

Signatures of Quantum Phase Transitions in Driven Dissipative Spin Chains

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Open driven quantum systems have defined a powerful paradigm of non-equilibrium phases and phase transitions; however, quantum phase transitions are generically not expected in this setting due to the decohering effect of dissipation. In this work, we show that a driven-dissipative quantum spin chain exhibits a peculiar sensitivity to the ground-state quantum phase transition. Specifically, we consider a quantum Ising model subject to bulk dissipation (at rate Γ) and show that, although the correlation length remains finite (hence no phase transition), it develops a pronounced peak close to the ground-state quantum critical point. While standard techniques seem to fail in this regime, we develop a versatile analytical approach that becomes exact with vanishing dissipation ($\Gamma \rightarrow 0$ but finite Γt). On a technical level, our approach builds on previous work where the state of the system is described by a slowly evolving generalized Gibbs ensemble that accounts for the integrability of the Hamiltonian (described by free fermions) while treating dissipation perturbatively which leads to nontrivial, nonlinear equations for fermionic correlators. Finally, we demonstrate a kind of *universality* in that integrability-breaking perturbations of the Hamiltonian lead to the same behavior.

Driven-dissipative quantum systems provide an exciting platform to investigate non-equilibrium many-body physics. Indeed, non-equilibrium steady states of driven-dissipative systems can host new phenomena and exotic states of matter that cannot be realized in their equilibrium counterparts [1, 2]. These systems naturally emerge in a plethora of experimental platforms including exciton-polariton fluids [3, 8], trapped ions [9, 10], Rydberg gases [11, 14], and superconducting qubits [15–17]. Furthermore, they can be efficiently realized in programmable quantum simulators [18–22].

These advances notwithstanding, dissipation generically leads to mixing and a loss of quantum coherence, leading to an effective classical or thermal behavior [23–26]. With the exception of a few (possibly fine-tuned) models [27–29], quantum phase transitions are generically not expected in driven-dissipative settings. Given the paradigmatic status of ground-state quantum phase transitions, it would be desirable to find their signatures in quantum simulators. While dynamical signatures of quantum phase transitions have been identified in isolated systems [30–35], dissipation, being unavoidable in quantum simulators, is expected to spoil those at long times. On the theoretical side, a major challenge is that the analytical toolbox for open quantum systems is rather limited compared to the standard setting of condensed matter physics. With dissipation taking the spotlight, in part due to the emergence of noisy quantum simulators, several exact results and exactly solvable models [36–42] have been discovered recently. However, even for paradigmatic models such as a spin chain subject to loss in the bulk (with a nontrivial steady state), analytical solutions are simply scarce.

In this work, we investigate a quantum spin chain in an external field, h , and subject to individual spin decay [see

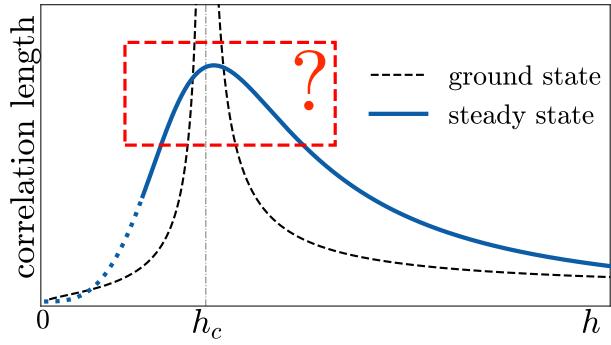


FIG. 1. Schematic presentation of the correlation length in the steady state of a driven-dissipative spin chain as a function of a tuning parameter h (e.g., an external field), compared to that of the ground state. While being very distinct, the steady state still features a peak close to the ground-state quantum critical point, h_c . An analytical approach is lacking in this regime (highlighted by the question mark).

Eq. (2)]. In the absence of dissipation, this model undergoes a ground-state quantum phase transition, accompanied by a diverging correlation length, at a critical value, h_c . In contrast, the steady state of the driven-dissipative model is always disordered. However, we show that, surprisingly, the correlation length peaks close to h_c ; see Fig. 1. With the exception of extreme limits $h \rightarrow 0, \infty$, a deeper, analytical understanding is simply unavailable. While the Hamiltonian dynamics can be mapped to free fermions, the addition of dissipation renders the model highly nonlinear. A naive approximation by simply dropping the nonlinear terms does not even correctly capture the extreme limits. Here, we develop an alternative analytical approach, inspired by the treatment of Bose gases in Ref. [43], in the limit of weak dissipation. In this limit, the system can be described by a generalized Gibbs ensemble (GGE) that accounts for the (free-fermion) inter-

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grability of the Hamiltonian, while treating the dissipation perturbatively [43–45]. Remarkably, this approach closely matches the numerical results obtained from numerical simulations at arbitrary h and small but finite Γ . Finally, we consider integrability-breaking perturbations of the Hamiltonian (beside dissipation) and show that the peak moves even closer to the corresponding quantum critical point (QCP), hinting at a kind of *universality*.

Model.— Here, we consider a driven-dissipative quantum Ising model where the Hamiltonian is given by

$$\hat{H} = \sum_{i=1}^L -\sigma_i^x \sigma_{i+1}^x + h \sigma_i^z, \quad (1)$$

where $\sigma_i^{x,y,z}$ denote the Pauli operators, and $h > 0$ the transverse field (with open boundary conditions, $\sigma_{L+1}^x = 0$, for now). Before considering dissipation, we remark that the ground state of this Hamiltonian makes a transition from a paramagnetic phase for $h > 1$ to a ferromagnetic phase at $h_c = 1$. As shown in Fig. 1, the phase transition is characterized by a diverging correlation length ξ associated with spin-spin correlations, $\langle \sigma_i^x \sigma_j^x \rangle \propto \exp \frac{-|i-j|}{\xi}$. In this work, we assume that individual spins decay at a rate Γ as described by the Lindblad operator $\hat{L}_i = \sqrt{\Gamma} \sigma_i^-$. The full dynamics of the system's density matrix $\hat{\rho}$ is then governed by the quantum master equation

$$\begin{aligned} \frac{d\hat{\rho}}{dt} &= -i[\hat{H}, \hat{\rho}] + \sum_{i=1}^L \mathcal{D}_i(\hat{\rho}), \\ \mathcal{D}_i(\hat{\rho}) &\equiv \hat{L}_i \hat{\rho} \hat{L}_i^\dagger - \frac{1}{2} \left\{ \hat{L}_i^\dagger \hat{L}_i, \hat{\rho} \right\}. \end{aligned} \quad (2)$$

Such models have been proposed in several quantum simulation platforms including superconducting circuits [17], trapped ions [21, 22], Rydberg atoms [46], and an array of coupled cavities [47, 48]. Unlike the ground state, the steady state of the driven-dissipative model does not host a phase transition as the quantum coherence is destroyed due to Markovian loss [46, 48–50]. Consequently, the correlation length always remains finite; however, as we discuss shortly, a pronounced feature in the *steady state* appears near the critical point of the *ground-state* phase transition.

We first present the results of an exact numerical simulation based on a variational Matrix Product State (MPS) approach [51] combined with a split-basis local Hilbert space [52, 53]; see the Supplemental Material (SM) [54] for more details. The correlation length for a chain of 40 spins is presented in Fig. 2 in the low dissipation regime. While the correlation length is always finite, we observe that it peaks at $h_{\text{peak}} \approx 1$ close to the ground-state critical point. Similar observations, but for different indicators such as entanglement negativity, have been reported in previous works [41, 50, 51, 55]; however, a proper understanding is simply lacking. This is in part because no analytical treatment is available in the regime

where $h \sim 1$. A simple spin-wave analysis or a mapping to free fermions (while exact in the absence of dissipation) simply fail in this regime. Prior to presenting our analytical approach, we briefly overview these simplistic methods.

Spin-wave analysis.— As a first attempt, we perform spin-wave analysis by mapping spins to bosons under the Holstein-Primakoff approximation $\sigma_i^- \rightarrow \hat{b}_i$, $\sigma_i^+ \rightarrow \hat{b}_i^\dagger$, and $\sigma_i^z \rightarrow 2\hat{b}_i^\dagger \hat{b}_i - 1$, where \hat{b}_i (\hat{b}_i^\dagger) is the bosonic annihilation (creation) operator. This approximation is expected to hold where the bosonic occupation is low, $\langle \hat{b}_i^\dagger \hat{b}_i \rangle \ll 1$. To solve for the steady state of the bosonic model, we utilize the Heisenberg-Langevin equation [50], work in the thermodynamic limit $L \rightarrow \infty$, and take periodic boundary conditions for convenience. Defining the spin-spin correlations in the steady state as $\mathcal{C}(r) = \langle \sigma_i^x \sigma_{i+r}^x \rangle_{\text{SS}}$, we find [54]

$$\mathcal{C}(r) = \frac{1}{4\pi i} \oint_{|z|=1} dz \frac{(z + z^{-1})(z^r + z^{-r})}{z^2 - [(\Gamma^2 + 16h^2)/8h]z + 1}, \quad (3)$$

where the integral is defined over the unit circle in the complex plane. The integrand in the expression above has two non-trivial poles Z_\pm with $|Z_-| \leq 1 \leq |Z_+|$. However, only the pole at Z_- contributes to the solution as Z_+ lies outside the boundaries of the unit disk. The correlation length ξ is then given by $\xi = -1/\log|Z_-|$. For $h \gg 1$, the latter pole approaches $Z_- \sim 1/h$, hence $\xi \sim 1/\log h$ in good agreement with the numerical results; see Fig. 2. On the other hand, spin-wave analysis predicts that $\xi \rightarrow \infty$ as $h \rightarrow 2$ in the limit $\Gamma \rightarrow 0$ and breaks down when $h \leq 2$, and thus fails to capture the physics except for large h .

