*) of g, denoted by $\bar{g}$ (respectively, $\mathrm{lc}(g)$, $\mathrm{lt}(g)$). We say g is *monic* if $\mathrm{lc}(g) = 1$.

Suppose $\mathcal{G} \subseteq \mathcal{T}(\mathcal{X})$. Let $\bar{\mathcal{G}} := \{\bar{g} : g \in \mathcal{G}\}$ and

$$\mathrm{Irr}(\mathcal{G}) := \{u \in \mathrm{TM}(\mathcal{X}) : u \text{ is not divisible by } \bar{g},\ \forall\, g \in \mathcal{G}\}.$$

A tree polynomial f is *reduced* with respect to $\mathcal{G}$ if $\mathrm{Supp}(f) \subseteq \mathrm{Irr}(\mathcal{G})$. We say $\mathcal{G}$ is *self-reduced* if, for all $g \in \mathcal{G}$, g is monic and reduced with respect to $\mathcal{G} \setminus \{g\}$.

Note that, for an ideal $\mathcal{I}$ of $\mathcal{T}(\mathcal{X})$, the vector space $\mathbb{F}\bar{\mathcal{I}}$ spanned by the leading monomials $\bar{\mathcal{I}}$ is also an ideal (see [BD16, Proposition 3.4.3.1]), which is exactly the ideal generated by $\bar{\mathcal{I}}$, that is, $(\bar{\mathcal{I}}) = \mathbb{F}\bar{\mathcal{I}}$.

Definition 3.4. Let $\mathcal{I}$ be an ideal of $\mathcal{T}(\mathcal{X})$. A subcollection $\mathcal{G}$ of $\mathcal{I}$ is called a *Gröbner-Shirshov basis* for $\mathcal{I}$ (or for the quotient operad $\mathcal{T}(\mathcal{X})/\mathcal{I}$) if the ideal generated by $\bar{\mathcal{G}}$ coincides with that generated by $\bar{\mathcal{I}}$, that is, $(\bar{\mathcal{G}}) = (\bar{\mathcal{I}})$.

It is clear that an ideal $\mathcal{I} \trianglelefteq \mathcal{T}(\mathcal{X})$ is a Gröbner-Shirshov basis for $\mathcal{I}$.

Proposition 3.5 ([BD16, Proposition 3.4.3.4]). *Let $\mathcal{I}$ be an ideal of $\mathcal{T}(\mathcal{X})$ and $\mathcal{G} \subseteq \mathcal{I}$. Then, $\mathcal{G}$ is a Gröbner-Shirshov basis for $\mathcal{I}$ if and only if* $\mathrm{Irr}(\mathcal{G})$ *forms an $\mathbb{F}$-basis for the quotient $\mathcal{T}(\mathcal{X})/\mathcal{I}$.*

4. Gelfand-Kirillov Dimension

We refer to [KL00] for basics about the Gelfand-Kirillov dimension of associative algebras. The Gelfand-Kirillov dimension of a locally finite operad is defined in [KP15, p. 400] and [BYZ20, Definition 4.1].

Definition 4.1 *(*[KP15, p. 400]*,* [BYZ20, Definition 4.1]*)***.** Let $\mathcal{P}$ be a locally finite operad. The *Gelfand-Kirillov dimension*, or *GK-dimension* for short, of $\mathcal{P}$ is defined to be

$$\mathrm{GKdim}(\mathcal{P}) := \limsup_{n\to\infty} \log_n \Big(\sum_{i=0}^{n} \dim \mathcal{P}(i) \Big)$$

where dim stands for $\dim_{\mathbb{F}}$.

When we talk about the GK-dimension of an operad $\mathcal{P}$, we always implicitly assume $\mathcal{P}$ is locally finite. By Lemma 4.2 below, we might only consider the GK-dimension of reduced (or reduced connected) operads from now on.

Given two subcollections $\mathcal{V}$ and $\mathcal{W}$ of $\mathcal{P}$, let $\mathcal{V} \circ \mathcal{W}$ be the subcollection of $\mathcal{P}$ spanned by all elements of the form $v \circ_i w$ for $v \in \mathcal{V}$, $w \in \mathcal{W}$ and $1 \le i \le \mathrm{Ar}(v)$. Given a subcollection $\mathcal{V}$ of $\mathcal{P}$, let $\mathcal{V}^0 = (0, \mathbb{F}, 0, 0, \dots)$, and inductively, let $\mathcal{V}^m = \mathcal{V}^{m-1} \circ \mathcal{V}$ for $m \ge 1$. It is clear that $\mathcal{V}^m = \{\mathcal{V}^m(n)\}_{n\ge 0}$ where $\mathcal{V}^m(n)$ is the subspace of $\mathcal{P}(n)$ spanned by all elements of arity n that have the following form:

$$((\cdots((a_1 \circ_{j_1} a_2) \circ_{j_2} a_3) \circ_{j_3} \cdots) \circ_{j_{m-1}} a_m), \quad a_i \in \mathcal{V}. \tag{4.1}$$

We call $\mathcal{V}$ a *generating subcollection* of $\mathcal{P}$ if

$$\mathcal{P} = \sum_{m\ge 0} \mathcal{V}^m := \Big\{ \sum_{m\ge 0} \mathcal{V}^m(n) \Big\}_{n\ge 0}.$$

We say $\mathcal{P}$ is a *finitely generated* operad with a finite generating alphabet $\mathcal{X}$, if $\mathcal{P}$ has a finite-dimensional generating subcollection $\mathcal{V} = \{\mathcal{V}(n)\}_{n\ge 0}$ where $\mathcal{V}(n)$ is the space spanned by $\mathcal{X}(n)$.

The following lemma is easy to prove and its proof is omitted.

Lemma 4.2. *Let $\mathcal{P}$ be a finitely generated locally finite operad. Then, the following hold:*

(1) *We define a reduced operad associated with $\mathcal{P}$:*

$$\mathcal{P}_r := (0, \mathcal{P}(1), \mathcal{P}(2), \mathcal{P}(3), \dots).$$

Then, $\mathcal{P}_r$ is finitely generated and locally finite.

(2) *We define a reduced connected operad*

$$\mathcal{P}_{rc} := (0, \mathbb{F}, \mathcal{P}(2), \mathcal{P}(3), \dots).$$

Then, $\mathcal{P}_{rc}$ is finitely generated and locally finite.
(3) $\mathrm{GKdim}(\mathcal{P}_r) = \mathrm{GKdim}(\mathcal{P}_{rc}) = \mathrm{GKdim}(\mathcal{P})$.

The lemma below gives a characterization of the GK-dimension of a finitely generated operad.

Lemma 4.3. *Suppose $\mathcal{P}$ is a locally finite operad generated by a finite-dimensional subcollection $\mathcal{V}$. Then,*

$$\mathrm{GKdim}(\mathcal{P}) = \limsup_{n\to\infty} \log_n \Big(\dim \Big(\sum_{i=0}^{n} \mathcal{V}^i \Big) \Big).$$

Proof. Let t be the maximal arity of nonzero elements in $\mathcal{V}$, and s be an integer such that

$$(\mathcal{P}(0), \mathcal{P}(1), \dots, \mathcal{P}(t), 0, 0, \dots) \subseteq \sum_{j=0}^{s} \mathcal{V}^j.$$

By the definition of t, we have $\mathcal{V} \subseteq (\mathcal{P}(0), \mathcal{P}(1), \dots, \mathcal{P}(t), 0, 0, \dots)$. Then, for every $n > 0$,

$$\sum_{i=0}^{n} \mathcal{V}^i \subseteq (\mathcal{P}(0), \mathcal{P}(1), \dots, \mathcal{P}(nt - (n-1)), 0, 0, \dots) \subseteq \sum_{i=0}^{ns} \mathcal{V}^i. \tag{4.2}$$

Consequently,

$$\dim \Big(\sum_{i=0}^{n} \mathcal{V}^i \Big) \le \dim \Big(\sum_{i=0}^{nt-(n-1)} \mathcal{P}(i) \Big) \le \dim \Big(\sum_{i=0}^{ns} \mathcal{V}^i \Big), \quad n > 0.$$

Thus, we have that

$$\begin{aligned} \mathrm{GKdim}(\mathcal{P}) &= \limsup_{n\to\infty} \log_n \Big(\dim \Big(\sum_{i=0}^{n} \mathcal{P}(i) \Big) \Big) \\ &= \limsup_{n\to\infty} \log_n \Big(\dim \Big(\sum_{i=0}^{n} \mathcal{V}^i \Big) \Big). \end{aligned}$$ ☐

The following example shows that, in general, the GK-dimension of an operad is not the supremum of the GK-dimensions of its finitely generated suboperads.

Example 4.4. Let α be any positive real number. Let $\mathcal{P}$ be the operad generated by infinitely many elements in $\mathcal{X}$ such that

$$\mathcal{X} = \{\mathcal{X}(n)\}_{n\geq 0},\ \mathcal{X}(0) = \mathcal{X}(1) = \emptyset,\ |\mathcal{X}(n)| = \lfloor n^{\alpha}\rfloor - \lfloor (n-1)^{\alpha}\rfloor,\ \forall\, n \geq 2,$$

and subject to relations

$$x_m \circ_i x_n = 0, \quad \forall\, x_m \in \mathcal{X}(m),\ x_n \in \mathcal{X}(n),\ m, n \geq 2,\ 1 \leq i \leq m.$$

Then, it is easy to check that $\operatorname{GKdim}(\mathcal{P}) = \alpha > 0$ but $\operatorname{GKdim}(\mathcal{P}') = 0$ for all finitely generated suboperads $\mathcal{P}'$ of $\mathcal{P}$.

We will use the following nice construction which is related to Warfield's example [War84].

Example 4.5. We fix a real number r strictly between 2 and 3 and let q be $(r-1)/2$ which is strictly between $\frac{1}{2}$ and 1. Let A be the quotient algebra $\mathbb{F}\langle x_1, x_2\rangle / J$ generated by two elements x_1, x_2 of degree 1 and modulo the monomial ideal J generated by monomials having degree ≥ 3 in x_2 together with all monomials of the form

$$x_1^i x_2 x_1^j x_2 x_1^{\ell}$$

satisfying $j < n - \lfloor n^q \rfloor$ where $n = i + j + \ell + 2$.

By an easy counting, $\dim A_0 = 1$, $\dim A_1 = 2$, and for each $n \geq 2$,

$$\dim A_n = 1 + n + \sum_{j=n-\lfloor n^q\rfloor}^{n-2} (n-1-j) = 1 + n + \sum_{p=1}^{\lfloor n^q\rfloor - 1} p.$$

As a consequence, we have the following:

(1) $\dim A_n$ is strictly increasing.

(2) We have

$$\dim A_n = 1 + n + \frac{1}{2}(\lfloor n^q\rfloor - 1)(\lfloor n^q \rfloor) \sim \frac{1}{2} n^{2q} = \frac{1}{2} n^{r-1}.$$

(3) $\operatorname{GKdim}(A) = r$.

Using the example above, we obtain the range of possible values for the GK-dimension of an associative algebra.

Lemma 4.6.

(1) *For every integer $d \in \mathbb{N}^*$, there is a graded algebra A such that*

$$\operatorname{GKdim}(A) = d \quad \text{and} \quad \{\dim A_i\}_{i=0}^{\infty} \text{ is weakly increasing.}$$

(2) *There is a graded algebra A such that* $\operatorname{GKdim}(A) = \infty$.

(3) *For $d \in R_{\mathrm{GKdim}} \setminus \{0\}$, there is a graded algebra A such that* $\mathrm{GKdim}(A) = d$ *and* $\{\dim A_i\}_{i=0}^{\infty}$ *is strictly increasing.*

(4) *All algebras in parts* (1), (2), (3) *can be taken to be monomial algebras* (*hence connected graded*) *finitely generated in degree* 1.

Proof. (1) We can take A as the commutative polynomial ring $\mathbb{F}[x_1, \dots, x_d]$.

(2) We can take A to be the free algebra $\mathbb{F}\langle x_1, x_2 \rangle$.

(3) If d is an integer, it follows from part (1). If $d = \infty$, it follows from part (2). Now, we let d be a finite non-integral real number > 2. Let n be $\lfloor d \rfloor - 2$ and $r = d - n$. Then, $n \geq 0$ and $2 < r < 3$. Let A be the algebra given in Example 4.5, and let $A' = A[x_1, \dots, x_n]$. Then, the assertion follows.

(4) It is well known (see, e.g., [Bel15, Remark 4.1]) that given a finitely generated associative algebra A, there is a finitely generated monomial algebra $B = \mathbb{F}\langle X \rangle / I$ such that $\mathrm{GKdim}(A) = \mathrm{GKdim}(B)$, where $\mathbb{F}\langle X \rangle$ is the free associative algebra generated by a finite set X and I is an ideal consisting of words in X. Since A is connected graded, we even have that A and B have the same Hilbert series. □

Below is a weak version of Theorem 1.2 (2).

Proposition 4.7. *For any $r \in R_{\mathrm{GKdim}}$, there exists a finitely generated operad $\mathcal{P}$ such that* $\mathrm{GKdim}(\mathcal{P}) = r$.

Proof. By Lemma 4.6 (3), there is a finitely generated connected graded monomial algebra A such that $\mathrm{GKdim}(A) = r$. By Construction 2.3, there is a finitely generated operad $\mathcal{P}$ such that $\dim \mathcal{P}(n) = \dim A_{n-1}$ for all $n \geq 1$. Therefore, by definition, $\mathrm{GKdim}(\mathcal{P}) = \mathrm{GKdim}(A) = r$. The assertion follows. □

Proposition 4.8. *Suppose $\mathcal{P}$ is a finitely generated and locally finite ns operad. We then have the following:*

(1) $\mathrm{GKdim}(\mathcal{P}) = 0$ *if and only if $\mathcal{P}$ is finite dimensional.*

(2) $\mathrm{GKdim}(\mathcal{P})$ *cannot be strictly between* 0 *and* 1.

Proof. (1) The "if" part is clear. For the "only if" part, suppose $\dim(\mathcal{P}) = \infty$. Since $\mathcal{P}$ is finitely generated, there is a finite-dimensional subcollection $(1_{\mathcal{P}} \in) \mathcal{V}$ that generates $\mathcal{P}$. We claim that $\mathcal{V}^{m+1} \neq \mathcal{V}^m$ for every m. Suppose to the contrary $\mathcal{V}^{m+1} = \mathcal{V}^m$ for some m. Then by induction, one sees that $\mathcal{V}^n = \mathcal{V}^m$ for every $n > m$. Thus, $\mathcal{P} = \bigcup_{n>m} \mathcal{V}^n = \mathcal{V}^m$, which is finite dimensional. This yields a contradiction. Therefore, we have proved the claim, and consequently, $\dim \mathcal{V}^m \geq m + 1$ for every m. By Lemma 4.3,

$$\mathrm{GKdim}(\mathcal{P}) = \limsup_{n \to \infty} \log_n \Big(\sum_{i=0}^{n} \dim \mathcal{V}^i \Big) \geq \lim_{n \to \infty} \log_n (n + 1) = 1,$$

a contradiction.

