PATHOLOGY AND ASYMMETRY: CENTRALIZER RIGIDITY FOR PARTIALLY HYPERBOLIC DIFFEOMORPHISMS

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Abstract. We discover a rigidity phenomenon within the volume-preserving partially hyperbolic diffeomorphisms with 1-dimensional center. In particular, for smooth, ergodic perturbations of certain algebraic systems – including the discretized geodesic flows over hyperbolic manifolds and certain toral automorphisms with simple spectrum and exactly one eigenvalue on the unit circle, the smooth centralizer is either virtually $\mathbb{Z}^\ell$ or contains a smooth flow.

At the heart of this work are two very different rigidity phenomena. The first was discovered in [2, 3]: for a class of volume-preserving partially hyperbolic systems including those studied here, the disintegration of volume along the center foliation is either equivalent to Lebesgue or atomic. The second phenomenon is the rigidity associated to several commuting partially hyperbolic diffeomorphisms with very different hyperbolic behavior transverse to a common center foliation [25].

We employ a variety of techniques, among them a novel geometric approach to building new partially hyperbolic elements in hyperbolic Weyl chambers using Pesin theory and leafwise conjugacy, measure rigidity via thermodynamic formalism for circle extensions of Anosov diffeomorphisms, partially hyperbolic Livšic theory, and nonstationary normal forms.

To the memory of Anatole Katok.

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1. Introduction

The *centralizer* of a diffeomorphism $f\colon M \to M$ is the set of diffeomorphisms g that commute with f under composition: $f \circ g = g \circ f$. Put another way, the centralizer of f is the group of symmetries of f, where "symmetries" is meant in the classical sense: coordinate changes that leave the dynamics of the system unchanged. The centralizer of

f always contains the integer powers of f and typically not more, at least conjecturally [82, 83]. By contrast, a diffeomorphism belonging to a smooth flow has large centralizer, containing a 1-dimensional Lie group.

To date, the study of smooth centralizers has mainly focused in two directions: showing that the typical map commutes only with its powers; and classifying the manifolds and/or dynamics that can support abelian centralizers of sufficiently high rank. In this paper we aim at describing the centralizers of *all diffeomorphisms* in a small neighborhood of a given map, for specific classes of maps. This relates to one of the classical questions in perturbation theory: if a diffeomorphism belongs to a smooth flow, which perturbations also belong to a smooth flow? We answer this question fully for algebraic geodesic flows in negative curvature in conservative setting.

More generally, we start with certain diffeomorphisms with *exceptionally* large centralizer – containing a 1-dimensional Lie group – and consider what happens when these diffeomorphisms are perturbed. We find that for such perturbed systems, if the centralizer gets large enough, as measured by the rank of its abelianization, then in fact it must be exceptionally large.

To fix notation, let $\mathcal{G}$ be a group: our central example will be the space $\mathrm{Diff}^r(M)$ of C^r diffeomorphisms of a closed manifold M under composition. For $f \in \mathcal{G}$, denote by $\mathcal{Z}_{\mathcal{G}}(f)$ the centralizer of f in $\mathcal{G}$:

$$\mathcal{Z}_{\mathcal{G}}(f) := \{g \in \mathcal{G} \,:\, gf = fg\}.$$

We say that $f \in \mathcal{G}$ has *trivial centralizer in* $\mathcal{G}$ if the centralizer of f consists of the iterates of f:

$$\mathcal{Z}_{\mathcal{G}}(f) =< f >:= \{f^n \,:\, n \in \mathbb{Z}\} \cong \mathbb{Z},$$

and *virtually trivial centralizer* if $\mathcal{Z}_{\mathcal{G}}(f)$ contains $< f >$ as a finite index subgroup.[1]

For $f \in \mathrm{Diff}^r(M)$ and M fixed, we will use the shorthand notation $\mathcal{Z}_r(f) := \mathcal{Z}_{\mathrm{Diff}^r(M)}(f)$. If $f \in \mathrm{Diff}^r_{\mathrm{vol}}(M)$ is a volume-preserving element of $\mathrm{Diff}^r(M)$, then we denote $\mathcal{Z}_{r,\mathrm{vol}}(f) := \mathcal{Z}_{\mathrm{Diff}^r_{\mathrm{vol}}(M)}(f)$. It is not hard to see (see Lemma 11) that if $f \in \mathrm{Diff}^r_{\mathrm{vol}}(M)$ is ergodic with respect to volume, then $\mathcal{Z}_r(f) = \mathcal{Z}_{r,\mathrm{vol}}(f)$. For $r = 1$ we drop r in the notation: $\mathrm{Diff}^1(M) = \mathrm{Diff}(M)$.

Discretized geodesic flows. The context in which our main results are easiest to state and prove is that of perturbations of discretized geodesic flows in negative curvature. Let X be a closed, negatively curved, locally symmetric manifold, for example, a compact hyperbolic manifold. Denote by T^1X the unit tangent bundle of X and by ψ_t the geodesic flow $\psi_t : T^1X \to T^1X$ over X. The flow ψ_t preserves the canonical Liouville probability measure on T^1X, which we denote by $\mathrm{vol} = \mathrm{vol}_{T^1X}$. Any element ψ_t of this flow commutes with any other element, and thus

$$\mathcal{Z}_\infty(\psi_t) \supseteq \{\psi_s : s \in \mathbb{R}\} \cong \mathbb{R}.$$

Our first result concerns volume-preserving perturbations of the *discretized flow:* the time-t_0 map ψ_{t_0}, for a fixed $t_0 \neq 0$. Such a perturbation $f \in \mathrm{Diff}^\infty_{\mathrm{vol}}(T^1X)$ will not necessarily

[1] If general, one says that a property holds *virtually* for a group G if G contains a finite index subgroup H with that property.

embed in a flow: for example, any perturbation with a hyperbolic periodic point cannot embed in a flow, and such perturbations are plentiful. The upshot of this result is that if such a perturbation does *not* embed in a flow, then it has virtually trivial centralizer.

Theorem 1. *Let X be a closed, negatively curved, locally symmetric manifold, and let $\psi_t\colon T^1X \to T^1X$ be the associated geodesic flow. Fix $t_0 \neq 0$, and suppose $f \in \mathrm{Diff}^\infty_{\mathrm{vol}}(T^1X)$ is a C^1-small perturbation of ψ_{t_0}. Then either f has virtually trivial centralizer in $\mathrm{Diff}^\infty(T^1X)$ or f embeds into a smooth, volume preserving flow (and thus $\mathcal{Z}_\infty(f) \supseteq \mathbb{R}$). Moreover, in the latter case, the centralizer $\mathcal{Z}_\infty(f)$ is virtually $\mathbb{R}$.*

The conclusions of Theorem 1 hold in considerably greater generality; see Theorem 3 and Remark 4. In particular, X can be any closed Riemannian manifold with pointwise 1/4-pinched negative curvature (such as a surface), or more generally any closed, negatively curved manifold whose geodesic flow satisfies either a 2-bunched or narrow band spectrum condition.

Thus for perturbations of these flows, up to finite index subgroups, the centralizer is either $\mathbb{Z}$ or $\mathbb{R}$. We do not know whether the same result holds for perturbations of discretized Anosov flows in general.

Question 1. *Do the same conclusions of Theorem 1 hold for the volume-preserving perturbations of the time-t_0 map of an arbitrary volume-preserving Anosov flow?*

A partial answer to this question has recently been found in dimension 3 by Barthelmé and Gogolev [4].

We remark that *virtually trivial* cannot be replaced by *trivial* in the conclusion of Theorem 1. Indeed for any $t_0 \in \mathbb{R}$, Burslem shows in [16, Theorem 1.3] that the time-$t_0/2$ map $\psi_{t_0/2}$ can be C^∞ approximated by $f \in \mathrm{Diff}^\infty_{\mathrm{vol}}(T^1X)$ with trivial centralizer. Then map f^2 has virtually trivial, but not trivial, centralizer and C^∞-approximates ψ_{t_0}.

Toral automorphisms. Linear automorphisms of tori present a rich family of algebraic systems with notable rigidity properties. Any orientation-preserving automorphism of the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ lifts to a linear automorphism of $\mathbb{R}^d$ preserving $\mathbb{Z}^d$, which can be represented by a matrix $C \in \mathrm{SL}(d,\mathbb{Z})$. For such a matrix C we write $T_C\colon \mathbb{T}^d \to \mathbb{T}^d$ to denote the associated toral automorphism. Since C has determinant 1, the map T_C preserves the Lebesgue-Haar measure on $\mathbb{T}^d$, which we again denote by $\mathrm{vol}(= \mathrm{vol}_{\mathbb{T}^d})$.

In the *hyperbolic* case where C has no eigenvalues on the unit circle, the automorphism T_C has a strong topological rigidity property known as structural stability: any perturbation of T_C in $\mathrm{Diff}^1(\mathbb{T}^d)$ is topologically conjugate to T_C. The centralizer of a perturbation $f \in \mathrm{Diff}^1(\mathbb{T}^d)$ *within* $\mathrm{Homeo}^+(\mathbb{T}^d)$ is thus isomorphic to the centralizer of T_C in $\mathrm{Homeo}^+(\mathbb{T}^d)$. It is well-known that when C is irreducible — meaning that its characteristic polynomial is irreducible over $\mathbb{Z}$ – both $\mathcal{Z}_{\mathrm{Homeo}^+(\mathbb{T}^d)}(T_C)$ and $\mathcal{Z}_{\mathrm{SL}(d,\mathbb{Z})}(C)$ are virtually finitely generated free abelian groups (see Lemma 15 or Proposition 3.7 in [45]). Of course for a perturbation $f \in \mathrm{Diff}^r(\mathbb{T}^d)$ of T_C, the centralizer $\mathcal{Z}_r(f)$ can be considerably smaller than $\mathcal{Z}_{\mathrm{Homeo}^+(\mathbb{T}^d)}(f)$: in fact, Palis and Yoccoz showed that, among the smooth Anosov

diffeomorphisms, there exists an open and dense subset of $f \in \mathrm{Diff}^\infty(\mathbb{T}^d)$ such that the centralizer $\mathcal{Z}_\infty(f)$ is trivial [60, 61].

From a dynamical point of view, perturbations of the non-hyperbolic automorphisms are considerably more interesting. When C has no eigenvalues that are roots of unity, then T_C is mixing with respect to vol, and in several cases of interest, *stably mixing:* any sufficiently smooth, volume-preserving perturbation of T_C is mixing if $d \leq 5$ [70].

We consider a case in which both structural stability and ergodicity are violated in a fairly dramatic fashion, where the generating matrix $C \in \mathrm{SL}(d, \mathbb{Z})$ has 1 as an eigenvalue, with multiplicity 1.[2] By conjugating by a toral automorphism, we may assume without loss of generality that $C = \begin{pmatrix} A & \\ & 1 \end{pmatrix}$. For such A, the map $T_C = T_A \times \mathrm{id}_{\mathbb{T}}$ admits non-conjugate *affine* perturbations of the form $f = T_A \times R_\theta$, where $R_\theta(z) = z + \theta$ is a rotation by $\theta \in \mathbb{T}$ in last factor in $\mathbb{T}^d = \mathbb{T}^{d-1} \times \mathbb{T}$, and so T_C is not structurally stable, even within the restricted class of affine transformations. By the same token, these affine perturbations also have large centralizer, commuting with any affine map of the form $T_B \times R_\theta$, with $B \in \mathcal{Z}_{\mathrm{SL}(d-1,\mathbb{Z})}(A)$, and $\theta \in \mathbb{T}$.

In the case that A is irreducible, the group $\mathcal{Z}_{\mathrm{SL}(d-1,\mathbb{Z})}(A)$ is virtually abelian and its rank is precisely the number $\ell_0 = \ell_0(A)$ which is defined as follows (see also Lemma 15 or Proposition 3.7 in [45]).

Definition 1. For an irreducible element A in $\mathrm{SL}(d-1, \mathbb{Z})$ define $\ell_0(A) := r + c - 1$, where r is the number of real eigenvalues of A and c is the number of pairs of complex eigenvalues of A.

We obtain the following classification result for the centralizer of perturbations of T_C.

Theorem 2. *Let $f \in \mathrm{Diff}^\infty_{\mathrm{vol}}(\mathbb{T}^d)$ be a C^1-small, ergodic perturbation of $T_A \times \mathrm{id}_{\mathbb{T}}$, where $A \in \mathrm{SL}(d-1, \mathbb{Z})$ is hyperbolic and irreducible. Let $\ell_0 = \ell_0(A)$. Then one of the following holds:*

(1) $\mathcal{Z}_\infty(f)$ is virtually $\mathbb{Z}^\ell$ for some $\ell \in [1, \ell_0]$. Furthermore, $\ell < \ell_0$ if $\ell_0 > 1$.
(2) $\mathcal{Z}_\infty(f)$ is virtually $\mathbb{Z} \times \mathbb{T}$.
(3) $\mathcal{Z}_\infty(f)$ is virtually $\mathbb{Z}^{\ell_0} \times \mathbb{T}$, $\ell_0 > 1$ and f is C^∞ conjugate to $T_A \times R_\theta$, $\theta \notin \mathbb{Q}/\mathbb{Z}$.

Remark 1. Theorem 2 has a stronger formulation for perturbations of isometric extensions of an irreducible toral automorphism, stated in Theorem 4 in the next section. For similar problems on nilmanifolds, cf. our upcoming paper [23].

Remark 2. Consider the simplest non-ergodic example of $f = T_A \times \mathrm{id}$ itself, for which $\mathcal{Z}_\infty(f)$ is virtually $\mathbb{Z}^{\ell_0} \times \mathrm{Diff}^\infty(\mathbb{T})$. This example illustrates the *a priori* possibility that the centralizer might not be virtually abelian, and thus part of the work in Theorem 2 is to establish that for an *ergodic* perturbation, the centralizer is virtually abelian. In particular, this shows that the ergodicity assumption in Theorem 2 is necessary. On the other hand, the ergodicity assumption is satisfied generically: it is proved by Burns

[2]The case where $C \in \mathrm{SL}(d, \mathbb{Z})$ has exactly one eigenvalue of modulus 1 can be treated by similar methods.

and Wilkinson in [13] and F. Rodriguez Hertz, M. A. Rodriguez Hertz and Ures in [72] that ergodicity (indeed, mixing) holds open and densely among the *partially hyperbolic diffeomorphisms with* 1*-dimensional center* in $\mathrm{Diff}^\infty_{\mathrm{vol}}(\mathbb{T}^d)$ (for precise definitions and more details, see Section 3.4). In particular, for any neighborhood U of T_C, there is a C^1-open set $U_0 \subset U$ such that every $f \in U_0$ is ergodic.

We conjecture that for any volume preserving (possibly non-ergodic) C^1-small perturbation f of $T_A \times \mathrm{id}$, the group $\mathcal{Z}_\infty(f)$ is either virtually trivial or contains a nontrivial Lie group.

Remark 3. We expect that the conclusions in Theorem 2 extend to the case when T_A is *reducible* hyperbolic toral automorphism as well. Moreover, we conjecture that for general hyperbolic T_A, the conclusion (1) should read: for every $g \in \mathcal{Z}_\infty(f)$, and on any $< f, g >$-invariant subtorus of $\mathbb{T}^d$, the action of $< f, g >$ is virtually a $\mathbb{Z}$-action.

Both Theorems 1 and 2 are consequences of more general results, which we state in Section 2.

The secret sauce. While it does not appear in the statements, there is a hidden concept behind Theorems 1 and 2: pathological foliations. Both the discretized geodesic flows and the toral automorphisms we discuss above preserve smooth, 1-dimensional foliations, in the first case, the foliation by orbits of the flow, and in the second, the foliation by circles tangent to the last factor in $\mathbb{T}^{d-1} \times \mathbb{T}$.

Transverse to the leaves of these foliations, the dynamics is hyperbolic, and so the theory of normally hyperbolic foliations developed in [35] applies. In particular, the perturbations of these examples considered in Theorems 1 and 2 also preserve 1-dimensional foliations with smooth leaves, homeomorphic as foliations to the unperturbed smooth foliations (see Section 3.4 for a detailed discussion). The measure-theoretic properties of these *center* foliations are well-studied and play a key role in our proofs.

By a standard procedure, the volume vol can be locally disintegrated along the leaves of a foliation $\mathcal{F}$ to obtain in each foliation chart a measurable family of measures, supported on the local leaves (or *plaques*) $\mathcal{F}_{loc}$ of the foliation. Each plaque $\mathcal{F}_{loc}(x)$ of a foliation, being a C^1 embedded disk, also carries a natural measure class $\mathrm{vol}_{\mathcal{F}_{loc}(x)}$ associated to leafwise volume, or length in the case of 1-dimensional leaves. If the foliation is C^1 (i.e. has C^1 foliation charts), then the disintegration of vol and leafwise volume are equivalent, meaning they have the same sets of measure zero.

When, as is typically the case in our perturbed examples, the foliation is not C^1, anything goes, at least *a priori*. The two extremal cases are:

- *Lebesgue disintegration,* where the disintegrated and leafwise volume are equivalent. A foliation $\mathcal{F}$ of M has Lebesgue disintegration if for every set $Z \subset M$:

$$\mathrm{vol}(Z) = 0 \iff \mathrm{vol}_{\mathcal{F}_{loc}(x)}(Z) = 0, \quad \text{for vol-a.e. } x \in M.$$

- *atomic disintegration,* where the disintegrated volume is atomic. A foliation $\mathcal{F}$ of M has atomic disintegration if there exists a set $Y \subset M$ and $k \geq 1$ such that

$$\text{vol}(M \setminus Y) = 0 \text{ and } \#\{Y \cap \mathcal{F}_{loc}(x)\} \leq k, \text{ for vol-a.e. } x \in M.$$

If a foliation fails to have Lebesgue disintegration with respect to volume, we call it *pathological*, a concept first considered by Shub and Wilkinson in [81]. This concept plays an important role in our paper. In brief, pathological disintegration is associated with small centralizer and Lebesgue disintegration with large centralizer (at least in the group of homeomorphisms).

Higher rank abelian actions. Another key role in our proofs is played by *higher rank abelian* group actions with some hyperbolicity. A smooth *Anosov action* is a homomorphism $\alpha\colon G \to \text{Diff}^\infty(M)$, where G is a finitely generated abelian group, and $\alpha(a)$ is an Anosov diffeomorphism, for some $a \in G$ (see Section 2.2 for definitions of Anosov and partially hyperbolic diffeomorphisms). For example, if f is Anosov, and the smooth centralizer $\mathcal{Z}_\infty(f)$ is finitely generated and abelian, then the action of $\mathcal{Z}_\infty(f)$ on M is Anosov. This is the case, for example, when M is the torus, and f is an irreducible hyperbolic automorphism.

An Anosov action has *higher rank* if it contains an Anosov $\mathbb{Z}^2$ subaction that does not have a topological factor (possibly on a different manifold) which is virtually a $\mathbb{Z}$-action. Anosov higher-rank actions often display a range of rigidity properties (cf. [48], [50]), most strikingly *global rigidity.* Katok and Spatzier conjectured that any higher rank Anosov $\mathbb{Z}^k$ action on a compact manifold is *essentially algebraic*, i.e. smoothly conjugate to an affine action on a nilmanifold, up to a finite cover of M and up to a finite index subgroup in $\mathbb{Z}^k$. (For more on the Katok-Spatzier conjecture, see for example [26] and references therein). The conjecture was proved for Anosov actions *on nilmanifolds* by F. Rodriguez Hertz and Wang [74] (for the statement on $\mathbb{T}^d$, see Theorem 10 in Section 3.7).

In particular, if T_A is an irreducible, hyperbolic automorphism of the torus $\mathbb{T}^d$, where the centralizer of T_A is virtually $\mathbb{Z}^\ell$, for some $\ell > 1$, then the result of Rodriguez Hertz and Wang implies the following dichotomy for the centralizer of *every* sufficiently small perturbation f of T_A, when $\ell \geq 2$ (Corollary 33 in Section 3.7): *either $\mathcal{Z}_\infty(f)$ is virtually trivial, or $\mathcal{Z}_\infty(f)$ is essentially algebraic, and its rank is the same as that of $\mathcal{Z}_\infty(T_A)$.* This has been the only existing situation where the smooth centralizer is completely locally classified. Our results in Theorems 1 and 2 give classification of the centralizer for *ergodic, conservative* perturbations of certain *partially hyperbolic* systems.

One of the main achievements in [74] is showing existence of *many* independent hyperbolic elements in an action given a *single* hyperbolic element. This is also one of the main obstacles to proving the Katok-Spatzier conjecture in full generality. In [74] it is shown that any higher-rank Anosov action on a nilmanifold has Anosov elements in *every* Weyl chamber; together with [29], this proves the Katok–Spatzier conjecture on nilmanifolds, and gives the dichotomy of the centralizer as mentioned above. The proof in [74] makes use of the Franks–Manning conjugacy on nilmanifolds and fine analytic properties of the dynamics of Anosov diffeomorphisms, in particular exponential rates of mixing.