Free fermions.— Another approximation is motivated by the fact that the Hamiltonian maps to free fermions under the Jordan-Wigner transformation, $\sigma_j^- = e^{i\pi \sum_{l=1}^{j-1} \hat{c}_l^\dagger \hat{c}_l} \hat{c}_j^\dagger$ and $\sigma_j^z = 1 - 2\hat{c}_j^\dagger \hat{c}_j$, where \hat{c}_j (\hat{c}_j^\dagger) is the fermionic annihilation (creation) operator; see Eq. (4). However, applying the same transformation to the dissipative terms (specifically the jump term $\hat{L}_i \hat{\rho} \hat{L}_i^\dagger$) generates long string operators. A naive approximation would be simply to drop the string operator, and write the Lindblad jump operators as $\hat{L}_i = \sqrt{\Gamma} \sigma_i^- \rightarrow \sqrt{\Gamma} \hat{c}_i^\dagger$ in Eq. (2). In the thermodynamic limit, the solution for this model in the steady state is also depicted in Fig. 2. Interestingly, the latter solution in the limit $\Gamma \rightarrow 0$ coincides with the quench dynamics of the Ising model [56], in the *absence* of dissipation, starting from an initial state with all spins pointing up [54]. The correlation length in this case is given by $\xi \sim 1/\log(2h)$ for $h \geq 1$, and $\xi = 1/\log 2$ when $h < 1$. This approximation features a “kink” (that softens for finite Γ) in the correlation length at the quantum critical point; however, it still fails to capture the peak. Moreover, it does not even produce the correlation length for large h due to the extra factor of 2 in the logarithm, in disagreement with the spin-wave analysis as well as the exact numerics; see Fig. 2.

Asymptotically exact solution.— The naive approximation we discussed disregards the non-locality of the spin operators by dropping the JW strings. Moreover, even the Hamiltonian dynamics alone cannot be considered as free fermions since the *parity* $e^{i\pi\sum_{j=1}^L \hat{c}_j^\dagger \hat{c}_j}$ is not conserved: dissipation can remove a fermion hence changing the parity. Here, we follow a different approach by assuming that dissipation is weak compared to the Hamiltonian and perform a Born approximation which becomes asymptotically exact as $\Gamma \rightarrow 0$ but for any finite Γt . Our approach is inspired by the treatment in Ref. [43], but is rather distinct since we consider a spin Hamiltonian on a lattice as opposed to a Bose gas in the continuum. We consider periodic boundary conditions for spins, $\sigma_{L+1}^\alpha = \sigma_1^\alpha$, and take the thermodynamic limit at the end. Just for convenience, we also apply a π -rotation around the x -axis, taking $\sigma_{y,z} \rightarrow -\sigma_{y,z}$. Under the JW transformation, the Hamiltonian becomes block diagonal $\hat{H} = \hat{H}_e \oplus \hat{H}_o$ in the even/odd (e/o) parity sectors where

$$\hat{H}_{e/o} = \sum_{j=1}^L (\hat{c}_j - \hat{c}_j^\dagger)(\hat{c}_{j+1} + \hat{c}_{j+1}^\dagger) + 2h\hat{c}_j^\dagger \hat{c}_j, \quad (4)$$

with anti-periodic/periodic boundary conditions $\hat{c}_{L+1} = \mp \hat{c}_1$ for even/odd sectors, respectively. For each parity sector, this Hamiltonian can be diagonalized as $\hat{H}_{e/o} = \sum_k \hat{h}_k \equiv \sum_k \epsilon_k \hat{a}_k^\dagger \hat{a}_k$ in terms of Bogoliubov fermions in momentum basis $\hat{a}_k = \cos(\theta_k/2)\hat{c}_k - i \sin(\theta_k/2)\hat{c}_{-k}^\dagger$ where $\tan \theta_k = \frac{\sin k}{h - \cos k}$ and $\epsilon_k = 2\sqrt{1 + h^2 - 2h \cos k}$. Note that, depending on the boundary conditions, momentum k is quantized as $k \in \frac{2\pi}{L}(\mathbb{Z}_L + \frac{1}{2}) \equiv \mathbb{Z}_{ap}$, and $k \in \frac{2\pi}{L}\mathbb{Z}_L \equiv \mathbb{Z}_p$ for even/odd parity sectors, respectively ($\mathbb{Z}_L = \mathbb{Z} \bmod L$). In the absence of dissipation and given the integrability of the Hamiltonian, the system quickly relaxes to a state described by a generalized Gibbs ensemble (GGE) with an extensive number of conserved quantities [57].

Naively, one would take the system's density matrix as $\hat{\rho}_{GGE} = \prod_k \hat{\rho}_k \sim e^{\sum_k \beta_k \hat{h}_k}$ which takes the form of a Gaussian operator with β_k the generalized inverse temperatures; however, properly incorporating the parity, the density matrix becomes a block diagonal matrix consisting of two Gaussian operators [43]: $\hat{\rho}_{GGE} = \prod_{k \in \mathbb{Z}_{ap}} \hat{\rho}_k \oplus \prod_{k \in \mathbb{Z}_p} \hat{\rho}_k$. Equivalently, the state can be fully characterized by the expectation value of the conserved charges $\hat{Q}_k \equiv \mathbb{P}_k \hat{a}_k^\dagger \hat{a}_k / 2\pi$; here, \mathbb{P}_k denotes a projector onto the even/odd parity sector if $k \in \mathbb{Z}_{ap}$ or $k \in \mathbb{Z}_p$, respectively. With dissipation present but small, one can still describe the system in terms of a GGE [43–45]; however, conservation becomes approximate and the charges $\langle \hat{Q}_k \rangle$ (or equivalently β_k) slowly evolve. Finally, using the Born approximation together with translation invariance, the dynamics for $Q_k \equiv \langle \hat{Q}_k \rangle$ is given by

$$\frac{dQ_k}{dt} = \sum_{i=1}^L \langle \mathcal{D}_i^\dagger(\hat{Q}_k) \rangle_{GGE} = L \langle \mathcal{D}_1^\dagger(\hat{Q}_k) \rangle_{GGE}, \quad (5)$$

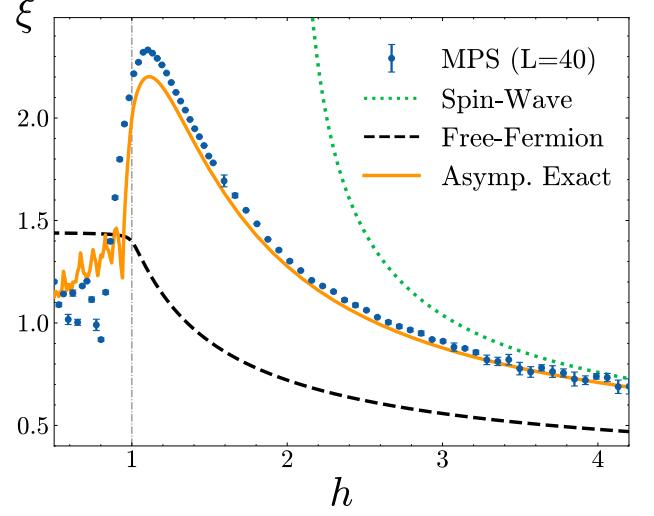


FIG. 2. Correlation length ξ calculated in the steady state of the driven-dissipative Ising model. MPS, spin-wave theory, and the free-fermion jump calculations are done at $\Gamma = 0.15$. The solution we developed, plotted in solid line, is asymptotically-exact in the limit $\Gamma \rightarrow 0$. The vertical dashed line marks the QCP.

where the adjoint superoperator \mathcal{D}^\dagger governs the dynamics in the Heisenberg picture, and the last equality follows from translation invariance. In the above equation, the expectation value is taken with respect to the GGE state, hence no contribution from the Hamiltonian. With the dissipator now acting on the first site only, we can simply make the substitution $\hat{L}_1 = \sqrt{\Gamma} \sigma_1^- \rightarrow \sqrt{\Gamma} \hat{c}_1^\dagger$ without the JW string, significantly simplifying the subsequent calculations. We can then use Wick's theorem to compute the rhs of this equation. The anti-commutator term $\{\hat{c}_1^\dagger \hat{c}_1, \hat{Q}_k\}$ is easy to treat since it does not mix the parities; however, the jump term $\hat{c}_1^\dagger \hat{Q}_k \hat{c}_1$ changes the parity, and thus the boundary conditions. Computing the latter term then requires a change of the basis. To this end, we note that either periodic ($\ell \in \mathbb{Z}_p$) or anti-periodic functions ($k \in \mathbb{Z}_{ap}$) form a complete basis and are related to each other by [58]

$$\hat{c}_k = -\frac{2}{L} \sum_{\ell \in \mathbb{Z}_p} \frac{1}{1 - e^{i(k-\ell)}} \hat{c}_\ell. \quad (6)$$