(2) (See the proof of part (1).) □

In part (2) of the above proposition, as in the case of associative algebras, there is a gap between 0 and 1 for the GK-dimensions of finitely generated operads. This is false if $\mathcal{P}$ is infinitely generated (see Example 4.4 above and [BYZ20, Corollary 6.12]).

A *monomial operad* means a quotient of free operad by an ideal generated by tree monomials. The following lemma implies that given an operad, there exists a monomial operad with the same GK-dimension.

Lemma 4.9. *Suppose $\mathcal{I} \trianglelefteq \mathcal{T}(\mathcal{X})$. Then,*

$$\operatorname{GKdim}(\mathcal{T}(\mathcal{X})/\mathcal{I}) = \operatorname{GKdim}(\mathcal{T}(\mathcal{X})/(\bar{\mathcal{I}})).$$

Proof. Note that both $\mathcal{I}$ and $(\bar{\mathcal{I}})$ are ideals and thus Gröbner-Shirshov bases. Now, the statement follows from Proposition 3.5 and $\operatorname{Irr}(\mathcal{I}) = \operatorname{Irr}((\bar{\mathcal{I}}))$. □

5. Bergman's Gap Theorem

5.1. Single-branched ns operads. In this subsection, we will introduce so-called single-branched tree monomials and study their properties, which will be used in the next subsection to prove Bergman's gap theorem for operads.

Let τ be a PRT. An internal vertex v is called a *top internal vertex* if its children are all leaves. A *branch* of τ is a path from the root to a top internal vertex v, denoted by $\operatorname{bran}(v)$. A PRT is called a *single-branched tree* if it has exactly one branch. The planar structure of τ induces an order on branches of τ:

$$\operatorname{bran}(v) > \operatorname{bran}(v') \quad \text{if } \operatorname{h}(v) > \operatorname{h}(v'), \text{ or } \operatorname{h}(v) = \operatorname{h}(v') \text{ and } v > v'.$$

The maximal branch of τ is called the *pivot branch* of τ. Denote by $\operatorname{Pivot}(\tau)$ the set of all vertices of τ that belong to the pivot branch of τ. The top internal vertex of τ belonging to $\operatorname{Pivot}(\tau)$ is called the *pivot top internal vertex*, denoted by $\operatorname{TIV}(\tau)$.

A tree monomial is said to be *single branched* if its underlying tree is single branched. An operad is called *single branched* if it has an $\mathbb{F}$-basis that consists of single-branched tree monomials.

A tree monomial T is called *right normal* if it can be written in terms of partial compositions of generating operations with parentheses from right to left, that is,

$$T = (x_1 \circ_{i_1} (\cdots (x_{n-2} \circ_{i_{n-2}} (x_{n-1} \circ_{i_{n-1}} x_n)) \cdots)). \tag{5.1}$$

Left normal tree monomials are defined similarly (see (4.1)). Note that every tree monomial is left normal, but not necessarily right normal. For a right normal tree monomial, we usually write without parentheses (and/or composition symbols $\circ_i$ sometimes) for short when no confusion arises, for example,

$$T = x_1 \circ_{i_1} x_2 \circ_{i_2} \cdots \circ_{i_{n-1}} x_n = x_1 x_2 \cdots x_n. \tag{5.2}$$

The following lemma is straightforward.

Lemma 5.1. *Let T be a tree monomial. Then, the following statements are equivalent:*

(i) *T is single branched.*
(ii) *T is right normal.*
(iii) $\mathrm{wt}(T) = \mathrm{h}(T)$.

Example 5.2. In Figure 2.3, T_1, T_2, and T_3 are single-branched and T_4 is not single branched.

A single-branched tree monomial $w = x_1 \circ_{i_1} x_2 \circ_{i_2} \cdots \circ_{i_{n-1}} x_n$ is called *periodic* if there exists a positive integer $p < n$ such that $x_j = x_{j+p}$ for all $1 \leq j \leq n-p$ and $i_{j'} = i_{j'+p}$ for all $1 \leq j' \leq n-p-1$. The integer p is called a *local period* of w and the smallest local period of w is called the *minimal period* of w. Given a periodic tree monomial $w = x_1 \circ_{i_1} x_2 \circ_{i_2} \cdots \circ_{i_{n-1}} x_n$ with minimal period p, by using its minimal period, we can extend w to the left and/or to the right to get a new single-branched tree monomial which contains w as a submonomial. More precisely, w can be extended to

$$w_{m,\ell} := x_{-m} \circ_{i_{-m}} \cdots \circ_{i_{-1}} x_0 \circ_{i_0} x_1 \circ_{i_1} x_2 \circ_{i_2} \cdots \circ_{i_{n-1}} x_n \circ_{i_n} \cdots \circ_{i_{\ell-1}} x_\ell$$

where $m \geq -1$, $\ell \geq n$, for each integer q between $-m$ and ℓ (say, $-m \leq q = ps + r \leq \ell$, $s, r \in \mathbb{Z}$, $1 \leq r \leq p$), $x_q = x_r$, and $\circ_{i_q} = \circ_{i_r}$ (except $\circ_{i_\ell}$). It is clear that p is still the minimal period of $w_{m,\ell}$. If a positive integer p' is a local period of all extensions $w_{m,\ell}$ of w, we call p' a *period* of w. For example, given

$$w = a \circ_1 a \circ_1 b \circ_1 a \circ_1 a \circ_1 b \circ_1 a \circ_1 a,$$

then 3 is the minimal period of w, 6 is a period of w, and 7 is a local period (but not a period) of w.

The following lemma on periods is easy to prove.

Lemma 5.3. *Suppose w is a single-branched tree monomial of minimal period p. If ℓ is a period of w, then p divides ℓ.*

Proof. Suppose to the contrary that $\ell = pq + r$ with $q, r \in \mathbb{N}$ and $0 < r < p$. Assume $w = x_1 \circ_{i_1} x_2 \circ_{i_2} \cdots \circ_{i_{n-1}} x_n$. Consider the extension

$$w_{\ell,n+p} = x_{-\ell} \circ_{i_{-\ell}} \cdots \circ_{i_{-1}} x_0 \circ_{i_0} x_1 \circ_{i_1} \cdots \circ_{i_{n-1}} x_n \circ_{i_n} \cdots \circ_{i_{n+p-1}} x_{n+p}.$$

For all $1 \leq i \leq n-r$, since ℓ and pq are local periods of $w_{\ell,n+p}$, we have that $x_{i+r} = x_{i+\ell-pq} = x_{i-pq} = x_i$. Thus, r is a local period of w, contradicting the minimality of p. $\square$

The following is an analogue of [KL00, Lemma 2.3].

Lemma 5.4. *Let w be a single-branched tree monomial of height $n > 0$. Suppose that w is periodic with minimal period $p < n$ and has two equal submonomials*

$$x_{i+1} \circ_{\alpha_{i+1}} \cdots \circ_{\alpha_{i+r-1}} x_{i+r} = x_{j+1} \circ_{\alpha_{j+1}} \cdots \circ_{\alpha_{j+r-1}} x_{j+r} \tag{5.3}$$

of height $r \geq p$, $0 \leq i < j$. Then, p divides $j-i$.

Proof. We modify the proof of [KL00, Lemma 2.3] to take care of the composition indices. For the manipulations described below, if the tree monomial w is too short, consider it in its extensions.

Since $r \geq p$, we have $i+r \geq i+p$. Similarly, $j+r \geq j+p$. Since p is the minimal period of w,

$$x_i = x_{i+p} = x_{j+p} = x_j.$$

By periodicity, we have the following equality of two sets:

$$\{\circ_{\alpha_i}, \circ_{\alpha_{i+1}}, \ldots, \circ_{\alpha_{i+p-1}}\} = \{\circ_{\alpha_j}, \circ_{\alpha_{j+1}}, \ldots, \circ_{\alpha_{j+p-1}}\}.$$

By (5.3), we also have

$$\{\circ_{\alpha_{i+1}}, \ldots, \circ_{\alpha_{i+p-1}}\} = \{\circ_{\alpha_{j+1}}, \ldots, \circ_{\alpha_{j+p-1}}\},$$

which forces that $\circ_{\alpha_i} = \circ_{\alpha_j}$. Thus, w has two equal submonomials

$$x_i \circ_{\alpha_i} x_{i+1} \circ_{\alpha_{i+1}} \cdots \circ_{\alpha_{i+r-1}} x_{i+r} = x_j \circ_{\alpha_j} x_{j+1} \circ_{\alpha_{j+1}} \cdots \circ_{\alpha_{j+r-1}} x_{j+r},$$

which have height $r+1$. Similarly, we can extend the above two submonomials to any height $\geq r$.

Next, we show that $j-i$ is a period of w. In any extension $w_{m,m'}$ of w (for some $m \geq 0$ and $m' \geq n$), let $-m \leq \ell \leq m' - (j-i)$ and $t \in \mathbb{Z}$ such that $\ell + tp = i + s$, $0 \leq s \leq p-1$. Then,

$$x_{\ell+(j-i)} = x_{\ell+(j-i)+tp} = x_{i+s+(j-i)} = x_{j+s} = x_{i+s} = x_{\ell+tp} = x_\ell$$

for $-m \leq \ell \leq m' - (j-i)$ and

$$\alpha_{\ell+(j-i)} = \alpha_{\ell+(j-i)+tp} = \alpha_{i+s+(j-i)} = \alpha_{j+s} = \alpha_{i+s} = \alpha_{\ell+tp} = \alpha_\ell$$

for $-m \leq \ell \leq m' - (j-i) - 1$. Hence, w has a period $j-i$, and now it follows from Lemma 5.3 that p divides $j-i$. ◻

We modify [KL00, Lemma 2.4] and obtain the following result.

Lemma 5.5. *Suppose $\mathcal{X}$ is a finite operation alphabet. Let W be a set of single-branched tree monomials on $\mathcal{X}$ such that all submonomials of elements in W still belong to W. Suppose that, for some positive integer $d \geq 3$, W contains at most $d-1$ tree monomials of height d. Then, W contains at most $(d-1)^3$ tree monomials of height h for all $h \geq d$.*

Proof. Let w be a single-branched tree monomial of height h of the form (5.2), and let W_h be the subset of W consisting of $w \in W$ of height h.

If $h \leq 2d - 1$, then w is completely determined by its top and bottom submonomials of height d. There are at most $(d-1)^2$ possibilities for this case. Thus,

$$|W_h| \leq (d-1)^2 < (d-1)^3.$$

If $2d \leq h \leq 3d - 2$, then w is completely determined by its top and bottom submonomials of height d and the submonomial $x_d \cdots x_{2d-1}$. There are at most $(d-1)^3$ possibilities for this case. Thus,

$$|W_h| \leq (d-1)^3.$$

For other cases, we need to use the following claim.

Claim 5.6. *Suppose w has height $\geq 2d-1$. Then, $w = w_1 w_2 w_3$ $(= w_1 \circ_{\beta_1} w_2 \circ_{\beta_2} w_3)$ where w_2 is periodic of minimal period $p \leq d-1$, $\mathrm{h}(w_2) \geq p + d$, $\mathrm{h}(w_1) \leq d - p$, and $\mathrm{h}(w_3) \leq d - p$. The periodic submonomial w_2 is $x_{j+1} \cdots x_r$ for some integers $j \leq d - p$ and $r \geq h - (p - d)$.*

Note that our lemma for $h \geq 3d - 1$ $(> 2d - 1)$ follows from the claim. In fact, by the claim, $w = x_1 \cdots x_h$ for $h \geq 3d - 1$ is uniquely determined by its bottom and top submonomials of height d (each has at most $d - 1$ possibilities by assumption), and the periodic submonomial $w_2' := x_{d-p+1} \cdots x_{h-(d-p)}$. Note that w_2' is a submonomial of w_2 with minimal period $p \leq d - 1$. Hence, w_2' has at most $d - 1$ possibilities, determined by its top submonomial of height d. As a result, there are at most $(d-1)^3$ possibilities for w, as desired.

Proof of Claim 5.6. We prove this claim by induction on h.

Initial step: Suppose $h = 2d - 1$. Then, $w = x_1 x_2 \cdots x_{2d-1}$ has d submonomials of height d, all belonging to W by assumption. Hence, two of these tree monomials must be equal, say,

$$x_{j+1} \cdots x_{j+d} = x_{j+p+1} \cdots x_{j+p+d}$$

where $j \geq 0$, $p \geq 1$, $j + p \leq d - 1$, p chosen as small as possible. Then, $x_{j+1} \cdots x_{j+d}$ and $x_{j+p+1} \cdots x_{j+p+d}$ have a nonempty overlap, and thus

$$w_2 := x_{j+1} \cdots x_{j+p+d}$$

is a periodic submonomial (of w) of minimal period $p \leq d - 1$ and of height $\mathrm{h}(w_2) = p + d$. The bottom submonomial $w_1 := x_1 \cdots x_j$ and the top submonomial $w_3 := x_{j+p+d+1} \cdots x_{2d-1}$ are of height $\leq d - p - 1 < d - p$. Thus, the claim holds for $h = 2d - 1$ and we finish the initial step.