The actions considered here (as in the setting of Theorem 2), have a hyperbolic part and a 1-dimensional nonhyperbolic, central part. The hyperbolic part is, on a topological level, a maximal Anosov action – considerably simpler than the actions considered in [29, 74]. On the other hand, the methods in these works are not available to us: the central part of our actions obstruct conjugacy to a linear system, and the dynamics of the systems are potentially not even mixing. What is available instead is a *leaf conjugacy* to a linear system, that is, a topological conjugacy modulo the center dynamics. Starting from the leaf conjugacy, and using maximality of the action, we build up the partial hyperbolicity of other elements in the action. Existence of many partially hyperbolic elements in the large rank centralizer in the conservative setting forces Lebesgue disintegration of the volume in the center direction.

Our arguments are geometric rather than analytic in nature and employ a range of techniques, including the theory of normally hyperbolic foliations, rigidity of 1-dimensional solvable group actions, Weyl chamber analysis, Pesin theory, normal forms, and Livšic theory. One important idea, also employed in [12], is to use Pesin theory and uniform estimates to upgrade a uniformly expanded topological foliation $\mathcal{W}^{\#}$ to a foliation with smooth leaves. To carry out such an argument requires precise control over the Hölder exponent of leaf conjugacies, something established relatively recently in [68].

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1.2. **Structure of this paper.** In Section 2 we state our main results in the more general context of partially hyperbolic diffeomorphisms with 1 dimensional center foliations and discuss prior results. Section 3 contains background information and some new techniques used in the proofs of our main results. In Section 4, we prove the main results about discretized geodesic flows (Theorems 3 and 5). Theorem 6 provides the disintegration dichotomy which is the driver behind one of our main results, Theorem 4. The proof of Theorem 6 occupies Section 5. Finally, in Section 7, we prove Theorem 4. The Appendix contains the statement of a result from another work that we use in this paper.

2. Statements of the main results and discussion

2.1. **The general formulations.** In this section we state the following more general versions of the rigidity results for centralizers, which immediately imply Theorems 1 and 2.

Theorem 3. *Let X be a closed, negatively curved manifold, and let $\psi_t\colon T^1X \to T^1X$ be the geodesic flow. Suppose that ψ_1 satisfies either the 2-bunched or narrow band spectrum condition given in Section 3.6, Definition 8.*

Then there exists a real number $r_0 = r_0(X) \geq 1$ such that for all $r > r_0$, and any $t_0 \neq 0$, if $f \in \mathrm{Diff}^\infty_{\mathrm{vol}}(T^1X)$ is sufficiently C^1 close to ψ_{t_0}, then either $\mathcal{Z}_r(f) = \mathcal{Z}_{r,\mathrm{vol}}(f)$ is virtually trivial, or $\mathcal{Z}_s(f) = \mathcal{Z}_{s,\mathrm{vol}}(f)$ is virtually $\mathbb{R}$ for any $s \geq 1$. In the latter case f embeds into a C^∞, volume preserving flow.

Remark 4. The hypotheses of Theorem 3 are satisfied by a large class of negatively curved manifolds X. In particular:

(1) The 2-bunched condition is satisfied if X has *pointwise (strictly) 1/4-pinched curvature:* the minimum and maximum sectional curvatures $K_{\min}(x) \leq K_{\max}(x) < 0$ at $x \in X$ satisfy

$$\zeta(X) := \inf_{x \in X} \frac{K_{\max}(x)}{K_{\min}(x)} > 1/4. \tag{1}$$

(See [51, Theorem 3.2.17] and the discussion following Definition 8). This holds for example, if X is a surface. In this case $r_0(X) = \sqrt{\zeta(X)^{-1}} \in [1, 2)$.

(2) The narrow band spectrum condition is satisfied by all locally symmetric X. If X is a real hyperbolic manifold, then $r_0(X) = 1$, and if X is locally symmetric but not real hyperbolic, then $r_0(X) = 2$ (Lemma 26).

Let $g\colon \mathbb{T}^{d-1} \to \mathbb{T}^{d-1}$ be a diffeomorphism. An *isometric (circle) extension* of g is a map $f = g_\rho\colon \mathbb{T}^{d-1} \times \mathbb{T} \to \mathbb{T}^{d-1} \times \mathbb{T}$ of the form

$$g_\rho(x, y) = (g(x), y + \rho(x)),$$

where $\rho\colon \mathbb{T}^{d-1} \to \mathbb{T}$ is a continuous map taking values in the circle $\mathbb{T} = \mathbb{T}^1$. If ρ is homotopic to a constant then it can be lifted to (and hence viewed as) a map taking values in $\mathbb{R}$. The map g_ρ is a C^r diffeomorphism if and only if g and ρ are C^r and preserves volume if and only if g does.

The simplest examples of isometric extensions are products $g \times R_\theta$, where $R_\theta(y) := y + \theta$ is a rotation. In this case $\rho \equiv \theta$ is a constant function. It is easy to check that there exists $\beta\colon \mathbb{T}^{d-1} \to \mathbb{T}$ such that $\mathrm{id}_\beta \circ g_\rho = (g \times R_\theta) \circ \mathrm{id}_\beta$ if and only if ρ satisfies the cohomological equation

$$\rho = -\beta \circ g + \beta + \theta.$$

In this case we say that ρ is cohomologous to a constant θ. If $g \in \mathrm{Diff}^2_{\mathrm{vol}}(\mathbb{T}^{d-1})$ is Anosov, then g_ρ is ergodic if and only if ρ is not cohomologous to a rational constant, and g_ρ is stably ergodic if and only if ρ is not cohomologous to a constant [14].

If T_A is an irreducible hyperbolic automorphism and $(T_A)_\rho$ is ergodic, then for all $s \geq 1$, the centralizer of $(T_A)_\rho$ in $\mathrm{Diff}^s(\mathbb{T}^d)$ contains $\mathbb{Z} \times \mathbb{T}$. In addition, it contains $\mathbb{Z}^{\ell_0(A)} \times \mathbb{T}$ if ρ is C^∞ cohomologous to a constant, where $\ell_0(A) > 0$ is as in Definition 1.

[3] Our first result addresses perturbations of these maps.

Theorem 4. *Suppose $A \in \mathrm{SL}(d-1,\mathbb{Z})$ is an irreducible hyperbolic matrix. Let $\ell_0 := \ell_0(A)$ be as in Definition 1 . Then there exists $r_0 \geq 1$ such that for any $r > r_0$ and any C^∞ function $\rho_0 \colon \mathbb{T}^{d-1} \to \mathbb{R}$, if $f \in \mathrm{Diff}^\infty_{\mathrm{vol}}(\mathbb{T}^d)$ is a C^1-small, ergodic perturbation of the isometric extension $f_0 := (T_A)_{\rho_0}$, then one of the following holds for the centralizer $\mathcal{Z}_s(f) = \mathcal{Z}_{s,\mathrm{vol}}(f)$:*

1. *(Small centralizer) $\mathcal{Z}_s(f)$ is virtually $\mathbb{Z}^\ell$ for some $\ell \in [1, \ell_0]$ and any $s \geq r$. Furthermore, $\ell < \ell_0$ if $\ell_0 > 1$.*
2. *(Isometric extension) $\mathcal{Z}_s(f)$ is virtually $\mathbb{Z} \times \mathbb{T}$ for all $s \geq r$. In this case f is smoothly conjugate to a smooth isometric extension g_ρ of an Anosov diffeomorphism $g \in \mathrm{Diff}^\infty_{\mathrm{vol}}(\mathbb{T}^{d-1})$. Moreover, either g is not C^∞ conjugate to A, or ρ_0 is not C^∞ cohomologous to a constant.*
3. *(Rigidity) $\mathcal{Z}_s(f)$ is virtually $\mathbb{Z}^{\ell_0} \times \mathbb{T}$ for all $s \geq 1$. In this case, f is C^∞ conjugate to the product $T_A \times R_\theta$ with $\theta \notin \mathbb{Q}/\mathbb{Z}$.*

Remark 5. The value r_0 in Theorem 4 is explicit: $r_0 = \max(\frac{\lambda^s}{\mu^s}, \frac{\lambda^u}{\mu^u})$, where λ^u, μ^u (resp. λ^s, μ^s) are the top and bottom unstable (resp. stable) Lyapunov exponents of A.

Remark 6. In the interests of space, Theorem 4 treats only isometric extensions homotopic to $T_A \times \mathrm{id}_{\mathbb{T}}$. For the general case where $\rho_0 \colon \mathbb{T}^{d-1} \to \mathbb{T}$ is not null-homotopic, similar results hold, up to finite factors. In particular, for an ergodic perturbation f of an arbitrary isometric extension $(T_A)_{\rho_0}$, conclusions (1) and (2) are the same, and in conclusion (3), f is smoothly conjugate to an ergodic affine map isotopic to $(T_A)_{\rho_0}$.

Remark 7. In the case $\ell_0 > 1$ the conclusion (1) gives that the rank of the centralizer of a perturbation is *strictly* less than ℓ_0. We conjecture that conclusion (1) should be much stronger: the centralizer should be virtually trivial. The main obstacle in obtaining virtually trivial centralizer in this case is that several techniques we use apply currently only to the *maximal* actions [4] defined in Section 3.7, as opposed to general higher rank actions.

In sufficiently low dimension, Theorems 2 and 4 establish the truth of this conjecture and give a dichotomy between virtually trivial centralizer and large centralizer. In particular, if we assume in addition to the hypotheses of Theorem 4 that $A \in \mathrm{SL}(d-1,\mathbb{Z})$ satisfies one of the following conditions:

- $d = 3$ or 4;
- $d = 5$ and A has at least one pair of complex roots;
- $d = 6$ and A has two pairs of complex roots; or
- $d = 7$ and A has three pairs of complex roots;

[3] In fact for s large enough, the centralizer is either virtually $\mathbb{Z} \times \mathbb{T}$ or virtually $\mathbb{Z}^{\ell_0(A)} \times \mathbb{T}$, in the latter case ρ is C^∞ cohomologous to a constant. This follows from Theorem 4 but also has a more elementary proof using cocycle rigidity of the centralizer of T_A.

[4] or more generally, for totally non-symplectic (TNS) action (cf. Proposition 58).

then $\ell_0(A) = 1$ or 2, and the dichotomy in Theorem 4 reduces to the following: if $f \in \mathrm{Diff}^\infty_{\mathrm{vol}}(\mathbb{T}^d)$ is a C^1-small, ergodic perturbation of f_0, then $\mathcal{Z}_s(f)$ is either virtually trivial, virtually $\mathbb{Z} \times \mathbb{T}$ for all $s \geq r$, or virtually $\mathbb{Z}^2 \times \mathbb{T}$ for all $s \geq 1$.

Before stating the rest of the main results in this paper, we define partial hyperbolicity and some related concepts.

2.2. **Partially hyperbolic diffeomorphisms and center foliations.** Let M be a complete Riemannian manifold, and let $h \in \mathrm{Diff}(M)$. A *dominated splitting* for h is a direct sum decomposition of the tangent bundle

$$TM = E^1 \oplus E^2 \oplus \cdots \oplus E^k$$

such that

- the bundles E^i are *Dh-invariant*: for every $i \in \{1, \ldots, k\}$ and $x \in M$, we have $D_x h(E^i(x)) = E^i(h(x))$; and
- *$Dh|_{E^i}$ dominates $Dh|_{E^{i+1}}$*: there exists $N \geq 1$ such that for any $x \in M$ and any unit vectors $u \in E^{i+1}$, and $v \in E^i$:

$$\|D_x h^N(u)\| \leq \frac{1}{2} \|D_x h^N(v)\|.$$

The property of a splitting being dominated is independent of choice of equivalent metric (and independent of choice of metric in the case where M is compact). A dominated splitting is always continuous. If M is compact and h' is C^1 close to h with a dominated splitting, then h' also has a dominated splitting, which varies continuously with h' in the C^1 topology.

A C^1 diffeomorphism $f : M \to M$ of a complete Riemannian manifold M is *partially hyperbolic* if there is a dominated splitting $TM = E^u \oplus E^c \oplus E^s$ and $N \geq 1$ such that for any $x \in M$, and any choice of unit vectors $v^s \in E^s(x)$ and $v^u \in E^u(x)$, we have

$$\max\{\|D_x f^N(v^s)\|, \|D_x f^{-N}(v^u)\|\} < 1/2.$$

We always assume the bundles E^s and E^u are nontrivial. If E^c is trivial then f is *Anosov*.

A flow $\varphi\colon M \times \mathbb{R} \to M$ is *Anosov* if for some $t_0 \neq 0$, the time-t_0 map φ_{t_0} is partially hyperbolic, with the center bundle $E^c = \mathbb{R}\dot{\varphi}$ tangent to the orbits of the flow. If φ is Anosov, then the time-t map φ_t is partially hyperbolic for every $t \neq 0$. An example of an Anosov flow is the geodesic flow over a closed, negatively curved manifold, such as those considered in Theorem 3.

Isometric circle extensions of Anosov diffeomorphisms, such as the diffeomorphisms considered in Theorem 4, are also partially hyperbolic, with E^c tangent to the vertical foliation by circles $\{\{x\} \times \mathbb{T} : x \in \mathbb{T}^{d-1}\}$ (see, e.g. [14]).

If M is a closed manifold, then partial hyperbolicity is open property in the C^1 topology on $\mathrm{Diff}^1(M)$. Thus the C^1-small perturbations considered in Theorems 3 and 4 are also partially hyperbolic.

If f is partially hyperbolic and C^r, $1 \leq r \leq \infty$, then the bundles E^s and E^u are tangent to foliations $\mathcal{W}^s$ and $\mathcal{W}^u$, known respectively as the stable and the unstable foliations of f. These foliations have C^r leaves but are typically only Hölder continuous. For a more detailed discussion of foliation regularity, see Section 3.1.

We say a $Df-$invariant distribution $E \subset TM$ is *integrable* if there exists an $f-$invariant foliation $\mathcal{W} = \{\mathcal{W}(x)\}_{x \in M}$ with C^1 leaves everywhere tangent to the bundle E, and *uniquely integrable* if every C^1 curve tangent to E lies in a single leaf of $\mathcal{W}$.

The bundles E^u and E^s are thus integrable, and are in fact uniquely integrable. The center bundle E^c is not always integrable (see [73]), but in many examples of interest, such as the time-one map of an Anosov flow and its perturbations, or perturbations of an isometric extension of Anosov map, the theory of normally hyperbolic foliations developed in [35] implies that E^c is integrable, as are the bundles $E^{cs} = E^c \oplus E^s$ and $E^{cu} = E^c \oplus E^u$. In particular, for those f considered in this paper, E^c is integrable, and in fact uniquely integrable, tangent to a *center foliation* $\mathcal{W}^c$, (see Theorem 7). Our main results can be recast in terms of the measure theoretic properties of center foliations, as follows.

2.3. **Lebesgue disintegration and large centralizer.** As mentioned in the introduction, some of the key ingredients in proofs of Theorems 3 and 4 are the following *dichotomy results* which link the disintegration of volume along the center foliation with the structure of the centralizer.

For volume-preserving perturbations of the discretized geodesic flow on any negatively curved manifold, we have

Theorem 5. *Let X be a closed, negatively curved Riemannian manifold, and let $\psi_t \colon T^1X \to T^1X$ be the geodesic flow. Fix $t_0 \neq 0$, and suppose $f \in \mathrm{Diff}^2_{\mathrm{vol}}(T^1X)$ is a C^1-small perturbation of ψ_{t_0}. Then either the volume* vol *has Lebesgue disintegration along $\mathcal{W}^c_f$, or f has virtually trivial centralizer in* $\mathrm{Diff}(T^1X)$.

For perturbations of an isometric extension of a hyperbolic toral automorphism, we have

Theorem 6. *Let $f_0 : \mathbb{T}^d \to \mathbb{T}^d$ and ℓ_0 be as in Theorem 4, and let $f \in \mathrm{Diff}^2_{\mathrm{vol}}(\mathbb{T}^d)$ be a C^1-small, ergodic perturbation of f_0. Then either the volume has Lebesgue disintegration along $\mathcal{W}^c_f$, or $\mathcal{Z}_2(f)$ is virtually $\mathbb{Z}^\ell$ for some $\ell \leq \ell_0$. Moreover $\ell < \ell_0$ if $\ell_0 > 1$.*

2.4. **Prior results.** As mentioned in the introduction, it is expected that the typical diffeomorphism has small centralizer. Indeed, Smale asked [82, 83] whether the set of C^r diffeomorphisms with trivial centralizer is generic in $\mathrm{Diff}^r(M)$. Several works have been devoted to this question in various contexts, going back to Kopell's solution [52] to the question in the smooth case on the circle: those diffeomorphisms with trivial centralizer contains a C^∞ open and dense set in $\mathrm{Diff}^\infty(\mathbb{T})$. The question has also been answered in full generality by Bonatti–Crovisier–Wilkinson in the C^1 topology: trivial centralizer is generic (but not open) in $\mathrm{Diff}^1(M)$ and $\mathrm{Diff}^1_{\mathrm{vol}}(M)$, for any closed manifold M [7, 8, 9]. See [7] for a discussion of the history of this problem.

In the restricted context of partially hyperbolic systems, stronger results are known in the smooth category: Palis–Yoccoz showed that the set of C^∞ diffeomorphisms with trivial centralizer contains an open and dense subset of the set of Axiom A diffeomorphisms in $\mathrm{Diff}^\infty(M)$ possessing at least one periodic sink or source [60, 61]. The conditions have subsequently been relaxed [28, 69]. In another direction, Burslem showed [16] that for a class of C^∞ partially hyperbolic systems, (including non-volume-preserving perturbations of the systems considered in this paper), there is a residual subset whose centralizer is trivial.

When it comes to (partially) hyperbolic diffeomorphisms whose centralizers contain large rank abelian subgroups of (partially) hyperbolic diffeomorphisms, the general philosophy has been that a rich variety of (partially) hyperbolic dynamics in an abelian group action should be a rare occurrence. Classes of algebraic examples of such abelian actions have been listed in [49] by Katok and Spatzier, who also proved in [50] that such *Anosov* abelian actions are *locally rigid*: small perturbations of such an action are all smoothly conjugate to unperturbed action. Further local rigidity results for classes of partially hyperbolic abelian actions are found in [21], [86]. Moreover, for Anosov diffeomorphisms, if the centralizer contains a $\mathbb{Z}^2$ subgroup that does not factor onto a virtually $\mathbb{Z}$-action, Katok and Spatzier conjectured that f is then smoothly conjugate to a hyperbolic (infra)nilmanifold automorphism, and in particular it has a full rank centralizer smoothly conjugate to a group of automorphisms. We refer to [26], [85] and references therein for the history and most recent results in the direction of this *global rigidity* conjecture.

In the case of volume preserving partially hyperbolic diffeomorphisms with *compact center foliation*, it is found in [25] that a large rank centralizer with sufficiently many partial hyperbolic elements also leads to global rigidity. In particular, it was first discovered in [25] that the bad disintegration of volume along the leaves of the center foliation should be the main obstacle to rigidity for higher rank partially hyperbolic actions. The forthcoming paper [23] exploits this further by obtaining in some cases stronger global rigidity results (see Appendix A). For the case of commuting isometric extensions over hyperbolic toral automorphisms, local rigidity results have been obtained earlier under Diophantine conditions, in [19].

Work of Avila, Viana and Wilkinson [2, 3] establishes a dichotomy for a class of partially hyperbolic diffeomorphisms with 1-dimensional center foliation: either the disintegration of Lebesgue is atomic on the center foliation or volume has Lebesgue disintegration on the center. [5] Moreover, for these maps, if volume has Lebesgue disintegration on the center, then there is a *continuous* volume-preserving flow commuting with the map. These results apply directly to the systems considered here, and we take them as a starting point. Otherwise, our methods are almost entirely disjoint from those in [2, 3].

3. Preliminaries

3.1. **Regularity of maps and foliations.** For $r \in (0,1)$, we say that map between metric spaces is C^r if it is Hölder continuous of exponent r. For $r \geq 1$ we say that a map

[5] Under an accessibility assumption. See Section 3.4.5.

between smooth manifolds is C^r if it is $C^{[r]}$ and the $[r]$th-order derivatives are $C^{r-[r]}$. For $r \geq 0$, a map is C^{r+} if it is $C^{r+\varepsilon}$ for some $\varepsilon > 0$.

