New Implementations of Complete Radiation Boundary Conditions for Maxwell's Equations

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***Abstract*—We describe a new approach for implementing arbitrary-order local radiation boundary condition sequences for Maxwell's equations designed to have a straightforward interface with standard element-based volume schemes. Our goal is to develop easy-to-use software utilizing complete radiation boundary conditions. These implicitly apply uniformly accurate exponentially convergent rational approximants to the exact radiation boundary conditions. Numerical experiments for waveguide and free space problems using high-order discontinuous Galerkin spatial discretizations are presented.**

***Index Terms*—radiation conditions, time-domain methods.**

I. Introduction

The radiation of waves to the far field is a central feature of electromagnetism. Therefore efficient, convergent domain truncation algorithms are a necessary component of any software for simulating electromagnetic waves in the time domain. Complete radiation boundary conditions (CRBC), presented for electromagnetic waves in [1], [2], are an ideal solution to the radiation boundary condition problem. They possess a number of theoretical and practical advantages over perfectly matched layers (PML) [3]. In particular, as proven in [4], they are spectrally convergent, optimal parameters can be chosen automatically to guarantee any prescribed accuracy, and the computational boundary can be placed arbitrarily close to scatterers or other inhomogeneities.

Despite these facts, PML is still far more popular than CRBC in the user community. One reason is that the original implementation of CRBC for first order hyperbolic equations involved the solution of nonstandard systems of equations on the radiation boundary faces, as well as on the edges connecting the faces and on corner points where multiple edges meet. To make CRBC easier to use, we will release a library of implementations designed to be used with standard volume discretization schemes [5]. To date the library is limited to an implementation for the well-known Yee scheme. However, it will include implementations appropriate for discontinuous Galerkin discretizations [6]. (Preliminary implementations in F90 in three space dimensions and, for TM formulations, in Matlab are available from the author on request.)

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In this paper we will describe the data that is passed between the volume solver and the radiation boundary faces, along with a brief description of the equations which our codes will solve for auxiliary variables defined on faces, edges, and corners. We emphasize that a user will not need to implement the latter, but simply evolve the functions defined there using the computed time derivatives.

II. CRBC Theory

We consider Maxwell's equations in a uniform dielectric. Suppose a radiation boundary is located at $x_1 = L_1$ with normal in the x_1 direction and waves are produced by sources or scatterers located in the region $x_1 \leq L_1 - \delta$. We construct CRBC to be the radiation condition which minimizes the maximum possible error up to a given simulation time T subject to a constraint on the complexity. Practically we constrain the degrees-of-freedom (DOF) devoted to the radiation condition, denoted by a parameter P which equals the ratio of the DOFs to the number of boundary nodes per field. In three space dimensions if there are N nodes on the boundary we will evolve $6NP$ degrees of freedom, similar to the complexity requirements of a PML with P nodes in the normal direction.

The construction of the optimal conditions for a given P is carried out using the Remez algorithm based on the dimensionless parameter $\nu = \frac{cT}{\delta}$ where $c = \frac{1}{\sqrt{\epsilon\mu}}$ is the wave speed. This is possible since, as shown in [4], we can reduce the problem to that of minmizing a weighted error in the rational approximation to the square root function. We prove that an error tolerance τ can be achieved with $P \propto \log \nu \cdot \log \frac{1}{\tau}$ poles in the rational interpolant. To meet a given tolerance we simply find the smallest P for which the tolerance is met. We remark that ν may be interpreted as the number of times a wave can travel between the source and the radiation boundary over the course of a simulation. In Figure 1 below we plot P versus the error tolerance for $\nu = 10^2$ and $\nu = 10^3$; the efficiency of the approximations is obvious. We emphasize that the results are based on rigorous error estimates, while methods for choosing the width of a PML are usually based on ad hoc formulas which ignore the effect of evanescent modes.