Inductive step: Suppose the claim is true for an $h \geq 2d-1$. Consider $w = x_1x_2\cdots x_{h+1} \in W$ of height $h+1$. By hypothesis,

$$w = x_1(x_2\cdots x_j)(x_{j+1}\cdots x_{j+r})(x_{j+r+1}\cdots x_hx_{h+1})$$

where $\mathrm{h}(x_2\cdots x_j) = j-1 \leq d-p$, $\mathrm{h}(x_{j+r+1}\cdots x_hx_{h+1}) = h-j-r+1 \leq d-p$, and $x_{j+1}\cdots x_{j+r}$ is periodic of minimal period $p \leq d-1$ and height $r \geq p+d$. If $j-1 < d-p$, that is, $j \leq d-p$, then the claim follows by taking

$$w_1 = x_1\cdots x_j, \quad w_2 = x_{j+1}\cdots x_{j+r}, \quad w_3 = x_{j+r+1}\cdots x_hx_{h+1}.$$

Now suppose $j-1 = d-p$. We will show that the periodicity of

$$x_{j+1}\cdots x_{j+r} = x_{j+1} \circ_{\alpha_{j+1}} \cdots \circ_{\alpha_{j+r-1}} x_{j+r}$$

can be extended down to include the term x_j, which completes our inductive step. Note that

$$j+r = d-p+1+r \geq d-p+1+d+p > 2d.$$

Thus, $x_2\cdots x_{2d}$ is a proper submonomial of $x_1\cdots x_{j+r}$ and contains two equal submonomials of height d, say,

$$x_{i+1}\cdots x_{i+d} = x_{i+n+1}\cdots x_{i+n+d},$$

where $i \geq 1$, $n \geq 1$, and $i+n \leq d$. Since $j = d-p+1 \leq d$, both of these submonomials have an overlap with the periodic tree monomial $x_{j+1}\cdots x_{j+r}$ and the overlap contains at least their top (i.e., right) $d-(j-1) = p$ terms. By Lemma 5.4, we have that p divides n (say, $n = cp \geq p$ for some integer $c > 0$), and hence

$$i+d > d \geq j = d-p+1 \geq d-n+1 \geq i+1.$$

Thus, $x_j = x_{j+n}$ and $\alpha_j = \alpha_{j+n}$. Note that $j+1 \leq j+p < j+r$ and $j+1 \leq j+n < j+d < j+r$. By the periodicity, we have that

$$\begin{aligned} x_{j+p} &= x_{j+n-(c-1)p} = x_{j+n} = x_j, \\ \alpha_{j+p} &= \alpha_{j+n-(c-1)p} = \alpha_{j+n} = \alpha_j, \end{aligned}$$

that is, the periodicity of $x_{j+1}\cdots x_{j+r}$ extends down to include x_j as desired.

This finishes the inductive step, then the claim, and finally the assertion in the lemma. ◻

The following example shows that the original [KL00, Lemma 2.4] is not true for the setting of operads, that is, if we replace $d-1$ by d in Lemma 5.5, then the statement does not hold any further.

Example 5.7. Let W be the set of single-branched tree monomials of the form

$$x \circ_{i_1} x \circ_{i_2} \cdots \circ_{i_{n-1}} x$$

where $\mathrm{Ar}(x) = 2$, $n \geq 1$, at most one composition index i_j equals 2, and the other indices are all 1. It is easy to see that all submonomials of elements in W are still in W and that, for $n \geq 1$, W contains exactly n distinct monomials of height n.

Lemma 5.8. *Let $\mathcal{P}$ be a locally finite operad with* $\mathrm{GKdim}(\mathcal{P}) < 2$. *Let a and b be two positive real numbers. The following hold:*

(1) *There are infinitely many integers n such that* $\dim \mathcal{P}(n) < an - b$.
(2) *Suppose $\mathcal{P}$ is finitely generated by $\mathcal{V}$. Then, there are infinitely many integers n such that* $\dim(\sum_{i=0}^{n} \mathcal{V}^i)/(\sum_{i=0}^{n-1} \mathcal{V}^i) < an - b$.

Proof. (1) Suppose to the contrary that $\dim \mathcal{P}(n) \geq an - b$ for all $n \gg 0$. Then, there is an m, such that

$$\mathrm{GKdim}(\mathcal{P}) = \limsup_{n\to\infty} \log_n \Big(\sum_{i=0}^{n} \dim \mathcal{P}(i) \Big) \geq \limsup_{n\to\infty} \log_n \Big(\sum_{i=m}^{n} (ai - b) \Big) = 2,$$

yielding a contradiction. The proof of part (2) is similar to the above. □

5.2. Gap theorem for ns operads. In this subsection, we first prove three lemmas and then use them to prove an analogue of Bergman's gap theorem for finitely generated ns operads.

Roughly speaking, the following lemma says that, if the GK-dimension of a finitely generated operad is less than 2, then there exists a uniform upper bound for the heights of maximal subtrees that are rooted at the pivot branch and do not contain the pivot top internal vertex.

Lemma 5.9. *Let $\mathcal{X}$ be a finite operation alphabet and $\mathcal{P} = \mathcal{T}(\mathcal{X})/\mathcal{I}$. Write $\mathcal{V} = \mathbb{F}\mathcal{X}$. Suppose* $\dim(\sum_{i=0}^{d} \mathcal{V}^i)/(\sum_{i=0}^{d-1} \mathcal{V}^i) \leq d - 2$ *for some d (or* $\mathrm{GKdim}(\mathcal{P}) < 2$*). Then, there exists a positive integer M_1 such that, for all $\tau \in \mathrm{Tr}(\mathrm{Irr}(\mathcal{I}))$,*

$$\mathrm{h}(\mathrm{MaxSub}_{\tau}(v, v')) < M_1$$

where $v \in \mathrm{Pivot}(\tau)$, $v' \in \mathrm{Int}(\tau) \setminus \mathrm{Pivot}(\tau)$.

Proof. Suppose to the contrary that for each $n \in \mathbb{N}$ there is a tree monomial $T_n \in \mathrm{Irr}(\mathcal{I})$ with underlying PRT τ_n and a maximal subtree

$$\tau'_n := \mathrm{MaxSub}_{\tau_n}(v_n, v'_n)$$

such that $h_n := \mathrm{h}(\tau'_n) \geq n$, where $v_n \in \mathrm{Pivot}(\tau_n)$ and $v'_n \in \mathrm{Int}(\tau_n) \setminus \mathrm{Pivot}(\tau_n)$. Assume that the vertices in $\mathrm{Pivot}(\tau'_n)$ are labelled from v_n to $\mathrm{TIV}(\tau'_n)$ by generating operations $x, y_1, y_2, \ldots, y_{h_n}$, and that the vertices in $\mathrm{Int}(\tau_n) \cap \mathrm{Pivot}(\tau_n)$

with height $\geq \mathrm{h}(v_n)$ are labelled from v_n to $\mathrm{TIV}(\tau_n)$ by $x, x_1, x_2, \ldots, x_m$ (see Figure 5.1, where only a subset of $\mathrm{Vert}(\tau_n)$ are drawn and the pivot branch is drawn vertically).

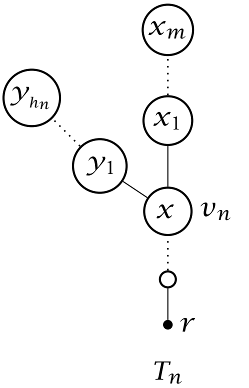


FIGURE 5.1. Tree monomial T_n

It follows from the definition of a pivot branch that $m \geq h_n \geq n$. Consider the submonomial T_n'' of T_n with underlying PRT $\tau_n'' \subseteq \tau_n$ for which $\mathrm{Root}(\tau_n'') = \mathrm{Parent}_{\tau_n}(v_n)$ and

$$\mathrm{Int}(\tau_n'') = \mathrm{Pivot}(\tau_n') \cup \{v \in \mathrm{Pivot}(\tau_n) \cap \mathrm{Int}(\tau_n) : \mathrm{h}(v) \geq \mathrm{h}(v_n)\}.$$

Without loss of generality, we assume T_n'' is of the form in Figure 5.1, and denote it by

$$T_n'' = (x \circ_j (x_1 x_2 \cdots x_m)) \circ_i (y_1 y_2 \cdots y_{h_n})$$

where i and j are proper composition indices and $i < j$. The following are distinct submonomials of T_n'' of weight n:

$$(x \circ_j (x_1 \cdots x_s)) \circ_i (y_1 \cdots y_{n-s-1}), \quad 1 \leq s \leq n-1.$$

These submonomials are pairwise distinct since their underlying PRTs are pairwise distinct. Thus, $\mathrm{Irr}(\mathcal{I})$ contains at least $n-1$ elements of weight n, namely, $\dim(\sum_{i=0}^{n} \mathcal{V}^i)/(\sum_{i=0}^{n-1} \mathcal{V}^i) \geq n-1$ for all n, yielding a contradiction. ◻

The following lemma says that, under suitable conditions, if an internal vertex v in the pivot branch has a child that is an internal vertex and not in the pivot branch, then v must be either "close" to the root or "close" to the top of the PRT.

Lemma 5.10. *Let $\mathcal{X}$ be a finite operation alphabet and $\mathcal{P} = \mathcal{T}(\mathcal{X})/\mathcal{I}$. Write $\mathcal{V} = \mathbb{F}\mathcal{X}$. Assume that* $\dim(\sum_{i=0}^{d} \mathcal{V}^i)/(\sum_{i=0}^{d-1} \mathcal{V}^i) \leq d-3$ *for some* $d \geq 3$ *(or* $\mathrm{GKdim}(\mathcal{P}) < 2$*). Suppose that*

$$W := \{T \in \mathrm{Irr}(\mathcal{I}) : T \text{ not single branched}\} \neq \varnothing.$$

Then, there is a positive integer M_2 *such that, for all* $\tau \in \mathrm{Tr}(W)$ *and all* $v \in \mathrm{Pivot}(\tau)$ *such that there is a maximal subtree* $\mathrm{MaxSub}(v, v')$ *for some* $v' \in \mathrm{Int}(\tau) \setminus \mathrm{Pivot}(\tau)$, *the following inequality holds:*

$$\min\{\mathrm{h}(v), \mathrm{h}(\tau) - \mathrm{h}(v)\} < M_2.$$

Proof. Suppose to the contrary that for each $n \in \mathbb{N}^*$ there exist $T_n \in W$ with $\mathrm{Tr}(T_n) = \tau_n$, $v_n \in \mathrm{Pivot}(\tau_n)$, and $v'_n \in \mathrm{Int}(\tau_n) \setminus \mathrm{Pivot}(\tau_n)$ such that τ_n has a maximal subtree $\mathrm{MaxSub}(v_n, v'_n)$ and $\min\{\mathrm{h}(v_n), \mathrm{h}(\tau_n) - \mathrm{h}(v_n)\} \geq n$. Without loss of generality, we suppose that $\mathrm{Parent}(v'_n) = v_n$. Consider the subtree $\tau'_n \subseteq \tau_n$ that has internal vertices

$$\mathrm{Int}(\tau'_n) = (\mathrm{Int}(\tau_n) \cap \mathrm{Pivot}(\tau_n)) \cup \{v'_n\}.$$

Assume that the vertex v'_n is labelled by generating operation x' and that the internal vertices in $\mathrm{Pivot}(\tau_n)$ are labelled from bottom to top by generating operations

$$x_m, x_{m-1}, \ldots, x_1, x, y_1, y_2, \ldots, y_q$$

for some integers $m \geq n - 1$, $q \geq n$ (see Figure 5.2, where only a subset of $\mathrm{Vert}(\tau'_n)$ are drawn).

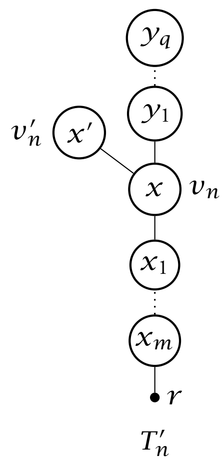


FIGURE 5.2. Submonomial T'_n of T_n

Consider the following submonomial of T'_n of weight $n + 1$:

$$(x_s x_{s-1} \cdots x_1 x y_1 y_2 \cdots y_{n-s-1}) \circ_{i_s} x', \quad 1 \leq s \leq n - 1$$

with suitable composition indices i_s. These submonomials are pairwise distinct since their underlying PRTs are as well. Thus, $\mathrm{Irr}(\mathcal{I})$ contains at least $n-1$ elements of weight $n+1$, namely,

$$\dim\Big(\sum_{i=0}^{n+1}\mathcal{V}^i\Big)/\Big(\sum_{i=0}^{n}\mathcal{V}^i\Big)\geq n-1,$$

yielding a contradiction. □

The following lemma is a generalization of Lemma 5.5.

Lemma 5.11. *Retain the hypotheses of Lemma 5.10. Then, there exist positive integers d_1 and d_2 such that* $\mathrm{Irr}(I)$ *contains at most d_1 tree monomials of weight h for all $h \geq d_2$.*

Proof. Let M_1 and M_2 be the same as in Lemmas 5.9 and 5.10. Suppose $T \in \mathrm{Irr}(I)$ and denote $\tau := \mathrm{Tr}(T)$ and $n := \mathrm{wt}(\tau)$. Assume that $\mathrm{h}(T)$ is large enough for the following decomposition (say, $\mathrm{h}(\tau) > 2(M_1 + 2M_2)$). Let T_1, T_2, and T_3 be the submonomials of T whose underlying PRTs τ_1, τ_2 and τ_3 have internal vertices

$$\begin{aligned}
\mathrm{Int}(\tau_1) &= \{v \in \mathrm{Int}(\tau) : \mathrm{h}(v) \leq M_1 + M_2\},\\
\mathrm{Int}(\tau_2) &= \{v \in \mathrm{Int}(\tau) : M_1 + M_2 \leq \mathrm{h}(v) \leq \mathrm{h}(\tau) - M_2\},\\
\mathrm{Int}(\tau_3) &= \{v \in \mathrm{Int}(\tau) : \mathrm{h}(v) \geq \mathrm{h}(\tau) - M_2\}.
\end{aligned}$$

These subtrees are well defined. In fact, by Lemmas 5.9 and 5.10, within each $\mathrm{Int}(\tau_i)$ vertices are connected by the parent function of τ and τ has a unique internal vertex v_1 of height $M_1 + M_2$ (respectively, v_2 of height $\mathrm{h}(\tau) - M_2$), whose parent is the root of τ_2 (respectively, τ_3). Note that, by Lemmas 5.9 and 5.10, all $v \in \mathrm{Int}(\tau) \setminus \mathrm{Pivot}(\tau)$ are contained in $\mathrm{Int}(\tau_1) \cup \mathrm{Int}(\tau_3)$, and thus τ_2 is single branched. See Figure 5.3 for an example of such decomposition, where only a subset of the internal vertices of τ are drawn and all leaves are omitted.

Set $a := \max\{\mathrm{Ar}(x) : x \in \mathcal{X}\}$ and $c := |\mathcal{X}|$. Then,

$$M_1 + M_2 = \mathrm{h}(\tau_1) \leq \mathrm{wt}(\tau_1) < M_3 := 1 + a + a^2 + \cdots + a^{M_1+M_2} \tag{5.4}$$

and

$$M_2 + 1 = \mathrm{h}(\tau_3) \leq \mathrm{wt}(\tau_3) < M_3. \tag{5.5}$$

There are only finitely many (say, M_4) PRTs of weight $< M_3$, and thus there are at most $c^{M_3} \cdot M_4$ tree monomials of weight $< M_3$. It follows from

$$\mathrm{wt}(\tau_1) + \mathrm{wt}(\tau_2) + \mathrm{wt}(\tau_3) = n + 2$$

and inequalities (5.4) and (5.5) that

$$\mathrm{h}(\tau_2) = \mathrm{wt}(\tau_2) > n - 2M_3 + 2.$$

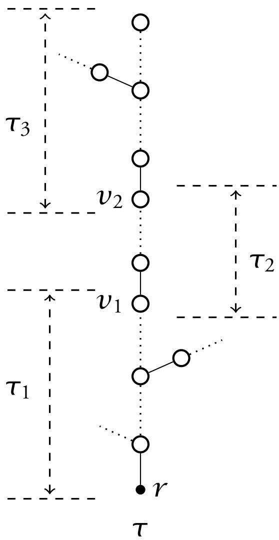


FIGURE 5.3. Decomposition of τ, where we have $\mathrm{h}(v_1) = M_1 + M_2$, $\mathrm{h}(v_2) = \mathrm{h}(\tau) - M_2$, $v_1 \in \mathrm{Int}(\tau_1) \cap \mathrm{Int}(\tau_2)$, and $v_2 \in \mathrm{Int}(\tau_2) \cap \mathrm{Int}(\tau_3)$.

