Student Understanding of Eigenvalue Equations in Quantum Mechanics: Symbolic Forms Analysis

Anthony Pina
University of Maine

Zeynep Topdemir
University of Maine

John R. Thompson
University of Maine

As part of an effort to examine students' mathematical sensemaking (MSM) in a spins-first quantum mechanics (QM) course, students were asked to construct an eigenvalue equation (EE) for a one-dimensional position operator. Sherin's symbolic forms were used in analysis. The data suggest three symbolic forms for an EE, all sharing a single symbol template but with unique conceptual schemata: a transformation which reproduces the original, an operation taking a measurement of state, and a statement about the potential results of measurement. These findings corroborate prior literature on a construction task rather than a comparison or deconstruction task, and with a continuous variable after instruction on discrete variables.

Keywords: Quantum Mechanics, Student Thinking, Eigentheory

Introduction

Quantum mechanics is one of a handful of topics in which every undergraduate physics major will take at least one course due to its pervasiveness in modern research and applications in physics and beyond. Despite the ubiquity of quantum mechanics courses and the significant amount of work that has gone into improving them, the subject has still proven difficult for students. Learning quantum mechanics has been shown to be a non-trivial task across both traditional (functions-first) (Singh and Marshman, 2014a; Singh and Marshman, 2015a; Emigh et al., 2018) and the more novel (spins-first) (Passante et al., 2015a, 2015b) approaches.

Eigentheory is central to the theory of quantum mechanics; it's baked into the second and third postulates of quantum mechanics. McIntyre (2012) presents the first three as follows:

1. The state of a quantum mechanical system, including all of the information you can know about it, is represented mathematically by a normalized ket $|\psi\rangle$.
2. A physical observable is represented by an operator $\hat{A}$ that acts on kets.
3. The only possible result of a measurement of an observable is one of the eigenvalues a_n of the corresponding operator $\hat{A}$.

The first postulate gives the first exposure to Dirac notation. The ket $|\psi\rangle$ is referred to as the state vector and, when projected into a basis, is often represented as either a column vector or a sum of other basis vectors in Dirac notation with appropriate coefficients. In general, any expression in Dirac notation has a direct translation to more standard linear algebra. Dirac notation however explicitly shows what basis the vector is written in, whereas the basis vectors are only implied in linear algebra. In Dirac notation an operator is denoted with a hat ($\hat{A}$); the linear algebra representation of this operator would be a matrix. The eigenvalues of this matrix are the possible results of measurement referenced in the third postulate. Standard in linear algebra, but not written into the postulates, is that each of the eigenvalues, a_n, of the operator $\hat{A}$, has an associated eigenvector, labeled $|a_n\rangle$. Therefore, one could argue that an eigenvalue equation in quantum mechanics has a fundamentally different interpretation than that of an eigenvalue equation in a mathematics context, and while they could be interpreted the same way, there are more productive interpretations for quantum mechanics.

The majority of research on student understanding of eigenvalue equations has come from the research in undergraduate mathematics education (RUME) community. Henderson and colleagues (2010) found that prior to instruction student reasoning about eigenvalue equations fell into one of three categories: superficial algebraic cancellation, correct solutions but an inability to interpret appropriately, and a correct solution with a correct interpretation. The correct interpretation in this case relates to an operation of a matrix acting on a vector resulting in the scaling of that vector by its eigenvalue. Thomas and Stewart (2011) studied student understanding of eigentheory over a two-year span and found that students tended to continue to think about linear algebra as a set of procedures rather than focus on the concepts. They target specific goals for instruction such as shifting focus toward the ideas of sets of eigenvectors and reinforcing a view that students were lacking: a geometric interpretation.

Physics education research (PER) has studied interpretations of the eigenvalue equation, often focusing on the Schrodinger equation or the eigenvalue equation for the Hamiltonian (total energy) operator, $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$, which is a major focus of QM courses, since time evolution of a system is associated with the Hamiltonian operator. In a study focused on resources students use to understand quantum mechanical operators, Gire and Manogue (2008) identified "quantum measurement as an agent," the idea that taking a measurement changes the system. Students also know that operating on a vector generally transforms the vector, which the researchers labeled "operating as agent." Accessing both of these resources in parallel can lead students to the idea that operating represents measuring. Gire and Manogue (2011) followed this up, noting that navigating unfamiliar language in addition to eigentheory presents additional challenges for students. Singh and Marshman (2013) reported student difficulties determining states for which the eigenvalue equation for the Hamiltonian would be true, some students going so far as to say that it is true for all states, including superposition states.