Let M be a manifold of dimension $d \geq 2$. A *$k-$dimensional topological foliation* $\mathcal{F}$ of M is a decomposition of M into path-connected subsets

$$M = \bigcup_{x \in M} \mathcal{F}(x)$$

called *leaves*, where $x \in \mathcal{F}(x)$, and two leaves $\mathcal{F}(x)$ and $\mathcal{F}(y)$ are either disjoint or equal, and a covering of M by coordinate neighborhoods $\{U_\alpha\}$ with local coordinates $(x_\alpha^1, \dots, x_\alpha^d)$ with the following property. For $x \in U_\alpha$, denote by $\mathcal{F}_{U_\alpha}(x)$ the connected component of $\mathcal{F}(x) \cap U_\alpha$ containing x. Then in coordinates on U_α the local leaf $\mathcal{F}_{U_\alpha}(x)$ is given by a set of equations of the form $x_\alpha^{k+1} = \cdots = x_\alpha^d = cst$. If the local coordinates $(x_\alpha^1, \dots, x_\alpha^d)$ can be chosen uniformly C^r along the local leaves (i.e., to have uniformly C^r overlaps on the sets $x_\alpha^{k+1} = \cdots = x_\alpha^d = cst$) then we say that $\mathcal{F}$ *has C^r-leaves*. If the $(x_\alpha^1, \dots, x_\alpha^d)$ can be chosen C^r on U_α then $\mathcal{F}$ is called a *C^r foliation.*

Note that the leaves of a foliation with C^r leaves are C^r, injectively immersed submanifolds of M.

The next lemma follows from an application of C^r-section theorem in [35]; for a precise proof cf. Corollary 5.6 in [24] or [68].

Lemma 8. *Let f be a C^{r+1} diffeomorphism of a closed Riemannian manifold M. Let $\mathcal{W}$ be an $f-$invariant foliation with uniformly C^r-leaves. For $x \in M$, let $\alpha_x := \|Df|^{-1}_{T\mathcal{W}(x)}\|$. Let E^1 and E^2 be continuous, $f-$invariant distributions on M such that the distribution $E = E^1 \oplus E^2$ is uniformly C^r along $\mathcal{W}$ leaves and $E^1 \oplus E^2$ is a dominated splitting in the sense that for any $x \in M$,*

$$k_x := \frac{\max_{v \in E^2(x), \|v\|=1} \|Df(v)\|}{\min_{v \in E^1(x), \|v\|=1} \|Df(v)\|} < 1.$$

If $\sup_{x \in M} k_x \alpha_x^r < 1$, then E^1 is uniformly C^r along the leaves of $\mathcal{W}$. In particular if $\alpha_x \leq 1$ for all $x \in M$ then E^1 is uniformly C^r along the leaves of $\mathcal{W}$.

Suppose $\mathcal{F}$ is a foliation of a closed manifold M with C^1 leaves, and μ is a Borel probability measure on M. Let B be a foliation box, and let μ_B be normalized Lebesgue measure on B. There is a unique family of conditional measures μ_x defined for μ_B-almost every x in B with the following properties (see [75]). First, for almost every x, the measure μ_x is supported on the plaque $\mathcal{F}_B(x)$; second, for every μ_B-integrable function $\psi \colon B \to \mathbb{R}$, we have

$$\int_B \psi(x)\, d\mu_B(x) = \int_B \int_{\mathcal{F}_B(x)} \psi(y) d\mu_x(y)\, d\mu_B(x).$$

We say μ has *Lebesgue disintegration* along $\mathcal{F}$ if for any foliation box B and μ_B-almost every x, the conditional measure of μ_B on $\mathcal{F}_B(x)$ is equivalent to the Riemannian measure on $\mathcal{F}_B(x)$. The measure μ has *atomic disintegration* (along $\mathcal{F}$) if there exists $k \geq 1$ such that for any foliation box B the conditional of μ_B measure on $\mathcal{F}_B(x)$ is atomic, with at most k atoms, for μ_B-almost every x.

Lemma 9. *Let $\mathcal{F}$ be an orientable topological foliation of a closed manifold M such that all leaves are circles. Suppose that there exists a full volume set $S \subset M$ and $k \in \mathbb{N}$ such that S meets almost every leaf of $\mathcal{F}$ in exactly k points. Let $\mathcal{G}_{\text{fix}}(\mathcal{F})$ be the set of $g \in \text{Diff}_{\text{vol}}(M)$ such that g preserves orientation on $\mathcal{F}$, and $g(\mathcal{F}(x)) = \mathcal{F}(x)$, for all $x \in M$. Then $\mathcal{G}_{\text{fix}}(\mathcal{F})$ is a finite cyclic group.*

Proof. Since the action of $\mathcal{G}_{\text{fix}}(\mathcal{F})$ fixes all the leaves of $\mathcal{F}$ and preserves the volume, on almost every leaf $\mathcal{F}(x)$, any element g of $\mathcal{G}_{\text{fix}}(\mathcal{F})$ maps atoms to atoms, which means that g induces a permutation on $S \cap \mathcal{F}(x)$. Moreover since g preserves the orientation of each circle leaf of $\mathcal{F}$ it induces a cyclic permutation (with respect to the circle ordering) of the atoms on almost every leaf.

Thus for every $x \in S$, the restriction of $g \in \mathcal{G}_{\text{fix}}(\mathcal{F})$ to $\mathcal{F}(x)$ has rotation number $k'(g,x)/k \pmod 1$, for some $k \in \mathbb{Z}^+$ and $k' = k'(g,x) \in \mathbb{Z}/k\mathbb{Z}$, where k is the number of atoms. Since the rotation number is a continuous function on diffeomorphisms, and S is dense, $k'(g,x)$ is independent of x. Therefore on **every** center leaf, g has rotation number $k'(g)/k \pmod 1$. Moreover for any other $h \in \mathcal{G}_{\text{fix}}(\mathcal{F})$ such that $k'(g) = k'(h)$, and every $x \in S$, h induces the same permutation on $S \cap \mathcal{F}(x)$ as g, which implies that $g = h$, by the density of S. Therefore k' induces an injective homomorphism from $\mathcal{G}_{\text{fix}}(\mathcal{F})$ to $\mathbb{Z}/k\mathbb{Z}$. □

3.2. **Lyapunov exponents and the Oseledec splitting.** Suppose M is a smooth manifold and $f \in \text{Diff}^1(M)$ is a diffeomorphism preserving a probability measure μ (for instance, volume). In analogy with the Birkhoff ergodic theorem, one can inquire about the asymptotic behavior of the composition of tangent maps of f

$$D_p f^n = D_{f^n(p)} f \circ \cdots \circ D_p f : T_p M \to T_{f^n(p)} M,$$

for μ-a.e. $p \in M$. An answer is given by the Oseledets Multiplicative Ergodic theorem, which we describe here in the setting of continuous cocycles.

Suppose X is a compact metric space and $E \to X$ is a (continuous) vector bundle. Let $T : X \to X$ be homeomorphism. A continuous linear *cocycle* over T is a bundle map $F \colon E \to E$ covering T. On the fibers, F is given by linear maps $F_x \colon E_x \to E_{Tx}$ that vary continuously with x. For simplicity we assume that each F_x is invertible, so that F is a bundle isomorphism.

Suppose that T preserves an ergodic probability measure μ on X, and E is equipped with a continuous Finsler structure $\{\|\cdot\|_x : x \in X\}$. Then the Oseledec theorem gives real numbers $\lambda_1 > \cdots > \lambda_k$ called *Lyapunov exponents* and a measurable, $F-$invariant splitting $E = E^{\lambda_1} \oplus \cdots \oplus E^{\lambda_k}$, such that for $v \in E_x \setminus \{0\}$,

$$v \in E_x^{\lambda_i} \iff \lim_{n \to \pm\infty} \frac{1}{n} \log \|F^n(v)\|_{T^n(x)} = \lambda_i.$$

The splitting $E = \oplus E^{\lambda_i}$ is called the Oseledets splitting for the cocycle.

The following well-known result allows one to deduce *uniform* growth of cocycles from knowledge about exponents for *every* invariant measure. The proof is a corollary of a classical result on subadditive sequences (cf. [80] or chapter 4 in [39].)

Lemma 10. *Let $f : X \to X$ be a continuous map of a compact metric space, and let $F\colon E \to E$ be a continuous linear cocycle over f, where $p : E \to X$ is a continuous vector bundle over X.*

(1) If for any $f-$invariant ergodic measure ν, the top Lyapunov exponent $\lambda^{\max}(F,\nu)$ is $\leq \lambda$, then for any $\epsilon > 0$, there exists $n \in \mathbb{Z}^+$ such that

$$\|F^n(x)\| \leq e^{n(\lambda+\epsilon)},\ \forall x \in X.$$

(2) If for any $f-$invariant ergodic measure ν, the bottom Lyapunov exponent $\lambda^{\min}(F,\nu)$ is $\geq \lambda'$, then for any $\epsilon > 0$, there exists $n \in \mathbb{Z}^+$ such that

$$\|F^n(x)^{-1}\|^{-1} \geq e^{n(\lambda'-\epsilon)},\ \forall x \in X.$$

3.3. **Some useful properties of commuting maps.** A basic principle in the study of abelian actions is the following: if f and g are commuting maps, and Υ is an f-invariant object, then $g_*(\Upsilon)$ is also f-invariant. For example, if $f(p) = p$, then $f(g(p)) = g(f(p)) = g(p)$. Thus $g(\mathrm{Fix}(f)) \subset \mathrm{Fix}(f)$; in other words, the set of f-periodic points of period k is a g-invariant set. Similar results hold for invariant sets of commuting homeomorphisms, such as the limit set and non-wandering set.

In the measurable context, if μ is an f-invariant measure, then $g_*\mu$ is also f-invariant, and so g_* preserves the set of f-invariant measures. When further assumptions are added, such as those in the present context, we get the following useful lemma.

Lemma 11. *Let M be a closed manifold, and suppose that $f, g \in \mathrm{Diff}(M)$ satisfy $fg = gf$. If f is topologically transitive and preserves a volume with continuous density (for example, if f is ergodic with respect to volume), then g is volume preserving as well.*

Proof. The commutativity implies that vol_M and $g_*(\mathrm{vol}_M)$ are both $f-$invariant measures. Since g is C^1, the induced Radon-Nikodym derivative $\frac{d(g_*\mathrm{vol}_M)}{d(\mathrm{vol}_M)}$ is an $f-$invariant continuous function. Transitivity of f implies that this derivative is constant and equal to the degree of g, which is 1. Thus $g_*(\mathrm{vol}_M) = \mathrm{vol}_M$. □

If f and g are commuting diffeomorphisms, then their derivatives commute as well. It follows that if $f(p) = p$, then the derivative of f at p is conjugate to its derivative at $g(p)$, and so D_pf and $D_{g(p)}f$ have the same eigenvalues. More generally:

Lemma 12. *Let M be a closed manifold, and suppose that $f, g \in \mathrm{Diff}(M)$ satisfy $fg = gf$. If μ is an ergodic invariant measure for f, then the Lyapunov exponents of μ are the same as the Lyapunov exponents of $g_*\mu$.*

Applying the same principle to the invariant subbundles in a dominated splitting, we obtain the following lemma, whose proof is straightforward.

Lemma 13. *Let M be a closed manifold, and suppose that $f, g \in \mathrm{Diff}(M)$ satisfy $fg = gf$. If f preserves a dominated splitting $TM = E^1 \oplus \cdots \oplus E^\ell$, then so does g. Moreover if, for some i, E^i is uniquely integrable, with integral foliation $\mathcal{W}^i$, then $g(\mathcal{W}^i) = \mathcal{W}^i$.*

Sufficiently high regularity of a map plus some hyperbolicity can force high regularity of its centralizer. A basic motivating example is a linear map on $\mathbb{R}$. If $f(x) = 2x$ and $fg = gf$, then $g(0) = 0$, and the commutativity of f and g implies that for all $x \neq 0$ and n:

$$\frac{g(x)}{x} = \frac{f^n g f^{-n}(x)}{x} = \frac{2^n}{x} g\left(\frac{x}{2^n}\right) = \frac{g\left(\frac{x}{2^n}\right) - g(0))}{\frac{x}{2^n}}.$$

If g is differentiable at 0, then the right hand side converges as $n \to \infty$ to $g'(0)$. Thus $g(x) = g'(0)x$ is linear.

A more sophisticated illustration of this principle in the setting of linear Anosov diffeomorphisms is the following result, due to Adler and Palais:

Lemma 14. *[[1]] Suppose $A \in \mathrm{SL}(k, \mathbb{Z})$ does not have a root of unity as an eigenvalue, and let T_A be the induced automorphism of $\mathbb{T}^k$. Suppose $h : \mathbb{T}^k \to \mathbb{T}^k$ is a homeomorphism such that $T_A h = h T_A$. Then h is affine and $h(0) \in \mathbb{Q}^k/\mathbb{Z}^k$.*

For such toral automorphisms T_A, we thus have

$$\mathcal{Z}_{\mathrm{Homeo}(\mathbb{T}^k)}(T_A) \subseteq \{x \mapsto T_L x + b : L \in \mathcal{Z}_{\mathrm{GL}(k,\mathbb{Z})}(A),\ b \in \mathbb{Q}^k/\mathbb{Z}^k\}.$$

When A is irreducible, the linear part of the right hand side can be computed using the following lemma, which a corollary of the Dirichlet unit theorem (cf. Proposition 3.7 in [45]).

Lemma 15. *Let $A \in \mathrm{GL}(k, \mathbb{Z})$ be a matrix with characteristic polynomial irreducible over $\mathbb{Z}$. Denote by $\mathcal{Z}_{\mathrm{GL}(k,\mathbb{Z})}(A)$ and $\mathcal{Z}_{\mathrm{SL}(k,\mathbb{Z})}(A)$ the centralizer of A in $\mathrm{GL}(k, \mathbb{Z})$ and $\mathrm{SL}(k, \mathbb{Z})$, respectively. Then $\mathcal{Z}_{\mathrm{GL}(k,\mathbb{Z})}(A)$ and $\mathcal{Z}_{\mathrm{SL}(k,\mathbb{Z})}(A)$ are abelian, and both are virtually $\mathbb{Z}^{r+c}$ where r is the number of real eigenvalues and c is the number of pairs of complex eigenvalues, $r + 2c = k$.*

3.4. **More on partial hyperbolicity.** In this section we discuss fundamental concepts in the study of partially hyperbolic diffeomorphisms: *normal hyperbolicity, leaf conjugacy, center bunching, and accessibility.* We also discuss some results of Avila–Viana–Wilkinson [2, 3] that we use in this paper.

3.4.1. *Normal hyperbolicity.* Suppose M is closed manifold, and let $f_1, f_2 \in \mathrm{Diff}(M)$. Assume that $\mathcal{F}_1, \mathcal{F}_2$ are foliations of M with C^1 leaves and that f_1 and f_2 respectively preserve $\mathcal{F}_1$ and $\mathcal{F}_2$.

Definition 2. A *leaf conjugacy* from $(f_1, \mathcal{F}_1)$ to $(f_2, \mathcal{F}_2)$ is a homeomorphism $h : M \to M$ sending $\mathcal{F}_1$ leaves diffeomorphically onto $\mathcal{F}_2$ leaves, equivariantly in the sense that

$$h(f_1(\mathcal{F}_1(p))) = f_2(\mathcal{F}_2(h(p))),\ \forall p \in M.$$

Definition 3. Suppose $f \in \mathrm{Diff}(M)$ and $\mathcal{F}$ is an $f-$invariant foliation of M with C^1 leaves. $\mathcal{F}$ is *normally hyperbolic* if there exists a $Df-$invariant dominated splitting $TM = E^u \oplus E^c \oplus E^s$, with at least two of the bundles nontrivial, such that Df uniformly expands E^u, uniformly contracts E^s, and such that $T\mathcal{F} = E^c$.

Note that a diffeomorphism with a normally hyperbolic foliation is partially hyperbolic, with $E^c = T\mathcal{F}$, but, as remarked above, the converse does not hold in general: the center bundle of a partially hyperbolic diffeomorphism is not necessarily tangent to a foliation, let alone an invariant foliation.

Definition 4. A partially hyperbolic diffeomorphism f is *dynamically coherent* if there exist $f-$invariant center stable and center unstable foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$, tangent to the bundles E^{cs} and E^{cu}, respectively; intersecting their leaves gives an invariant center foliation $\mathcal{W}^c$.

3.4.2. *Fibered partially hyperbolic systems.* In many of the cases of interest here, the center foliation $\mathcal{W}^c$ of a partially hyperbolic diffeomorphism f has compact leaves that form a fibration. We distinguish between several cases of such fibered systems.

Definition 5. Let f be a partially hyperbolic diffeomorphism of a closed manifold M. Assume that there exists an $f-$invariant center foliation $\mathcal{W}^c_f$ with compact leaves.

- If $\mathcal{W}^c_f$ is a topological fibration of M, i.e. the quotient space $M/\mathcal{W}^c_f$ is a topological manifold[6], then f is called a *fibered partially hyperbolic system*, and the map $\bar{f} : M/\mathcal{W}^c \to M/\mathcal{W}^c$ canonically induced by f is called the *base map*.
- A fibered partially hyperbolic system f is *smoothly fibered* (or C^r-*fibered*, for $r \geq 1$) if $\mathcal{W}^c_f$ is a C^∞ (respectively C^r) foliation, and $\bar{f}$ is C^∞ (resp. C^r).
- A fibered partially hyperbolic system f is *isometrically fibered* if there is a continuous Riemannian metric on E^c such that $Df|_{E^c_f}$ is an isometry.
- An isometrically fibered partially hyperbolic system f is an *isometric extension* (or smoothly isometrically fibered) if f is smoothly fibered.

3.4.3. *Leafwise structural stability.* A central result in [35] concerns perturbations of normally hyperbolic systems. It provides techniques to study integrability of the central distribution and robustness of the central foliation for partially hyperbolic systems.

To study the precise smoothness of the leaves of a normally hyperbolic foliation, we refine the definition of normal hyperbolicity. For $r \geq 1$ we say that $(f, \mathcal{F})$ is *r-normally hyperbolic* if there exists $k \geq 1$ such that

$$\sup_p \|D_p f^k|_{E^s}\| \cdot \|(D_p f^k|_{T\mathcal{F}})^{-1}\|^r < 1, \text{ and } \sup_p \|(D_p f^k|_{E^u})^{-1}\| \cdot \|D_p f^k|_{T\mathcal{F}}\|^r < 1.$$

Note that 1-normally hyperbolic = normally hyperbolic, and r-normal hyperbolicity is a C^1-open condition.

The proof of the following theorem can be found in [35, Theorems 7.5 and 7.6] (see also Remark 4 on p. 117), [6, Theorem 1.26], and [18], [35, Theorem 7.1], and [68, Theorems A and B]. See the discussion in [68, Section 3].

Theorem 7 (Foliation Stability and Hölder continuity of the leaf conjugacy)**.** *Let M be a closed manifold, and let $(f, \mathcal{F})$ be an r-normally hyperbolic foliation of M, for some $r \geq 1$, with Df-invariant splitting $E^u \oplus (T\mathcal{F} = E^c) \oplus E^s$. Then the leaves of $\mathcal{F}$ are uniformly C^r.*

[6]Or, equivalently, if $\mathcal{W}^c_f$ has trivial holonomy; see [6]

The bundles E^u and E^s are uniquely integrable and the leaves of their integral foliations $\mathcal{W}^u$ and $\mathcal{W}^s$ are as smooth as f.

Suppose in addition that one of the following holds:

(a) f is a fibered system, with 1-dimensional center fibration $\mathcal{F}$, or
(b) the restriction $Df|_{T\mathcal{F}}$ is an isometry.

Then

(1) f is dynamically coherent, and the foliations $\mathcal{W}^{cu}$, $\mathcal{W}^{cs}$ and $\mathcal{F} = \mathcal{W}^{cu} \cap \mathcal{W}^{cs}$ are r-normally hyperbolic and uniquely integrable.
(2) Every diffeomorphism g that C^1-approximates f is dynamically coherent and the foliations $\mathcal{W}^{cs}_g$, $\mathcal{W}^{cu}_g$ and $\mathcal{F}_g = \mathcal{W}^{cu}_g \cap \mathcal{W}^{cs}_g$ are r-normally hyperbolic near $\mathcal{F}$. Moreover, $(f, \mathcal{F})$ is leaf conjugate to $(g, \mathcal{F}_g)$ by a homeomorphism $h^c \colon M \to M$ close to the identity.
(3) In case (a) the conjugacy in (2) is bi–Hölder continuous (that is, Hölder continuous with a Hölder continuous inverse).

By combining Theorem 7 with Lemma 13 gives the next proposition as an immediate corollary.

Proposition 16. *Let $f \colon M \to M$ satisfy one of the following conditions.*

(a) $M = \mathbb{T}^d$, and f is a C^1-small perturbation of an isometric extension of an Anosov diffeomorphism of $\mathbb{T}^{d-1}$;
(b) $M = T^1X$, where X is a closed, negatively curved manifold, and f is a C^1-small perturbation of the discretized geodesic flow ψ_{t_0}, for some $t_0 \neq 0$ (or more generally any Anosov flow).

Then f is dynamically coherent, and for any $g \in \mathcal{Z}_1(f)$ we have $g\mathcal{W}^*_f = \mathcal{W}^*_f$, for $* \in \{c, s, u, cs, cu\}$.

Finally, we have a lemma that we will use in Section 4.