Thus, for all $n \geq d + 2M_3 - 2$, we have $\mathrm{h}(\tau_2) > n - 2M_3 + 2 \geq d$. By Lemma 5.5, there are at most $(d-1)^3$ possibilities for T_2 for all $n \geq d + 2M_3 - 2$. Since T is uniquely determined by its submonomials T_1, T_2, and T_3, there are at most $(c^{M_3} \cdot M_4)^2 (d-1)^3$ possibilities for T. Therefore, there exist positive integers

$$d_1 := (c^{M_3} \cdot M_4)^2 (d-1)^3 \quad \text{and} \quad d_2 := d + 2M_3 - 2$$

such that $\mathrm{Irr}(I)$ contains at most d_1 tree monomials of weight h for all $h \geq d_2$. ◻

Now we are ready to prove our main result.

Theorem 5.12. *Let $\mathcal{X}$ be a finite operation alphabet and $\mathcal{P} = \mathcal{T}(\mathcal{X})/\mathcal{I}$. Write $\mathcal{V} = \mathbb{F}\mathcal{X}$. Assume that* $\dim(\sum_{i=0}^{d} \mathcal{V}^i)/(\sum_{i=0}^{d-1} \mathcal{V}^i) \leq d - 3$ *for some* $d \geq 3$ *(or* $\mathrm{GKdim}(\mathcal{P}) < 2$*). Then, there exist positive real numbers a, b such that*

$$\dim \sum_{i=0}^{n} \mathcal{P}(i) \leq an + b \quad \text{for all } n \geq 0.$$

As a consequence, $\mathrm{GKdim}(\mathcal{P}) \leq 1$.

Proof. Define

$$d_{\mathcal{V}}(n) := \dim \sum_{i \leq n} \mathcal{V}^i \quad \text{for all } n \in \mathbb{N}.$$

By Lemma 5.11, there exist positive integers d_1 and d_2 such that $\mathrm{Irr}(I)$ contains at most d_1 tree monomials of weight h for all $h \geq d_2$. Thus, for all $n \geq d_2$, setting $n := d_2 + q$, we have that

$$d\gamma(n) \leq d\gamma(d_2) + qd_1 = d\gamma(d_2) + (n - d_2)d_1,$$

and the function on the righthand side is linear in n. The assertion now follows from (4.2). The consequence is clear. ◻

Theorem 1.1 follows from Theorem 5.12 easily.

6. SINGLE-GENERATED NS OPERADS

We say an operad $\mathcal{P}$ is *single generated* (or *single-element generated*) if it is generated by an operation alphabet $\mathcal{X}$ that consists of a single element. In this section, we will describe an approach that can be used to construct a single-generated single-branched operad from a finitely generated monomial algebra. This procedure is called the *operadization* of a finitely generated monomial algebra, which is different from that given in Construction 2.3.

Fix an integer $d \geq 2$. Denote $X_0 = \{x_1, x_2, \ldots, x_d\}$ and X_0^* the free monoid generated by X_0. (Note that X_0 is not the generating alphabet $\mathcal{X}$ of an operad.) Let $\mathbb{F}\langle X_0\rangle := \mathbb{F}\langle x_1, x_2, \ldots, x_d\rangle = \bigoplus_{\ell\geq 0} \mathbb{F}\langle X_0\rangle_\ell$ be the free graded algebra generated by X_0 with $\deg(x_i) = 1$ $(1 \leq i \leq d)$.

We always assume that the tree monomials T_d and $R^{\vee}_{i,j}$ $(1 \leq i < j \leq d)$ are defined as in Figure 6.1. Suppose that $\tilde{\mathcal{Q}}$ is the operad presented by generator

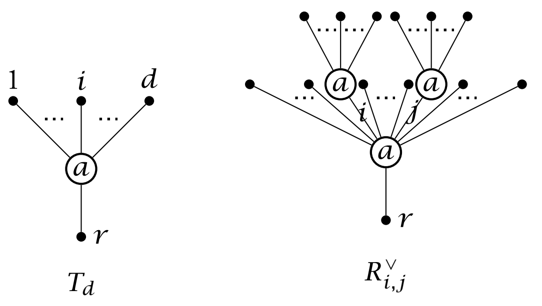


FIGURE 6.1

$\mathcal{X} = \{a\}$ and relations $\{(a \circ_j a) \circ_i a,\ 1 \leq i < j \leq d\}$; then, $\tilde{\mathcal{Q}}$ is a single-branched operad. We also say $\tilde{\mathcal{Q}}$ is the operad generated by T_d and subject to the relations $R^{\vee}_{i,j}$ $(1 \leq i < j \leq d)$.

Identifying the tree monomial T_d and its labelling a, we define the following $\mathbb{F}$-linear *operadization map*, which is an isomorphism of vector spaces

$$\mathcal{O} : \mathbb{F}\langle X_0\rangle \longrightarrow \tilde{\mathcal{Q}}_{\geq 2}$$

by setting

$$\mathcal{O}(w) = \mathcal{O}(x_{i_1}x_{i_2}\cdots x_{i_k}) = a \circ_{i_1} a \circ_{i_2} \cdots \circ_{i_k} a := (a \circ_{i_1} (a \circ_{i_2} (\cdots \circ_{i_k} a)))$$

for any $w = x_{i_1}x_{i_2}\cdots x_{i_k} \in X_0^*$. (See (5.1) and (5.2).)

We are now ready to give the following construction, in the proof of which we describe an operadization of a finitely generated monomial algebra.

Construction 6.1. For any finitely generated monomial algebra $A = \mathbb{F}\langle X_0\rangle/I$ with $I \subseteq \bigoplus_{\ell\geq 2}\mathbb{F}\langle X_0\rangle_\ell$ being a monomial ideal of the free algebra, there exists a single-generated single-branched ns operad $\mathcal{Q}_A$ such that $\mathrm{GKdim}(\mathcal{Q}_A) = \mathrm{GKdim}(A)$.

Proof. Since I is a monomial ideal of $\mathbb{F}\langle X\rangle$, it is routine to check that $\mathcal{O}(I)$ is an ideal of $\tilde{\mathcal{Q}}$.

Define the operad $\mathcal{Q}_A$ to be the quotient operad $\tilde{\mathcal{Q}}/\mathcal{O}(I)$. Noting that

$$\dim \mathcal{Q}_A(n) = \begin{cases} 1, & \text{when } n = 1 \text{ or } d, \\ \dim A_\ell, & \text{when } n = (\ell+1)d - \ell \text{ and } \ell \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

we obtain that $\mathrm{GKdim}(\mathcal{Q}_A) = \mathrm{GKdim}(A)$. ◻

Definition 6.2. Suppose that $d := |X_0| \geq 2$. The single-generated single-branched ns operad $\mathcal{Q}_A = \tilde{\mathcal{Q}}/\mathcal{O}(I)$ defined in Construction 6.1 is called the *operadization* of the finitely generated monomial algebra $A = \mathbb{F}\langle X_0\rangle/I$, where $I \subseteq \bigoplus_{\ell\geq 2}\mathbb{F}\langle X_0\rangle_\ell$ is a monomial ideal of the free algebra.

Note that Construction 6.1 is related to some ideas presented in [DMR20, DT20]. For example, a special case of this construction is given in Section 7.1.4 of [DT20].

Next, we present some examples of single-generated single-branched operads by using the operadization procedure described above.

Example 6.3. Let $d = 2$. Suppose $T_2, R^{\vee}_{1,2}, R_{1,1}, R_{1,2}, R_{2,1}, R_{2,2}$ are defined as in Figure 6.2. We have the following:

(1) If $\mathcal{Q}$ is the operad generated by T_2 and subject to the relation $R^{\vee}_{1,2}$, then we can check that $\dim \mathcal{Q}(n) = 2^{n-2}$ for $n \geq 2$. Hence, in this case $G_{\mathcal{Q}}(z) = z + z^2/(1-2z)$ and $\mathrm{GKdim}(\mathcal{Q}) = \infty$.

(2) Define $\mathcal{Q}$ to be the operad generated by T_2 and subject to the relations $R^{\vee}_{1,2}$, $R_{1,1}$. Then, we can check that $\mathcal{Q}$ is an operadization of the graded algebra $A := \mathbb{F}\langle x_1, x_2\rangle/\langle x_1^2\rangle$, and

$$\dim \mathcal{Q}(n) = \dim \mathcal{Q}(n-1) + \dim \mathcal{Q}(n-2) \quad \text{for } n \geq 2.$$

Hence, in this case $G_{\mathcal{Q}}(z) = z/(1-z+z^2)$, $\mathrm{GKdim}(\mathcal{Q}) = \infty$, and we call $\mathcal{Q}$ the *Fibonacci operad*.

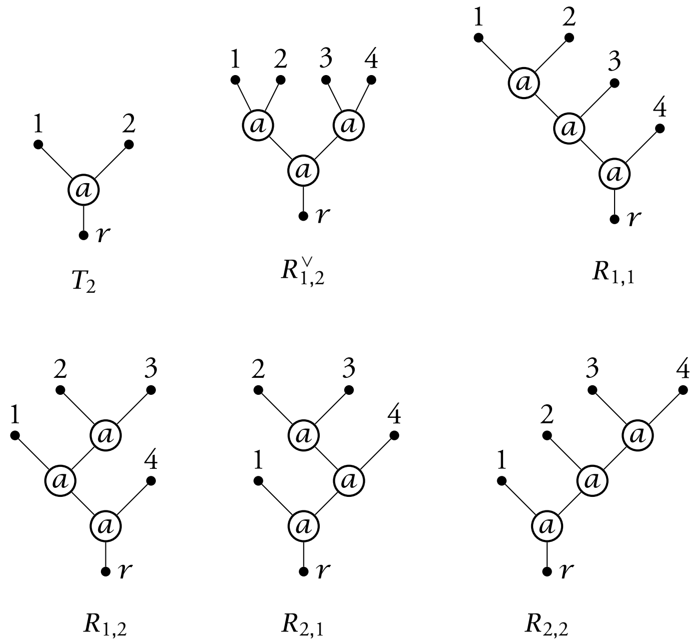


FIGURE 6.2

(3) Define $\mathcal{Q}$ to be the operad generated by T_2 and subject to the relations $R^{\vee}_{1,2}, R_{2,1}, R_{2,2}$. Then, we see that $\mathcal{Q}$ is an operadization of the graded algebra $\mathbb{F}\langle x_1, x_2\rangle/\langle x_2x_1, x_2^2\rangle$, and

$$\dim \mathcal{Q}(n) = \begin{cases} 1, & \text{when } n = 1 \text{ or } 2, \\ 2, & \text{when } n \geq 3, \\ 0, & \text{when } n = 0. \end{cases}$$

Hence, $G_{\mathcal{Q}}(z) = z + z^2 + 2z^3/(1-z)$ and $\operatorname{GKdim}(\mathcal{Q}) = 1$.

If we consider a graded algebra $A = \mathbb{F}\langle x_1, x_2\rangle/\langle x_1x_2 - x_1^2\rangle$ which is not a monomial algebra, it is easy to check that $\mathcal{O}(I)$ is not an ideal of $\tilde{\mathcal{Q}}$. Thus, it requires more work to understand the quotient operad $\tilde{\mathcal{Q}}/\langle \mathcal{O}(I)\rangle$ in this case.

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. (1) Let $\mathcal{P}$ be a finitely generated locally finite ns operad. It is clear that $\operatorname{GKdim}(\mathcal{P}) \geq 0$. By Proposition 4.8, $\operatorname{GKdim}(\mathcal{P})$ is not strictly between 0 and 1. By Theorem 1.1, $\operatorname{GKdim}(\mathcal{P})$ is not between 1 and 2. Therefore, $\operatorname{GKdim}(\mathcal{P}) \in R_{\mathrm{GKdim}}$ (see (1.1) for the definition of R_{GKdim}).

(2) Let $r \in R_{\mathrm{GKdim}}$. Let A be a finitely generated algebra of GK-dimension r. By [Bel15, Remark 4.1], we can assume A is a monomial algebra finitely generated

in degree 1. By Construction 6.1, $\mathcal{Q}_A$ is a single-generated single-branched locally finite operad with $\mathrm{GKdim}(\mathcal{Q}_A) = r$. The assertion follows. □

7. Generating Series and Exponential Generating Series

The generating series of an operad is defined in (1.2). By [KP15, (0.1.1)], the *exponential generating series* of an operad $\mathcal{P}$ is defined to be

$$E_{\mathcal{P}}(z) := \sum_{n\geq 0} \frac{\dim \mathcal{P}(n)}{n!} z^n. \tag{7.1}$$

Clearly, $G_{\mathcal{P}}(z)$ (respectively, $E_{\mathcal{P}}(z)$) contains more information than $\mathrm{GKdim}(\mathcal{P})$. A list of (exponential) generating series of operads is given in [Zin12, KP15]. Several authors have recently been studying the holonomic and differential algebraic properties of $G_{\mathcal{P}}(z)$ (respectively, $E_{\mathcal{P}}(z)$) (see Definition 1.3).

By [Sta80, Theorem 1.5], $F(z) := \sum_{n=0}^{\infty} f(n) z^n$ is holonomic if and only if the sequence $\{f(n)\}_{n\geq 0}$ satisfies a recurrence relation of the form

$$f(n) = q_1(n) f(n-1) + \cdots + q_{n-m}(n) f(n-m), \qquad n \gg 0 \tag{7.2}$$

for some fixed m and rational functions $q_1(x), \ldots, q_m(x)$. For all examples given in Example 6.3, $G_{\mathcal{P}}(z)$ are rational, and consequently, holonomic. By using (7.2), it is easy to see that

$$G_{\mathcal{P}}(z) \quad \text{is holonomic if and only if so is } E_{\mathcal{P}}(z). \tag{7.3}$$

For two integers a and b, let $[a, b]_{\mathbb{N}} := [a, b] \cap \mathbb{N}$. Suppose the sequence $\{f(n)\}_{n\geq 0}$ has infinitely many nonzeros. Let $\Phi_F = \{n \mid f(n) = 0\}$ and write Φ as $[i_1, j_1]_{\mathbb{N}} \cup [i_2, j_2]_{\mathbb{N}} \cup \cdots$ where $i_s \leq j_s \leq i_{s+1} - 2$ for all $s \geq 1$. Then, the following is true.