More recently, researchers have looked at how physics students reason about eigenvalue equations in different formats. Wawro, Thompson, and Watson (2020) investigated how students in a quantum mechanics course were thinking about both traditional mathematical eigenvalue equations ($\hat{A}\,\vec{x} = \lambda\vec{x}$) and a quantum mechanical eigenvalue equation ($\hat{S}_x|+\rangle_{\mathrm{x}} = \frac{\hbar}{2}|+\rangle_x$). They found that the mathematical equation typically elicited a scaling model, which carried over to the QM equation in some cases. They also noted that some students thought of the quantum mechanical equation in terms of representing a physical measurement, similar to prior PER findings. A novel finding was that some students used conditional language describing the equation in terms of potential or possible measurements rather than directly linking operating to measurement; this more subtle interpretation is consistent with the expert interpretation of a QM EE. Notably, some students who seemingly initially equated operating with measuring expanded their interpretation to encompass potential measurement.

The primary goal of this study is to identify cognitive resources students access when constructing and reasoning about eigenvalue equations in quantum mechanics (symbolic forms). This was done by analyzing written data from student responses to an eigenvalue equation construction task. We identified three different symbolic forms for an eigenvalue equation in the data. One is consistent with the mathematical interpretation. Another is reflective of a common misconception in quantum mechanics. The third is connected to the interpretation of eigentheory presented in the postulates of quantum mechanics. The primary delineating factor of the first, and other two are the connections the second and third make to the physical systems being modeled.

Symbolic Forms

Symbolic forms (Sherin, 2001) can be seen as an extension of the knowledge in pieces framework for student understanding (diSessa, 1993), developed as a means of examining how students think about (physics) equations. Sherin intentionally modeled this framework after diSessa's phenomenological primitives (p-prims), which were another set of intuitive chunks of knowledge or ideas (diSessa, 1993). These p-prims were each self-contained and relatively simple; small knowledge structures originating from nearly superficial interpretations of reality. By comparison, symbolic forms are larger structures consisting of two pieces: a pattern of symbols in an expression, the symbol template, and the rough idea expressed therein, labeled the conceptual schema. A conceptual schema is intended to have a fairly simple structure and is not inherently connected to any physical system or reasoning. Given that the schema contains all the meaning, it is possible for a single symbol template to be a part of several different symbolic forms, depending on the interpretation and/or context. To make clear the difference between them, some examples can be quickly explored. A common expression for the total energy of a system in intermediate mechanics is $H = T + U$, where T is the total kinetic energy in the system and U is the total potential energy. The sum on the right side is an example of the *parts-of-a-whole* symbolic form, as the individual pieces being summed compose the total energy of the system. In a senior-level quantum mechanics, class this expression would look quite different, $\widehat{H} = \frac{\hat{p}^2}{2m} + V(\hat{r})$ but could be interpreted in much the same way. In both cases here the symbol template is $\square + \square$ and the schema is "elements that combine to make a whole quantity.'"

Given their intended nature as building blocks, it is possible and likely to find more than one symbolic form in a physics equation. In the examples discussed above the focus was on the meaning of the sum, but other symbolic forms could be used to determine the meaning of the different equal signs. Due to the nature of the framework, it has been used in several contexts to analyze the intersection of math and physics. An example of its more traditional implementation is seen in the work of Schermerhorn and Thompson, who used symbolic forms to investigate student construction of differential length elements (Schermerhorn, 2019a).

Dreyfus and colleagues engaged in a theory-building effort meant to use data to illustrate their conjectures and to explore a method for analyzing student mathematical sensemaking in quantum mechanical problem solving (Dreyfus et al., 2017). They also argue the MSM tools learned in introductory physics are necessary but insufficient for MSM in quantum mechanics; this argument would be consistent with the findings of Kuo and colleagues (2013). The third and final claim of Dreyfus and colleagues' paper is that when students do not succeed in MSM in QM it is not because they are not engaging in the process, but that they are not using particular cognitive machinery (i.e., symbolic forms) needed to engage in expert MSM in QM. They highlight episodes from a study (Bing and Redish, 2012) to illustrate problems students could have interpreting equations with unproductive symbolic.