Lemma 17. *Let $\psi_t \colon M \to M$ be an Anosov flow with the property that the lift $\tilde{\psi}_t$ of ψ_t to the universal cover $\widetilde{M}$ has no closed orbits, and let f be a C^1-small perturbation of ψ_{t_0}, for some $t_0 \neq 0$.*

Then for any $g \in \mathcal{Z}_1(f)$, and for any closed leaf $\mathcal{W}^c_f(x)$, there exists $k \geq 1$ such that $g^k(\mathcal{W}^c_f(x)) = \mathcal{W}^c_f(x)$.

Proof. Consider the lifts $\tilde{\psi}_t, \tilde{f}$ of ψ_t, f respectively to $\widetilde{M}$, where $\tilde{f}$ is uniformly C^1-close to $\tilde{\psi}_t$, and $\tilde{f}$ preserves the lift $\widetilde{\mathcal{W}}^c_f$ of W^c_f. On each $\widetilde{\mathcal{W}}^c_f-$leaf, the action of $\tilde{f}$ is uniformly close to a translation by t_0 on $\mathbb{R}$ and thus is topologically conjugate to a translation. Thus there exist $0 < \tau_{min} \leq \tau_{max}$ such that for every $x \in \widetilde{M}$ and every $N \geq 1$, we have $d^c(x, \tilde{f}^N(x)) \in [N\tau_{min}, N\tau_{max}]$, where d^c is the distance measured along $\widetilde{\mathcal{W}}^c_f$ leaves.

Let $g \in \mathcal{Z}_{\mathrm{Diff}(M)}(f)$, and fix an arbitrary lift $\tilde{g}\colon \widetilde{M} \to \widetilde{M}$. Fix an arbitrary $\tilde{x}_0 \in \widetilde{M}$, and let $\gamma\colon [0,T] \to \widetilde{M}$ be a unit-speed, C^1 path tangent to $\widetilde{\mathcal{W}}^c_f(\tilde{x}_0)$ with $\gamma(0) = \tilde{x}_0$ and $\gamma(T) = \tilde{f}^N(\tilde{x}_0)$, for some $N \geq 1$. Note that $T \in [N\tau_{min}, N\tau_{max}]$.

Proposition 16 implies that $\tilde{g}$ preserves the foliation $\widetilde{\mathcal{W}}^c_f$, and so for any $m \geq 0$, $\tilde{g}^m(\gamma)$ is a path tangent to $\widetilde{\mathcal{W}}^c_f$ from $\tilde{g}^m(\tilde{x}_0)$ to $\tilde{f}^N(\tilde{g}^m(\tilde{x}_0))$. It follows that the length of $\tilde{g}^m(\gamma)$ also lies in the interval $[N\tau_{min}, N\tau_{max}]$. Now suppose that $\mathcal{W}^c_f(x)$ is a closed center leaf in M of length $R \in [N\tau_{min}, N\tau_{max}]$. Then there are lifts z_1, z_2 of x to $\widetilde{M}$ connected by a unit-speed path in $\widetilde{\mathcal{W}}^c_f$ of length R. This path is contained in a unit-speed path connecting z_1 to $\tilde{f}^{N+1}(z_1)$, whose length lies in $[(N+1)\tau_{min}, (N+1)\tau_{max}]$. Thus, for all $m \geq 1$, the distance between $g^m(z_1)$ and $g^m(z_2)$ is bounded by $(N+1)\tau_{max}$.

Since g is a diffeomorphism preserving $\mathcal{W}^c_f$, it permutes the closed leaves. Thus $g^m(\mathcal{W}^c_f(x))$ is a closed leaf whose length is at most CR, where C does not depend on R or m. Since f is a perturbation of ψ_{t_0}, its periodic center leaves of bounded length are isolated, and there are only finitely many of length $\leq CR$. If follows that every closed center leaf of $\mathcal{W}^c_f$ is g-periodic. □

3.4.4. *Bunching conditions.* For $r \geq 1$, we say that a partially hyperbolic diffeomorphism f of a Riemannian manifold M is *center $r-$bunched* if there exists $k \geq 1$ such that:

$$\sup_p \left\{ \|D_p f^k|_{E^s}\| \cdot \|(D_p f^k|_{E^c})^{-1}\|^r, \|(D_p f^k|_{E^u})^{-1}\| \cdot \|D_p f^k|_{E^c}\|^r \right\} < 1,$$

$$\sup_p \|D_p f^k|_{E^s}\| \cdot \|(D_p f^k|_{E^c})^{-1}\| \cdot \|D_p f^k|_{E^c}\|^r < 1, \text{ and}$$

$$\sup_p \|(D_p f^k|_{E^u})^{-1}\| \cdot \|D_p f^k|_{E^c}\| \cdot \|(D_p f^k|_{E^c})^{-1}\|^r < 1.$$

When f is C^r and dynamically coherent, the first of these three inequalities is r-normal hyperbolicity and implies that the leaves of $\mathcal{W}^c, \mathcal{W}^{cs}, \mathcal{W}^{cu}$ are C^r. If f is C^{r+1} and dynamically coherent they also imply the stable and unstable *holonomy* and E^s, E^u are C^r along $\mathcal{W}^c$, cf. [68, 87]. We say that f is *center bunched* if it is center 1-bunched. If E^c is 1-dimensional, then f is automatically center bunched. All systems we consider here have 1-dimensional center and thus are center bunched.

Unfortunately, the term "bunching" is also used in a completely different way, to describe stable (and unstable) expansion rates for contracted (and expanded) subbundles.

Definition 6. Let $f \in \mathrm{Diff}(M)$, and suppose that $E \subseteq TM$ is a continuous Df-invariant, subbundle. For $r > 0$, we say that $Df|_E$ is *r-bunched* if there exists $k \geq 1$ such that:

$$\sup_{p \in M} \max\{\|D_p f^k|_E\|, \ \|(D_p f^k|_E)^{-1}\| \cdot \|D_p f^k|_E\|^r\} < 1.$$

The smaller r is, the harder it is to satisfy r-bunching (as opposed to *center* r-bunching, which is easier to satisfy for small r) . If $Df|_E$ is conformal, then it is r-bunched, for all $r > 1$. If f is partially hyperbolic, we say that the stable (resp unstable) spectrum of f is r-bunched if $Df|_{E^s_f}$ (resp. $Df^{-1}|_{E^u_f}$) is r-bunched.

3.4.5. *Accessibility.* The foliations $\mathcal{W}^u_f$ and $\mathcal{W}^s_f$ of a partially hyperbolic diffeomorphism $f\colon M \to M$ induce an equivalence relation on M: we say that $x, y \in M$ are in the same *accessibility class* if they can be joined by an $su-$path, that is, a piecewise C^1 path such that every piece is contained in a single leaf of $\mathcal{W}^s_f$ or a single leaf of $\mathcal{W}^u_f$. Then f is *accessible* if M consists of a single accessibility class. At the opposite extreme of accessibility is joint integrability: E^u_f and E^s_f are *jointly integrable* if there exists an $f-$invariant foliation $\mathcal{W}^H$ with C^1 leaves everywhere tangent to the bundle $E^u \oplus E^s$. In this case, unique integrability of E^u, E^s implies that accessibility classes are the leaves of the foliation $\mathcal{W}^H$.

Pugh and Shub conjectured that if $f \in \mathrm{Diff}^2_{\mathrm{vol}}(M)$ is partially hyperbolic and accessible, then f is ergodic. This was proved for center bunched f by Burns–Wilkinson [13]. In particular, acessibility implies ergodicity for systems with 1-dimensional center bundle, and *stable accessibility* — i.e., accessibility that persists under C^1-small perturbations — implies stable ergodicity.

Pugh and Shub also conjectured that stable accessibility is a dense property among C^r partially hyperbolic diffeomorphisms, volume-preserving or not. Dolgopyat–Wilkinson [27] proved C^1 density of stable accessibility among all C^r partially hyperbolic diffeomorphisms, and Hertz-Hertz-Ures [72] proved C^r density (for any r) among the systems with 1-dimensional center foliation.

The next proposition will be used in the proofs of in Theorems 4 and 6.

Proposition 18. *Let f_0, A be as in Theorem 4, and let $f \in \mathrm{Diff}^2_{\mathrm{vol}}(\mathbb{T}^d)$ be a C^1-small, ergodic perturbation of f_0. Then*

1. *f is a fibered partially hyperbolic system. There is an equivariant fibration $\pi : \mathbb{T}^d \to \mathbb{T}^{d-1}$ such that $\pi \circ f = T_A \circ \pi$. The fibers of π are the leaves of the center foliation $\mathcal{W}^c_f$ by circles, where $\mathcal{W}^c_f$ is given by Theorem 7.*
2. *One of the following holds:*
   - (a) *there exists a full volume set $S \subset \mathbb{T}^d$ and $k \in \mathbb{N}$ such that S meets every leaf of $\mathcal{W}^c_f$ in exactly k points, i.e. volume has atomic disintegration along $\mathcal{W}^c_f$;*
   - (b) *f is accessible, $\mathcal{W}^c$ is absolutely continuous, and the disintegration of $\mathrm{vol}_{\mathbb{T}^d}$ has a continuous density function on the leaves of $\mathcal{W}^c$;*
   - (c) *f is topologically conjugate to $T_A \times R_\theta$, for some $\theta \notin \mathbb{Q}/\mathbb{Z}$ by a homeomorphism that is C^1 along the leaves of $\mathcal{W}^c_f$.*

Proof of Proposition 18. (1) follows from Theorem 7.

The proof of (2) involves an analysis of the accessibility classes of f. The first possibility is that f has an open accessibility class $U \neq \emptyset$. Since f is an ergodic, fibered partially hyperbolic system, with one dimensional fibers, [3, Theorem C (2)] implies that either f is accessible and vol has absolutely continuous disintegration, or vol has atomic disintegration along the leaves of $\mathcal{W}^c_f$. The conclusions follow immediately.

The second possibility is that there is no open accessibility class; that is, the extreme case of joint integrability holds [73]. Assume then that $E^s \oplus E^u$ is integrable, tangent to a foliation $\mathcal{W}^H$.

Recall that f is a C^1-small perturbation of an isometric extension $f_0 := (T_A)_{\rho_0}$, where $\rho_0 : \mathbb{T}^{d-1} \to \mathbb{R}$. If ρ_0 is not cohomologous to a constant function, then it is stably accessible [14]. Since we are assuming there is no open accessibility class, we may assume that ρ_0 is cohomologous to a constant function. Livšic's theorem implies that by conjugating by a C^∞ diffeomorphism of $\mathbb{T}^d$ covering the identity on $\mathbb{T}^{d-1}$, we may assume that $\rho_0 = \theta_0$ is constant. This implies that $E^s_{f_0} \oplus E^u_{f_0}$ is integrable, the leaves of the integral foliation $\mathcal{W}^H_{f_0}$ are compact, and f_0 is conjugate to the product of T_A with a rotation. We show that the same holds for f.

Lemma 19. *If the distribution $E^u_f \oplus E^s_f$ is integrable then the leaves of its integral foliation $\mathcal{W}^H$ are compact. Each leaf of $\mathcal{W}^H$ intersects each leaf of $\mathcal{W}^c$ in exactly one point.*

Proof of Lemma 19. We show that the monodromy representation on the circle bundle $\mathbb{T}^d \to \mathbb{T}^{d-1}$ induced by the foliation $\mathcal{W}^H$, combined with the action of f on an invariant $\mathcal{W}^c_f$ fiber, gives a C^1 action of an abelian-by-cyclic group. These actions have well-known rigidity properties, which we exploit to show that the monodromy part of the representation must have finite image.

To this end, fix $x_0 \in \mathbb{T}^d$ such that $f\left(\mathcal{W}^c_f(x_0)\right) = \mathcal{W}^c_f(x_0)$, and consider the map

$$H\colon \pi_1(\mathbb{T}^{d-1}, \pi(x_0)) \cong \mathbb{Z}^{d-1} \to \mathrm{Homeo}^+(\mathcal{W}^c_f(x_0))$$

defined by $\mathcal{W}^H_f$-holonomy along lifted paths: for $y \in \mathcal{W}^c(x_0)$ and $\gamma : [0,1] \to \mathbb{T}^{d-1}$ in the class $[\gamma]$, consider the unique continuous lift $\gamma^y : [0,1] \to \mathbb{T}^d$ such that $\gamma^y(0) = y$, $\gamma^y[0,1] \subset \mathcal{W}^H(y)$, and $\pi \circ \gamma^y = \gamma$. We then define

$$H([\gamma])(y) := \gamma^y(1).$$

Then H is a homomorphism, which we call the *monodromy representation.*

We remark that for $f = f_0$, where the leaves of $\mathcal{W}^H$ are compact, the map H is trivial.

Lemma 20. *For any $[\gamma] \in \pi_1(\mathbb{T}^{d-1})$:*

$$H(T_{A*}[\gamma]) = f \circ H([\gamma]) \circ f^{-1}.$$

Proof. (cf. [59]) Fix $y \in \mathcal{W}^c(x_0)$ and $\gamma : [0,1] \to \mathbb{T}^{d-1}$ in the class $[\gamma]$. Consider the lift $\gamma^{f^{-1}(y)}$ of γ starting at $f^{-1}(y)$. Note that the path $f \circ \gamma^{f^{-1}}(y)$ is a lift of $T_A \circ \gamma$ starting at y and tangent to $\mathcal{W}^H$ (by f-invariance of $\mathcal{W}^H$). But $(T_A \circ \gamma)^y$ is the unique such lift. It follows that $(T_A \circ \gamma)^y = f \circ \gamma^{f^{-1}(y)}$; evaluating both paths at their endpoint gives the desired conclusion. This completes the proof of Lemma 20. □

Lemma 21. *If $E^u_f \oplus E^s_f$ is integrable, then its integral foliation $\mathcal{W}^H$ is a C^1 foliation.*

Proof. Since E^c is 1-dimensional, f is center bunched. Then [67, Theorem B] implies that the leaves of $\mathcal{W}^s_f$ and $\mathcal{W}^u_f$ uniformly C^1 subfoliate the leaves of $\mathcal{W}^{cs}_f$ and $\mathcal{W}^{cu}_f$, respectively.

This implies that the stable and unstable holonomy maps between $\mathcal{W}^c_f$ leaves is C^1. The holonomy maps along $\mathcal{W}^H$ between $\mathcal{W}^c$ leaves can be written as a composition of stable and unstable holonomies, and thus are uniformly C^1.

A foliation with unifomly C^1 leaves and uniformly C^1 holonomy maps is C^1 (see [67]), and thus $\mathcal{W}^H$ is a C^1 foliation. □

Lemma 21 implies that the monodromy representation H above has C^1 image:

$$H(\pi_1(\mathbb{T}^{d-1}, \pi(x_0))) \subset \mathrm{Diff}^1(\mathcal{W}^c_f(x_0)).$$

Note that the induced action of T_A on $\pi_1(\mathbb{T}^{d-1}, \pi(x_0))$ is just matrix multiplication by A under the natural identification $\pi_1(\mathbb{T}^{d-1}, \pi(x_0)) \cong \mathbb{Z}^{d-1}$. Consider the abelian-by-cyclic group $\Gamma_A = \mathbb{Z} \ltimes_A \mathbb{Z}^{d-1}$ defined by

$$\Gamma_A := \left\langle a, e_1, \ldots, e_{d-1} : e_i e_j = e_j e_i, \quad a e_i a^{-1} = e_1^{\alpha_{1,i}} \cdots e_{d-1}^{\alpha_{d-1,i}} \right\rangle,$$

where $A = (\alpha_{i,j})$. Lemma 20 implies that we have a representation $\eta \colon \Gamma_A \to \mathrm{Diff}^1(\mathcal{W}^c_f(x_0))$ defined by

$$\eta(a) := f|_{\mathcal{W}^c_f(x_0)}; \ \ \eta(e_i) := H([e_i]).$$

Such representations are quite rigid. In particular, we have

Theorem 22. [10, Theorems 1.3, 1.7 and 1.10] *For any representation $\eta \colon \Gamma_A \to \mathrm{Diff}^1(\mathbb{T})$, either the image $\eta(\Gamma_A)$ is abelian or there exists an integer $m \geq 1$, a real eigenvalue λ of A, and a point $x \in \mathbb{T}$ such that $\eta(a^m)(x) = x$, and $\eta(a^m)'(x) = \lambda^m$.*

Applying Theorem 22 to the situation at hand, we obtain that either $\eta(\Gamma_A)$ is abelian, or there exists $m \geq 1$ such that $\eta(a^m) = f^m|_{\mathcal{W}^c_f(x_0)}$ has fixed point with derivative λ^m, for some real eigenvalue λ of A. But since A is hyperbolic, the eigenvalues of A are bounded in absolute value away from 1. Since $f_0|_{\mathcal{W}^c_{f_0}(0)}$ is a rotation by θ_0, whose derivative is everywhere 1, if f is sufficiently C^1-close to f_0, this is impossible.

Hence $\eta(\Gamma_A)$ is abelian, which implies that

$$\eta(a e_i a^{-1}) = \eta(e_i) = \eta(e_1)^{\alpha_{1,i}} \cdots \eta(e_{d-1})^{\alpha_{d-1,i}}, \text{ for } i = 1, \ldots, d-1. \tag{2}$$

Since 1 is not an eigenvalue of A, it follows that $A - \mathrm{id}$ is invertible over $\mathbb{Q}$, and so equations (2) imply that the group generated by $\eta(e_1), \ldots, \eta(e_d)$ is finite, of order $k \leq |\det(A - \mathrm{id})|$.

Thus the image of H is isomorphic to group of order k, and the leaves of $\mathcal{W}^H$ are compact, meeting each leaf of $\mathcal{W}^c_f$ in exactly k points. We claim that $k = 1$. As observed above, for f_0, the image of H is trivial, and the leaves of $\mathcal{W}^H_{f_0}$ are horizontal. Since f is close to f_0, the leaves of $\mathcal{W}^H_f$ are nearly horizontal; in particular if $d_{C^1}(f_0, f)$ is sufficiently small, either $k = 1$ or all the orbits of H on $\mathcal{W}^c_f(x_0)$ have arbitrarily small diameter (since k is bounded by $|\det(A - \mathrm{id})|$). Thus by the following theorem of Newman [58], we have $k = 1$.

Theorem 8. *Let N be a connected topological manifold endowed with a metric. Then there is $\epsilon > 0$ such that any non-trivial action of a finite group on N has an orbit of diameter larger than ϵ.*

This completes the proof of Lemma 19. □

To finish the proof of Proposition 18, we construct a C^1 projection $\mathrm{Pr}^c\colon \mathbb{T}^d \to \mathcal{W}^c_f(x_0)$ sending x to the unique point of intersection of $\mathcal{W}^H_f(x)$ and $\mathcal{W}^c_f(x_0)$. Let $\sigma\colon \mathcal{W}^c_f(x_0) \to \mathbb{T}$ be a C^1 diffeomorphism and define $\zeta\colon \mathbb{T}^d \to \mathbb{T}^d$ by $\zeta(x) := (\pi(x), \sigma \circ \mathrm{Pr}^c(x))$. Then ζ conjugates f to $T_A \times g$, where $g\colon \mathbb{T} \to \mathbb{T}$ is a diffeomorphism preserving a smooth ergodic measure. By a further C^1 change of coordinates, we may assume that g is an irrational rotation R_θ. We are thus in case 2c. This completes the proof of Proposition 18. □

3.4.6. *Estimate of the Hölder exponents of leaf conjugacies in the presence dominated splittings.* As in Theorems 4 and 6 let $f_0 \in \mathrm{Diff}^\infty_{\mathrm{vol}}(\mathbb{T}^d)$ be an isometric extension of an automorphism T_{A_f} on $\mathbb{T}^{d-1}$, where $A_f \in \mathrm{SL}(d-1,\mathbb{Z})$ is hyperbolic (we do not need irreducibility here). We denote by $P : \mathbb{T}^d \to \mathbb{T}^{d-1}$ the projection along the $\mathcal{W}^c_{f_0}$ leaves, which is just projection onto the first factor in $\mathbb{T}^{d-1}\times\mathbb{T}$. Under the identification $T\mathbb{T}^{d-1} \cong \mathbb{T}^{d-1}\times\mathbb{R}^{d-1}$, the action of DT_{A_f} is just $T_{A_f} \times A_f$, and $T\mathbb{T}^{d-1} = \mathbb{T}^{d-1} \times (\oplus V^i)$ is the $T_{A_f}-$invariant dominated splitting, where $\mathbb{R}^{d-1} = \oplus V^i$ is the decomposition into Lyapunov subspaces of A_f.

There is a Df_0-invariant dominated splitting $TM = \oplus E^i_{f_0}$ projecting to the dominated splitting for T_{A_f}, so that $D_pP(E^i_{f_0}) = \{P(p)\} \times V^i$, for each i. Moreover the Lyapunov exponent of $Df_0|_{E^i_{f_0}}$ is equal to Lyapunov exponent of $A_f|_{V^i}$.