Lemma 7.1. *Let $F(z) = \sum_{n=0}^{\infty} f(n) z^n$ with infinitely many nonzero coefficients. Write $\Phi_F = \bigcup_{s\geq 1} [i_s, j_s]_{\mathbb{N}}$. Suppose that $\limsup_{s\to\infty}(j_s - i_s) = \infty$. Then, $F(z)$ is not holonomic.*

Proof. Suppose to the contrary $F(z)$ is holonomic. Then, (7.2) holds for some m. Pick any s such that $j_s - i_s \geq m + 1$. By the definition of Φ_F, $f(n) = 0$ for all $i_s \leq n \leq j_s$. By (7.2) and induction, we have that $f(n) = 0$ for all $n \geq i_s + m$. Therefore, $F(z)$ has only finitely many nonzero coefficients, a contradiction. □

The next example shows we can easily construct a monomial algebra such that its Hilbert series is not holonomic.

Example 7.2. Let U be the monomial algebra $\mathbb{F}\langle x_1, x_2\rangle / I$ where $\deg x_1 = \deg x_2 = 1$ and I is a span of monomials satisfying the following:

(a) All monomials have degree ≥ 3 in x_2.

(b) $x_1^i x_2 x_1^j$ with $i > 0$ or $j > 0$.

(c) $x_2 x_1^i x_2$ where $(2m+1)^{2m+1} - 1 \leq i \leq (2m+2)^{2m+2} - 1$ for some integer $m \geq 0$.

Let Λ be $\bigcup_{m=0}^{\infty}[(2m+1)^{2m+1}+1, (2m+2)^{2m+2}+1]_{\mathbb{N}}$, which is a subset of $\mathbb{N}$. Let $\Lambda^c = \mathbb{N} \setminus \Lambda$. Then,

$$\Lambda^c = \{0\} \cup \{1\} \cup \bigcup_{m=0}^{\infty} [(2m+2)^{2m+2}+2, (2m+3)^{2m+3}]_{\mathbb{N}}.$$

Define

$$\delta_\Lambda(n) = \begin{cases} 0 & n \in \Lambda, \\ 1 & n \in \Lambda^c. \end{cases}$$

Then, U has an $\mathbb{F}$-basis $\{x_1^i\}_{i\geq 0} \cup \{x_1^i x_2\}_{i\geq 0} \cup \{x_2 x_1^i\}_{i\geq 0} \cup \{x_2 x_1^i x_2\}_{i+2\in\Lambda^c}$, and its Hilbert series is

$$H_U(z) = 1 + 2z + \sum_{n=2}^{\infty}(3+\delta_\Lambda(n))z^n = 1 + 2z + \frac{3z^2}{(1-z)} + V(z)$$

where $V(z) = \sum_{n=2}^{\infty}\delta_\Lambda(n)z^n$. Since $V(z)$ satisfies the hypothesis of Lemma 7.1, it is not holonomic.

Let $\mathcal{P}_U$ be the construction given in Construction 2.3. It is easy to see that

$$\operatorname{GKdim}(\mathcal{P}_U) = \operatorname{GKdim}(U) = 1$$

and that

$$G_{\mathcal{P}_U}(z) = zH_U(z) = z\left(1 + 2z + \frac{3z^2}{(1-z)}\right) + zV(z)$$

where $z(1+2z+3z^2/(1-z))$ is holonomic, but $zV(z)$ is not (by Lemma 7.1 again). Therefore, $G_{\mathcal{P}_U}(z)$ is not holonomic by Holonomic Theorem 2 in [Ber14].

Let $\mathcal{Q}_U$ be the construction given in Construction 6.1. Then, $\operatorname{GKdim}(\mathcal{Q}_U) = 1$ and

$$\begin{aligned} G_{\mathcal{Q}_U}(z) &= z + z^d H_U(z^{d-1}) = z + z^2 H_U(z) \\ &= z + z^2\left(1 + 2z + \frac{3z^2}{(1-z)}\right) + z^2 V(z) \end{aligned}$$

as $d = \dim U_1 = 2$. By an argument similar to the previous paragraph, $G_{\mathcal{Q}_U}(z)$ is not holonomic.

We are now ready to prove Proposition 1.5.

Proof of Proposition 1.5. Let $r \in R_{\mathrm{GKdim}} \setminus \{0\}$. We claim there is a monomial algebra B finitely generated in degree 1 such that $\operatorname{GKdim}(B) = r$ and $H_B(z)$ is not holonomic. By [Bel15, Remark 4.1], there is a monomial algebra A finitely generated in degree 1 such that $\mathrm{GKdim}(A) = r$. If $H_A(z)$ is not holonomic, then we are done by setting $B = A$. If $H_A(z)$ is holonomic, then we let $B = \mathbb{F} \oplus (A_{\geq 1} \oplus U_{\geq 1})$. Then, B is a monomial algebra finitely generated in degree 1 such that $\mathrm{GKdim}(B) = r$ and $H_B(z) = H_A(z) + H_U(z) - 1$. Since $H_U(z)$ is not holonomic, by Holonomic Theorem 2 in [Ber14], $H_B(z)$ is not holonomic. Thus, we have proved the claim.

Let $\mathcal{P} = \mathcal{Q}_B$ as in Construction 6.1. Then, $\mathcal{P}$ is single generated, $\mathrm{GKdim}(\mathcal{P}) = \mathrm{GKdim}(B) = r$, and $G_{\mathcal{P}} = z + z^d H_B(z^{d-1})$. Since $H_B(z)$ is not holonomic, it is routine to show that $G_{\mathcal{P}}(z)$ is not holonomic. □

The following lemma is easy and the proof is analogous to the one given in the quadratic case (see the proof of [Dot19, Corollary 4.2 (i)]).

Lemma 7.3. *Let A be a connected graded locally finite algebra, and let $\mathcal{P}_A$ be the operad constructed in Construction 2.3. We have the following:*

(1) *A is finitely generated if and only if so is $\mathcal{P}_A$.*
(2) *A is finitely presented if and only if so is $\mathcal{P}_A$.*

For the rest of this section, we construct a finitely presented locally finite ns operad such that its generating series is not holonomic. Therefore, we give a "non-generic" counterexample to [KP15, Expectation 2] (see Expectation 1.4).

Let us recall some history in the setting of noncommutative graded algebra. In 1972 Govorov conjectured that the Hilbert series of a finitely presented graded algebra is rational [Gov72]. This was shown to be false, for example, by Shearer in [She80] by constructing an irrational (but algebraic) Hilbert series of a finitely generated graded algebra. Shearer mentioned a similar construction giving also an example with a transcendental (but still holonomic) Hilbert series. His third example, involving the generating function of the number of partitions, indeed has its Hilbert series not holonomic. A similar example was given in [Smi76], which contains the following example as a special case.

Example 7.4. Suppose $\mathrm{char}\,\mathbb{F} = 0$. Let L be the graded Lie algebra with basis $\{e_1, e_2, \ldots, e_n, \ldots\}$ with $\deg e_i = i$ for all i and Lie bracket determined by

$$[e_i, e_j] = (i-j)e_{i+j}, \quad \forall\ i \neq j.$$

This is a subalgebra of the Witt algebra. Let A be the universal enveloping algebra of L. Then, A has intermediate growth and is generated by e_1 and e_2 ($\deg e_i = i$) and subject to the relations $e_3e_2 - e_2e_3 = e_5$ and $e_5e_2 - e_2e_5 = 3e_7$. Thus, A is finitely presented, but not generated in degree 1.

It is easy to see that

$$H_A(z) = \prod_{i=1}^{\infty} \frac{1}{(1-z^i)}.$$

Note that $H_A(z)$ is equal to $P(z) := \sum_{n=0}^{\infty} p(n)z^n$ where $p(n)$ is the number of partitions of n.

Now we list some facts about $P(z)$:

(P1) [Sta80, p. 187] $P(z)$ is not holonomic.

(P2) $zP(z)$ is not holonomic.

(P3) $Q(z) := z(zP(z))'$ $(= \sum_{n=0}^{\infty} p(n)(n+1)z^{n+1})$ is not holonomic.

Note that parts (P2), (P3) follow from part (P1) immediately.

Now, let $\mathcal{P}_A$ be the operad given by Construction 2.3. By Lemma 7.3, $\mathcal{P}_A$ is finitely generated. By (2.4),

$$G_{\mathcal{P}_A}(z) = zH_A(z) = zP(z),$$

which is not holonomic by (P2). Therefore, $\mathcal{P}_A$ is a "non-generic" counterexample to [KP15, Expectation 2].

Note there is a quadratic algebra with similar properties on its Hilbert series, but with 14 generators and 96 quadratic relations (see Theorem 1 (iv) in [Koc15]). The algebra A could be replaced by Koçak's example.

Remark 7.5. Using the same proof as above, one sees that a version of Proposition 1.5 holds for exponential generating series. (See (7.1) for the definition of the exponential generating series.)

8. COMMENTS ON SYMMETRIC OPERADS

In this final section we make some comments on symmetric operads and prove the main result of this section, namely, Theorem 1.7.

First of all, we refer to [LV12, Chapter 5] or [BYZ20, Definition 1.2] for the definition of a symmetric operad.

Given a graded algebra with augmentation, there is an easy construction of a symmetric operad similar to the one given in Construction 2.3.

Construction 8.1. Let $A := \bigoplus_{i\geq 0} A_i$ be a locally finite $\mathbb{N}$-graded algebra with unit 1_A. Suppose that A has a graded augmentation $\varepsilon : A \to \mathbb{F}$ such that $\mathfrak{m} := \ker \varepsilon$ is a maximal graded ideal of A. We let C_n be the cyclic group of order n, and elements in C_n are denoted by $\{1, \ldots, n\}$.

Let $S_A(0) = 0$ and $S_A(n) = A_{n-1} \otimes \mathbb{F}C_n$ for all $n \geq 1$. Elements in $S_A(n)$ are $\mathbb{F}$-linear combinations of (a, i) where $a \in A_{n-1}$ and $1 \leq i \leq n$. The $\mathbb{S}$-action on $S_A(n)$ is determined by

$$(a, i) * \sigma = (a, \sigma^{-1}(i))$$

for all $\sigma \in \mathbb{S}_n$.

Define partial compositions as follows, for $1 \leq s \leq m$:

$$\circ_s : S_A(m) \otimes S_A(n) \to S_A(n+m-1),$$

$$(a_{m-1}, i) \otimes (a_{n-1}, j) \mapsto \begin{cases} (ca_{m-1}, i) & a_{n-1} = c1_A,\ c \in \mathbb{F}, \\ (a_{m-1}a_{n-1}, i+j-1) & a_{n-1} \in \mathfrak{m}, s = i, \\ 0 & a_{n-1} \in \mathfrak{m},\ s \neq i. \end{cases}$$

We claim that $S_A := \{S_A(n)\}_{n\geq 0}$ is a symmetric operad with identity $\mathrm{id} = (1_A, 1)$. We give a sketch proof below.

We refer to [LV12, Section 5.3.4] for the partial definition of a symmetric operad. In fact, a symmetric operad $\mathcal{P}$ is an $\mathbb{S}$-module satisfying the axioms of a nonsymmetric operad [Definition 2.1] and the following two additional equations:

$$\mu \circ_s (\nu * \sigma) = (\mu \circ_s \nu) * \sigma', \tag{8.1}$$

$$(\mu * \phi) \circ_s \nu = (\mu \circ_{\phi(s)} \nu) * \phi'', \tag{8.2}$$

where $\mu \in \mathcal{P}(m)$, $\nu \in \mathcal{P}(n)$, $1 \leq s \leq m$, $\sigma \in \mathbb{S}_n$, $\phi \in \mathbb{S}_m$, and where $\sigma' = \mathbf{1}_m \circ_s \sigma$ and $\phi'' = \phi \circ_s \mathbf{1}_n$. (See [LV12, Section 5.3.4] and (E1.2.1), (E8.1.3) in [BYZ20] for the explanation of σ' and ϕ''.) We first verify (8.1) and (8.2) and then the rest of axioms given in Definition 2.1 for $\mathcal{P} := S_A$.

Verification of (8.1): Write $\mu = (a_{m-1}, i)$ and $\nu = (a_{n-1}, j)$. If $n = 1$ and $a_{n-1} = c1_A$ (or if $n = 1$ and $a_{n-1} \in \mathfrak{m}$), then $j = 1$ and $\sigma = \mathbf{1}_1 \in \mathbb{S}_1$ and

$$\sigma' = \mathbf{1}_{n+m-1} \in \mathbb{S}_{n+m-1}.$$

Clearly, (8.1) holds. If $n \geq 2$ and $s \neq i$, both sides of (8.1) are zero. It remains to consider the case when $n \geq 2$ and $s = i$. Write

$$\sigma = \begin{pmatrix} k_1 & k_2 & \cdots & k_n \\ 1 & 2 & \cdots & n \end{pmatrix} \tag{8.3}$$

where by convention $k_i = \sigma^{-1}(i)$ for all i. Then, by definition,

$$\sigma' = \begin{pmatrix} 1 & \cdots & i-1 & k_1+i-1 & k_2+i-1 & \cdots & k_n+i-1 & i+n & \cdots & n+m-1 \\ 1 & \cdots & i-1 & i & i+1 & \cdots & n+i-1 & i+n & \cdots & n+m-1 \end{pmatrix}.$$

In this case, we have

$$\begin{aligned} \text{LHS of (8.1)} &= (a_{m-1}, i) \circ_i [(a_{n-1}, j) * \sigma] = (a_{m-1}, i) \circ_i (a_{n-1}, \sigma^{-1}(j)) \\ &= (a_{m-1}a_{n-1}, i + \sigma^{-1}(j) - 1) = (a_{m-1}a_{n-1}, i + k_j - 1), \\ \text{RHS of (8.1)} &= [(a_{m-1}, i) \circ_i (a_{n-1}, j)] * \sigma' = (a_{m-1}a_{n-1}, i + j - 1) * \sigma' \\ &= (a_{m-1}a_{n-1}, (\sigma')^{-1}(i + j - 1)) = (a_{m-1}a_{n-1}, k_j + i - 1). \end{aligned}$$

Therefore, (8.1) holds.