In principle, the application of Wick's theorem together with the basis transformation allows us to compute the rhs of Eq. (5). It turns out that the resulting equations take a simpler form in terms of the original fermions $\hat{c}_k, \hat{c}_k^\dagger$ rather than Bogoliubov fermions $\hat{a}_k, \hat{a}_k^\dagger$. Indeed, the system can be fully characterized by the expectation value of the operators $\hat{A}_k = \mathbb{P}_k \hat{c}_k^\dagger \hat{c}_k / 2\pi$ and $i\hat{B}_k = \mathbb{P}_k \hat{c}_k \hat{c}_{-k} / 2\pi$. Ultimately, we aim to compute the corresponding expectation values A_k, B_k (denoted by c numbers with no hat). But the functions A_k, B_k are not independent as there is only a single conserved charge per k : the GGE constrains

them as $(A_k - 1/4\pi) \sin \theta_k - B_k \cos \theta_k = 0$, or more explicitly,

$$C_k \equiv (A_k - \frac{1}{4\pi}) \sin k - (h - \cos k) B_k = 0. \quad (7)$$

For operators A (B), not conserved under the Hamiltonian dynamics, the full dynamics is then subject to the above constraint: $dA_k/dt = -F_A(k) + \Lambda_k \partial C_k / \partial A_k$ (and similarly for B) where Λ_k defines a Lagrange multi-

plier, and the terms $F_{A,B}$ are computed from Eq. (5) upon the substitution $\hat{Q} \rightarrow \hat{A}, \hat{B}$, respectively. Applying Wick's theorem then yields a nonlinear equation for A_k, B_k which is also nonlocal in momentum (due to the basis transformation). In the thermodynamic limit $L \rightarrow \infty$, this equation can be written in a compact form by introducing the Hilbert transform \mathcal{H} on the unit circle defined as $(\mathcal{H}f)_k = \frac{1}{2\pi} P \int_0^{2\pi} d\lambda f_\lambda \cot(\frac{k-\lambda}{2})$ with P the principal part. The resulting equations take the form [54]

$$\begin{aligned} F_A(k) &= A + \frac{n^2}{2\pi} - 2\pi [A^2 - (\mathcal{H}A)^2 - (A \rightarrow B)] + 2n(\mathcal{H}A)', \\ F_B(k) &= B - 4\pi[AB - (\mathcal{H}A)(\mathcal{H}B)] + 2n(\mathcal{H}B)', \end{aligned} \quad (8)$$

where $n = \int \frac{dk}{2\pi} \langle \hat{c}_k^\dagger \hat{c}_k \rangle$ is the fermionic density, and the prime indicates partial derivative with respect to k which itself is implicit on the rhs. The resulting integro-differential equations are rather difficult to work with. Instead, one can make an analytical continuation $e^{ik} \rightarrow z$ to the unit disk $|z| \leq 1$ in the complex plane as

$$\mathcal{G}_A(z) = \int \frac{dk}{2\pi} A_k \frac{e^{ik} + z}{e^{ik} - z}, \quad (9)$$

and similarly for $\mathcal{G}_B, \mathcal{G}_\Lambda$. Note that these functions are analytic everywhere inside the unit disk, and $\mathcal{G}_A(e^{ik-0^+}) = A_k + i(\mathcal{H}A)_k$. With this representation, the dynamics become local in z ; see the SM [54] for details. A further simplification occurs by defining $\mathcal{G}_\pm(z) = \mathcal{G}_A(z) \mp i\mathcal{G}_B(z) \mp n/2\pi$. The full dynamics is then given by [54]

$$\frac{1}{\Gamma} \frac{d\mathcal{G}_\pm(z)}{dt} = -[1 \mp 2n - 2\pi\mathcal{G}_\pm(z) + 2nz\partial_z]\mathcal{G}_\pm(z) \pm i[h - z^{\pm 1}]\mathcal{G}_\Lambda(z). \quad (10)$$

Together with Eq. (7), these equations allow solving for the \mathcal{G} functions, and thus the fermionic correlations A_k, B_k . We can then determine spin correlations by computing the determinant of a matrix consisting of real-space fermionic correlators, and finally extract the correlation length ξ ; see [54] for details.

In Fig. 2 we present the correlation length obtained from our analytical approach; here, we have numerically solved the nonlinear equation Eq. (10) together with the constraint (7) by making a multipole expansion, e.g., $\mathcal{G}_A(z) = \sum_{n=0}^{\infty} a_n z^n$. Our asymptotically exact results closely match the MPS simulations for $\Gamma = 0.15$. Interestingly, we also observe that the main conclusions of our work extend to a wide range of Γ ; see [54]. Our analytical approach clearly reproduces a peak close to, but slightly above, the quantum critical point $h_c = 1$; more precisely, we find the peak location in the limit $\Gamma \rightarrow 0$ as

$$h_{\text{peak}} \approx 1.11, \quad (11)$$

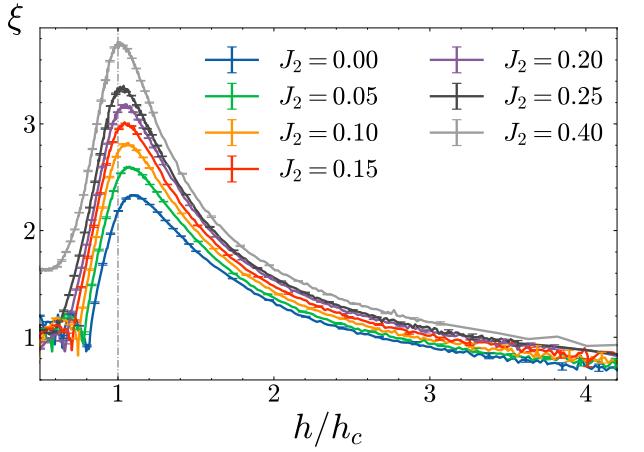


FIG. 3. Correlation length ξ computed in the steady state of the driven-dissipative NNN Ising model ($\Gamma = 0.15$) obtained from MPS numerical results. The corresponding QCPs for $0 \leq J_2 \leq 0.4$ are at $1 \leq h_c \lesssim 1.63$

thus confirming that $h_{\text{peak}} \gtrsim 1$ is not just an artifact of numerics at finite L or Γ . Furthermore, our solution accurately predicts the spin correlations for $h < 1$ as well; however, in this regime, correlations do not simply decay exponentially but also exhibit oscillating spatial features, hence the irregular behavior of ξ in this regime. In the limit $h \rightarrow 0$, an exact solution becomes available and correlations identically vanish where spins are separated by two or more sites [41].

Beyond integrable Hamiltonians.— It might seem that the pronounced peak near the QCP is specific to the nearest-neighbor Ising chain, which is an integrable model (aside from dissipation). Here, we consider integrability-breaking perturbations of the Hamiltonian

by adding next-to-nearest-neighbor interactions,

$$\hat{H}_2 = \sum_{i=1}^L -\sigma_i^x \sigma_{i+1}^x - J_2 \sigma_i^x \sigma_{i+2}^x + h \sigma_i^z, \quad (12)$$

and show that, surprisingly, our main result is quite robust. We first note that this model undergoes a similar ground-state phase transition at $h_c(J_2) > 1$ when $J_2 > 0$. In Fig. 3, we depict the correlation length in the steady state obtained from the MPS simulation of a chain of 40 spins. We observe that, as we depart from the nearest-neighbor Ising model, the correlation length still peaks near the corresponding QCPs, and, rather strikingly, the peak moves closer to $h/h_c = 1$ as J_2 increases. While a phase transition is absent in these models, our observation hints at a kind of *universality* where driven-dissipative Ising chains exhibit the same pronounced feature regardless of their microscopic details.

Conclusion.— In this work, we have studied the intricate interplay of quantum phase transitions and dissipation in driven-dissipative spin chains. Specifically, we have shown that the *steady state* of driven-dissipative Ising models exhibit a pronounced feature near the *ground state* quantum critical point. For the nearest-neighbor Ising Hamiltonian, we have exploited (free-fermion) integrability as the basis for an asymptotically

exact solution. Furthermore, we have numerically observed the same features when integrability-breaking perturbations are included. On a conceptual level, our results are significant in that they show that quantum features could survive despite dissipation. On a practical level, our work opens the door to identifying quantum phase transitions in the context of noisy quantum simulators. Finally, on a technical level, our approach is immediately applicable to a large class of spin chains subject to decay in the bulk. The latter distinguishes our work from exact approaches focused on hermitian jump operators (e.g., dephasing) [50], boundary dissipation [60, 61], or collective models [42, 62].