As in Theorem 6, we now assume that $f \in \mathrm{Diff}^2_{\mathrm{vol}}(\mathbb{T}^d)$ is a C^1-small perturbation of f_0. Then Df also preserves a dominated splitting $TM = \oplus E^i_f$. By Theorem 7, f is a fibered partially hyperbolic system, and $(f_0; \mathcal{W}^c_{f_0})$ is leaf conjugate to $(f; \mathcal{W}^c_f)$ by a *bi-Hölder* continuous homeomorphism $h^c : \mathbb{T}^d \to \mathbb{T}^d$. The leaf conjugacy h^c is canonical in the sense that

$$\pi \circ h^c = P, \tag{3}$$

where π is the equivariant fibration from Proposition 18, whose fibers are the leaves of the center foliation $\mathcal{W}^c_f$. For the estimate of the bi-Hölder exponents of h^c, cf. [68]. In this context we can give a concrete description of how h^c is constructed. Fixing a smooth normal bundle $\mathcal{N}$ to E^c, the map $h^c = h^c_{\mathcal{N}}$ is defined by

$$h^c(x) = \pi^{-1}(P(x)) \cap \mathcal{D}_\epsilon(x),$$

where $\{\mathcal{D}_\epsilon(x) : x \in M\}$ is the smooth family of embedded disks defined by

$$\mathcal{D}_\epsilon(x) = \exp_x\left(\{tv : t \in [0,\epsilon), v \in \mathcal{N}(x)\}\right).$$

If f is sufficiently C^1 close to f_0 and $\epsilon > 0$ is sufficiently small, then h^c is a well-defined homeomorphism that is smooth along the leaves of $\mathcal{W}^c_{f_0}$ (as smooth as the leaves of $\mathcal{W}^c_{f_0}$).

It is easy to see that $E^i_{f_0}, E^i_{f_0} \oplus E^c_{f_0}$ are integrable; we denote by $\mathcal{W}^i_{f_0}, \mathcal{W}^{ic}_{f_0}$ their integral manifolds respectively. In general, E^i_f and $E^i_f \oplus E^c_f$ might not be integrable.

Lemma 23. *Suppose $E^i_f, E^i_f \oplus E^c_f$ are integrable and their integral manifolds are denoted by $\mathcal{W}^i_f, \mathcal{W}^{ic}_f$ respectively. Then for every $\alpha \in (0,1)$, there exists $\epsilon > 0$ such that for every smooth normal bundle $\mathcal{N}$ to E^c, if $d_{C^1}(f, f_0) < \epsilon$ and the map $h^c_{\mathcal{N}}$ sends $\mathcal{W}^{ic}_{f_0}$ to $\mathcal{W}^{ic}_f$, for each i, then $h^c_{\mathcal{N}}$ and its inverse are α-Hölder continuous.*

Proof. Fix i and consider the foliation $\mathcal{W}^{ic}_f$. Its leaves are jointly foliated by $\mathcal{W}^i_f$ and the uniformly compact foliation $\mathcal{W}^c_f$. By taking f^{-1} if necessary, we may assume that the leaves of $\mathcal{W}^i_f$ are uniformly contracted by the dynamics. Let $\lambda_i < 0$ be the corresponding Lyapunov exponent for $A_f|_{L_i}$. Since f_0 is an isometric extension of a linear map, for any $\epsilon > 0$ we may choose a continuous adapted metric on $\mathbb{T}^d$ such that for all f sufficiently C^1-close to f_0, for all $p \in \mathbb{T}^d$, and all $v \in E^i_f(p)$:

$$e^{\lambda_i - \epsilon}\|v\| \leq \|D_p f(v)\| \leq e^{\lambda_i + \epsilon}\|v\|.$$

Let $\mu_i = e^{\lambda_i - \epsilon}$ and $\nu_i = e^{\lambda_i + \epsilon}$. If f is sufficiently C^1-close to f_0, then for any $w \in M$ and $w' \in W^i_f(w, loc)$, if $f^{-j}(w') \in \mathcal{W}^i_f(f^{-j}(w'), loc)$ for $j = 0, \dots n$, then

$$\nu_i^{-n}\, d(w, w') \leq d_{\mathcal{W}^i_f}(f^{-n}(w), f^{-n}(w')) \leq \mu_i^{-n}\, d(w, w').$$

(This is easily proved by induction on n).

Consider the restriction of h^c to $\bigsqcup \mathcal{W}^{ic}_{f_0}$, whose image is $\bigsqcup \mathcal{W}^{ic}_f$, sending $\mathcal{W}^c_{f_0}$ leaves smoothly to $\mathcal{W}^c_f$ leaves. Now h^c does not necessarily send $\mathcal{W}^i_{f_0}$ leaves to $\mathcal{W}^i_f$, but we can estimate the Hölder exponent of h^c restricted to $\mathcal{W}^i_{f_0}$ leaves via a standard argument, which we now describe.

Fix $\eta > 0$ such that for all $w, w' \in \bigsqcup \mathcal{W}^{ic}_{f_0}$, with $d(w, w') < \eta$, then for any $z \in \mathcal{W}^c_f(w)$, there is a unique point z' in $\mathcal{W}^i_f(z, loc) \cap \mathcal{W}^c_f(w')$ and the distance between z and z' is uniformly comparable to the distance between z and z' as measured along $\mathcal{W}^i_f(z, loc)$. This is possible because the foliation $\mathcal{W}^c_f$ has uniformly compact leaves. Next fix a small constant $\delta > 0$ such that $d(w, w') < \delta$ implies $d(h^c(w), h^c(w')) < \eta$.

Now let $x \in M$ and $x' \in \mathcal{W}^i_{f_0}(x)$. Let $y = h^c(x)$ and $y' = h^c(x')$. We want to estimate $d(y, y')$ in terms of $d(x, x')$. Let $z = \mathcal{W}^i_f(y) \cap \mathcal{W}^c_f(y')$. By the construction of h^c using the smooth normal bundle $\mathcal{N}$, we have that $d(y', z) = O(d(y, z))$, so it suffices to estimate $d(y, z)$ in terms of $d(x, x')$.

We may assume that $d(x, x') < \delta$. Fix $n \geq 0$ such that $d(x, x') \in [\delta \mu_i^{n+1}, \delta \mu_i^n)$. Since $x' \in \mathcal{W}^i_{f_0}(x, loc)$, we have $d(f_0^{-n}(x), f_0^{-n}(x')) < \mu_i^{-n} d(x, x') < \delta$. By our choice of δ, we have that $d(h^c(f^{-n}(x)), h^c(f^{-n}(x'))) < \eta$. Since h^c is a leaf conjugacy, $h^c(f_0^{-n}(x)) \in \mathcal{W}^c_f(f^{-n}(y))$ and $h^c(f_0^{-n}(x')) \in \mathcal{W}^c_f(f^{-n}(y')) = \mathcal{W}^c_f(f^{-n}(z))$. Since $f^{-n}(y) \in \mathcal{W}^i(f^{-n}(z), loc)$, our choice of η implies that $d(f^{-n}y, f^{-n}z)$ is comparable to the distance measured along $\mathcal{W}^i_f$, which is at least $\nu_i^{-n} d(y, z)$. Thus $d(y, z) = O(\nu_i^{-n}) = O(\mu_i^{-n\beta}) = O(d(x, x')^\beta)$, where

$$\beta = \frac{\log \mu_i}{\log \nu_i} = \frac{\lambda_i + \epsilon}{\lambda_i - \epsilon}.$$

Since we may choose $\epsilon > 0$ arbitrarily small by setting $d_{C^1}(f_0, f)$ small enough, this shows that we may choose β arbitrarily close to 1.

This shows that h^c is uniformly β-Hölder continuous along $\mathcal{W}^{ic}_{f_0}$-leaves, for all i. It is thus β-Hölder continuous.

A similar argument (reversing the roles of f_0 and f_0^{-1}) shows that $(h^c)^{-1}$ is β-Hölder continuous. □

3.5. **Some Pesin theory.** We will also use the following well-known corollaries of Pesin theory. Let f be a $C^r (r > 1)$ diffeomorphism of a closed $d-$manifold M, let ν be an $f-$invariant ergodic probability measure, and let $\lambda^{\max} = \lambda_1 \geq \cdots \geq \lambda_d \geq \lambda^{\min}$ be the Lyapunov exponents of Df with respect to ν.

For $x \in M$, $\delta > 0$, and $\lambda < 0$, we define the *local stable set*

$$\mathcal{W}^s(x, \lambda, \delta) := \{y \in M : d(f^n(x), f^n(y)) \leq \delta \exp(\lambda n),\ \forall n \geq 0\}.$$

The set of *regular* points for (f, ν) in M (also called the Lyapunov–Perron regular points, cf. [5]) have full ν-measure in M and the following important property.

Proposition 24 (Stable manifold theorem)**.** *Fix $\lambda < 0$ such that $\lambda_{k+1} < \lambda < \lambda_k$ holds for some k. Then for any regular point x, the local stable set $\mathcal{W}^s(x, \lambda, \delta)$ is a C^r embedded disk in M for small enough δ. The dimension of the disk is $d - k$.*

We call the set $\mathcal{W}^s(x, \lambda, \delta)$ defined by the Proposition the *local Pesin stable manifold*, and we denote it by $\mathcal{W}^s(x, \lambda, loc)$ (cf. [63] for a concrete estimate on δ). Suppose x is a regular point and $\mathcal{W}^s(x, \lambda, loc)$ is defined as above. The *global Pesin manifold* (of $\mathcal{W}^s(x, \lambda, loc)$) is defined by

$$\mathcal{W}^s(x, \lambda) = \cup_{n=0}^{\infty} f^{-n}(\mathcal{W}^s(f^n(x), \lambda, loc)).$$

We also obtain the following criterion for a diffeomorphism to contract an invariant bundle.

Lemma 25. *Let $f\colon M \to M$ be a C^{1+} diffeomorphism, and let $\mathcal{W}$ be an f-invariant foliation with C^{1+} leaves. Suppose there exist $\kappa_1 < \kappa_2 < 0$ such that for all $x \in M$, there exist $\delta > 0$ and $N \in \mathbb{N}$ such that for all $y \in \mathcal{W}(x)$:*

$$d(x, y) < \delta \implies e^{\kappa_1 n} \leq d(f^n(x), f^n(y)) \leq e^{\kappa_2 n},$$

for all $n \geq N$.

Then for all $\epsilon > 0$, there exists $N' \in \mathbb{N}$ such that for all $v \in T\mathcal{W}$, and all $n \geq N'$, we have

$$e^{(\kappa_1 - \epsilon)n}\|v\| \leq \|Df^n v\| \leq e^{(\kappa_2 + \epsilon)n}\|v\|.$$

Proof. Assume the hypotheses; we show how to establish the upper inequality $\|Df^n v\| \leq e^{(\kappa_2+\epsilon)n}\|v\|$ (the lower inequality is similarly proved).

By Lemma 10, it suffices to show that for every f-invariant, ergodic measure ν, the top Lyapunov exponent of the cocycle $Df|_{T\mathcal{W}}$ with respect to ν is at most κ_2. To this end,

fix ν, and let $\beta_k < \beta_{k-1} < \cdots < \beta_1 < 0$ be the Lyapunov exponents of the cocycle $Df|_{T\mathcal{W}}$ with respect to ν. Let

$$T\mathcal{W} = E_k \oplus \cdots \oplus E_1$$

be the corresponding Oseledec decomposition.

Using the graph transform argument in [66, Theorem 3.16] in restriction to the leaves of $\mathcal{W}$, one can construct for each i a measurable family of C^{1+} disks $\mathcal{D}_i(x)$, defined over a full ν-measure set $S_\nu \in M$, with the following properties, for all $x \in S_\nu$:

- $\mathcal{D}_i(x) \subset \mathcal{W}(x)$,
- $T_x\mathcal{D}_i(x) = E_i(x)$,
- $f(\mathcal{D}_i(x)) \subset \mathcal{D}_i(f(x))$, and
- for every $\epsilon > 0$, there exists $N = N_x$ such that for all $y \in \mathcal{D}_i(x)$ and all $n \geq N$, we have
$$e^{(\beta_i-\epsilon)n} \leq d(f^n(x), f^n(y)) \leq e^{(\beta_i+\epsilon)n}.$$

Fix $\epsilon > 0$ and a Pesin regular point x for ν, and consider $\mathcal{D}_1(x)$. For n sufficiently large, the action of f^n in restriction to $\mathcal{D}_1(x)$ is a uniform contraction by a factor bounded below by $e^{(\beta_1-\epsilon)n}$. On the other hand, the hypothesis implies that this contraction factor is bounded above by $e^{\kappa_2 n}$. It follows that $\beta_1 < \kappa_2 + \epsilon$. Since $\epsilon > 0$ was arbitrary, we conclude that $\beta_1 \leq \kappa_2$. □

3.6. **Normal forms for uniformly contracting foliations.** We will use non-stationary normal form theory to upgrade the regularity of certain homeomorphisms in the centralizer of the partially hyperbolic systems under consideration.

Let f be a diffeomorphism of a closed manifold M, and let $\mathcal{W}$ be an $f-$invariant foliation of M with uniformly C^1 leaves. We assume that f uniformly contracts the leaves $\mathcal{W}$. Let $E = T\mathcal{W}$ be the tangent bundle to $\mathcal{W}$. We denote by $F\colon E \to E$ the bundle automorphism induced by the derivative of f: $F_x = Df|_{T_x\mathcal{W}}\colon E_x \to E_{fx}$. Then F induces a bounded linear operator F^* on the space of continuous sections of E by $F^*v(x) = F(v(f^{-1}x))$. The spectrum of the complexification of F^* is called the *Mather spectrum* of F. If the non-periodic points of f are dense in M, then the Mather spectrum consists of finitely many closed annuli A_i, $i = 1, \ldots, \ell$, centered at 0 and bounded by circles of radii e^{λ_i} and e^{μ_i}, with $\lambda_i = \lambda_i(F)$ and $\mu_i = \mu_i(F)$ satisfying

$$\lambda_1 \leq \mu_1 < \lambda_2 \leq \mu_2 < \cdots < \lambda_\ell \leq \mu_\ell < 0; \tag{4}$$

see [55, 64].

The spectral intervals $\{[\lambda_i(F), \mu_i(F)] : i = 1, \ldots, \ell\}$ correspond to a splitting of the bundle E into a direct sum

$$E = E^1 \oplus \cdots \oplus E^\ell$$

of continuous, $F-$invariant sub-bundles such that Mather spectrum of $F|_{E^i}$ is contained in the annulus A_i (this splitting is thus dominated and invariant under perturbations of F). This can be expressed using the Lyapunov metric [32]: for each $i = 1, \ldots, \ell$ and $\epsilon > 0$, there exists a continuous metric $\|\cdot\|_{x,\epsilon}$ on E^i such that

$$e^{\lambda_i-\epsilon} \cdot \|v\|_{x,\epsilon} \leq \|F_x(v)\|_{f(x),\epsilon} \leq e^{\mu_i+\epsilon} \cdot \|v\|_{x,\epsilon}, \forall v \in E^i_x.$$

Definition 7. We say that the bundle automorphism F has *narrow band spectrum* if $\mu_i(F) + \mu_\ell(F) < \lambda_i(F)$, for $i = 1, \dots, \ell$.

For vector spaces E and $\bar{E}$ we say that a map $P : E \to \bar{E}$ is *polynomial* if with respect to some bases of E and $\bar{E}$, each component of P is a polynomial. A polynomial map P is *homogeneous of degree* n if $P(av) = a^n P(v)$ for all $v \in E$ and $a \in \mathbb{R}$. More generally, for a given splitting $E = E^1 \oplus \cdots \oplus E^\ell$ we say that $P\colon E \to \bar{E}$ has *homogeneous type* $s = (s_1, \dots, s_\ell)$ if for any real numbers $a_1, \dots, a_\ell$ and vectors $t_j \in E^j$, $j = 1, \dots, \ell$, we have

$$P(a_1 t_1 + \cdots + a_\ell t_\ell) = a_1^{s_1} \cdots a_\ell^{s_\ell} P(t_1 + \cdots + t_\ell).$$

Suppose $E = E^1 \oplus \cdots \oplus E^\ell$, $\bar{E} = \bar{E}^1 \oplus \cdots \oplus \bar{E}^\ell$ and $P\colon E \to \bar{E}$ is a polynomial map. Split P into components $P_i\colon E \to \bar{E}^i$ and write $P = (P_1, \dots, P_\ell)$. Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ and $\mu = (\mu_1, \dots, \mu_\ell)$ with $\lambda_1 \le \mu_1 < \cdots < \lambda_\ell \le \mu_\ell < 0$. We say that P is of (λ, μ) *sub-resonance type* if for each $i = 1, \dots, \ell$, there exists $s = s(i) = (s_1, \dots, s_\ell)$ satisfying the sub-resonance relation

$$\lambda_i \le \sum_{j=1}^{\ell} s_j \mu_j,$$

such that P_i has homogeneous type s.

We denote by $S^{\lambda,\mu}(E, \bar{E})$ the space of all polynomials $E \to \bar{E}$ of (λ, μ) sub-resonance type. It follows from the definition that polynomials in $S^{\lambda,\mu}(E, \bar{E})$ have degree at most $d = d(\lambda, \mu) = \lfloor \frac{\lambda_1}{\mu_l} \rfloor$. If (λ, μ) satisfies the narrow band condition, they generate (under composition) a finite-dimensional Lie group which we denote by $G^{\lambda,\mu}(E)$. The maps in $G^{\lambda,\mu}(E)$ are called *sub-resonance generated* and can be described by adding finitely many relations to the set of sub-resonance ones.

Now we can state the main results in this section.

Theorem 9. [40, 41, 78] *Let f be a C^r diffeomorphism of a closed manifold M, and let $\mathcal{W}$ be an $f-$invariant topological foliation of M with uniformly C^r leaves. Suppose that the leaves of $\mathcal{W}$ are contracted by f and that either: the spectrum of $F = Df|_{T\mathcal{W}}$ is $r_{\mathcal{W}}$-bunched, for some $r_{\mathcal{W}} \le 2$. (See Definition 6); or F has narrow band spectrum (see Definition 7).*

Fix $r > r_{\mathcal{W}}$ (in the bunched case) or $r > \lambda_1(F)/\mu_\ell(F)$, setting $\lambda = (\lambda_1(F), \dots, \lambda_\ell(F))$ and $\mu = (\mu_1(F), \dots, \mu_\ell(F))$ (in the narrow band case). Then there exists a family $\{H_x\}_{x \in M}$ of C^r diffeomorphisms $H_x\colon \mathcal{W}_x \to E_x = T_x\mathcal{W}$ such that

1. *$P_x = H_{fx} \circ f \circ H_x^{-1}\colon E_x \to E_{fx}$ is a linear map (in the bunched case) or a polynomial map of (λ, μ) sub-resonance type (in the narrow band case) for each $x \in M$;*
2. *$H_x(x) = 0$ and $D_x H_x$ is the identity map for each $x \in M$;*
3. *H_x depends continuously on $x \in M$ in the C^r topology and is jointly C^r in x and $y \in \mathcal{W}_x$ along the leaves of $\mathcal{W}$;*

(4) *$H_y \circ H_x^{-1} : E_x \to E_y$ is a linear map (in the bunched case) or a polynomial map of (λ, μ) sub-resonance generated (in the narrow band case) for each $x \in M$ and each $y \in \mathcal{W}_x$; and*
(5) *if g is a homeomorphism of M that commutes with f, preserves $\mathcal{W}$, and is C^s along the leaves of $\mathcal{W}$, with $s > r_{\mathcal{W}}$ (in the bunched case) or $s > \lambda_1(F)/\mu_\ell(F)$ (in the narrow band case), then the maps H_x bring g to a normal form as well, i.e. the map $Q_x = H_{fx} \circ g \circ H_x^{-1}$ is a linear map (in the bunched case) or a polynomial of (λ, μ) sub-resonance type.*

Definition 8. Let f be a C^∞ partially hyperbolic diffeomorphism of a closed manifold M. We say that f has *r-bunched spectrum* if the cocycles $F^s = Df|_{E^s_f}$ and $F^u = Df^{-1}\big|_{E^u_f}$ are r-bunched. (see Definition 6); we call the infimum of such r the *critical regularity* $r(f)$ *of* f.

We say that f has *narrow band spectrum* if the cocycles F^s and F^u have narrow band spectrum. In this case, we define the *critical regularity* $r(f)$ of f by

$$r(f) := \max\left(\frac{\lambda_1^s(f)}{\mu_{\ell_s}^s(f)}, \frac{\lambda_1^u(f)}{\mu_{\ell_u}^u(f)}\right), \tag{5}$$

where $\mu_i^*(f) := \mu_i(F^*)$, $\lambda_i^*(f) := \lambda_i(F^*)$, for $i = 1, \ldots, \ell_*$, for $* \in \{s, u\}$.