Verification of (8.2): Recycle the letter k_i and write

$$\phi = \begin{pmatrix} k_1 & k_2 & \cdots & k_m \\ 1 & 2 & \cdots & m \end{pmatrix}$$

using the convention of (8.3). If $a_{n-1} = c1_A$, then we have $n = j = 1$ and $\phi'' = \phi \circ_s \mathbf{1}_1 = \phi$. Consequently, $\phi^{-1}(i) = k_i = (\phi'')^{-1}(i)$. In this case,

$$\begin{aligned}
\text{LHS of (8.2)} &= [(a_{m-1}, i) * \phi] \circ_s (c1_A, 1) = (a_{m-1}, \phi^{-1}(i)) \circ_s (c1_A, 1)\\
&= (ca_{m-1}, \phi^{-1}(i)) = (ca_{m-1}, k_i),\\
\text{RHS of (8.2)} &= [(a_{m-1}, i) \circ_{\phi(s)} (c1_A, 1)] * \phi'' = (ca_{m-1}, i) * \phi''\\
&= (ca_{m-1}, i) * \phi = (ca_{m-1}, k_i).
\end{aligned}$$

Hence, (8.2) holds. Next, we assume that $a_{n-1} \in \mathfrak{m}$. If $s \neq \phi^{-1}(i)$, then both sides of (8.2) are zero. It remains to consider the case when $s = \phi^{-1}(i)$ or $i = \phi(s)$. By definition,

$$\phi'' = \begin{pmatrix} k'_1 & \cdots & k'_{i-1} & k'_i & k'_i + 1 & \cdots & k'_i + n - 1 & k'_{i+1} & \cdots & k'_m \\ 1 & \cdots & i-1 & i & i+1 & \cdots & i+n-1 & i+n & \cdots & m+n-1 \end{pmatrix}$$

where

$$k'_t = \begin{cases} k_t + n - 1 & k_t > s, \\ k_t & k_t \leq s. \end{cases}$$

In particular,

$$(\phi'')^{-1}(i + j - 1) = k'_i + j - 1 = s + j - 1 = k_i + j - 1 = \phi^{-1}(i) + j - 1.$$

Now, we compute

$$\begin{aligned}
\text{LHS of (8.2)} &= [(a_{m-1}, i) * \phi] \circ_s (a_{n-1}, j)\\
&= [(a_{m-1}, i) * \phi] \circ_{\phi^{-1}(i)} (a_{n-1}, j)\\
&= (a_{m-1}, \phi^{-1}(i)) \circ_{\phi^{-1}(i)} (a_{n-1}, j)\\
&= (a_{m-1}a_{n-1}, \phi^{-1}(i) + j - 1),\\
\text{RHS of (8.2)} &= [(a_{m-1}, i) \circ_{\phi(s)} (a_{n-1}, j)] * \phi''\\
&= [(a_{m-1}, i) \circ_i (a_{n-1}, j)] * \phi''\\
&= (a_{m-1}a_{n-1}, i + j - 1) * \phi''\\
&= (a_{m-1}a_{n-1}, (\phi'')^{-1}(i + j - 1))\\
&= (a_{m-1}a_{n-1}, \phi^{-1}(i) + j - 1)
\end{aligned}$$

which implies that (8.2) holds in this case. Thus, we have verified (8.2) for all cases.

Verification of (2.3): It follows easily from the definition that

$$(1_A, 1) \circ_1 (a_{n-1}, j) = (a_{n-1}, j) = (a_{n-1}, j) \circ_s (1_A, 1)$$

for all $1 \le s \le n$, which is (2.3).

Verification of (2.2): Recall that (2.2) is equivalent to

$$(\lambda \circ_s \mu) \circ_{t-1+m} \nu = (\lambda \circ_t \nu) \circ_s \mu \tag{8.4}$$

for $\lambda \in \mathcal{P}(\ell)$, $\mu \in \mathcal{P}(m)$, $\nu \in \mathcal{P}(n)$, and $1 \le s < t \le \ell$. Write $\lambda = (a_{\ell-1}, i)$, $\mu = (a_{m-1}, j)$, and $\nu = (a_{n-1}, k)$. If the LHS of (8.4) were nonzero, we must have $s = i$ and $s + j - 1 = t - 1 + m$. But $s < t$ and $j \le m$ which contradict the equation $s + j - 1 = t - 1 + m$. Thus, the LHS of (8.4) is zero. For a similar reason, the RHS of (8.4) is zero. Hence, (8.4) holds.

Verification of (2.1): Recall that (2.1) is equivalent to

$$(\lambda \circ_s \mu) \circ_{s-1+t} \nu = \lambda \circ_s (\mu \circ_t \nu) \tag{8.5}$$

for $\lambda \in \mathcal{P}(\ell)$, $\mu \in \mathcal{P}(m)$, $\nu \in \mathcal{P}(n)$, $1 \le s \le \ell$, and $1 \le t \le m$. Write $\lambda = (a_{\ell-1}, i)$, $\mu = (a_{m-1}, j)$, and $\nu = (a_{n-1}, k)$ as before. If the LHS of (8.5) is nonzero, then we must have both $s = i$ and $t = j$. Similarly, if the RHS of (8.5) is nonzero, then $s = i$ and $t = j$. Thus, it suffices to consider the case when $s = i$ and $t = j$. In this case, both sides of (8.5) are equal to $(a_{\ell-1}a_{m-1}a_{n-1}, i + j + k - 2)$. Hence, (8.5) holds.

We have now verified all axioms of a symmetric operad for $\mathcal{P} := S_A$, and therefore S_A is a symmetric operad.

It is obvious that

$$G_{S_A}(z) = z(zH_A(z))'. \tag{8.6}$$

Lemma 8.2. *Let A be a connected graded algebra and let S_A be the symmetric operad provided in Construction 8.1. The following hold:*

1. *If A is finitely generated, then S_A is finitely generated as a symmetric operad (respectively, as a nonsymmetric operad).*
2. *If A is finitely presented, then S_A is finitely presented as a nonsymmetric operad.*
3. *If A is finitely presented, then S_A is finitely presented as a symmetric operad.*

Proof. We continue to use the notation introduced in Construction 8.1.

(1) Let $\mathfrak{m} = \bigoplus_{i \ge 1} A_i$. Suppose $V \subseteq \mathfrak{m}$ is a finite-dimensional graded subspace that generates A. Let $\{v_1, \ldots, v_e\}$ be a basis of V. We claim that S_A is generated by $E := \sum_{i=1}^{e} (\sum_{j=1}^{\deg(v_i)+1} \mathbb{F}(v_i, j))$ where $\deg(v_i)$ is as defined in the graded algebra A. Note that $\mathrm{Ar}((v_i, j)) = \deg(v_i) + 1$. Since every element in S_A is of the form (a_{n-1}, d) where $1 \le d \le n$ and a_{n-1} is generated by V in A, it can be generated by (v_i, j) by partial compositions. Therefore, S_A is finitely generated as a nonsymmetric operad (respectively, as a symmetric operad).

(2) Suppose A is generated by a finite-dimensional graded subspace V and subject to a finite-dimensional relation subspace $W := \sum_{j=1}^{w} \mathbb{F} f_j$. Then, A is the factor algebra $\mathbb{F}\langle V\rangle/(W)$ where $\mathbb{F}\langle V\rangle$ is the free algebra generated by V and where (W) denotes the relation ideal generated by W. Note that $\mathbb{F}\langle V\rangle$ is connected graded. Let E be defined as in the proof of part (1), and let $\mathcal{F}^{ns}(E)$ be the free nonsymmetric operad generated by E (see [LV12, 5.9.6] for the definition).

Our first step is to define the relation subspace R of the $\mathcal{F}^{ns}(E)$ such that $S_A \cong \mathcal{F}^{ns}(E)/(R)$. For each homogeneous element g in the free algebra $\mathbb{F}\langle V\rangle$, fix an *expression* of g as a linear combination of possibly repeated monomials

$$g = \sum c_\bullet v_{i_1} \cdots v_{i_s} \cdots v_{i_u} \tag{8.7}$$

where the sum is over $\bullet := (i_1, \ldots, i_s, \ldots, i_u)$ and where $c_\bullet$ are scalars in $\mathbb{F}$. Note that monomials $\{v_{i_1} \cdots v_{i_s} \cdots v_{i_u}\}$ in expression (8.7) are not assumed to be distinct. For each term $v_{i_1} \cdots v_{i_s} \cdots v_{i_u}$ appearing in (8.7), let $\{k_{i_s}\}_{s=1}^{u}$ be a sequence of integers such that $1 \leq k_{i_s} \leq \deg(v_{i_s}) + 1$ for all $1 \leq s \leq u$. We will use $\mathbf{k}_\bullet$ to denote $\{k_{i_s}\}_{s=1}^{u}$. Define

$$|\mathbf{k}_\bullet| := \sum_{s=1}^{u} k_{i_s} - u + 1.$$

Fix any integer d between 1 and $\deg(g) + 1$. For each $\bullet := (i_1, \ldots, i_u)$ appearing in (8.7), pick any sequence $\{k_{i_s}\}_{s=1}^{u}$ such that $|\mathbf{k}_\bullet| = d$. Such a sequence $\{k_{i_s}\}_{s=1}^{u}$ is called a *d-sequence*. Let $\mathbf{k}_d$ be the collection of d-sequences associated with expression (8.7). Now, we define the following element in the free operad $\mathcal{F}^{ns}(E)$

$$r_{\mathbf{k}_d}(g) := \sum_{\mathbf{k}_\bullet \in \mathbf{k}_d} c_\bullet (v_{i_1}, k_{i_1}) \circ_{k_{i_1}} (v_{i_2}, k_{i_2}) \circ_{k_{i_2}} \cdots \circ_{k_{i_{u-1}}} (v_{i_u}, k_{i_u}),$$

where each $(v_{i_1}, k_{i_1}) \circ_{k_{i_1}} (v_{i_2}, k_{i_2}) \circ_{k_{i_2}} \cdots \circ_{k_{i_{u-1}}} (v_{i_u}, k_{i_u})$ is a right normal tree monomial defined in (5.1)–(5.2).

For example, let

$$h = v_1 v_4 - 2v_2 v_3 + 3v_1 v_4$$

be in $\mathbb{F}\langle V\rangle$ where $\deg v_i = i$ for $1 \leq i \leq 4$. Note that the first monomial in the above expression of h equals the third one. Let $d = 4$, and pick three d-sequences $\{1, 4\}$, $\{3, 2\}$, and $\{2, 3\}$ corresponding to three monomials in the expression of h. Then,

$$r_{\mathbf{k}_4}(h) = (v_1, 1) \circ_1 (v_4, 4) - 2(v_2, 3) \circ_3 (v_3, 2) + 3(v_1, 2) \circ_2 (v_4, 3).$$

Let g' be another homogeneous element of degree equal to $\deg(g)$, and fix an expression of g' similar to (8.7). Let c and c' be scalars in $\mathbb{F}$, and let $f = cg + c'g'$ with expression of f induced by the expression of g and g'. Let $\mathbf{k}_d$ (respectively,

$\mathbf{k}'_d$) be a collection of d-sequences of g (respectively, g') corresponding to the monomials appeared in (8.7). Then, the disjoint union $\mathbf{k}_d \uplus \mathbf{k}'_d$ is a collection of d-sequences corresponding to the expression of f. It follows from the definition that

$$r_{\mathbf{k}_d \uplus \mathbf{k}'_d}(f) = c r_{\mathbf{k}_d}(g) + c' r_{\mathbf{k}'_d}(g'). \tag{8.8}$$

For any given d, there are only finitely many possibilities for d-sequences $\{k_{i_s}\}_{s=1}^{u}$, and consequently, for $r_{\mathbf{k}_d}(g)$.

Now, let f_i be an element in the relation space W, and fix an expression of f_i as in (8.7). It is easy to check that all of

$$r_{\mathbf{k}_d}(f_j) = 0 \tag{8.9}$$

are relations of S_A for $1 \le d \le \deg f_j$ and different choices of d-sequences. By the definition of S_A, the following are also relations of S_A:

$$(v_{i_1}, j_1) \circ_\ell (v_{i_2}, j_2) = 0 \tag{8.10}$$

for all $(v_{i_s}, j_s) \in E$ and for all $\ell \neq j_1$ and

$$(v_{i_1}, j_1) \circ_{j_1} (v_{i_2}, j_2) - (v_{i_1}, j_1 - 1) \circ_{j_1 - 1} (v_{i_2}, j_2 + 1) = 0 \tag{8.11}$$

for all $(v_{i_s}, j_s) \in E$ and for all $1 < j_1 \le \deg v_{i_1} + 1$ and $1 \le j_2 < \deg(v_{i_2}) + 1$. Let R be the graded vector subspace generated by the lefthand side of equations (8.9), (8.10), and (8.11).

By part (1), there is a surjective morphism of nonsymmetric operads

$$\Phi : \mathcal{F}^{ns}(E) \to S_A$$

sending $(v_i, j) \in E \subseteq \mathcal{F}^{ns}(E)$ to $(v_i, j) \in S_A$. It is clear that Φ maps relations defined in (8.9)–(8.11) to zero. Therefore, Φ induces naturally a surjective morphism of nonsymmetric operads

$$\phi : \mathcal{F}^{ns}(E)/(R) \to S_A,$$

such that $\Phi = \phi \circ \pi$ where π is the canonical quotient map $\mathcal{F}^{ns}(E) \to \mathcal{F}^{ns}(E)/(R)$.

Our next step is to show that the natural map ϕ is injective (consequently, bijective).

We now need the following notion of leading form. For each tree monomial $t \in \mathcal{F}^{ns}(E)/(R)$ of the form

$$t = (v_{i_1}, j_1) \circ_{j_1} (v_{i_2}, j_2) \circ_{j_2} \cdots \circ_{j_{m-1}} (v_{i_m}, j_m),$$

there exist finitely many elements of the form

$$t' = (v_{i_1}, j'_1) \circ_{j'_1} (v_{i_2}, j'_2) \circ_{j'_2} \cdots \circ_{j'_{m-1}} (v_{i_m}, j'_m),$$

such that $j'_1 + \cdots + j'_m = j_1 + \cdots + j_m$ and $1 \le j'_\ell \le \deg(v_{i_\ell}) + 1$ for $1 \le \ell \le m$. Call the largest tree monomial (under the path-lexicographic order) among these elements the *leading form* of t, and denote it by $L(t)$. Let $(R_{8.11})$ be the ideal of $\mathcal{F}^{ns}(E)$ generated by relations given in (8.11). It follows from (8.11) that $L(t) - t \in (R_{8.11})$, and it is clear that $L(t') - L(t) = 0$ for any choice of $(j'_1, \dots, j'_m)$. As a consequence, modulo the ideal $(R_{8.11})$, $r_{\mathbf{k}_d}(g)$ is independent of the expressions of g given in (8.7) and the choices of $\mathbf{k}_d$—that is, the collection of d-sequences. By pre-composing with π, the argument in this paragraph together with (8.8) implies that $\pi r_{\mathbf{k}_d}(g)$ is independent of the expressions of g and the choices of d-sequences and that $\pi r_{\mathbf{k}_d}$ is a well-defined $\mathbb{F}$-linear map from $\mathbb{F}\langle V\rangle \to \mathcal{F}^{ns}(E)/(R)$.