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Supplemental Material for “Signatures of Quantum Phase Transitions in Driven Dissipative Spin Chains”

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In this Supplemental Material, we provide additional details on the results stated in the main text. In Secs. S.I and S.II, we discuss the numerical solution provided in the main text and show the results obtained in a wide section of the parameter space. In Sec. S.III, we provide the details of the spin-wave theory calculations done under the Holstein-Primakoff approximation. In Sec. S.IV, we layout the asymptotically-exact treatment of the model under the assumption that the state of the driven dissipative Ising model assumes a GGE in the $\Gamma \rightarrow 0$ limit. We show how spin correlations can be recovered from the resulting fermionic correlations in Sec. S.V. In Sec. S.VI, we highlight the convergence of the free-fermion approximation to the results of quench dynamics in the limit $\Gamma \rightarrow 0$. In Sec. S.VII, we discuss the Binder cumulant calculations used to obtain the quantum critical point for non-integrable Hamiltonians.

S.I. Numerical Solution

The numerical solution provided in the main text is based on a Matrix Product States (MPS) representation of the vectorized density matrix obtained under the mapping

$$\rho = \sum_{i,j} \rho_{ij} |i\rangle \langle j| \rightarrow |\rho\rangle\rangle = \sum_{i,j} \rho_{ij} |i\rangle \otimes |j\rangle. \quad (\text{S.1})$$

The MPS ansatz takes the form

$$|\rho\rangle\rangle = \sum_s W[1]^{s'_1 s''_1} W[2]^{s'_2 s''_2} \dots W[L]^{s'_L s''_L} |s'_1 s''_1, s'_2 s''_2, \dots, s'_L s''_L\rangle, \quad (\text{S.2})$$

where $W^{s'_i s''_i}$ is a rank-2 tensor of maximum dimensions $\chi^2 d^2$, where $d = 2$ is the local Hilbert space dimension, and χ is the MPS bond dimension. Furthermore, we split the local Hilbert space of $|\rho\rangle\rangle$, as described in [S1, S2], reducing the dimensionality of the tensors from d^2 to d . The final ansatz takes the form

$$|\rho\rangle\rangle = \sum_s A[1]^{s'_1} \tilde{A}[1]^{s''_1} A[2]^{s'_2} \tilde{A}[2]^{s''_2} \dots A[L]^{s'_N} \tilde{A}[L]^{s''_L} |s'_1, s''_1, s'_2, s''_2, \dots, s'_L, s''_L\rangle, \quad (\text{S.3})$$

where the tensors $W[i]$ are now recast into the tensors $A[i]$, and $\tilde{A}[i]$. We then iteratively optimize each tensor $A[i]$ ($\tilde{A}[i]$) to the tensor corresponding to the zero eigenvalue of the effective Liouvillian operator acting on the site resulting in an approximate form of the steady state density matrix [S3].

S.II. Correlation length in (h, Γ) plane

Figure S.1 shows the correlation length obtained in the steady state of the driven-dissipative nearest-neighbor Ising model for a chain of length $L = 40$, for a range of h , and Γ values. The peak of the correlation length is found to be close to the quantum critical point even for intermediate values of Γ .

S.III. Spin-wave Theory

Here we provide the details for the spin-wave analysis reported in the main text. We find the steady state solution by applying the Heisenberg-Langevin equation [S4, S5] to the relevant operators. In the derivation, we consider a more general transverse-field Ising Hamiltonian, where $J_{m,n}$ is the Ising interaction only depending on the distance $|n - m|$,

$$\hat{H} = \sum_{n>m} J_{m,n} \sigma_m^x \sigma_n^x + h \sum_m \sigma_m^z. \quad (\text{S.4})$$

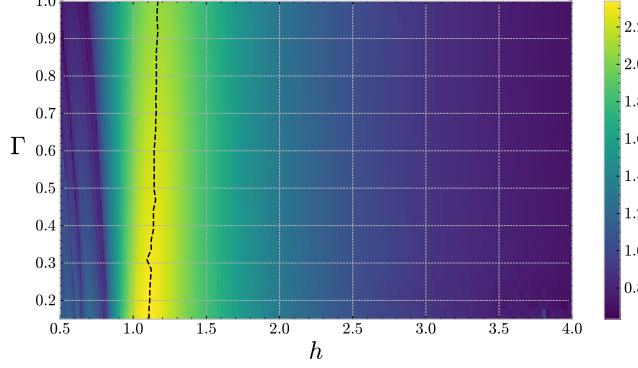


FIG. S.1. Correlation Length in the steady state of the driven dissipative nearest-neighbor Ising model against h and Γ . Numerical results obtained from MPS calculations for $L = 40$. The dashed line indicates the position of the maximum correlation length for a given value of Γ . The peak slightly deviates away from the QCP as Γ increases.

We apply a Holstein-Primakoff approximation under which we take

$$\begin{aligned}\sigma_m^- &\rightarrow \hat{b}_m, \\ \sigma_m^+ &\rightarrow \hat{b}_m^\dagger, \\ \sigma_m^x &\rightarrow \hat{b}_m + \hat{b}_m^\dagger, \\ \sigma_m^z &\rightarrow 2\hat{b}_m^\dagger \hat{b}_m - 1,\end{aligned}\tag{S.5}$$

where $[\hat{b}_m, \hat{b}_n] = 0$, and $[\hat{b}_m, \hat{b}_n^\dagger] = \delta_{m,n}$. With that, we arrive at the following quadratic bosonic Hamiltonian

$$\hat{H}_b = \sum_{n>m} J_{m,n} (\hat{b}_m + \hat{b}_m^\dagger)(\hat{b}_n + \hat{b}_n^\dagger) + h \sum_m (2\hat{b}_m^\dagger \hat{b}_m - 1).\tag{S.6}$$

Assuming periodic boundary conditions, we utilize the lattice translation-invariance symmetry to define the Fourier transformation $\hat{b}_m = \frac{1}{\sqrt{L}} \sum_k e^{-ikm} \hat{b}_k$, where the momentum modes k take the values $k = \frac{2\pi l}{L}$ with $1 \leq l \leq L$. In Fourier space, momentum k only couples to $-k$, and the Hamiltonian takes the simple form

$$\hat{H}_b = \sum_k \hat{H}_k + \text{const},\tag{S.7}$$

where

$$\hat{H}_k = (\hat{b}_k^\dagger \hat{b}_{-k}) \begin{pmatrix} J_k + h & J_k \\ J_k & J_k + h \end{pmatrix} \begin{pmatrix} \hat{b}_k \\ \hat{b}_{-k}^\dagger \end{pmatrix},\tag{S.8}$$

with $J_k \equiv \sum_{n-m>0} J_{m,n} \cos(k(n-m))$. Similarly, the dissipative part of the dynamics takes the form

$$\mathcal{D}_b(\rho) = \Gamma \sum_k \hat{b}_k \rho \hat{b}_k^\dagger - \frac{1}{2} \hat{b}_k^\dagger \hat{b}_k \rho - \frac{1}{2} \rho \hat{b}_k^\dagger \hat{b}_k.\tag{S.9}$$

Under these conditions, the bosonic operators evolve in time under the Heisenberg-Langevin equation given by

$$\begin{aligned}\frac{d\hat{b}_k(t)}{dt} &= 2i [\hat{H}_k, \hat{b}_k] - \frac{\Gamma}{2} \hat{b}_k + \sqrt{\Gamma} \hat{b}_k^{in}(t), \\ \frac{d\hat{b}_{-k}^\dagger(t)}{dt} &= 2i [\hat{H}_k, \hat{b}_{-k}^\dagger(t)] - \frac{\Gamma}{2} \hat{b}_{-k}^\dagger(t) + \sqrt{\Gamma} \hat{b}_{-k}^{in}(t),\end{aligned}\tag{S.10}$$

where we have used the fact that $\hat{H}_k = \hat{H}_{-k}$. $\hat{b}_k^{in}(t)$ are input noise operators satisfying the white-noise condition $\langle \hat{b}_k^{in}(t) \hat{b}_{k'}^{in}(t') \rangle = \delta_{k,k'} \delta(t - t')$. Other input noise correlators are identically zero. The above equations can be combined in the matrix form as

$$\dot{f}(t) = \mathcal{M}f(t) + m(t),\tag{S.11}$$

where $f(t) = \begin{pmatrix} \hat{b}_k \\ \hat{b}_{-k}^\dagger \end{pmatrix}$, $m(t) = \begin{pmatrix} \sqrt{\Gamma} \hat{b}_k^{in} \\ \sqrt{\Gamma} \hat{b}_{-k}^{in} \end{pmatrix}$ and

$$\mathcal{M} = \begin{pmatrix} -2i(J_k + h) - \frac{\Gamma}{2} & -2iJ_k \\ 2iJ_k & 2i(J_k + h) - \frac{\Gamma}{2} \end{pmatrix}, \quad (\text{S.12})$$

with the formal solution given by $f(t) = e^{\mathcal{M}t} f(0) + \int_0^t e^{\mathcal{M}(t-t')} m(t') dt'$. In the steady state, the first term vanishes, and we are left with the second term only. The matrix $e^{\mathcal{M}\tau}$ has the structure

$$e^{\mathcal{M}\tau} = \begin{pmatrix} g_1(\tau) & g_2(\tau) \\ g_2^*(\tau) & g_1^*(\tau) \end{pmatrix}. \quad (\text{S.13})$$