We remark that if $f = \psi_{t_0}$, where ψ_t is the geodesic flow over a negatively curved X, then transverse symplecticity of the flow implies that it suffices to check that one of F^s or F^u has r-bunched (resp. narrow band) spectrum to verify that f itself has r-bunched (resp. narrow band) spectrum in the sense of Definition 8.

Hasselblatt [34] also defines an α-bunched condition for Anosov flows, for $\alpha \in (0, 2)$. For a transversely symplectic Anosov flow ψ_t, we have that ψ_t is α-bunched in the sense of [34] if and only if ψ_1 has $2/\alpha$-bunched spectrum, in the sense of Definition 8. The connection between α-bunching and pointwise pinching of the curvature in (1) is discussed in [34].

Lemma 26. *If ψ_t is the geodesic flow over a locally symmetric space X, then for any $t_0 \neq 0$, the partially hyperbolic map ψ_{t_0} has narrow band spectrum. If X is a real hyperbolic manifold, then $r(\psi_{t_0}) = 1$, and if X is locally symmetric but not real hyperbolic, then $r(\psi_{t_0}) = 2$.*

Proof. The geodesic flow on a locally symmetric space has constant expansion and contraction factors on one or two invariant subbundles, depending on whether X is real hyperbolic or not. In particular, the Mather spectrum of $D\varphi_{t_0}|_{E^s}$ and $D\varphi_{t_0}^{-1}|_{E^u}$ has either one or two bands, and either $\lambda_1^s = \mu_1^s = -1 = \lambda_1^u = \mu_1^u$,. in the case where X is real hyperbolic, or

$$\lambda_1^s = \mu_1^s = -2 = \lambda_1^u = \mu_1^u;\ \lambda_2^s = \mu_2^s = -1 = \lambda_2^u = \mu_2^u,$$

otherwise. Thus $D\varphi_{t_0}|_{E^s}$ and $D\varphi_{t_0}^{-1}|_{E^u}$ have point Mather spectrum (i.e., $\lambda_i^{u/s} = \mu_i^{u/s}$), and the conclusions follow. □

The following lemma follows immediately from the continuity of dominated splittings.

Lemma 27. *Suppose that $f_0 \in \mathrm{Diff}^1(M)$ is partially hyperbolic and has bunched spectrum, (resp. narrow band spectrum) with critical regularity $r_0 = r(f_0)$. Then for any $r > r_0$, if $f \in \mathrm{Diff}^1(M)$ is sufficiently C^1-close to f_0, then f has r-bunched spectrum (resp. narrow band spectrum), and $r(f) < r$.*

Here is our central application of Theorem 9.

Proposition 28. *Let f be a C^∞ partially hyperbolic diffeomorphism of a closed manifold M. Assume that f has a $1-$dimensional center foliation $\mathcal{W}^c_f$ with C^∞ leaves. Suppose that $\varphi = \varphi_t : M \times \mathbb{R} \to M$ is a flow generated by a continuous vector field X such that $\varphi_t \circ f = f \circ \varphi_t$, for all t. Assume that f, φ_t, and X satisfy the following conditions.*

1. *f has 2-bunched spectrum, or f has narrow band spectrum.*
2. *The vector field X is tangent to E^c_f and uniformly C^∞ along the leaves of $\mathcal{W}^c_f$.*
3. *There exists a dense set $D \subset \mathbb{R}$ such that for all $t \in D$, $\varphi_t \in \mathrm{Diff}^r(M)$, for some $r > r(f)$.*

Then φ_t is a C^∞ flow.

Proof. Hypothesis (1) implies that for $r > f(f)$, the cocycle $Df|_{T\mathcal{W}^s_f}$ satisfies the hypotheses of Theorem 9, and so there exists a non-stationary normalization $\{H_x, x \in M\}$ for $f|_{\mathcal{W}^s_f}$ such that for any $g \in \mathcal{Z}_r(f)$, the map $H_{gx} \circ g \circ H_x^{-1}$ is a sub-resonance polynomial (with fixed type) as well.

Thus $\{H_x\}$ is also a normalization for φ_t on $\mathcal{W}^s_f$, for all $t \in D$. Now consider the homeomorphism φ_t for an arbitrary fixed $t \in \mathbb{R}$. Pick $t_k, k = 1, 2, \dots$ in D such that $\lim_{k\to\infty} t_k = t$. Then the sequence

$$H_{\varphi_{t_k} x} \circ \varphi_{t_k} \circ H_x^{-1} : E^s_f(x) \to E^s_f(\varphi_{t_k}(x))$$

uniformly converges to $H_{\varphi_t x} \circ \varphi_t \circ H_x^{-1} : E^s_f(x) \to E^s_f(\varphi_t(x))$.

But each of $H_{\varphi_{t_k} x} \circ \varphi_{t_k} \circ H_x^{-1}$ is a sub-resonance polynomial (with fixed type), so their C^0-limit is a sub-resonance polynomial as well. Thus $H_{\varphi_t x} \circ \varphi_t \circ H_x^{-1}$ is uniformly smooth along E^s_f, which means φ_t is uniformly smooth along $\mathcal{W}^s_f$. A similar argument shows that φ_t is uniformly smooth along $\mathcal{W}^s_f$.

Assumption (2) of Proposition 28 implies that φ_t is uniformly smooth along $\mathcal{W}^c$, and the evaluation map $t \mapsto \varphi_t(x), x \in M$ is smooth, uniformly in x. Applying Journé's Lemma as in [2], we obtain that $\{\varphi_t\}$ is a smooth flow and $D = \mathbb{R}$. □

3.7. **Partially hyperbolic higher rank abelian actions.** A detailed ground treatment of Anosov and partially hyperbolic abelian higher rank actions, including a variety of techniques and examples, can be found in [47]. For a detailed discussion of smooth ergodic theory of general abelian actions, see [11].

An action $\alpha : \mathbb{Z}^k \to \mathrm{Diff}(M)$ on a closed manifold M is *partially hyperbolic* if it contains a partially hyperbolic diffeomorphism $\alpha(a)$, for some $a \in \mathbb{Z}^k$, and *Anosov* if it contains an

Anosov diffeomorphism. Some basic questions and difficulties related to partially hyperbolic actions are described in [21], [22]. For background on partially hyperbolic abelian actions with compact center foliation we refer to [25] and the references therein.

Oseledec's theorem for a cocycle over an ergodic transformation has a version for abelian actions [11, Theorem 2.4]. Let $E \to M$ be a continuous vector bundle, and let $\mathcal{A}$ be a linear $\mathbb{Z}^k$-cocycle on E over an ergodic, μ-preserving action α of M, i.e. $\mathcal{A}\colon \mathbb{Z}^k \to \mathrm{Aut}(E)$ is a $\mathbb{Z}^k$-action by bundle isomorphisms projecting to the action of α on M. The higher-rank Oseledec theorem implies the existence of finitely many linear functionals $\chi\colon \mathbb{R}^k \to \mathbb{R}$, called (*Lyapunov functionals for* $\mathcal{A}$), and an $\mathcal{A}$- invariant measurable splitting $\oplus E_\chi$ of E, called the (*Oseledec decomposition for* $\mathcal{A}$), on a full μ-measure set, such that for $a \in \mathbb{Z}^k$ and $v \in E_\chi(x)$:

$$\lim_{a\to\infty} \frac{\log \|\mathcal{A}(a,x)(v)\| - \chi(a)}{\|a\|} = 0.$$

The hyperplanes $\ker_\chi \subset \mathbb{R}^k$ are called *Weyl chamber walls*, and the connected components of $\mathbb{R}^k - \cup_\chi \ker_\chi$ are the *Weyl chambers* for $\mathcal{A}$ (with respect to μ). Even though elements of the Weyl chambers are vectors in $\mathbb{R}^k$, we will often say that the diffeomorphism $\alpha(a)$ is in the Weyl chamber $\mathcal{C}$ if $a \in \mathcal{C}$.

Two nonzero Lyapunov functionals χ_i and χ_j are *coarsely equivalent* if they are positively proportional: there exists $c > 0$ such that $\chi_i = c \cdot \chi_j$. This is an equivalence relation on the set of Lyapunov functionals, and a *coarse Lyapunov functional* is an equivalence class under this relation. Given a fixed ordering of non-zero coarse Lyapunov functionals $(\chi_1, \dots, \chi_r)$, each Weyl chamber $\mathcal{C}$ can be labelled by its *signature*: $(\mathrm{sgn}\chi_1(a), \dots, \mathrm{sgn}\chi_r(a))$, where a is any element in $\mathcal{C}$. The Weyl chambers of $\mathcal{A}$ in $\mathbb{R}^k$ together with their assigned signatures we call *the Weyl chamber picture of* $\mathcal{A}$ *over* α. Two $\mathbb{Z}^k$ cocycles (over possibly two distinct $\mathbb{Z}^k$ actions), have *the same Weyl chamber picture* if the walls in $\mathbb{R}^k$ coincide and the signatures of each Weyl chamber coincide. If for two Lyapunov functionals χ^1, χ^2, we have $\ker \chi^1 = \ker \chi^2$ and $\chi^1(a)\chi^2(a) > 0$ for some a, then χ^1, χ^2 are positively proportional. This implies the following:

Lemma 29. *Suppose that the Lyapunov functionals* $\{\chi^i\}, \{\chi'^i\}$ *of two ergodic cocycles* $\mathcal{A}$ *and* $\mathcal{A}'$ *have the same Weyl chambers, and suppose that for any* i*, there is an element* $a \in \mathbb{Z}^k$ *such that* $\chi^i(a)\chi'^i(a) > 0$*. Then* $\mathcal{A}$ *and* $\mathcal{A}'$ *have the same the Weyl chamber picture.*

For Anosov actions, the higher-rank Oseledec theorem is applied to the derivative cocycle $D\alpha$, and the *Weyl chamber picture* depends only on α and on the invariant measure. In the presence of sufficiently many Anosov elements of the action (for example, one Anosov element in each Weyl chamber), and an ergodic measure of full support, even the dependence on the measure can be removed. Moreover, in this case the coarse Lyapunov distributions are intersections of stable distributions for finitely many elements of the action, they are well defined everywhere and tangent to foliations with smooth leaves. (For more details see Section 2.2 in [43]) The same holds for actions that have many elements normally hyperbolic to a common center foliation [22].

Suppose $\alpha : \mathbb{Z}^k \to \mathrm{Diff}^2_{\mathrm{vol}}(M)$ is a conservative ergodic partially hyperbolic action on a compact manifold M and let $\alpha(a)$ be a partially hyperbolic element. By the discussion in Section 3.3 the sum $E^H := E^u_a \oplus E^s_a$ of the stable and unstable distributions of $\alpha(a)$ is $\alpha-$invariant. We will apply the higher-rank Oseledec theorem to the cocycle $D\alpha|_{E^H}$ and to stress this restriction of the derivative cocycle to the smaller bundle, we call the corresponding picture the *hyperbolic* Weyl chamber picture for α.

A $\mathbb{Z}^k$-action α is *maximal* if there are exactly $k+1$ coarse Lyapunov functionals corresponding to $k+1$ distinct Lyapunov hyperspaces, and if the Lyapunov hyperspaces are in *general position*, i.e. if no Lyapunov hyperspace contains a non-trivial intersection of two other Lyapunov hyperspaces. Maximality implies a special property of Weyl chambers: there is any combination of signs of Lyapunov functionals among the Weyl chambers, except all positive, and all negative. Prime examples of maximal Anosov actions are actions by toral automorphisms. Namely

Lemma 30. *[[45]] Suppose $A \in \mathrm{SL}(d,\mathbb{Z})$ is a hyperbolic irreducible matrix. Then $\mathcal{Z}_{\mathrm{SL}(d,\mathbb{Z})}(A)$ induces a maximal abelian Anosov action on $\mathbb{T}^d$ if $\ell_0(A) > 1$.*

Results of Franks and Manning [30, 54] imply that every Anosov action $\alpha\colon \mathbb{Z}^k \to \mathrm{Diff}(\mathbb{T}^d)$ is topologically conjugate to an action $\kappa\colon \mathbb{Z}^k \to \mathrm{Aff}(\mathbb{T}^d)$ by affine automorphisms of the torus. Such an action κ is called a *linearization* of α. The *linear part* of κ is the action $\kappa_0\colon \mathbb{Z}^k \to \mathrm{Aut}(\mathbb{T}^d)$ that sends g to T_{A_g}, where $\kappa(g) = T_{A_g} + v(g)$. The linear part does not depend on the choice of linearization of α.

An affine $\mathbb{Z}^k$-action κ' on $\mathbb{T}^{d'}$, is called an (algebraic) factor of an affine $\mathbb{Z}^k$-action κ on $\mathbb{T}^d$ if there exists a surjective homomorphism $\varphi : \mathbb{T}^d \to \mathbb{T}^{d'}$ such that $\varphi \circ \kappa = \kappa' \circ \varphi$. An affine action κ is said to have a *rank one factor* if its linear part κ_0 has a nontrivial factor $\kappa'\colon \mathbb{Z}^k \to \mathrm{Aut}(\mathbb{T}^{d'})$ such that the image $\kappa'(\mathbb{Z}^k)$ is virtually cyclic. A smooth $\mathbb{Z}^k-$action on $\mathbb{T}^d$ is *higher rank* if its linearizations have no rank one factor. In particular, when one element of a linear action is an irreducible toral automorphism, the action is called *irreducible* and we have the following easy lemma:

Lemma 31. *[[45], Section 3.1.] Suppose $A, B \in \mathrm{GL}(n,\mathbb{Z})$ satisfies $AB = BA$. Assume that A is irreducible and the group generated by A and B is not virtually $\mathbb{Z}$. Then the action generated by $< T_A, T_B >$ on $\mathbb{T}^n$ is a higher rank action.*

One important feature of higher rank Anosov actions is cocycle rigidity, which has the following application to isometric extensions:

Lemma 32. *[[49], Theorem 2.9] Suppose $A, B \in \mathrm{GL}(n,\mathbb{Z})$ commute and generate a higher rank Anosov action $< T_A, T_B >$ on $\mathbb{T}^n$. Let ρ_A, ρ_B be Hölder functions on $\mathbb{T}^n$. Then the isometric extensions $(T_A)_{\rho_A}, (T_B)_{\rho_B}$ commute iff there exist a Hölder function β on $\mathbb{T}^n$ and $\theta_A, \theta_B \in \mathbb{R}$ such that $\rho_A = -\beta \circ T_A + \beta + \theta_A$, and $\rho_B = -\beta \circ T_B + \beta + \theta_B$.*

We state the global rigidity result [74] and its corollaries concerning centralizers.

Theorem 10. *[[74]] Let $\alpha : \mathbb{Z}^k \to \mathrm{Diff}^\infty(\mathbb{T}^d)$ be an Anosov action, and let κ be a linearization of α. If κ is higher rank, then α is C^∞ conjugate to κ.*

As a corollary we have following result about centralizers:

Corollary 33. *Let $A \in \mathrm{SL}(d-1,\mathbb{Z})$ and let r_0 be as in Theorem 4. Fix $r > r_0$. Suppose $g \in \mathrm{Diff}^\infty(\mathbb{T}^{d-1})$ is a C^1-small perturbation of T_A (or, more generally, has narrow band spectrum). Then either g is C^∞ conjugate to T_A or $\mathcal{Z}_{\mathrm{Diff}^s(\mathbb{T}^d)}(g)$ is virtually trivial for any $s \geq r$.*

Proof. Clearly T_A has narrow band spectrum. Fix $r' \in (r_0, r)$; Lemma 27 implies that any g sufficiently close to T_A has narrow band spectrum, and $r(g) < r'$. Corollary 33 then follows from the lemma that follows. □

Lemma 34. *Let $g\colon \mathbb{T}^n \to \mathbb{T}^n$ be a C^∞ Anosov diffeomorphism with narrow band spectrum, let $\kappa(g)$ be a linearization of g, and let $\kappa_0(g) \in \mathrm{Aut}(\mathbb{T}^n)$ be its linear part. If $\kappa_0(g)$ is irreducible, then either g is C^∞ conjugate to $\kappa(g)$ (equivalently, to $\kappa_0(g)$) or $\mathcal{Z}_s(g)$ is virtually trivial for any $s > r(g)$.*

Proof. The narrow band spectrum assumption and Theorem 9 imply that g preserves some C^∞ normal forms on $\mathcal{W}^*_g, * = s, u$, which are also preserved by any $h \in \mathcal{Z}_s(g)$ for $s > r(g)$. Since h is smooth along the tranverse foliations $\mathcal{W}^s_g$ and $\mathcal{W}^u_g$, Journé's lemma implies that h is smooth. Thus $\mathcal{Z}_s(g) = \mathcal{Z}_\infty(g) \subset \mathcal{Z}_{\mathrm{Homeo}(\mathbb{T}^n)}(g)$, which has a finite index subgroup $G \cong \mathbb{Z}^\ell$, by irreducibility of $\kappa_0(g)$, and Lemmas 14 and 15.

Suppose that g is not C^∞ conjugate to $\kappa(g)$. Applying Theorem 10 to the action of G gives a rank one factor for a linearization of G. By the irreducibility of $\kappa_0(g)$ and Lemma 31, the rank of G must be 1. Therefore $\mathcal{Z}_s(g) = \mathcal{Z}_\infty(g)$ is virtually trivial. □

4. Proofs of Theorems 3 and 5

We begin with a general discussion of perturbations of discretized geodesic flows in negative curvature. Let X be a closed, negatively curved Riemannian manifold of any dimension, and let $\psi_t\colon T^1X \to T^1X$ be the geodesic flow on the unit tangent bundle T^1X.

The centralizer of the flow ψ_t (and hence any element of the flow) contains the flow itself. If X admits an isometry h, then the derivative Dh preserves the unit tangent bundle T^1X and commutes with the flow. While the flow fixes its own orbits, the derivative of a nontrivial isometry permutes the orbits nontrivially.

Suppose $g\colon T^1X \to T^1X$ is an arbitrary continuous map, and let $g_*\colon \pi_1(T^1X) \to \pi_1(T^1X)$ be the induced map on the fundamental group. We claim that g induces a homomorphism $\bar{g}_*\colon \pi_1(X) \to \pi_1(X)$ such that $\bar{g}_* p_* = p_* g_*$, where $p\colon T^1X \to X$ is the canonical projection. When $\dim(X) \geq 3$, this is immediate, because the fibers of T^1X are simply connected. When X is a surface, this follows from the fact that $\pi_1(T^1X)$ is a central extension of the simple group $\pi_1(X)$.

Note that since ψ_t is isotopic to the identity, it induces a trivial map on $\pi_1(X)$, whereas the derivative of a nontrivial isometry h induces a nontrivial automorphism $\overline{Dh}_*$ of $\pi_1(X)$, namely h_* itself. The latter automorphism h_* induces a nontrivial *outer* automorphism; that is, it is not induced by a conjugacy on $\pi_1(X)$. This is because, as we shall see,

homeomorphisms of T^1X that leave invariant the orbit foliation of ψ_t and that induce inner automorphisms of $\pi_1(X)$ must fix the leaves of the orbit foliation.

Proposition 35. *Let X be a closed, negatively curved manifold, and suppose that $g\colon T^1X \to T^1X$ is a homeomorphism that leaves invariant the orbit foliation of the geodesic flow ψ_t. The following are equivalent:*

(1) there exists $\hat{\gamma} \in \pi_1(X)$ such that $\bar{g}_*(\gamma) = \hat{\gamma}\gamma\hat{\gamma}^{-1}$, for every $\gamma \in \pi_1(X)$.
(2) g leaves invariant each orbit of ψ_t.

Proof. (1): Since g preserves the orbits of the geodesic flow, the map $\bar{g}_*$ has a simple discription: given $\gamma \in \pi_1(X)$, represent γ by a closed, unit-speed geodesic c_γ in X (here we are using free homotopy equivalence): this representation is unique up to reparametrization, because X is negatively curved. The lift c'_γ to T^1X is a closed orbit of φ_t and is taken to a closed orbit $\hat{c}'$ by g; the projection of this orbit to X is a closed geodesic $\hat{c} = c_{\bar{g}_*(\gamma)}$ representing the class $\bar{g}_*(\gamma)$.

Now suppose that there exists $\hat{\gamma} \in \pi_1(X)$ such that for every $\gamma \in \pi_1(X)$, $\bar{g}_*(\gamma) = \hat{\gamma}\gamma\hat{\gamma}^{-1}$. The group $\Gamma = \pi_1(X)$ acts freely on the universal cover $\widetilde{X}$ on the left by isometries. Since X is closed and negatively curved, each $\gamma \in \Gamma$ has a unique axis α_γ, which is a geodesic in $\widetilde{X}$, invariant under γ and on which γ acts by translations.