Fix any n, and let $\{a_\alpha\}_{\alpha \in I_n}$ be a monomial basis of A_{n-1} for some index set I_n. Then, $\{(a_\alpha, i)\}_{\alpha \in I_n, 1 \le i \le n}$ is an $\mathbb{F}$-linear basis of $S_A(n)$. For any $a_\alpha = v_{\alpha,1} v_{\alpha,2} \cdots v_{\alpha,m_\alpha}$, let $b'_{\alpha,i} \in \mathcal{F}^{ns}(E)/(R)$ be a monomial of the following form:

$$b'_{\alpha,i} = (v_{\alpha,1}, j'_1) \circ_{j'_1} (v_{\alpha,2}, j'_2) \circ_{j'_2} \cdots \circ_{j'_{m_\alpha - 1}} (v_{\alpha,m_\alpha}, j'_{m_\alpha}),$$

where $j'_1 + \cdots + j'_{m_\alpha} = i + m_\alpha - 1$ and $1 \le j'_\ell \le \deg(v_{\alpha,\ell}) + 1$ for $1 \le \ell \le m_\alpha$. Then, $b'_{\alpha,i} \in \phi^{-1}((a_\alpha, i))$. Take $b_{\alpha,i}$ to be the largest one among the monomials of this form, that is, $b_{\alpha,i} := L(b'_{\alpha,i})$, which is independent of the choice of $b'_{\alpha,i}$. By definition, both $b_{\alpha,i}$ and $b'_{\alpha,i}$, considered as elements in $\mathcal{F}^{ns}(E)$, are of the form $r_{\mathbf{k}_i}(a_\alpha)$ for some choices of i-sequences $\mathbf{k}_\bullet$. By (8.11), $b_{\alpha,i} = b'_{\alpha,i}$ in $\mathcal{F}^{ns}(E)/(R)$. The conclusion of this paragraph is that $b_{\alpha,i} = \pi r_{\mathbf{k}_i}(a_\alpha)$ for $1 \le i \le n$ and any collection of i-sequences.

By definition, any monomial $s \in \mathcal{F}^{ns}(E)/(R)(n)$ (or in $\mathcal{F}^{ns}(E)(n)$) is left normal, namely,

$$s = (\cdots((v_{i_1}, j_1) \circ_{k_1} (v_{i_2}, j_2)) \circ_{k_2} \cdots) \circ_{k_{m-1}} (v_{i_m}, j_m).$$

Using the relations of the form (8.10), s in $\mathcal{F}^{ns}(E)/(R)(n)$ is either zero or equal to

$$s = (v_{i_1}, j_1) \circ_{j_1} (v_{i_2}, j_2) \circ_{j_2} \cdots \circ_{j_{m-1}} (v_{i_m}, j_m), \tag{8.12}$$

namely, it is right normal (see (5.1)–(5.2)). For the rest of the proof of part (2), we will only use right normal tree monomials in $\mathcal{F}^{ns}(E)$ and these will simply be called *monomials*. For each monomial s, we will freely replace s by $L(s)$ and vice versa as $L(s) = s$ in $\mathcal{F}^{ns}(E)/(R)$ by (8.11).

To prove ϕ is injective, it suffices to show that

$$\dim_{\mathbb{F}}(S_A(n)) \ge \dim_{\mathbb{F}} \mathcal{F}^{ns}(E)/(R)(n) \quad \text{for all } n.$$

Then, it is enough to show that every s in (8.12) can be presented as a linear combination of $\{b_{\alpha,i}\}_{\alpha\in I_n,\, 1\le i\le n}$ in $\mathcal{F}^{ns}(E)/(R)(n)$.

Since

$$\mathrm{Ar}((v_{i_1}, j_1)) + \cdots + \mathrm{Ar}((v_{i_m}, j_m)) = n + m - 1,$$

the element $v_{i_1}v_{i_2}\cdots v_{i_m} \in A_{n-1}$, and it can be presented in the free algebra generated by V as follows:

$$v_{i_1}v_{i_2}\cdots v_{i_m} = \sum_{\alpha} c_\alpha a_\alpha + \sum_{\gamma} c_\gamma v_{\gamma,1}\cdots v_{\gamma,p_\gamma} g_\gamma v_{\gamma,p_\gamma+1}\cdots v_{\gamma,q_\gamma},$$

where $c_\alpha, c_\gamma \in \mathbb{F}$ and $g_\gamma \in \{f_j \mid 1 \le j \le w\}$. Rewrite the above equation as

$$v_{i_1}v_{i_2}\cdots v_{i_m} - \sum_{\alpha} c_\alpha a_\alpha = \sum_{\gamma} c_\gamma v_{\gamma,1}\cdots v_{\gamma,p_\gamma} g_\gamma v_{\gamma,p_\gamma+1}\cdots v_{\gamma,q_\gamma}. \tag{8.13}$$

We claim that $r_{\mathbf{k}_d}(g) \in (R)$, or equivalently, $\pi r_{\mathbf{k}_d}(g) = 0$, where g is the righthand side of (8.13). Note that each monomial in the righthand side of (8.13) has degree $n-1$. If the claim holds, then, for $d := \sum_{w=1}^{m} j_w - m + 1$,

$$s = \pi r_{\mathbf{k}_d}(v_{i_1}v_{i_2}\cdots v_{i_m}) = \pi r_{\mathbf{k}_d}\Big(\sum_{\alpha} c_\alpha a_\alpha\Big) = \sum_{\alpha} c_\alpha \pi r_{\mathbf{k}_d}(a_\alpha) = \sum_{\alpha} c_\alpha b_{\alpha,d}$$

in $\mathcal{F}^{ns}(E)/(R)$. We now prove the claim. Since $\pi r_{\mathbf{k}_d}(-)$ is additive, we may assume g only has one term, namely,

$$g = v_1\cdots v_p f v_{p+1}\cdots v_q, \tag{8.14}$$

where f is one of f_i in W. Write f as a linear combination of monomials, say, $\{v_{f,1}\cdots v_{f,m_f}\}$. Then, (8.14) can be considered as an expression of g which is a linear combination of monomials of the form

$$s_h := v_1\cdots v_p v_{f,1}\cdots v_{f,m_f} v_{p+1}\cdots v_q. \tag{8.15}$$

For any d-sequence $\mathbf{k}_\bullet$ corresponding to (8.15), it can be decomposed into

$$\begin{aligned}\mathbf{k}(b)_\bullet &:= \{k_\ell\}_{i=1}^{p},\\ \mathbf{k}(m)_\bullet &:= \{k_{f,\ell}\}_{\ell=1}^{m_f},\\ \mathbf{k}(e)_\bullet &:= \{k_\ell\}_{k=p+1}^{q}\end{aligned}$$

such that $1 \le k_\ell \le \deg(v_\ell) + 1$ for $1 \le \ell \le q$, $1 \le k_{f,\ell} \le \deg(v_{f,\ell}) + 1$ for $1 \le \ell \le m_f$, and

$$\sum_{\ell=1}^{q} k_\ell + \sum_{\ell=1}^{m_f} k_{f,\ell} - q - m_f + 1 = d.$$

Note that $\mathbf{k}(b)_\bullet$ (respectively, $\mathbf{k}(f)_\bullet$, $\mathbf{k}(e)_\bullet$) is the beginning part (respectively, the middle part, the ending part) of $\mathbf{k}_\bullet$. For different s_h, as $\deg v_{f,1}\cdots v_{f,m_f} = \deg f$ which is independent of the individual monomial $v_{f,1}\cdots v_{f,m_f}$, one can easily choose $\mathbf{k}_\bullet$ such that $\mathbf{k}(b)_\bullet$ and $\mathbf{k}(e)_\bullet$ are independent of the middle part $v_{f,1}\cdots v_{f,m_f}$. Therefore,

$$r_{\mathbf{k}_d}(s_h) = r_{\mathbf{k}(b)_{d_b}}(v_1\cdots v_p)\circ_{d_b} r_{\mathbf{k}(f)_{d_f}}(v_{f,1}\cdots v_{f,m_f})\circ_{d_f} r_{\mathbf{k}(e)_{d_e}}(v_{p+1}\cdots v_q)$$

for some fixed $\mathbf{k}(b)_{d_b}$, $\mathbf{k}(e)_{d_e}$, and d_f. This fact implies that

$$r_{\mathbf{k}_d}(g) = r_{\mathbf{k}(b)_{d_b}}(v_1\cdots v_p)\circ_{d_b} r_{\mathbf{k}(f)_{d_f}}(f)\circ_{d_f} r_{\mathbf{k}(e)_{d_e}}(v_{p+1}\cdots v_q) \in (R).$$

Thus, we proved that $r_{\mathbf{k}_d}(g) \in (R)$ as desired.

(3) By part (1), there is a surjective morphism of symmetric operads

$$\Psi : \mathcal{F}^{sy}(E) \to S_A$$

where $\mathcal{F}^{sy}(E)$ is the free symmetric operad generated by E. Let R be the relation subspace defined in the proof of part (2). It is clear that Ψ maps R to 0. Hence, Ψ induces naturally a surjective morphism of symmetric operads

$$\psi : \mathcal{F}^{sy}(E)/(R * \mathbb{S}) \to S_A$$

where $R * \mathbb{S}$ is the space generated by $f * \sigma$ for all $f \in R$ and $\sigma \in \mathbb{S}_{\mathrm{Ar}(f)}$. By the universal property, there is a morphism of nonsymmetric operads

$$\mathcal{F}^{ns}(E) \to \mathcal{F}^{sy}(E)/(R * \mathbb{S})$$

which induces a morphism of nonsymmetric operads

$$\mathcal{F}^{ns}(E)/(R) \to \mathcal{F}^{sy}(E)/(R * \mathbb{S}).$$

Thus, we have the following sequence of morphisms

$$\mathcal{F}^{ns}(E)/(R) \xrightarrow{\theta} \mathcal{F}^{sy}(E)/(R * \mathbb{S}) \xrightarrow{\psi} S_A.$$

By part (2), the composition is an isomorphism, and consequently θ is injcetive. For simplicity, we consider θ as an inclusion and identity $f \in \mathcal{F}^{ns}(E)/(R)$ with $\theta(f) \in \mathcal{F}^{sy}(E)/(R * \mathbb{S})$.

It follows from the equivariance axiom [BYZ20, Definition 1.2(OP3')] that every element in $\mathcal{F}^{sy}(E)/(R * \mathbb{S})$ is a linear combination of elements of the form $s * \sigma$ where $s \in \mathcal{F}^{ns}(E)/(R)$ and $\sigma \in \mathbb{S}$. It remains to show the claim that every element of the form $s * \sigma$ is in fact in $\mathcal{F}^{ns}(E)/(R)$. By the proof of part (2), we may further assume that

$$s = (v_{i_1}, j_1) \circ_{j_1} (v_{i_2}, j_2) \circ_{j_2} \cdots \circ_{j_{m-1}} (v_{i_m}, j_m)$$

where $v_{i_s} \in E$ and $1 \leq j_s \leq \deg(v_{i_s}) + 1$. Let d be $\sum_{s=1}^{m} j_s - m + 1$. Note that $\{j_s\}_{s=1}^{m}$ can be replaced by any d-sequence in the above formula. Let σ be any permutation in $\mathbb{S}_{\mathrm{Ar}(s)}$. Using induction on m, relations of form (8.11), and that

$$(v_{i_s}, j) * \tau = (v_{i_s}, \tau^{-1}(j)) \quad \text{for all } \tau \in \mathbb{S}_{\mathrm{Ar}(v_{i_s}, j)},$$

we obtain that

$$s * \sigma = (v_{i_1}, j_1') \circ_{j_1'} (v_{i_2}, j_2') \circ_{j_2'} \cdots \circ_{j_{m-1}'} (v_{i_m}, j_m')$$

where $d' := \sum_{s=1}^{m} j_s' - m + 1$ is equal to $\sigma^{-1}(d)$. (The above equation can also be seen from the tree presentation of s and $s * \sigma$.) Again, $\{j_s'\}_{s=1}^{m}$ can be any d'-sequence by relations (8.11). Hence, $s * \sigma \in \mathcal{F}^{ns}(E)/(R)$ as desired. □

Note that Construction 8.1 can be viewed as a symmetric version of Construction 2.3. We are wondering if there is a symmetric version of Construction 6.1. If A is a commutative graded algebra, there is another construction of a symmetric operad associated with A (see [DK10, Section 4.2]).

Suppose a symmetric operad $\mathcal{P}$ is finitely generated by a finite alphabet $\mathcal{X}$. Let $\mathcal{V} = \mathbb{F}\mathcal{X}$, and for every $m \geq 0$, $\mathcal{V}^m$ is defined as in (4.1). For any subcollection $\mathcal{W} \subseteq \mathcal{P}$, let $\mathcal{W}_{\mathbb{S}} = \{\mathcal{W}(n) \otimes \mathbb{S}_n\}_{n \geq 0}$. Now, let $\mathcal{V}\mathbb{S}^m$ denote $(\mathcal{V}^m)_{\mathbb{S}}$, so $\mathcal{V}\mathbb{S}^m(n)$ is a right $\mathbb{S}_n$-module for all n. We have a symmetric version of Lemmas 4.2 and 4.3.

Lemma 8.3. *Suppose $\mathcal{P}$ is a locally finite symmetric operad generated by a finite-dimensional subcollection $\mathcal{V}$. Then, the following hold:*

(1) $\mathrm{GKdim}(\mathcal{P}) = \limsup_{n \to \infty} \log_n \left(\dim \left(\sum_{i=0}^{n} \mathcal{V}\mathbb{S}^i \right) \right).$
(2) *The reduced symmetric operad that is associated with $\mathcal{P}$, and defined as in Lemma 4.2 (1), is finitely generated and locally finite.*
(3) *The reduced connected symmetric operad associated with $\mathcal{P}$, defined as in Lemma 4.2 (2) is finitely generated and locally finite.*

We have a version of Proposition 4.8 for symmetric operads.

Lemma 8.4. *Suppose $\mathcal{P}$ is a finitely generated locally finite symmetric operad. The following hold:*

(1) $\mathrm{GKdim}(\mathcal{P}) = 0$ *if and only if $\mathcal{P}$ is finite dimensional.*
(2) $\mathrm{GKdim}(\mathcal{P})$ *cannot be strictly between* 0 *and* 1.

We are now ready to show Theorem 1.7.