Therefore, the solution in the limit $t \rightarrow \infty$ takes the form

$$f(t) = \sqrt{\Gamma} \int_0^t dt' \begin{pmatrix} g_1(t-t') \hat{b}_k^{in}(t') + g_2(t-t') \hat{b}_{-k}^{in}(t') \\ g_1^*(t-t') \hat{b}_{-k}^{\dagger in}(t') + g_2^*(t-t') \hat{b}_k^{\dagger in}(t') \end{pmatrix}. \quad (\text{S.14})$$

Using the following definitions,

$$\begin{aligned} \epsilon_k &= 2(J_k + h), \\ \eta_k &= 2J_k, \\ \xi_k &= \sqrt{\epsilon_k^2 - \eta_k^2}, \end{aligned} \quad (\text{S.15})$$

we can explicitly write the Green functions g as

$$\begin{aligned} g_1(\tau) &= e^{-\frac{\Gamma}{2}\tau} [\cos(\tau\xi_k) - i\epsilon_k \frac{\sin(\tau\xi_k)}{\xi_k}], \\ g_2(\tau) &= -i\eta_k e^{-\frac{\Gamma}{2}\tau} \frac{\sin(\tau\xi_k)}{\xi_k}. \end{aligned} \quad (\text{S.16})$$

Bosonic correlation functions in the steady state can thus be obtained using the solution in Eq. (S.14).

Spin correlations.—Under the HP approximation, the spin correlations at time t at distance l are given by

$$\langle \sigma_j^x(t) \sigma_{j+l}^x(t) \rangle = \langle \hat{b}_j^\dagger(t) \hat{b}_{j+l}(t) \rangle + \langle \hat{b}_j(t) \hat{b}_{j+l}(t) \rangle + \text{c.c.} \quad (\text{S.17})$$

In the steady state ($t \rightarrow \infty$), they can easily be found by applying an inverse Fourier transformation to the bosonic correlators. The final results are given by

$$\langle \sigma_j^x \sigma_{j+l}^x \rangle_{\text{SS}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \cos kl \frac{\eta_k (\eta_k - \epsilon_k)}{(\frac{\Gamma}{2})^2 + \xi_k^2} + \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \sin kl \frac{\eta_k (\frac{\Gamma}{2})}{(\frac{\Gamma}{2})^2 + \xi_k^2}. \quad (\text{S.18})$$

As we showed in the main text, this result can be further simplified by analytically continuing k to the unit disk in the complex plane ($e^{ik} \rightarrow z$) which gives Eq. (3).

S.IV. Born approximation & GGE

For convenience, we first apply the transformation $\sigma_i^{y,z} \rightarrow -\sigma_i^{y,z}$. This transformation flips the sign of h in the Hamiltonian

$$\hat{H} = - \sum_{i=1}^L \sigma_i^x \sigma_{i+1}^x + h \sigma_i^z, \quad (\text{S.19})$$

and takes $\hat{L}_i = \sqrt{\Gamma} \sigma_i^- \rightarrow \sqrt{\Gamma} \sigma_i^+$.

1. Jordan-Wigner transformation

By performing a Jordan-Wigner transformation

$$\sigma_i^- = \prod_{l=1}^{i-1} (1 - 2\hat{c}_l^\dagger \hat{c}_l) \hat{c}_i^\dagger, \quad \sigma_i^+ = (\sigma_i^-)^\dagger, \quad \sigma_i^z = 1 - 2\hat{c}_i^\dagger \hat{c}_i, \quad (\text{S.20})$$

the Hamiltonian takes the form [S6]

$$\hat{H} = - \sum_{i=1}^{L-1} (\hat{c}_i^\dagger - \hat{c}_i)(\hat{c}_{i+1} + \hat{c}_{i+1}^\dagger) - h \sum_{i=1}^L (\hat{c}_i \hat{c}_i^\dagger - \hat{c}_i^\dagger \hat{c}_i) - e^{i\pi \hat{N}} (\hat{c}_L - \hat{c}_L^\dagger)(\hat{c}_1 + \hat{c}_1^\dagger), \quad (\text{S.21})$$

where $\hat{N} = \sum_{i=1}^L \hat{c}_i^\dagger \hat{c}_i$ is the total fermionic number operators. Because the Hamiltonian is number-conserving, the even- and odd-parity sectors of \hat{H} can be treated independently. One can write the Hamiltonian in even/odd sectors as

$$\begin{aligned} \hat{H}_{\text{e/o}} &= - \sum_{i=1}^L (\hat{c}_i^\dagger - \hat{c}_i)(\hat{c}_{i+1} + \hat{c}_{i+1}^\dagger) - h \sum_{i=1}^L \hat{c}_i \hat{c}_i^\dagger - \hat{c}_i^\dagger \hat{c}_i, \\ \text{with } \hat{c}_{L+1} &= \mp \hat{c}_1. \end{aligned} \quad (\text{S.22})$$

That is, the Hamiltonian is that of free fermions but with anti-periodic/periodic boundary conditions for even/odd sectors, respectively. Next, one can diagonalize the Hamiltonian in terms of Bogoliubov fermions defined as [S6, S7]

$$\hat{\alpha}_k = \cos(\theta_k/2) \hat{c}_k - i \sin(\theta_k/2) \hat{c}_{-k}^\dagger, \quad (\text{S.23})$$

where

$$e^{i\theta_k} = \frac{h - e^{-ik}}{\sqrt{1 + h^2 - 2h \cos k}}, \quad (\text{S.24})$$

or $\tan \theta_k = \frac{\sin k}{h - \cos k}$ as stated in the main text. With this set of definitions, we find¹

$$\hat{H}_{\text{e/o}} = \sum_{k \in \mathbb{Z}_{\text{ap},p}} \epsilon_k [\hat{\alpha}_k^\dagger \hat{\alpha}_k - 1], \quad (\text{S.25})$$

where with the dispersion relation

$$\epsilon_k = 2\sqrt{1 + h^2 - 2h \cos k}. \quad (\text{S.26})$$

2. Basis transformation

For a closed system and for even observables (i.e., those not changing the parity), we can limit ourselves to the even sector [S6]; however, introducing dissipation violates parity conservation, and thus requires a change of basis states. Since both c_k for $k \in \mathbb{Z}_{\text{ap}}$ and c_p for $p \in \mathbb{Z}_p$ span the one-particle Hilbert space on a ring, they can be related to each other. Indeed, the corresponding momentum eigenstates are related via

$$|k\rangle = \sum_{p \in \mathbb{Z}_p} \langle p|k\rangle |p\rangle, \quad (\text{S.27})$$

where

$$\langle p|k\rangle = \frac{1}{L} \sum_{j=1}^L e^{i(k-p)j} = -\frac{2}{L} \frac{1}{1 - e^{-i(k-p)}}, \quad (\text{S.28})$$

¹ More precisely, defining the zero mode in the odd parity sector requires a particle-hole transformation for $h < 1$ [S6].

Identifying $|k\rangle = \hat{c}_k^\dagger |0\rangle$, we find

$$\hat{c}_k = -\frac{2}{L} \sum_{p \in \mathbb{Z}_p} \frac{1}{1 - e^{i(k-p)}} \hat{c}_p, \quad (\text{S.29})$$

Up to a relative sign, this expression is given in Ref. [S8]. Taking the thermodynamic limit $L \rightarrow \infty$, we recover the basis transformation $\hat{c}_k = -\frac{2i}{L} \sum_{p_m \in \mathbb{Z}_p} \frac{1}{k_n - p_m} \hat{c}_p$ reported in [S9]. Finally, from the completeness relation

$$\sum_{p \in \mathbb{Z}_p} \langle k' | p \rangle \langle p | k \rangle = \delta_{k,k'}, \quad (\text{S.30})$$

we arrive at the condition

$$\sum_{p \in \mathbb{Z}_p} \frac{1}{1 - e^{-i(k-p)}} \frac{1}{1 - e^{i(k'-p)}} = \frac{L^2}{4} \delta_{k,k'}. \quad (\text{S.31})$$