Dreyfus and colleagues observed two students (electrical engineering (EE) majors) recruited from an upper-level EE course that included a fair bit of quantum mechanics, working on a quantum mechanics tutorial. The researchers identified instances where the mathematics was consequential to the students' reasoning, labeling them as occurrences of potential MSM; they used these instances to conjecture what potential quantum mechanical symbolic forms could be. The primary segment of this interview entails the students reasoning about the time-independent Schrodinger equation or energy eigenvalue equation. The students struggled to determine whether the energy of the ground state of the infinite square well was constant due to the inclusion of the wave function on both sides of equation. The authors suggest that the students were interpreting the energy eigenvalue equation through the *dependence* symbolic form ([...*x*...]; a whole dependent on a quantity associated with an individual symbol) rather than seeing it as an eigenvalue equation, leading to their conceptual difficulties.

These observations were the starting point for the suggestion of new symbolic forms that would be productive for quantum mechanical MSM. The first symbolic form they posit is the *transformation* symbolic form, whose symbol template is $\hat{\square}|\ \rangle$ (an operator acting on a state) and whose conceptual schema is “reshaping” (the idea of stuff getting molded into a different shape). The only other posited symbolic form is the *eigenvector-eigenvalue* symbolic form, whose symbol template is $\hat{\square}|\ \rangle = \mathrm{C}|\ \rangle$, and conceptual schema is “a transformation which reproduces the original”. This is a compound symbolic form, containing their *transformation* symbolic form, as well as a new interpretation of the equal sign, and Sherin’s *coefficient* symbolic form. In this case the equal sign denotes a relationship between the operator, scalar, and eigenvector, as opposed to other previously proposed meanings of the equal sign (Dreyfus et al., 2017). Expert-level reasoning on quantum mechanical eigenvalue equations adds layers of physical interpretation on top of conceptual understanding, where the operator corresponds to a physical quantity and the solutions are physical states for which that quantity has a definite value, given by the eigenvalue.

Methods

Data were collected in a senior-level, spins-first quantum mechanics course that is required for the completion of physics and engineering physics B.S. degrees at the institution but is also taken by B.A. physics majors, physics minors, and non-physics majors. While not the students' first introduction to quantum mechanics in physics or engineering physics programs, this is the most in-depth study of the topic available to undergraduate students at this institution.

Students were asked to construct an eigenvalue equation for an operator that represents the position of a particle constrained to a one-dimensional system (Fig. 1) as an ungraded quiz. This task falls more in line with the mathematization oriented tasks used to develop the framework, making it better suited to identification of forms than prior, interpretive tasks. An adequate answer to this prompt would use the same symbol template as the *eigenvector-eigenvalue* symbolic form proposed by Dreyfus and colleagues (2017) but specified to the position context.

Consider a quantum mechanical system that is physically constrained to be located along a straight line, as shown below.

Position of object/system constrained to a line

a. Write down an **eigenvalue equation** for an operator that represents the position of this system.
b. Briefly explain what each term in the equation represents.
    i. How do each of these relate to the physical system?
    ii. What, if any, connections exist between the terms in your equation?

Figure 1. Eigenvalue equation construction task.

The authors engaged in collaborative qualitative analysis to refine the codebook until a consensus was reached (Richards, 1981). Student responses to this task were coded with the symbolic forms framework in mind. Due to the nature of the task, most student responses did not fit into existing symbolic forms. As a result, student responses were first coded for a symbol template, and then an associated conceptual schema. The first pass was essentially a binary coding identifying which students provided an expression utilizing the template $\hat{\square}|\cdot\rangle = C|\cdot\rangle$, where the dots inside the ket symbols indicate identical symbols inside the kets, and thus identical kets. This symbol template is the same as that of the *eigenvector-eigenvalue* symbolic form proposed by Dreyfus and colleagues with the exception that the "empty" kets have been given dots to denote that it must be the same ket on either side of the equal sign.