Denote by $\pi\colon \widetilde{X} \to X$ the covering projection. It is easy to see that

$$\pi^{-1}(c_\gamma) = \bigsqcup_{\eta\in\Gamma} \eta\alpha_\gamma = \bigsqcup_{\eta\in\Gamma} \alpha_{\eta\gamma\eta^{-1}}.$$

Denote by $\tilde{g}$ the action of g on lifted geodesics in $\widetilde{X}$, which is well-defined up to deck transformations. Then

$$\tilde{g}\left(\pi^{-1}(c_\gamma)\right) = \pi^{-1}(c_{\bar{g}_*\gamma}) = \bigsqcup_{\eta\in\Gamma} \alpha_{\eta\hat{\gamma}\gamma(\eta\hat{\gamma})^{-1}} = \pi^{-1}(c_\gamma).$$

Thus $g(c'_\gamma(\mathbb{R})) = c'_\gamma(\mathbb{R})$, for every closed ψ_t-orbit $c'_\gamma(\mathbb{R})$. Since X is closed and negatively curved, ψ_t-periodic orbits are dense in T^1X, and so g fixes all ψ_t-orbits.

(2) If g fixes all ψ_t-orbits, then by the argument for (1), we obtain that $\bar{g}_*$ preserves the conjugacy classes in $\pi_1(X)$ and thus must act by conjugation. □

Suppose that $f \in \mathrm{Diff}^r(T^1X), r \geq 1$ is a C^1-small perturbation of ψ_{t_0}. By Theorem 7, f is dynamically coherent, and $(f, \mathcal{W}^c)$ is leaf conjugate to $(\psi_{t_0}, \mathcal{W}^c_{\psi_{t_0}})$. Proposition 16 implies that for any $g \in \mathcal{Z}_1(f)$, $g(\mathcal{W}^*) = \mathcal{W}^*$, for $* \in \{u, c, s, cu, cs\}$.

Let $\mathcal{Z}_r^+(f)$ be the subgroup of $\mathcal{Z}_r(f)$ consisting of the elements that preserve the orientation of $\mathcal{W}^c$. Clearly $\mathcal{Z}_r^+(f)$ has finite index in $\mathcal{Z}_r(f)$. We denote by $\mathcal{Z}_r^c(f)$ the set of $g \in \mathcal{Z}_r^+(f)$ fixing the leaves of $\mathcal{W}^c(f)$. Observe that $\mathcal{Z}_r^c(f)$ is a normal subgroup of $\mathcal{Z}_r^+(f)$.

Proposition 36. *Let ψ_{t_0} be the discretized geodesic flow over a closed, negatively curved manifold X. There exists $\epsilon > 0$ such that for any $r \geq 1$, if $f \in \mathrm{Diff}^r(T^1X)$, and $d_{C^1}(f, \psi_{t_0}) < \epsilon$, then $\mathcal{Z}_r^+(f)/\mathcal{Z}_r^c(f)$ is isomorphic to a subgroup of the outer automorphism group* $\mathrm{Out}(\pi_1(X))$.

Proof. Consider the map that sends $g \in \mathcal{Z}_r^+(f)$ to $[\bar{g}_*] \in \mathrm{Out}(\pi_1(X))$. It suffices to prove that the kernel of this map is $Z_r^c(f)$. Suppose then that g lies in the kernel, i.e. that there exists $\hat{\gamma} \in \pi_1(X)$ such that $\bar{g}_*(\gamma) = \hat{\gamma}\gamma\hat{\gamma}^{-1}$, for all $\gamma \in \pi_1(X)$.

Let $h\colon T^1X \to T^1X$ be the leaf conjugacy between $(\mathcal{W}_f^c, \psi_f)$ and $(\mathcal{W}_{\psi_{t_0}}^c, \psi_{t_0})$, satisfying

$$h\left(\mathcal{W}_f^c(v)\right) = \mathcal{W}_{\psi_{t_0}}^c(h(v)),$$

for all $v \in T^1X$, and let $g_1 = h \circ g \circ h^{-1}$, which is a homeomorphism preserving the orbit foliation of ψ_t. Since h is homotopic to the identity, the induced maps $\bar{g}_*$ and $\bar{g_1}_*$ are the same (i.e., conjugacy by $\hat{\gamma}$). Proposition 35 implies that g_1 fixes the ψ_t orbits, and so g fixes the leaves of $\mathcal{W}^c$, i.e. $g \in \mathcal{Z}_r^c(f)$. Similarly, if $g \in \mathcal{Z}_r^c(f)$, then g lies in the kernel. □

Proposition 37. *Let X be a closed, negatively curved manifold. There exists $\epsilon > 0$ such that for any $r \geq 1$, if $f \in \mathrm{Diff}^r(T^1X)$, and $d_{C^1}(f, \psi_{t_0}) < \epsilon$, then the quotient $\mathcal{Z}_r^+(f)/\mathcal{Z}_r^c(f)$ is finite.*

Proof. The argument splits into two cases according to the dimension of X. In the first case, $\dim(X) \geq 3$, the outer automorphism group of $\pi_1(X)$ is finite, which immediately gives the conclusion. In the second case, $\dim(X) = 2$, the outer automorphism group is infinite, isomorphic to the extended mapping class group $\mathrm{Mod}^{\pm}(X)$, which contains the mapping class group $\mathrm{Mod}(X)$ as an index 2 subgroup. A further analysis of the dynamics of centralizer is required.

The case $\dim(X) \geq 3$**.** Work of Paulin and Sela [62, 79] shows that if X is closed and negatively curved, of dimension at least 3, then $\mathrm{Out}(\pi_1(X, p(v)))$ is finite: the fundamental group of X is a torsion-free hyperbolic group that does not admit an essential small action on a real tree (see [79, Corollary 0.2] and the discussion that follows). Thus Proposition 37 follows immediately from Proposition 36.

The case $\dim(X) = 2$**.** If X is a closed, negatively curved surface, then $\mathrm{Out}(\pi_1(X, p(v)))$ is isomorphic to the extended mapping class group, which since X is a surface, is the group of diffeomorphisms of X modulo homotopy equivalence. The following lemmas are well-known; we sketch their proofs for completeness.

Lemma 38. *Let X be a closed, negatively curved surface. Suppose that $h \in \mathrm{Mod}^{\pm}(X) \cong \mathrm{Out}(\pi_1(X, p(v)))$ has the property that for every conjugacy class $[\gamma]$ of $\gamma \in \pi_1(X, p(v))$, there exists $k \geq 1$ such that*

$$h^k[\gamma] = [\gamma].$$

Then h has finite order.

Proof. Represent h by a diffeomorphism $\hat{h}\colon X \to X$, and take a system of filling curves $\gamma_1, \ldots, \gamma_n$ in X. (These are closed curves with minimal intersection that separate X into a union of disks). Then some power of $\hat{h}$ fixes these curves (up to homotopy). Iterating further, some power $\hat{h}^L$ leaves invariant the disks bounded by the curves (up to homotopy). But then by coning off $\hat{h}^L$ in each disk, we get that $\hat{h}^L$ is homotopic to the identity in each disk, and so $\hat{h}^L$ is homotopic to the identity. Thus h^L is trivial. □

Lemma 39. *Let X be a closed, negatively curved surface, and let G be a subgroup of $Mod^{\pm}(X)$ with the property that every $h \in G$ has finite order. Then G is finite.*

We remark that there is no assumption that G be finitely generated in Lemma 39.

Proof of Lemma 39. Since $\mathrm{Mod}(X)$ has index 2 in $\mathrm{Mod}^{\pm}(X)$, it suffices to prove the statement for $G < \mathrm{Mod}(X)$. The Torelli group $\mathrm{Tor}(X)$ is the set of $g \in \mathrm{Mod}(X)$ that induce a trivial action on first homology $H^1(X, \mathbb{Z})$. We have the short exact sequence

$$1 \to \mathrm{Tor}(X) \to \mathrm{Mod}(X) \to \mathrm{Sp}\left(H^1(X, \mathbb{Z})\right) \cong \mathrm{Sp}_{2g}(\mathbb{Z}) \to 1,$$

where g is the genus of X, and $\mathrm{Sp}_{2g}(\mathbb{Z})$ is the integer symplectic group.

It is well-known that $\mathrm{Tor}(X)$ is torsion-free. Thus if $G < \mathrm{Mod}(X)$ is a torsion group, it is isomorphic to a subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$. But $\mathrm{Sp}_{2g}(\mathbb{Z})$ is arithmetic and thus contains a finite index torsion free normal subgroup H (for example, $H = \Gamma(3) = \{A \in \mathrm{Sp}_{2g}(\mathbb{Z}) : A \equiv I \mod 3\}$). But this implies that G injects into $\mathrm{Sp}_{2g}(\mathbb{Z})/H$, which is finite. Hence G is finite. □

We return to the proof of Proposition 37 in the case $\dim(X) = 2$. Suppose that $g \in \mathrm{Diff}^1(T^1X)$ commutes with f, a perturbation of the discretized geodesic flow ψ_{t_0}. Lemma 17 implies that every closed leaf of $\mathcal{W}^c_f$ is periodic under g. Thus $h = [\bar{g}_*] \in \mathrm{Out}(\pi_1(X))$ satisfies the hypotheses of Lemma 38 and hence has finite order. The image of the quotient $\mathcal{Z}^+_r(f)/\mathcal{Z}^c_r(f)$ in $\mathrm{Out}(\pi_1(X))$ is thus a torsion group, and so by Lemma 39 is finite. □

We remark that Proposition 36 and the discussion above also imply that for X negatively curved and locally symmetric, of dimension at least 3,

$$\mathcal{Z}^+_r(\psi_{t_0})/\mathcal{Z}^c_r(\psi_{t_0}) \cong \mathrm{Out}(\pi_1(X))/ < \pm\mathrm{id} >,$$

since by Mostow rigidity, every outer automorphism is represented by a unique isometry. With a little more work (see, e.g., [38]), one can show that for any $t_0 \neq 0$ the centralizer of ψ_{t_0} in $\mathrm{Diff}^1(T^1X)$ is precisely the group generated by the flow itself and the isometry group of X. The same holds for hyperbolic surfaces. Details are left to the reader.

Proof of Theorem 5. Let f be a diffeomorphism satisfying all the hypotheses of Theorem 5. By [46] and [72], ψ_{t_0} in Theorem 5 is stably accessible and hence stably ergodic (by, e.g. [13]), and so we may assume that f is accessible and ergodic. Lemma 11 then implies that $\mathcal{Z}_1(f) \subset \mathrm{Diff}_{\mathrm{vol}}(M)$. Proposition 37 implies that $\mathcal{Z}^+_1(f)$, and hence $\mathcal{Z}_1(f)$, is virtually $\mathcal{Z}^c_1(f)$.

Assume that the disintegration of vol along $\mathcal{W}^c_f$ leaves is not Lebesgue; we show that $\mathcal{Z}^c_1(f)$ is virtually $< f^n >$, which will complete the proof of Theorem 5. First, since $(f, \mathcal{W}^c_f)$ is leaf conjugate to ψ_{t_0}, all but countably many $\mathcal{W}^c_f-$leaves are noncompact. For any noncompact $\mathcal{W}^c_f-$leaf, we consider the total order " $<$ " induced by the canonical orientation on $\mathcal{W}^c_f$. The action of f on every non-compact $\mathcal{W}^c_f-$leaf is uniformly close to a translation by t_0 on $\mathbb{R}$, and therefore is topologically conjugate to a translation.

Theorem F in [3] implies that the disintegration of vol along $\mathcal{W}^c_f$ leaves is atomic: there is a full volume set $S \subset T^1X$ and $k \in \mathbb{Z}^+$ such that for almost every $v \in T^1X$, $\mathcal{W}^c_f(v)$ is non-compact,

$$S \cap \mathcal{W}^c_f(v) = \{x_{i,j}(v), i \in \mathbb{Z}, 1 \leq j \leq k\}, \tag{6}$$

and

$$f^i(v) \leq x_{i,1}(v) < x_{i,2}(v) < \cdots < x_{i,k}(v) < f^{i+1}(v),\ f(x_{i,j}(v)) = x_{i+1,j}(v). \tag{7}$$

Fix an arbitrary $g \in \mathcal{Z}^c_1(f)$. Lemma 11 implies that g is volume preserving, which implies that, modulo a zero set, $gS = S$. As a consequence, there is an $f-$invariant full volume set $\Omega \subset T^1X$ such that for any $v \in \Omega$,

- $\mathcal{W}^c_f(v)$ is noncompact;
- S meets $\mathcal{W}^c_f(v)$ in exactly k orbits and (6), (7) hold, i.e. we can define $x_{i,j}(v)$ associated to v;
- $g(S \cap \mathcal{W}^c_f(v)) = S \cap \mathcal{W}^c_f(v)$; and
- $f(S \cap \mathcal{W}^c_f(v)) = S \cap \mathcal{W}^c_f(v)$.

Since g preserves the orientation on $\mathcal{W}^c_f-$leaves, for any $v \in \Omega$, the restriction of g to $\mathcal{W}^c_f(v) \cap S(= \{x_{i,j}(v), i \in \mathbb{Z}, 1 \leq j \leq k\})$ is an order preserving transformation. By (7), for any $v \in \Omega$, both $g|_{\mathcal{W}^c_f(v) \cap S}, f|_{\mathcal{W}^c_f(v) \cap S}$ are conjugate to a translation on $\mathbb{Z}$.

In particular, for any $v \in \Omega$, there exists $k'(g, v) \in \mathbb{Z}$ such that on $W^c_f(v) \cap S$, we have $g^k = f^{k'(g,v)}$. Moreover by the construction of $x_{i,j}$, the fact that $fg = gf$ implies $k'(g, v)$ is an $f-$invariant function on v. Ergodicity of f then implies that $k'(g, v)$ is almost everywhere a constant $k'(g)$, and on a full measure subset of S, $g^k = f^{k'(g)}$. But any full measure subset of S is dense in T^1X, and hence $g^k = f^{k'(g)}$ on all of T^1X. In addition, any $g_1, g_2 \in \mathcal{Z}^c_1(f)$ satisfying $k'(g_1) = k'(g_2)$ must induce the same transformation on $S \cap \mathcal{W}^c_f(v)$ for almost every $v \in T^1X$, which implies that $g_1 = g_2$. Therefore k' induces a group embedding

$$k' : \mathcal{Z}^c_1(f) \to \mathbb{Z},$$

and $k'(< f^n >) = k\mathbb{Z}$. Then $\mathcal{Z}^c_1(f)$ is virtually $< f^n >$, proving Theorem 5 □

Proof of Theorem 3. Returning to the proof of Theorem 3, if the volume has singular disintegration along $\mathcal{W}^c_f$, then Theorem 3 is just a corollary of Theorem 5.

Suppose now the volume has Lebesgue disintegration along $\mathcal{W}^c_f$. Theorem F in [3] implies that there is a continuous vector field Y tangent to $\mathcal{W}^c_f$ such that the continuous flow (a priori it might not be smooth) φ_t generated by Y satisfies the following:

- $\varphi_1 = f$, and
- Y, and hence φ_t, is uniformly smooth along the leaves of $\mathcal{W}^c_f$.

By assumption, ψ_{t_0} has either 2-bunched or narrow band spectrum. Let $r_0 = r(\psi_{t_0}) \geq 1$. Fix $r > r_0$; we may assume, by Lemma 27, that f has either 2-bunched or narrow band spectrum, and $r(f) < r$. Consider $h \in \mathcal{Z}^c_r(f)$.

By ergodicity of f, h preserves the disintegration of volume along $\mathcal{W}^c_f$. Therefore $h = \varphi_t$ for some $t \in \mathbb{R}$. If follows that

$$\mathcal{Z}^c_r(f) = \{\varphi_t, t \in D\}, \text{ where } D := \{t \in \mathbb{R} : \varphi_t \in \mathrm{Diff}^r(T^1X)\}. \tag{8}$$

Since $f = \varphi_1$ is C^∞, it follows that D is a non-empty subgroup of $\mathbb{R}$, and by Proposition 37, $\mathcal{Z}_r(f)$ contains $\{\varphi_t : t \in D\}$ as a finite index subgroup.

Case 1: D is discrete. Then, since $f = \varphi_1$, it follows that $< f >$ has finite index in $\{\varphi_t : t \in D\}$, and hence in $\mathcal{Z}_r(f)$. Thus f has virtually trivial centralizer in $\mathrm{Diff}^r(T^1X)$.

Case 2: D is dense in $\mathbb{R}$. We use the normal form theory from Section 3.6 to show that the C^∞ smoothness of the φ_t with $t \in D$ extends to all $t \in \mathbb{R}$, as follows. Applying Proposition 28 to the triple (f, φ_t, Y), we obtain that $D = \mathbb{R}$, Y is a C^∞ vector field and φ_t is a C^∞ flow. As a consequence, by (8) for any $s \geq r$ we have

$$\mathcal{Z}^c_r(f) = \{\varphi_t : t \in \mathbb{R}\} \subset \mathcal{Z}^c_s(f) \subset \mathcal{Z}^c_r(f),$$

which implies $\mathcal{Z}^c_s(f) = \{\varphi_t : t \in \mathbb{R}\}$. Thus by Proposition 37 for any $s \geq 1$, $\mathcal{Z}^+_s(f)$ hence $\mathcal{Z}_s(f) = \mathcal{Z}_{\mathrm{Diff}^s_{\mathrm{vol}}(T^1X)}(f)$ is virtually $\{\varphi_t : t \in \mathbb{R}\} \cong \mathbb{R}$. □

5. Proof of Theorem 6

As mentioned in the introduction, the key idea in the proof of Theorem 6 is to show existence of many partially hyperbolic elements commuting with f, an argument that we now detail.

5.1. **The groups G and G_0.** Two central players in the proof of Theorem 6 are groups G and G_0, which we define in this subsection. We start with an easy observation.

For f_0 as in Theorem 6, we denote by $\lambda^1(f_0) > \cdots > \lambda^i(f_0) > \cdots$ the *distinct* Lyapunov exponents of f_0 and the corresponding Df_0-invariant Lyapunov splitting by

$$T\mathbb{T}^d = \oplus E^i_{f_0} \oplus E^c_{f_0}. \tag{9}$$

Since f is C^1-close to f_0, it follows that there is a corresponding $Df-$invariant dominated splitting

$$T\mathbb{T}^d = \oplus E^i_f \oplus E^c_f \tag{10}$$

and f-invariant foliations $\mathcal{W}^s, \mathcal{W}^u, \mathcal{W}^{cs}, \mathcal{W}^{cu}$, and $\mathcal{W}^c$.

Consider an arbitrary element $g \in \mathcal{Z}_2(f)$. Proposition 16 implies that $g\mathcal{W}^c = \mathcal{W}^c$. Thus f, g induce homeomorphisms $\bar{f}, \bar{g}$ on the topological manifold $\mathbb{T}^d/\mathcal{W}^c$ such that $\bar{f}\bar{g} = \bar{g}\bar{f}$. Moreover $\bar{f}$ is Hölder conjugate to the hyperbolic automorphism T_A on $\mathbb{T}^{d-1}$. By Lemma 14, $\bar{g}$ is conjugate to an affine map by the same conjugacy. For $g \in \mathcal{Z}_2(f)$, we denote the linear part of this affine map by T_{A_g} , where $A_g \in \mathrm{GL}(d-1, \mathbb{Z})$. In particular we have $A_f = A$.

Let $\pi \colon \mathbb{T}^d \to \mathbb{T}^{d-1}$ be the fibration given by Proposition 18, which satisfies $\pi \circ f = T_{A_f} \circ \pi$. Then the center leaf $\pi^{-1}(0)$ is invariant under f; denote it by $\mathcal{W}^c_f(x_0)$. We use this leaf to define G and G_0.

Definition 9. Let G_0 be the group of all the elements $g \in \mathcal{Z}_2(f)$ such that g fixes $\mathcal{W}^c_f(x_0)$ and preserves the orientation of $\mathcal{W}^c$ and $\mathbb{T}^d/\mathcal{W}^c$. Let $G < \mathrm{SL}(d-1,\mathbb{Z})$ be the group generated by $\{A_g : g \in G_0\}$.

The next proposition lays out the properties of G and G_0 that we will use here.

Proposition 40. *Suppose f, ℓ_0 satisfy the hypotheses of Theorem 6. Then*

(1) $\mathcal{Z}_2(f)$ is virtually G_0.
(2) G_0, G are abelian groups. If the disintegration of vol *along $\mathcal{W}^c$ is not Lebesgue, then G_0 is finitely generated.*
(3) One or both of the following cases holds:
  I. *G is virtually $\mathbb{Z}^\ell$ for some $\ell \leq \ell_0$, where $\ell < \ell_0$ if $\ell_0 > 1$.*
  II. *G is a finite index subgroup of $\mathcal{Z}_{\mathrm{SL}(d-1,\mathbb{Z})}(A_f)$. In particular, G induces a maximal Anosov action on $\mathbb{T}^{d-1}$ if $\ell_0 > 1$.*

Proof of Proposition 40. (1) Let $\mathcal{Z}^+$ be the group of all the elements $g \in \mathcal{Z}_2(f)$ such that g preserves the orientation of $\mathcal{W}^c$ and $\mathbb{T}^d/\mathcal{W}^c$. Clearly $\mathcal{Z}^+$ has finite index in $\mathcal{Z}_2(f)$. Denote by $\mathcal{Z}^c$ the set of elements of $\mathcal{Z}^+$ which fix the leaves of the center foliation $\mathcal{W}^c$.