3. Wick's theorem

As stated in the main text, it is more convenient to work with the c, c^\dagger fermions (rather than Bogoliubov fermions). To this end, we observe that $A_k = \langle \hat{c}_k^\dagger \hat{c}_k \rangle / 2\pi$ and $B_k \equiv \langle \hat{c}_k \hat{c}_{-k} \rangle / 2\pi i$ are constrained together as the only nontrivial correlator is given by $\langle \hat{\alpha}_k^\dagger \hat{\alpha}_k \rangle$. Inverting Eq. (S.23), we obtain the expectation values

$$2\pi A_k = \frac{1}{2} + \cos \theta_k p_k, \quad 2\pi B_k = \sin \theta_k p_k, \quad (\text{S.32})$$

where $\langle \hat{\alpha}_k^\dagger \hat{\alpha}_k \rangle = \frac{1}{2} + p_k$. We thus find

$$(A_k - 1/4\pi) \sin \theta_k - B_k \cos \theta_k = 0, \quad (\text{S.33})$$

which reduces to Eq. (7) of the main text, and is repeated here for completeness:

$$(A_k - 1/4\pi) \sin k - B_k (h - \cos k) = 0. \quad (\text{S.34})$$

We treat the constraint which is linear in A_k, B_k via Lagrange multipliers. Combined with the Born approximation, this yields

$$\frac{1}{\Gamma} \frac{d}{dt} A_k = -F_A(k) + \Lambda_k \sin k, \quad (\text{S.35a})$$

$$\frac{1}{\Gamma} \frac{d}{dt} B_k = -F_B(k) - \Lambda_k (h - \cos k), \quad (\text{S.35b})$$

where we have defined

$$-\frac{1}{L} F_A \equiv \langle \hat{c}_1^\dagger \hat{A}_k \hat{c}_1 \rangle - \langle \hat{c}_1^\dagger \hat{c}_1 \hat{A}_k \rangle, \quad (\text{S.36})$$

and similarly for F_B ; here the expectation value is computed with respect to the GGE state (the label dropped for notational ease). Now, the last term in the above equation does not involve a change of parity ($\hat{c}_1^\dagger \hat{c}_1$ conserves the particle number), and can be easily computed by noting $\hat{c}_1 = \frac{1}{\sqrt{L}} \sum_{\lambda} e^{ik} \hat{c}_k$ and using Wick's theorem as

$$\frac{L}{2\pi} \langle \mathbb{P}_k \hat{c}_1^\dagger \hat{c}_1 \hat{c}_k^\dagger \hat{c}_k \rangle = n L A_k + A_k (1 - 2\pi A_k) + 2\pi B_k^2. \quad (\text{S.37})$$

On the other hand, the first term on the rhs of Eq. (S.36) involves a change of the sector due to the jump term. We can then write $\hat{A}_k = c_k^\dagger c_k$ in the new basis using Eq. (S.29) and then using Wick's theorem. Some algebra then yields

$$\langle \hat{c}_1^\dagger \mathbb{P}_k \hat{c}_k^\dagger \hat{c}_k \hat{c}_1 \rangle = \frac{4}{L^2} \left(n \sum_{\lambda} \frac{2\pi A_{\lambda}}{|e^{ik} - e^{i\lambda}|^2} - \frac{1}{L} \left| \sum_{\lambda} \frac{2\pi A_{\lambda}}{e^{ik} - e^{i\lambda}} \right|^2 + \frac{1}{L} \left| \sum_{\lambda} \frac{2\pi B_{\lambda}}{e^{ik} - e^{i\lambda}} \right|^2 \right). \quad (\text{S.38})$$

For large L , sums can be replaced by integrals. To avoid a divergence in $\sum_\lambda A_\lambda / |e^{ik} - e^{i\lambda}|^2$, we can write

$$\sum_\lambda \frac{A_\lambda}{|e^{ik} - e^{i\lambda}|^2} = \sum_\lambda \frac{A_\lambda - A_k}{|e^{ik} - e^{i\lambda}|^2} + \sum_\lambda \frac{1}{|e^{ik} - e^{i\lambda}|^2} A_k = \sum_\lambda \frac{A_\lambda - A_k}{|e^{ik} - e^{i\lambda}|^2} + \frac{L^2}{4} A_k, \quad (\text{S.39})$$

using the completeness relation, Eq. (S.31).

Finally we take the thermodynamic limit via the substitution $\sum_\lambda \cdot = \frac{L}{2\pi} \int_{-\pi}^{\pi} dk \cdot$. Putting everything together, we find

$$\begin{aligned} F_A(k) &= A_k - 2\pi \left((A_k)^2 - \left| \frac{1}{\pi} \oint d\lambda \frac{A_\lambda}{e^{ik} - e^{i\lambda}} \right|^2 \right) + \frac{2n}{\pi} \oint d\lambda \frac{A_k - A_\lambda}{|e^{ik} - e^{i\lambda}|^2} \\ &\quad + 2\pi \left((B_k)^2 - \left| \frac{1}{\pi} \oint d\lambda \frac{B_\lambda}{e^{ik} - e^{i\lambda}} \right|^2 \right), \end{aligned} \quad (\text{S.40})$$

where the crossed integral denotes the Cauchy principal value of the integral. A similar analysis gives the expression for F_B :

$$F_B(k) = B_k - 2\pi \left(2A_k B_k - \left[\oint \frac{d\lambda}{\pi} \frac{A_\lambda}{e^{ik} - e^{i\lambda}} \oint \frac{d\lambda'}{\pi} \frac{B_{\lambda'}}{e^{-ik} - e^{-i\lambda'}} + \text{c.c.} \right] \right) + \frac{2n}{\pi} \oint d\lambda \frac{B_k - B_\lambda}{|e^{ik} - e^{i\lambda}|^2}. \quad (\text{S.41})$$

The above expressions can be written in a more suggestive form as follows. We first observe that

$$\begin{aligned} \left| \oint \frac{d\lambda}{\pi} \frac{A_\lambda}{e^{ik} - e^{i\lambda}} \right|^2 &= \left(\frac{n}{2\pi} \right)^2 + \left| \oint \frac{d\lambda}{2\pi} A_\lambda \cot \frac{k - \lambda}{2} \right|^2, \\ \left| \oint \frac{d\lambda}{\pi} \frac{B_\lambda}{e^{ik} - e^{i\lambda}} \right|^2 &= \left| \oint \frac{d\lambda}{2\pi} B_\lambda \cot \frac{k - \lambda}{2} \right|^2, \end{aligned} \quad (\text{S.42})$$

where we have used the fact the fermion density is given by $n = \int dk A_k$, and $\int dk B_k = 0$ together with the identity $\frac{1}{1 - e^{i\lambda}} = \frac{1}{2} (1 + i \cot \frac{\lambda}{2})$. We can then write Eq. (S.40) as

$$F_A(k) = A_k + \frac{n^2}{2\pi} - 2\pi \left((A_k)^2 - \left| \oint \frac{d\lambda}{2\pi} A_\lambda \cot \left(\frac{p - \lambda}{2} \right) \right|^2 - (A \rightarrow B) \right) + 2n \oint \frac{d\lambda}{2\pi} \frac{A_k - A_\lambda}{1 - \cos(k - \lambda)}. \quad (\text{S.43})$$

Furthermore, we note that

$$-\partial_p \cot \left(\frac{p - \lambda}{2} \right) = \frac{1}{1 - \cos(p - \lambda)}. \quad (\text{S.44})$$

The last term in Eq. (S.43) is the *Hadamard* finite part and can be written as

$$\oint \frac{d\lambda}{2\pi} \frac{A_p - A_\lambda}{1 - \cos(p - \lambda)} = \partial_p \oint \frac{d\lambda}{2\pi} A_\lambda \cot \left(\frac{p - \lambda}{2} \right). \quad (\text{S.45})$$

Putting these equations together and using the definition of the Hilbert transform, we recover the first line of Eq. (8) in the main text. One can similarly simplify the expression for $F_B(k)$. We note that

$$F_B(k) = B_k - 4\pi \left(A_k B_k - \oint \frac{d\lambda}{2\pi} A_\lambda \cot \frac{k - \lambda}{2} \oint \frac{d\lambda'}{2\pi} B_{\lambda'} \cot \frac{k - \lambda'}{2} \right) + 2n \oint \frac{d\lambda}{2\pi} \frac{B_k - B_\lambda}{1 - \cos(k - \lambda)}, \quad (\text{S.46})$$

and the last term can be written in a similar fashion to Eq. (S.45). One can then similarly recover the second line of Eq. (8) of the main text.