The optimal conditions are implemented by introducing P auxiliary variables per field. These satisfy recursion relations which enforce the interpolation conditions. The new feature

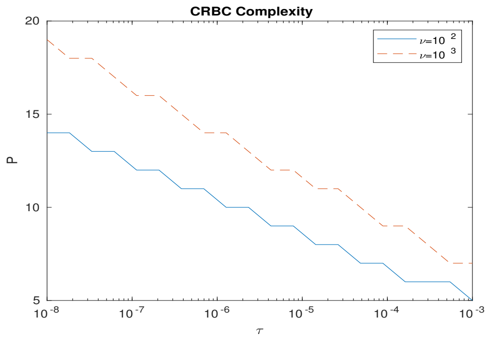

Fig. 1. Required DOFs/field/boundary point plotted versus tolerance.

compared with our original implementations is that we associate the **normal derivatives** of the auxiliary variables with the volume variables, which in turn simplifies the data input from the volume. We must also evolve auxiliary variables on edges and at corner points. Solving these recursions requires the use of sparse linear algebra solvers for systems of size proportional to P^2 at edge nodes and P^3 at corner points.

III. Interface with the Volume

To interface with the CRBC system the following steps are required:

i. Input ν and τ or ν and P into the optimizer. This will determine the optimal parameters.
ii. Input the face meshes and normal (the faces must be flat), the requested spatial discretization order and the impedance at any physical termination boundaries.
iii. At each time step or stage input the tangential fields into the CRBC solver. The solver will return the outside state at the boundary faces as well as the time derivatives of all auxiliary variables associated with faces, edges, and corners. These can be evolved with any time-stepping scheme.

So far we have only experimented with DG but we note that the required data is also compatible with edge elements.

IV. Second Order Formulations

We are developing an alternative implementation for second order formulations which we term the double absorbing boundary (DAB) [7]. As demonstrated in [8], the method can be used in conjunction with upwinded interior penalty DG discretizations for the scalar wave equation [9] and we will apply these to Maxwell's equations written as a system of second order wave equations for the electric field as in [10]. The DAB formulation involves the evolution of the auxiliary variables in a one or two element layer normal to the radiation boundary with analogous edge and corner layers. The equations themselves are simply copies of the second order Maxwell system coupled at the layer ends by the recursions, avoiding any linear algebra. However, the cost is an increase in the number of degrees-of-freedom.

V. Numerical Experiments

We demonstrate the accuracy of the method with DG discretizations of the TM system in two space dimensions. We consider initial value problems in a waveguide of width 2 and in free space. The computational domain is $(-1,1)\times(-1,1)$ with PEC boundary conditions imposed at $x_2=-1,1$ and the CRBCs imposed at $x_1=\pm1$ for the waveguide. For the free space problem the CRBCs are imposed at $x_1=\pm1$ and $x_2=\pm1$ - in this case corner variables must also be evolved. The initial electric field is $E_x=10e^{-40(x_1^2+x_2^2)}$ with the magnetic fields set to 0. We use upwind fluxes with degree 9 polynomials, an explicit fourth order Runge-Kutta method in time, and fix the CRBC order to be $P=7$. We use 20×20 elements in space and choose $\Delta t=2\times10^{-3}$. The dielectric parameters are $\epsilon=1.5$, $\mu=\epsilon^{-1}$. Errors are computed by comparison with solutions computed on domains so large that the effect of the external boundaries can't be felt in the interior. We plot the errors relative to the initial pulse in Figure 2. We see that the errors remain below τ for all time. (Here we set $\delta=1$ and $T=20$ and find $\tau=2.98\times10^{-6}$.)

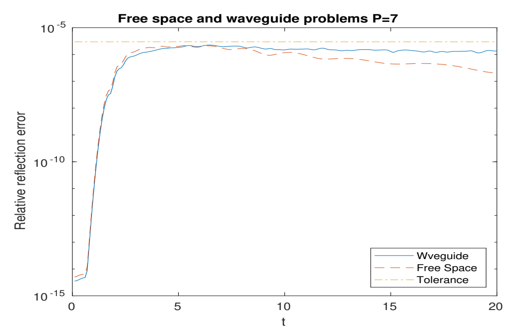

Fig. 2. Relative errors for the TM problem.

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