For those students that used this template, a variety of conceptual schemata were identified, derived from the portions of student responses where they interpreted their expressions. Coding only for expressions that used an eigenvalue equation template proved inadequate however, as a variety of other expressions provided by students did not conform to this symbol template and therefore required additional categorization. Grouping similar student responses by the structure of their equation, and subsequently by ideas presented by the students, resulted in the identification of other potential symbolic forms in addition to those for eigenvalue equations.

Results

Student responses were indicative of three different ways of thinking about eigenvalue equations which are summarized in Table 1. Each of these different symbolic forms shares a single template but has a distinct conceptual schema.

Table 1. Summary of identified symbolic forms for the eigenvalue equation in quantum mechanics.

Symbol Template	Conceptual Schema	Symbolic Form
$\hat{\square}|\cdot\rangle = c|\cdot\rangle$	A transformation which reproduces the original	*Reproductive transformation*
	An operation taking a measurement of state	*Operating as measuring*
	A statement about the potential results of measurement	*Potential measurement outcome*

Reproductive Transformation

The first symbolic form discussed for an eigenvalue equation has the symbol template $\hat{\square}|\cdot\rangle = c|\cdot\rangle$, and the conceptual schema related to a geometric scaling but not rotation of the vector. This is consistent with the traditional mathematical interpretation of the eigenvalue equation (Henderson et al., 2010), and documented in a QM context by Dreyfus and colleagues as the conceptual schema "a transformation which reproduces the original" and labeled *eigenvector-eigenvalue* (Dreyfus et al., 2017). However, because we identify three distinct symbolic forms that all share the eigenvalue equation symbol template, we refer to this form as *reproductive transformation* as opposed to *eigenvector-eigenvalue* for clarity.

These student responses all had the appropriate terms, and some even labeled the elements appropriately (e.g., Fig. 2) but did not provide any physical reasoning or explanation for their expressions. Responses in this group give us insight into the symbol template students are using for eigenvalue equations. These students are showing evidence of the proposed *reproductive transformation* symbol template. Given that this is all some students provided however, it is difficult to determine the exact nature of the schema associated with these students' understanding of the eigenvalue equation. While they are not demonstrating any additional

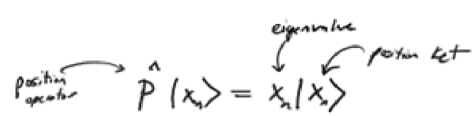


Figure 2. Example student response for reproductive transformation *symbolic form.*

quantum mechanical knowledge in their responses, these students are at least demonstrating that they know that the same ket needs to be on both sides of the equation, consistent with the form.

Operating as Measuring

Some student responses to the eigenvalue equation construction task are indicative of their conflating an operator acting on a state with the taking of a measurement of that state. These students seem to be thinking that the position operator acting on the eigenstate of position represents a measurement of position. While this and *reproductive transformation* share a symbol template, a more appropriate conceptual schema for these students may be “an operator taking a measurement of a state,” which would yield an *operating as measuring* symbolic form for eigenvalue equations.

In one such response and explanation, shown in Figure 3, the student’s expression has all the correct elements, and they are able to appropriately identify the different elements of the expression. However, in addressing how each element of the expression relates to the physical system, the student says that the operator represents “the operation of measuring position,” indicative of *operating as measuring*, a distinct interpretation of an eigenvalue equation.

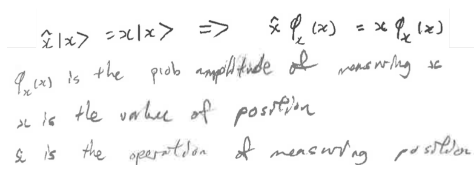
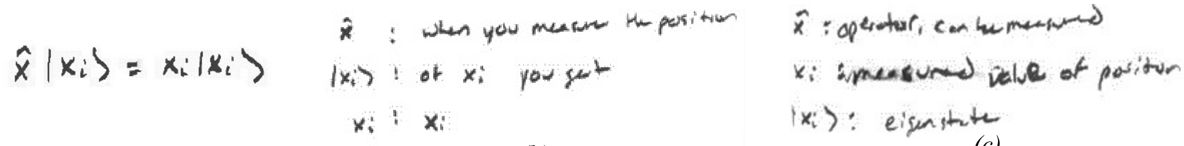