4. Analytic continuation

Next we consider the analytic continuation to the unit disk via

$$\mathcal{G}_A(z) = \int_0^{2\pi} \frac{dk}{2\pi} A_k \frac{e^{ik} + z}{e^{ik} - z} \Rightarrow \mathcal{G}_A(z) \text{ is analytic in } |z| < 1, \text{ and } \text{Re } \mathcal{G}_A(e^{ik-0^+}) = A_k. \quad (\text{S.47})$$

A Kramers-Kronig relation follows as well:

$$\text{Im } \mathcal{G}_A(e^{ik-0^+}) = \int_0^{2\pi} \frac{d\lambda}{2\pi} A(\lambda) \cot\left(\frac{k-\lambda}{2}\right). \quad (\text{S.48})$$

We first consider Eq. (S.43) and focus on the expression in parentheses (considering just the A term). Using the analytic properties of $\mathcal{G}_A(z)$, one can easily identify it as the real part of $(\mathcal{G}_A(z = e^{ip-0^+}))^2$. The B term in parentheses can be treated identically. Finally, the last term, written in terms of the partial derivative, can be identified as the real part of $z\partial_z \mathcal{G}_A(z)$. One can similarly analytically continue the expression in Eq. (S.46) for $F_B(k)$. A similar argument shows that the expression in parentheses can be identified as the real part of the analytic function $\mathcal{G}_A(z)\mathcal{G}_B(z)$, and the last term in $F_B(k)$ can be dealt with similarly to the last term in $F_A(k)$. Putting all the different pieces together, we can then write

$$\frac{1}{\Gamma} \frac{d}{dt} \mathcal{G}_A(t, z) = -[\mathcal{G}_A + \frac{n^2}{2\pi} - 2\pi\mathcal{G}_A^2 + 2\pi\mathcal{G}_B^2 + 2nz\partial_z \mathcal{G}_A] + \mathcal{G}_{\Lambda_1}(z), \quad (\text{S.49a})$$

$$\frac{1}{\Gamma} \frac{d}{dt} \mathcal{G}_B(t, z) = -[\mathcal{G}_B - 4\pi\mathcal{G}_A\mathcal{G}_B + 2nz\partial_z \mathcal{G}_B] + \mathcal{G}_{\Lambda_2}(z). \quad (\text{S.49b})$$

Note that the Lagrange multiplier terms are also promoted to analytic functions corresponding to the functions $\Lambda_1(k) = \Lambda_k \sin k$ and $\Lambda_2(k) = -\Lambda_k(h - \cos k)$. The above equations can be simplified by defining

$$\mathcal{G}_{\pm}(z) = \mathcal{G}_A(z) \mp i\mathcal{G}_B(z). \quad (\text{S.50})$$

Now combining the two equations in Eq. (S.49), we find

$$\frac{1}{\Gamma} \frac{d}{dt} \mathcal{G}_{\pm}(t, z) = -\left[\mathcal{G}_{\pm} + \frac{n^2}{2\pi} - 2\pi\mathcal{G}_{\pm}^2 + 2nz\partial_z \mathcal{G}_{\pm}\right] \pm ih\mathcal{G}_{\Lambda}(z) \mp i\mathcal{G}_{\Lambda_{\pm}}(z), \quad (\text{S.51})$$

where we have defined $\Lambda_{\pm}(k) \equiv e^{\pm ik}\Lambda_k$. Note that, apart from the constraint, the equations for \mathcal{G}_{\pm} are now decoupled from each other. Next we observe

$$\mathcal{G}_{\Lambda_{\pm}} = i\lambda_1 + z^{\pm 1}\mathcal{G}_{\Lambda}(z) \quad \text{with} \quad \lambda_1 = \int dk \sin k \Lambda_k \quad (\text{S.52})$$

where we have also used the fact that Λ_k is odd in k (while A_k is even, and B_k is also odd).

Next, we expand Eq. (S.51) to the zeroth order in powers of z . It turns out that both equations give the relation

$$\dot{n} = -n + \lambda_1, \quad \text{where} \quad \lambda_1 = \int dk \Lambda(k) \sin k. \quad (\text{S.53})$$

With this equation together with Eq. (S.52), we can write Eq. (S.51) as

$$\frac{1}{\Gamma} \frac{d}{dt} \left(\mathcal{G}_{\pm} \mp \frac{n}{2\pi} \right) = \pm \frac{n}{2\pi} - \left[\mathcal{G}_{\pm} + \frac{n^2}{2\pi} - 2\pi\mathcal{G}_{\pm}^2 + 2nz\partial_z \mathcal{G}_{\pm} \right] \pm i(h - z^{\pm 1})\mathcal{G}_{\Lambda}(z). \quad (\text{S.54})$$

This motivates a change of variables for $\mathcal{G}_{\pm} \rightarrow \mathcal{G}_{\pm} \pm n/(2\pi)$. The above equation can be then written as

$$\frac{1}{\Gamma} \frac{d}{dt} \mathcal{G}_{\pm} = -[\mathcal{G}_{\pm} \mp 2n\mathcal{G}_{\pm} - 2\pi\mathcal{G}_{\pm}^2 + 2nz\partial_z \mathcal{G}_{\pm}] \pm i(h - z^{\pm 1})\mathcal{G}_{\Lambda}(z). \quad (\text{S.55})$$

Thus, we recover the main result of the paper stated in Eq. (10) of the main text.

5. Multipole expansion

For a numerical solution, we consider a series expansion

$$\begin{aligned} A_k &= \sum_{n=0}^{\infty} A_n \cos nk, & B_k &= \sum_{n=1}^{\infty} B_n \sin nk, \\ A_n &= c_n \int_0^{2\pi} \frac{dk}{\pi} \cos(kn) A_k, & B_n &= \int_0^{2\pi} \frac{dk}{\pi} \sin(kn) B_k, \end{aligned} \quad (\text{S.56})$$

with the coefficient $c_n = 1/2$ for $n = 0$ and $c_n = 1$ for $n \geq 1$. The above series expansion corresponds to the multipole expansion of the analytic functions:

$$\mathcal{G}_A(z) = \sum_{n=0}^{\infty} A_n z^n, \quad \mathcal{G}_B(z) = -i \sum_{n=1}^{\infty} B_n z^n. \quad (\text{S.57})$$

Finally in terms of the analytic functions \mathcal{G}_{\pm} , we have

$$\mathcal{G}_+ = \sum_{n \geq 1}^{\infty} (A_n - B_n) z^n, \quad \mathcal{G}_- = 2A_0 + \sum_{n \geq 1}^{\infty} (A_n + B_n) z^n, \quad \text{with} \quad A_0 = \frac{n}{2\pi}. \quad (\text{S.58})$$

Finally we point out that the coefficients A_n, B_n are constrained by Eq. (S.34)

$$\begin{aligned} A_0 - \frac{1}{4\pi} &= - \sum_{m=1}^{\infty} B_{2m} + h \sum_{m=0}^{\infty} B_{2m+1}, \\ A_n &= -B_n + \sum_{\substack{m > n \\ m+n \text{ even}}}^{\infty} -2B_m + \sum_{\substack{m > n \\ m+n \text{ odd}}}^{\infty} 2hB_m, \quad n \geq 1. \end{aligned} \quad (\text{S.59})$$

To numerically solve these equations, we first eliminate the function $\mathcal{G}_{\Lambda}(z)$ in Eq. (9) to obtain (setting $\Gamma = 1$):

$$\frac{d}{dt} [(-h + \frac{1}{z})\mathcal{G}_+ - (h - z)\mathcal{G}_-] = -(-h + \frac{1}{z})[1 - 2\pi\mathcal{G}_+ - 2n + 2nz\partial_z]\mathcal{G}_+ + (h - z)[1 - 2\pi\mathcal{G}_- + 2n + 2nz\partial_z]\mathcal{G}_-. \quad (\text{S.60})$$

Next, with the aid of the condition in Eq. (S.59), Eq. (S.60) is then expanded in powers of z to find the coefficients B_n which can be used to write all non-vanishing fermionic correlation functions. More precisely, we can write the correlations between two fermions at a distance l using the inverse Fourier transformation

$$\langle \hat{c}_i^\dagger \hat{c}_{i+l} \rangle = \frac{1}{2\pi} \int_0^{2\pi} dk e^{ikl} \langle \hat{c}_k^\dagger \hat{c}_k \rangle = \int_0^{2\pi} dk \cos kl [A_0 + \sum_{n=1}^{\infty} A_n \cos kn] = 2\pi A_0 \delta_{l,0} + \pi \sum_{n=1}^{\infty} A_n \delta_{|l|,n}. \quad (\text{S.61})$$

Similarly,

$$\langle \hat{c}_i \hat{c}_{i+l} \rangle = \text{sgn}(l) \pi \sum_{n=1}^{\infty} B_n \delta_{|l|,n}. \quad (\text{S.62})$$

We solve Eq. (S.60) by time-evolving B_n starting from an infinite temperature state [$A_k(t=0) = 1/4\pi$, $B_k(t=0) = 0$] and keeping up to $n_{\max} = 80$ terms in the expansion.