Proof of Theorem 1.7. (1) Let $\mathcal{P}$ be a finitely generated and locally finite symmetric operad. It is clear that $\operatorname{GKdim}(\mathcal{P}) \geq 0$. The assertion then follows from Lemma 8.4 (2).

(2) It is well known that, if $r \in \mathbb{N}$, then there are finitely generated and locally finite symmetric operads $\mathcal{P}$ such that $\operatorname{GKdim}(\mathcal{P}) = r$. Now suppose that r is not an integer and $r \in R_{\mathrm{GKdim}} \setminus (2,3)$. Then, $r > 3$. By Lemma 4.6 (3), there is a connected graded algebra A such that $\operatorname{GKdim}(A) = r - 1$ and $f(n) := \dim A_n$ is increasing. By Construction 8.1 and Lemma 8.2, there is a locally finite and finitely generated symmetric operad S_A such that $\dim S_A(n) = nf(n-1)$ for all $n \in \mathbb{N}$. Then, using the fact that $f(n)$ are increasing,

$$\begin{aligned}
\operatorname{GKdim}(S_A) &= \limsup_{n\to\infty} \log_n \Big(\sum_{i=0}^{n} \dim S_A(i)\Big) = \limsup_{n\to\infty} \log_n \Big(\sum_{i=0}^{n} if(i-1)\Big) \\
&\leq \limsup_{n\to\infty} \log_n \Big(n \sum_{i=0}^{n} f(i-1)\Big) = 1 + \operatorname{GKdim}(A), \\
\operatorname{GKdim}(S_A) &= \limsup_{n\to\infty} \log_n \Big(\sum_{i=0}^{n} if(i-1)\Big) \\
&\geq \limsup_{n\to\infty} \log_n \Big(\sum_{i=\lfloor n/2\rfloor+1}^{n} if(i-1)\Big) \\
&\geq \limsup_{n\to\infty} \log_n \Big(\frac{n}{2} \sum_{i=\lfloor n/2\rfloor+1}^{n} f(i-1)\Big) \\
&\geq \limsup_{n\to\infty} \log_n \Big(\frac{n}{2} \sum_{i=1}^{\lfloor n/2\rfloor} f(i-1)\Big) = 1 + \operatorname{GKdim} A.
\end{aligned}$$

Therefore, $\operatorname{GKdim}(S_A) = \operatorname{GKdim}(A) + 1 = r$.

(3) Let U be the finitely generated graded algebra given in Example 7.2. Thus, $\operatorname{GKdim}(U) = 1$. Let S_U be the finitely generated locally finite symmetric operad constructed in Construction 8.1 and Lemma 8.2. By the proof of part (2), $\operatorname{GKdim}(S_U) = 2$. In fact, its generating series is

$$G_{S_U}(z) = z(zH_U(z))' = \frac{z(1+2z+2z^2-2z^3)}{(1-z)^2} + \bar{V}(z),$$

where

$$\bar{V}(z) = \sum_{n=2}^{\infty} (n+1)\delta_\Lambda(n) z^{n+1}.$$

By Lemma 7.1, $\bar{V}(z)$, and hence $G_{S_U}(z)$ are not holonomic.

Let $r \geq 2$ be any real number such that there is a finitely generated locally finite symmetric operad $\mathcal{P}$ with $\mathrm{GKdim}(\mathcal{P}) = r$. If $G_{\mathcal{P}}$ is not holonomic, then we are done. Otherwise, $G_{\mathcal{P}}$ is holonomic. Now consider a new operad $\mathcal{Q} := \mathcal{P} \oplus \mathcal{S}_U$ with $G_{\mathcal{Q}} = G_{\mathcal{P}} + G_{\mathcal{S}_U}$. By Holonomic Theorem 2 in [Ber14], $G_{\mathcal{Q}}$ is not holonomic. Since $\mathrm{GKdim}(\mathcal{S}_U) = 2$, we have $\mathrm{GKdim}(\mathcal{Q}) = \mathrm{GKdim}(\mathcal{P}) = r$. The assertion follows. ◻

We finish with a potential counterexample to Expectation 1 in [KP15].

Example 8.5. Let A be the connected graded algebra in Example 7.4, and let $\mathcal{S}_A$ be the operad given in Construction 8.1. Then, $\mathcal{S}_A$ is a finitely presented symmetric operad by Lemma 8.2. By (8.6), its generating series is

$$G_{\mathcal{S}_A}(z) = z(zH_A(z))' = z(zP(z))'$$

which is not holonomic by property (P3) in Example 7.4. Therefore, $\mathcal{S}_A$ is a "non-generic" counterexample to the symmetric version of Expectation 2 in [KP15].

Since $G_{\mathcal{S}_A}(z)$ is not holonomic, by (7.3), $E_{\mathcal{S}_A}(z)$ is not holonomic. We conjecture that $E_{\mathcal{S}_A}(z)$ is not differential algebraic. If this is the case, then $\mathcal{S}_A$ is a "non-generic" counterexample to [KP15, Expectation 1].

Acknowledgements. The authors thank the referee for the careful reading and valuable comments. Many thank are due to Yanhong Bao, Yu Li, Yu Ye, and Zerui Zhang for reading an earlier version of the paper, to Jason Bell for pointing out the fact that the partition generating function is not holonomic and for providing several references, and to Vladimir Dotsenko for email exchanges and his interesting remarks on the subject.

The first, second, and fourth authors would like to thank J.J. Zhang and the department of Mathematics at the University of Washington for the hospitality during their visits.

The first author was partially supported by National Science Foundation of China (grant nos. 11971460, 12071137) and the Science and Technology Commission of Shanghai Municipality (grant no. 18dz2271000). The second author was partially supported by the National Science Foundation of China (grant nos. 11501317, 11871301), and the China Postdoctoral Science Foundation (grant no. 2016M600530). The third author was partially supported by the US National Science Foundation (grant nos. DMS-1700825 and DMS-2001015). The fourth author was partially supported by the Natural Science Foundation of Huizhou University (grant nos. HZU202001, HZU201804) and the Characteristic Innovation Project of Guangdong Provincial Department of Education (grant no. 2020KTSCX145).

References

[BV73] J. M. Boardman and R. M. Vogt, *Homotopy Invariant Algebraic Structures on Topological Spaces*, Lecture Notes in Mathematics, vol. 347, Springer-Verlag, Berlin-New York, 1973. MR0420609.

[BD16] M. R. BREMNER AND V. DOTSENKO, *Algebraic Operads: An Algorithmic Companion*, CRC Press, Boca Raton, FL, 2016. MR3642294.

[Bel15] J. P. BELL, *Growth functions*, Commutative Algebra and Noncommutative Algebraic Geometry. Vol. I, Math. Sci. Res. Inst. Publ., vol. 67, Cambridge Univ. Press, NY, 2015, pp. 1–24. MR3525466.

[BYZ20] Y.-H. BAO, Y. YE, AND J. J. ZHANG, *Truncation of unitary operads*, Adv. Math. **372** (2020), 107290, 75. https://dx.doi.org/10.1016/j.aim.2020.107290. MR4125516.

[BBY12] J. P. BELL, S. N. BURRIS, AND K. YEATS, *On the set of zero coefficients of a function satisfying a linear differential equation*, Math. Proc. Cambridge Philos. Soc. **153** (2012), no. 2, 235–247. https://dx.doi.org/10.1017/S0305004112000114. MR2981925.

[Ber14] A. BERELE, *Cocharacter sequences are holonomic*, J. Algebra **412** (2014), 150–154. https://dx.doi.org/10.1016/j.jalgebra.2014.02.030. MR3215949.

[Dot19] V. DOTSENKO, *A Quillen adjunction between algebras and operads, Koszul duality, and the Lagrange inversion formula*, Int. Math. Res. Not. IMRN **14** (2019), 4281–4301. https://dx.doi.org/10.1093/imrn/rnx257. MR4023760.

[DK10] V. DOTSENKO AND A. KHOROSHKIN, *Gröbner bases for operads*, Duke Math. J. **153** (2010), no. 2, 363–396. https://dx.doi.org/10.1215/00127094-2010-026. MR2667136.

[DMR20] V. DOTSENKO, M. MARKL, AND E. REMM, *Veronese powers of operads and pure homotopy algebras*, Eur. J. Math. **6** (2020), no. 3, 829–863. https://dx.doi.org/10.1007/s40879-019-00351-6. MR4151721.

[DT20] V DOTSENKO AND P. TAMAROFF, *Tangent complexes and the diamond lemma* (2020), preprint. https://doi.org/10.48550/arXiv.2010.14792.

[Get94] E. GETZLER, *Batalin-Vilkovisky algebras and two-dimensional topological field theories*, Comm. Math. Phys. **159** (1994), no. 2, 265–285. http://dx.doi.org/10.1007/BF02102639. MR1256989.

[Get95] ———, *Operads and moduli spaces of genus* 0 *Riemann surfaces*, The Moduli Space of Curves (Texel Island, 1994), Progr. Math., vol. 129, Birkhäuser Boston, Boston, MA, 1995, pp. 199–230. https://dx.doi.org/10.1007/978-1-4612-4264-2_8. MR1363058.

[GK94] V. GINZBURG AND M. KAPRANOV, *Koszul duality for operads*, Duke Math. J. **76** (1994), no. 1, 203–272. https://dx.doi.org/10.1215/S0012-7094-94-07608-4. MR1301191.

[Gov72] V. E. GOVOROV, *Graded algebras*, Mat. Zametki **12** (1972), no. 2, 197–204 (Russian). http://dx.doi.org/10.1007/BF01095017. MR318199.

[KP15] A. KHOROSHKIN AND D. PIONTKOVSKI, *On generating series of finitely presented operads*, J. Algebra **426** (2015), 377–429. https://dx.doi.org/10.1016/j.jalgebra.2014.12.012. MR3301915.

[Koc15] D. KOÇAK, *Finitely presented quadratic algebras of intermediate growth*, Algebra Discrete Math. **20** (2015), no. 1, 69–88. MR3431952.

[KL00] G. R. KRAUSE AND T. H. LENAGAN, *Growth of Algebras and Gelfand-Kirillov Dimension*, Revised edition, Graduate Studies in Mathematics, vol. 22, American Mathematical Society, Providence, RI, 2000. https://dx.doi.org/10.1090/gsm/022. MR1721834.

[Kon03] M. KONTSEVICH, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. **66** (2003), no. 3, 157–216. https://dx.doi.org/10.1023/B:MATH.0000027508.00421.bf. MR2062626.

[LV12] J.-L. LODAY AND B. VALLETTE, *Algebraic Operads*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 346, Springer, Heidelberg, 2012. https://dx.doi.org/10.1007/978-3-642-30362-3. MR2954392.

[May72] J. P. MAY, *The Geometry of Iterated Loop Spaces*, Lecture Notes in Mathematics, vol. 271, Springer-Verlag, Berlin-New York, 1972. MR0420610.

[MZ96] C. Martinez and E. Zelmanov, *Jordan algebras of Gelfand-Kirillov dimension one*, J. Algebra **180** (1996), no. 1, 211–238. https://dx.doi.org/10.1006/jabr.1996.0063. MR1375577.

[Pet97] V. M. Petrogradsky, *On Lie algebras with nonintegral q-dimensions*, Proc. Amer. Math. Soc. **125** (1997), no. 3, 649–656. https://dx.doi.org/10.1090/S0002-9939-97-03679-4. MR1363180.

[PS19] V. Petrogradsky and I. P. Shestakov, *On Jordan doubles of slow growth of Lie superalgebras*, São Paulo J. Math. Sci. **13** (2019), no. 1, 158–176. https://dx.doi.org/10.1007/s40863-019-00122-x. MR3947400.

[Pio17] D. Piontkovski, *Growth in varieties of multioperator algebras and Gröbner bases in operads*, ISSAC'17—ACM International Symposium on Symbolic and Algebraic Computation, ACM, NY, 2017, pp. 373–379. https://dx.doi.org/10.1145/3087604.3087655. MR3703709.

[She80] J. B. Shearer, *A graded algebra with a nonrational Hilbert series*, J. Algebra **62** (1980), no. 1, 228–231. https://dx.doi.org/10.1016/0021-8693(80)90213-6. MR561125.

[Smi76] M. K. Smith, *Universal enveloping algebras with subexponential but not polynomially bounded growth*, Proc. Amer. Math. Soc. **60** (1976), 22–24 (1977). https://dx.doi.org/10.2307/2041103. MR419534.

[Sta80] R. P. Stanley, *Differentiably finite power series*, European J. Combin. **1** (1980), no. 2, 175–188. https://dx.doi.org/10.1016/S0195-6698(80)80051-5. MR587530.

[Tam98] D. Tamarkin, *Another proof of M. Kontsevich formality theorem* (1998), preprint. https://doi.org/10.48550/arXiv.math/9803025.

[War84] R. B. Warfield Jr., *The Gelfand-Kirillov dimension of a tensor product*, Math. Z. **185** (1984), no. 4, 441–447. https://dx.doi.org/10.1007/BF01236255. MR733766.

[Zei90] D. Zeilberger, *A holonomic systems approach to special functions identities*, J. Comput. Appl. Math. **32** (1990), no. 3, 321–368. https://dx.doi.org/10.1016/0377-0427(90)90042-X. MR1090884.

[ZCY20] Z. R. Zhang, Y. Q. Chen, and B. Yu, *No dialgebra has Gelfand-Kirillov dimension strictly between* 1 *and* 2, Linear Multilinear Algebra **70** (2022), no. 1, 42–52. https://dx.doi.org/10.1080/03081087.2019.1710101. MR4364386.

[Zin12] G. W. Zinbiel, *Encyclopedia of types of algebras 2010*, Int. Conf. Nankai Ser., Operads and Universal Algebra, Pure Appl. Math. Theoret. Phys., vol. 9, World Sci. Publ., Hackensack, NJ, 2012, pp. 217–297. https://dx.doi.org/10.1142/9789814365123_0011. MR3013090.

Zihao Qi:
Department of Mathematics
Fudan University
Shanghai 200433, P.R. China
E-mail: qizihao@foxmail.com

Yongjun Xu:
School of Mathematical Sciences
Qufu Normal University
P.R. China
E-mail: yjxu2002@163.com

James J. Zhang:
Department of Mathematics
Box 354350
University of Washington
Seattle, WA 98195, USA
E-mail: zhang@math.washington.edu

Xiangui Zhao:
School of Mathematics and Statistics
Huizhou University
Huizhou, Guangdong 516007
P.R. China
E-mail :zhaoxg@hzu.edu.cn

Key words and phrases: Nonsymmetric operad, Gelfand-Kirillov dimension, generating series, Groebner basis, Groebner-Shirshov basis.

2010 Mathematics Subject Classification: 18M70, 16P90, 16Z10, 17A61, 13P10, 18M65.

Received: November 25, 2020.