Figure 2. Example of operating as measuring *symbolic form.*

Potential Measurement Outcome

The final interpretation of eigenvalue equations with this symbol template comes from one student. Their eigenvalue equation contained all the correct elements, written as one would expect from convention (Fig. 4a). Figure 4b shows their explanation of what each term represents. When the text alone is read this is a fairly sophisticated statement: “When you measure the position of [the eigenstate] x_i you get x_i.” (The portion in brackets is an addition by the author for coherence, which is supported by the student’s response to the question, “How does each of these relate to the physical system?”, shown in Figure 4c.) This student’s interpretation of an eigenvalue equation goes beyond a geometric interpretation and does not include the notion that operating is the act of taking a measurement. This is indicated by the student’s use of the phrase, “can be measured,” which stands in contrast to the student in Figure 3 who stated that the operator represented “the operation of measuring position.” This is a subtle but important distinction as it separates the idea of taking a measurement from the idea that the operator represents a quantity which can be measured. This student presents a more sophisticated interpretation of the eigenvalue equation than the previous ones shown: that it is a statement about the *possible* outcome of measurement of the position, or more generally, the quantity being represented by the operator. A more concise version for use as a conceptual schema would be “a statement about the potential results of measurement”. This schema is directly connected to the 3rd postulate of quantum mechanics: “The only possible results of measurement of an observable $\hat{A}$ are one of the eigenvalues of the observable a_n” (e.g., McIntyre, 2012). While a single student is not indicative of the greater study population, the sophistication of this student’s response and its alignment with the meaning of a quantum mechanical eigenvalue equation warrant its inclusion as an example of a student-generated expert-level form for an eigenvalue equation.



(a) (b) (c)

Figure 3. Response related to potential measurement *symbolic form. (a) Student constructed eigenvalue equation. (b) An explanation of individual terms. (c) Relation of terms back to physical system.*

Discussion and Conclusion

Dreyfus and colleagues posited that there may be symbolic forms specific to quantum mechanics that had yet to be seen in student work. Their eigenvalue equation symbolic form, *reproductive transformation,* which is consistent with the desired outcome of mathematics instruction on eigenvalue equations (Henderson et al., 2010) and a productive conceptualization for students (Wawro, Watson, & Zandieh, 2019), is one of three forms identified that all use the same eigenvalue equation template, which follows the canonical structure.

The other two forms, *operating as measuring* and *potential measurement outcome*, both illuminate the physical meaning of the QM eigenvalue equation rather than a mathematical interpretation. *Operating as measuring* instead focuses on the act of taking a measurement "operator as agent" (Gire and Manogue, 2008, 2011) as a lens for interpreting an eigenvalue equation. Wawro and colleagues observed this conceptualization and proposed an explanation for this form: that physics equations typically represent relationships between physical quantities, documenting covariation among the quantities; *operating as measuring* conceptually reflects this relationship and is consistent with physics students' experience.

The third symbolic form, *potential measurement outcome*, is significantly more relevant to and meaningful for the physical interpretation of a QM eigenvalue equation. One could argue it falls into the expert-like additional layered meaning discussed by Dreyfus and colleagues; however, the student response discussed in relation to this form seems completely void of the ideas that form the schema of *reproductive transformation* (Dreyfus et al., 2017). Students could hold these two forms simultaneously (Wawro et al., 2020), and invoke them as needed or convenient, consistent with evoked concept images or resource activation, but *potential measurement outcome* is a more expert-like interpretation of a QM eigenvalue equation.

The way students mathematize a physics problem has been of interest to the PER community since the introduction of symbolic forms. Student construction of equations meant to describe physical systems, both in this and Sherin's (2001) tasks, provides insight into resources accessed by students in mathematizing physics problems. This analysis also opens the door for mathematical sensemaking analysis (Kuo et al., 2013). Some students explicitly attempt to utilize mathematics from other quantum contexts to generalize to this novel system. It is also noteworthy that both interpreting and constructing eigenvalue equations was not a trivial task for students. Some explicitly wrote about the difficulty of the tasks in their responses, while others showed the non-triviality of the task through their failure to provide a classifiable expression in response to the construction task. These data can provide insight into the ways students are reasoning through the discrete-to-continuous transition in a spins-first quantum mechanics course and will be a topic of further exploration.

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