S.V. Spin correlations from fermions

Under the Jordan-Wigner transformation defined by $\sigma_j^- = e^{i\pi \sum_{i \geq 1}^j \hat{c}_i^\dagger \hat{c}_i} \hat{c}_j$, we can write the spin correlator $\langle \sigma_i^x \sigma_j^x \rangle$ (assuming $j > i$) as

$$\begin{aligned} \langle \sigma_i^x \sigma_j^x \rangle &= \left\langle \hat{N}_i \left(\prod_{i < l < j} \hat{M}_l \hat{N}_l \right) \hat{M}_j \right\rangle \\ &= \left\langle \hat{N}_i \hat{M}_{i+1} \hat{N}_{i+1} \dots \hat{M}_{j-1} \hat{N}_{j-1} \hat{M}_j \right\rangle, \end{aligned} \quad (\text{S.63})$$

where we define

$$\hat{M}_i \equiv \hat{c}_i^\dagger + \hat{c}_i, \quad \hat{N}_i \equiv \hat{c}_i^\dagger - \hat{c}_i. \quad (\text{S.64})$$

as well as the fermionic correlations

$$\begin{aligned} S_{i,j} &\equiv \langle \hat{N}_i \hat{N}_j \rangle, \\ Q_{i,j} &\equiv \langle \hat{M}_i \hat{M}_j \rangle, \\ G_{i,j} &\equiv \langle \hat{N}_i \hat{M}_j \rangle. \end{aligned} \quad (\text{S.65})$$

Since both function A_k and B_k are real, one can see that the correlation functions $Q_{i,j} = -S_{i,j} = \delta_{i,j}$ for any i and j . The spin correlations at a distance R are then given by the determinant

$$\langle \sigma_i^x \sigma_{i+R}^x \rangle = \begin{vmatrix} G_{-1} & G_{-2} & \dots & \dots & \dots & \dots & G_{-R} \\ G_0 & G_{-1} & \dots & \dots & \dots & \dots & G_{-R+1} \\ 0 & \ddots & \ddots & & & & \\ \vdots & & \ddots & & & & \ddots \\ G_{R-2} & G_{R-3} & \dots & \dots & \dots & \dots & G_{-1} \end{vmatrix}, \quad (\text{S.66})$$

where

$$G_l = 2 \operatorname{Re} \left\{ \langle \hat{c}_i^\dagger \hat{c}_{i+l} \rangle - \langle \hat{c}_i \hat{c}_{i+l} \rangle \right\} - \delta_{l,0}. \quad (\text{S.67})$$

Alternatively, using the expressions Eqs. (S.61) and (S.62), we have

$$\begin{aligned} G_0 &= 4\pi A_0 - 1, \\ G_l &= 2\pi(A_{|l|} - \operatorname{sgn}(l)B_{|l|}) \quad \text{for } l \neq 0. \end{aligned} \quad (\text{S.68})$$

S.VI. Contrast against free fermions & quench dynamics

In this section, we provide a solution to the quadratic fermionic problem where the Lindblad operators take the form of independent fermionic losses, $\hat{L}_i = \hat{c}_i$, can be solved exactly assuming periodic boundary conditions $\hat{c}_{L+1} = \hat{c}_1$; our presentation closely follows Ref. [S10]. Applying a Fourier transformation to H_o in Eq. (S.22), we get

$$\hat{H}_o = 2 \sum_{k>0} (h - \cos k) [\hat{c}_k^\dagger \hat{c}_k + \hat{c}_{-k}^\dagger \hat{c}_{-k}] - i \sin k [\hat{c}_k \hat{c}_{-k} + \hat{c}_k^\dagger \hat{c}_{-k}^\dagger]. \quad (\text{S.69})$$

For a single momentum mode, the basis is spanned by the states

$$\begin{aligned} |\phi_1^k\rangle &= |0,0\rangle, \\ |\phi_2^k\rangle &= \hat{c}_k^\dagger |0,0\rangle = |k,0\rangle, \\ |\phi_3^k\rangle &= \hat{c}_{-k}^\dagger |0,0\rangle = |0,-k\rangle, \\ |\phi_4^k\rangle &= \hat{c}_{-k}^\dagger \hat{c}_k^\dagger |0,0\rangle = |k,-k\rangle, \end{aligned} \quad (\text{S.70})$$

where we can write the Hamiltonian on the form $\hat{H}_o = \sum_{k>0} \hat{h}_k$, as we define the operators

$$\hat{c}_k = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{c}_{-k} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{h}_k = 2 \begin{pmatrix} -(h - \cos k) & 0 & 0 & -i \sin k \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i \sin k & 0 & 0 & (h - \cos k) \end{pmatrix}. \quad (\text{S.71})$$

Since the modes k are decoupled, we can then write a Lindblad master equation for each mode

$$\begin{aligned} \frac{d\hat{\rho}_k}{dt} &= -i [\hat{h}_k, \hat{\rho}_k] + \mathcal{D}_k(\hat{\rho}_k), \\ \mathcal{D}_k(\hat{\rho}_k) &= \Gamma (\hat{c}_k \hat{\rho}_k \hat{c}_k^\dagger - \frac{1}{2} \{ \hat{c}_k^\dagger \hat{c}_k, \hat{\rho}_k \} + \hat{c}_{-k} \hat{\rho}_k \hat{c}_{-k}^\dagger - \frac{1}{2} \{ \hat{c}_{-k}^\dagger \hat{c}_{-k}, \hat{\rho}_k \}). \end{aligned} \quad (\text{S.72})$$

In the steady state, we find the only non-vanishing elements of the density matrix to be

$$\hat{\rho}_k(t \rightarrow \infty) = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & 0 & 0 \\ 0 & 0 & \rho_{33} & 0 \\ \rho_{14}^* & 0 & 0 & \rho_{44} \end{pmatrix}, \quad (\text{S.73})$$

where, with the aid of the condition $\text{Tr}\{\rho_k\} = 1$, we find

$$A_k = \frac{1}{2\pi}(\rho_{22} + \rho_{44}) = \frac{1}{4\pi} \frac{\sin^2 k}{1 + h^2 - 2h \cos k + (\frac{\Gamma}{4})^2}, \quad (\text{S.74a})$$

$$B_k = \frac{1}{2\pi i} \rho_{14}^* = \frac{-1}{4\pi} \frac{\sin k(h - \cos k + \frac{i\Gamma}{4})}{1 + h^2 - 2h \cos k + (\frac{\Gamma}{4})^2}. \quad (\text{S.74b})$$

Alternatively, in the limit $\Gamma \rightarrow 0$, we can simply drop the nonlinear terms in Eq. (S.35) and just keep the leading linear term in $F_{A,B}$. Then the steady state is described by

$$\frac{dA_k}{dt} = -A_k + \Lambda_k \sin \theta_k = 0, \quad (\text{S.75a})$$

$$\frac{dB_k}{dt} = -B_k - \Lambda_k \cos \theta_k = 0, \quad (\text{S.75b})$$

which then yields

$$A_k \cos \theta_k + B_k \sin \theta_k = 0. \quad (\text{S.76})$$

Together with the constraint in Eq. (S.33), we can solve the two equations for A_k , and B_k in the steady state which yields

$$A_k = \frac{1}{2} \sin^2 \theta_k, \quad B_k = -\frac{1}{2} \sin \theta_k \cos \theta_k. \quad (\text{S.77})$$

Upon substituting for θ_k in terms of h , we find

$$A_k = \frac{1}{4\pi} \frac{\sin^2 k}{1 + h^2 - 2h \cos k}, \quad (\text{S.78a})$$

$$B_k = \frac{-1}{4\pi} \frac{\sin k(h - \cos k)}{1 + h^2 - 2h \cos k}. \quad (\text{S.78b})$$

These equations simply reproduce Eq. (S.78) in the limit $\Gamma \rightarrow 0$.

We can also find these expressions in terms of the occupation of the Bogoliubov fermions. Recalling Eq. (S.32), we can identify the occupation number of Bogoliubov fermions as

$$N_k = \frac{1}{2} + p_k = \frac{1}{2}(1 - \cos \theta_k) \quad (\text{S.79})$$

It turns out that the result is identical to a quench from the ground state corresponding to $h_0 = \infty$ (corresponding to a state with all spins pointing up) to h [S11, S12]. The correlation function in this model is described by [S11]

$$C_r = \begin{cases} 2^{-r} \cos[r \arccos h], & h \leq 1 \\ (2h)^{-r} & h \geq 1 \end{cases} \quad (\text{S.80})$$

Therefore, the correlation length is given by

$$\xi = \begin{cases} 1/\log 2, & h \leq 1 \\ 1/\log(2h) & h \geq 1 \end{cases} \quad (\text{S.81})$$

While the correlation length does not change below the critical point, we note the oscillatory nature of the correlation function. In some sense, the correlations are *best formed* at the critical point [S11].

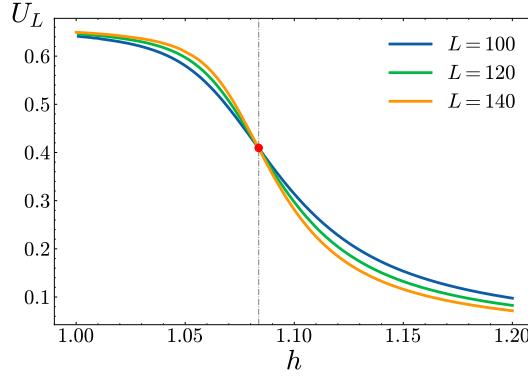


FIG. S.2. The Binder cumulant U_L calculated in the ground state of \hat{H}_2 with $J_2 = 0.05$ for multiple system sizes. The red dot marks, h_c , the point where the curves cross.

S.VII. Quantum Critical Point for Non-Integrable Hamiltonians

For the non-integrable next-nearest-neighbor Ising model considered in Fig. 3 in the main text, we resort to finding the quantum critical point, h_c , by calculating the Binder cumulant [S13] in the ground state

$$U_L = 1 - \frac{\langle S_x^4 \rangle_L}{3 \langle S_x^2 \rangle_L^2}, \quad (\text{S.82})$$

where $S_x = \sum_{i=1}^L \sigma_i^x$ is the total magnetization operator on the chain. The ground state is obtained using standard MPS techniques for $100 \leq L \leq 140$. The critical point h_c is where the different curves for large system sizes.

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