Consider the map from $\mathcal{Z}^+$ to $\mathcal{Z}_{\mathrm{Homeo}^+(\mathbb{T}^{d-1})}(T_{A_f})$, sending g to the map induced by g. The kernel is $\mathcal{Z}^c$, and so $\mathcal{Z}^+/\mathcal{Z}^c$ is isomorphic to a subgroup of $\mathcal{Z}_{\mathrm{Homeo}^+(\mathbb{T}^{d-1})}(T_{A_f})$. By Lemmas 14 and 15, the group $\mathcal{Z}_{\mathrm{Homeo}^+(\mathbb{T}^{d-1})}(T_{A_f})$ is virtually $\mathbb{Z}^m$, for some m, and hence $\mathcal{Z}^+/\mathcal{Z}^c$ is virtually $\mathbb{Z}^{m'}$, for some m'.

Note that since there are finitely many center leaves fixed by f, and each element of $\mathcal{Z}^+$ permutes the fixed center leaves, there exists $k \geq 1$ such that for every element $g \in \mathcal{Z}^+/\mathcal{Z}^c$, we have $g^k \in G_0/\mathcal{Z}^c$. Thus the finitely generated, abelian quotient

$$\frac{\mathcal{Z}^+/\mathcal{Z}^c}{G_0/\mathcal{Z}^c} \cong \mathcal{Z}^+/G_0$$

has the property that every element has order at most k, and is therefore finite. This proves that G_0 has finite index in $\mathcal{Z}^+$, as claimed.

(2) Since A_f is irreducible, Lemma 15 implies that $\mathcal{Z}_{\mathrm{SL}(d-1,\mathbb{Z})}(A_f)$ (hence G) is a finitely generated abelian group.

To study the group G_0, first we consider the group $\mathcal{Z}^c$ defined as in the proof of (1). For any $h \in \mathcal{Z}^c$, the rotation number $\rho(h,x) \in \mathbb{T}$ is well-defined for $h|_{\mathcal{W}^c(x)}$ for any $x \in \mathbb{T}^d$. By commutativity, it is not hard to get $\rho(h,x) = \rho(h,f(x))$. Since the rotation number is a continuous funtion on diffeomorphisms, the ergodicity of f implies $\rho(h) = \rho(h,x)$ is independent of x. Moreover we have

Lemma 41. *The map $\rho : \mathcal{Z}^c \to \mathbb{T},\ h \mapsto \rho(h)$ is a group embedding. In particular for any $h \in \mathcal{Z}^c$, if there exists $x \in \mathbb{T}^d$ such that $\rho(h,x) = 0$ then $h = \mathrm{id}$.*

Proof of Lemma 41. By Proposition 18 we have three possibilities:

Case 1: The volume $\mathrm{vol}_{\mathbb{T}^d}$ has atomic disintegration along $\mathcal{W}^c$. Lemma 9 implies that $\mathcal{Z}^c < \mathcal{G}_{\mathrm{fix}}(\mathcal{W}^c_f)$ is an abelian group, and therefore $\rho : \mathcal{Z}^c \to \mathbb{T}$ is a group homomorphism. Moreover for $h \in \mathcal{Z}^c$, if $\rho(h) = 0$, then by the proof of Lemma 9, h fixes all the atoms, which are dense in $\mathbb{T}^d$. Thus $h = \mathrm{id}$.

Case 2: f is topologically conjugate to $T_{A_f} \times R_\theta$ for some $\theta \notin \mathbb{Q}/\mathbb{Z}$. Let ζ be the conjugacy, so that $\zeta^{-1} \circ f \circ \zeta(x, y) = (T_{A_f}(x), y + \theta)$. Fix $h \in \mathcal{Z}^c$. Since h is center-fixing, there exists a continuous function $R(x, y) = R(h, x, y)$ such that $\zeta^{-1} \circ h \circ \zeta(x, y) = (x, y + R(x, y))$.

Since h commutes with f, $R(x, y)$ is $T_{A_f} \times R_\theta$-invariant. Transitivity of $T_{A_f} \times R_\theta$ implies that $R(x, y)$ is a constant function. Therefore for any $h \in \mathcal{Z}^c$, we have $\zeta^{-1} \circ h \circ \zeta = \mathrm{id} \times R_{\rho(h)}$, which implies Lemma 41.

Case 3: f is accessible, and the disintegration of $\mathrm{vol}_{\mathbb{T}^d}$ has a continuous density function on the leaves of $\mathcal{W}^c$. Then [3, Theorem C] implies that f is topologically conjugate to a rotation extension over $(T_{A_f})_r$, i.e. there exist a continuous function $r(x) = r(x, y)$ and a homeomorphism $\zeta : \mathbb{T}^d \to \mathbb{T}^d$ such that $\zeta^{-1} \circ f \circ \zeta(x, y) = (T_{A_f}(x), y + r(x))$.

For any $h \in \mathcal{Z}^c$, as in Case 2. we can assume that there exists a $(T_{A_f})_r-$invariant, continuous function $R(x, y) = R_h(x, y)$ such that $\zeta^{-1} \circ h \circ \zeta(x, y) = (x, y + R(x, y))$. Then by transitivity of $(T_{A_f})_r$ (which follows from transitivity of f), we have $\zeta^{-1} \circ h \circ \zeta = \mathrm{id} \times R_{\rho(h)}$, for any $h \in \mathcal{Z}^c$, which implies Lemma 41.

□

Returning to the proof of item (2) of Proposition 40, we obtain from Lemma 41 that $\mathcal{Z}^c$ is an abelian group. Observe that the map $h \mapsto A_h$ is a surjective homomorphism from G_0 to G, with kernel $\mathcal{Z}^c$, and therefore G_0 is a group extension of G by $\mathcal{Z}^c$. By commutativity of $\mathcal{Z}^c$ and G, we have that G_0 is a solvable group and $[G_0, G_0] \subset \mathcal{Z}^c$.

Now we claim that $[G_0, G_0]$ is trivial, and so G_0 is abelian. Suppose there exists $h \in [G_0, G_0] \subset \mathcal{Z}^c$, $h \neq \mathrm{id}$. Lemma 41 implies that $h|_{\mathcal{W}^c_f(x_0)}$ has non-zero rotation number, where $\mathcal{W}^c_f(x_0)$ is the G_0-fixed center leaf we defined in Section 5.1. On the other hand, $G_0|_{\mathcal{W}^c_f(x_0)}$ is a solvable, orientation-preserving action on a circle. It is known (cf. [57]) that rotation number induces a group homomorphism from any solvable subgroup of $\mathrm{Homeo}^+(S^1)$ to $\mathbb{T}^1$, and so the kernel contains $[G_0, G_0]$. Thus $h|_{\mathcal{W}^c_f(x_0)}$ must have rotation number 0, which is a contradiction.

To show that G_0 is finitely generated if the disintegration of $\mathrm{vol}_{\mathbb{T}^d}$ along $\mathcal{W}^c_f$ is not Lebesgue, we only need to show the following lemma, since G_0 is a group extension of G by $\mathcal{Z}^c$ and G is finitely generated.

Lemma 42. *The group $\mathcal{Z}^c$ is finite if the disintegration of $\mathrm{vol}_{\mathbb{T}^d}$ along $\mathcal{W}^c_f$ is not Lebesgue.*

Proof. Proposition 18 gives two possibilities.

If conclusion 2a holds, i.e. the volume $\mathrm{vol}_{\mathbb{T}^d}$ has atomic disintegration along $\mathcal{W}^c$, then the finiteness follows directly from Lemma 9.

If 2c holds, then f is conjugate to $T_{A_f} \times R_\theta$ for some $\theta \notin \mathbb{Q}/\mathbb{Z}$. In this case E^u_f and E^s_f are jointly integrable. Let $\mathcal{W}^H$ be the compact foliation tangent to the distribution $E^u_f \oplus E^s_f$.

Let ζ be the conjugacy satisfying $\zeta^{-1} \circ f \circ \zeta = T_{A_f} \times R_\theta$. Then as in the proof of Lemma 41, for any $h \in \mathcal{Z}^c$, we have $\zeta^{-1} \circ h \circ \zeta = \mathrm{id} \times R_{\rho(h)}$. Let

$$D := \{\rho \in \mathbb{T} : \zeta \circ (\mathrm{id} \times R_\rho) \circ \zeta^{-1} \in \mathcal{Z}^c\}.$$

If D is discrete, then $\mathcal{Z}^c$ is finite. If D is dense, we will prove that in this case $\mathrm{vol}_{\mathbb{T}^d}$ has Lebesgue disintegration along $\mathcal{W}^c$, which contradicts our assumption above. By density of D, any measure on $\mathbb{T}$ invariant under $\{R_\rho : \rho \in D\}$ is the Lebesgue measure $\mathrm{vol}_{\mathbb{T}}$. Recall that $\mathrm{vol}_{\mathbb{T}^d}$ is $\mathcal{Z}^c-$invariant, therefore $\mathrm{vol}_{\mathbb{T}^d}$ has the form $\zeta_*(\nu \times \mathrm{vol}_{\mathbb{T}})$, where ν is some probability measure on $\mathbb{T}^{d-1}$.

In particular, if we denote by Pr^c the projection from $\mathbb{T}^d$ to $\mathcal{W}^c(x_0)$ along $\mathcal{W}^H$ and Pr^H the canonical projection from $\mathbb{T}^d$ to $\mathbb{T}^d/\mathcal{W}^c$, we have that any $\mathcal{Z}^c-$invariant measure μ is the product of $\mathrm{Pr}^c_*(\mu)$ with $\mathrm{Pr}^H_*(\mu)$. In particular, for almost every x, the conditional measure m^c_x on $\mathcal{W}^c(x)$ of $\mathrm{vol}_{\mathbb{T}^d}$ has the following form

$$m^c_x = \mathrm{Pr}^c|^*_{\mathcal{W}^c(x)}(\mathrm{Pr}^c_*(\mathrm{vol}_{\mathbb{T}^d}));$$

that is, m^c_x is the pullback of $\mathrm{Pr}^c_*(\mathrm{vol}_{\mathbb{T}^d})$ on $\mathcal{W}^c(x)$ by $\mathrm{Pr}^c|_{\mathcal{W}^c(x)}$.

By Lemma 21, we have the key fact that $\mathcal{W}^H$ is a C^1 foliation. It follows that Pr^c is C^1, and so $\mathrm{Pr}^c|^*_{\mathcal{W}^c(x)}(\mathrm{Pr}^c_*(\mathrm{vol}_{\mathbb{T}^d}))$ has continuous density function for any x. This implies that $\mathrm{vol}_{\mathbb{T}^d}$ has Lebesgue disintegration along $\mathcal{W}^c$ leaves. □

This completes the proof of item (2). Item (3) is a corollary of Lemma 15: for more details, see [71]. The proof of Proposition 40 is complete. □

Having defined the groups G and G_0 and established their essential properties, we return to the proof of Theorem 6. We are given f sufficiently C^1 close to f_0 and aim to prove that either the vol Lebesgue disintegration along $\mathcal{W}^c_f$, or $\mathcal{Z}_2(f)$ is virtually $\mathbb{Z}^\ell$ for some $\ell \leq \ell_0$, with $\ell < \ell_0$ if $\ell_0 > 1$.

By Proposition 40, item (3), there are two possibilities:

I. G is virtually $\mathbb{Z}^\ell$ for some $\ell \leq \ell_0$, where $\ell < \ell_0$ if $\ell_0 > 1$.
II. G is a finite index subgroup of $\mathcal{Z}_{\mathrm{SL}(d-1,\mathbb{Z})}(A_f)$. In particular, G induces a maximal Anosov action on $\mathbb{T}^{d-1}$ if $\ell_0 > 1$.

Suppose that conclusion I. holds. Lemma 42 implies that either $\mathrm{vol}_{\mathbb{T}^d}$ has Lebesgue disintegration along $\mathcal{W}^c$; or $\mathrm{vol}_{\mathbb{T}^d}$ has singular disintegration along $\mathcal{W}^c$ and $\mathcal{Z}^c$ is finite. In the former case, we are finished. In the latter case, item (2) of Proposition 40 implies that G_0 is a finitely generated abelian group and also a group extension of G by $\mathcal{Z}^c$. As we are assuming that G is virtually $\mathbb{Z}^\ell$ for some $\ell \leq \ell_0 = \ell_0(A_f)$ ($\ell < \ell_0$ if $\ell_0 > 1$), tt is not hard to check that there is a subgroup G_1 of G_0 isomorphic to the torsion free part of

G, which is $\mathbb{Z}^\ell$. Therefore by finiteness of $\mathcal{Z}^c$, G_0 is virtually $\mathbb{Z}^\ell$. Thus Theorem 6 follows from conclusion (1) of Proposition 40.

Suppose on the other hand that conclusion II of Proposition 40, item (3), holds. The case $\ell_0 = 1$ is contained in conclusion I, so we may assume that $\ell_0 > 1$. We have the following key proposition.

Proposition 43. *Suppose f is as in Theorem 6, and G_0, G are as in Definition 9. If G induces a maximal Anosov action on $\mathbb{T}^{d-1}$, then $\mathrm{vol}_{\mathbb{T}^d}$ has Lebesgue disintegration along $\mathcal{W}^c_f$.*

Assuming this proposition, the proof of Theorem 6 is complete. The proof of Proposition 43 is lengthy and occupies the next section.

6. Proof of Proposition 43

This section is devoted to the proof of Proposition 43. We continue to assume that $f_0 : \mathbb{T}^d \to \mathbb{T}^d$ and ℓ_0 are as in Theorem 6, and that $f \in \mathrm{Diff}^2_{\mathrm{vol}}(\mathbb{T}^d)$ is a C^1-small, ergodic perturbation of f_0. In addition, we assume the hypothesis of Proposition 43, that G induces a maximal Anosov action on $\mathbb{T}^{d-1}$. Our goal is to show that $\mathrm{vol}_{\mathbb{T}^d}$ has Lebesgue disintegration along $\mathcal{W}^c_f$.

Without loss of generality we may assume that G and G_0 are finitely generated abelian groups (otherwise Proposition 43 follows from Proposition 40). Then there is a subgroup G_1 of G_0 isomorphic to G, through the map $g \mapsto A_g$. Replacing G_0 with G_1, we may thus assume that G_0 is isomorphic to G through the map $g \mapsto A_g$. Moreover we may assume G, G_0 are torsion free (otherwise we consider their free parts instead).

The following proposition is the key step in the proof of Proposition 43. Recall that by linearity of the action of G, we can define the Lyapunov functionals and associated (hyperbolic) Weyl chamber picture as in Section 3.7, independently of the invariant measure. Consider the G_0-invariant dominated splitting $\oplus_i E^i \oplus E^c$ given by (10), ordered in i by decreasing values of the Lyapunov exponents.

Proposition 44. *Assume that G induces a maximal Anosov action on $\mathbb{T}^{d-1}$. For every i, we have the following.*

1. *The bundle E^i_f is uniquely integrable, tangent to an absolutely continuous foliation $\mathcal{W}^i_f$ with C^2 leaves.*
2. *The restriction of $\pi\colon \mathbb{T}^d \to \mathbb{T}^{d-1}$ to each leaf of $\mathcal{W}^i_f$ is a bi-Hölder homeomorphism onto the leaf of the affine foliation tangent to $E^i_{A_f}$, with exponent δ, where $\delta \to 1$ as $d_{C^1}(f, f_0) \to 0$. Consequently π itself is δ-Hölder continuous as well.*
3. *For any $h \in G_0$ such that A_h is not in any Weyl chamber wall of the action of G, Dh uniformly contracts or expands E^i.*

The rest of Section 5 is dedicated to the proofs of Propositions 43 and 44. The plan of the proofs is as follows: in Section 6.1 we prove the fundamental property of G_0, namely

that G and G_0 share the same Weyl chamber picture. Then in Sections 6.2 and 6.3 we prove Proposition 44. In Section 6.4 we derive from Proposition 44 an important corollary: the joint integrability of E^s_f and E^u_f. In particular, this implies that f is Hölder conjugate to $T_{A_f} \times R_\theta$ for some $\theta \notin \mathbb{Q}/\mathbb{Z}$; see Proposition 55.

In Section 6.5 we consider a partially hyperbolic generalization of a classical result of thermodynamic formalism in the Anosov setting. Combining a cocycle rigidity result (see Section 6.6) over a partially hyperbolic abelian action, using Proposition 55 we complete the proof of Proposition 43 in Section 6.7.

6.1. G, G_0 have the same hyperbolic Weyl chamber picture.

Lemma 45. *Any G_0-invariant ergodic measure ν has the same hyperbolic Weyl chamber picture as G.*

Proof of Lemma 45. First we prove that the action of (G_0, ν) has the same Weyl chamber walls as G. Proposition 16 implies that the foliations $\mathcal{W}^u_f$ and $\mathcal{W}^s_f$ are G_0-invariant. Moreover $\pi(\mathcal{W}^*_f) = \mathcal{W}^*_{A_f}$, for $* \in \{u, s\}$. Therefore to analyze the hyperbolic Weyl chamber walls of G_0 we need only consider the action of G_0 on $\mathcal{W}^u_f$, $\mathcal{W}^s_f$ separately. We show this for $\mathcal{W}^u$; the proof for $\mathcal{W}^s$ is analogous.

Recall that by Lemma 25, to prove Lemma 45, we only need to establish the following claim: for any $A_h \in G$ that is not in any Weyl chamber wall, if A_h has d^u_-, d^u_+- dimensional stable and unstable distributions respectively within $\mathcal{W}^u_{A_f}$, then h has d^u_-, d^u_+- dimensional stable and unstable *topological foliations* (with exponential contracting or expanding speed) respectively within $\mathcal{W}^u_f$.

In fact if the claim holds, then Lemma 25 implies that for typical $h \in G$, the map h and the matrix A_h have the same number of positive (resp. negative) Lyapunov exponents with respect to any G_0-invariant ergodic measure ν. From this it follows that G and (G_0, ν) have the same Weyl chamber walls.

Notice that for any $x \in M$, the restriction $\pi : \mathcal{W}^{u(s)}_f(x) \to \mathcal{W}^{u(s)}_{A_f}(\pi(x))$ is a homeomorphism. To finish the proof of the claim, we use the following classical bi-Hölder estimate on the projection π, which we will use repeatedly to lift hyperbolicity of elements acting in $\mathbb{T}^{d-1}$ to hyperbolicity in $\mathbb{T}^d$.

Lemma 46. *There exist $C, \delta > 0$ such that for any $x \in \mathbb{T}^d$ and $y \in \mathcal{W}^{u(s)}_f(x, loc)$,*

$$d_{\mathbb{T}^{d-1}}(\pi(x), \pi(y)) \le C \cdot d_{\mathbb{T}^d}(x, y)^\delta, \quad \text{and} \quad d_{\mathbb{T}^d}(x, y) \le C \cdot d_{\mathbb{T}^{d-1}}(\pi(x), \pi(y))^\delta.$$

Lemma 46 is a simple consequence of Theorem 7. This bi-Hölderness of π implies that the hyperbolicity of $T_{A_h}|_{\mathcal{W}^u_{T_{A_f}}}$ lifts under π to uniform hyperbolicity of $h|_{\mathcal{W}^u_f}$. Consequently, G and (G_0, ν) have the same Weyl chamber walls.

Next consider the Lyapunov functionals $\{\lambda^{u,i}_{G_0}(\cdot, \nu), i = 1, \dots, \dim E^u_f\}$ associated to the action of G_0 on $\mathcal{W}^u_f$ with respect to an ergodic measure ν, and the Lyapunov functionals

$\{\lambda_G^{u,i}(\cdot), i = 1, \dots, \dim E^u_{T_{A_f}}\}$ associated to the action of G on $\mathcal{W}^u_{T_{A_f}}$. By our discussion above, without loss of generality we may assume that Weyl chamber wall $\ker \lambda^{u,i}_{G_0}(\cdot, \nu)$ coincides with that of $\lambda^{u,i}_G(\cdot)$. Moreover

$$\lambda^{u,i}_{G_0}(f, \nu) > 0, \text{ and } \lambda^{u,i}_G(T_{A_f}) > 0.$$

then by Lemma 29 (identifying G, G_0 with $\mathbb{Z}^k$ in the obvious way), the Weyl chamber picture of the action of G_0 on $\mathcal{W}^u_f$ with respect to ν is the same as that of G on $\mathcal{W}^u_{T_{A_f}}$. The same argument applied to the action of G_0 on $\mathcal{W}^s_f$ gives that the Weyl chamber picture of the action of G_0 on $\mathcal{W}^s_f$ with respect to μ is the same as that of G on $\mathcal{W}^s_{T_{A_f}}$. In conclusion, (G_0, ν) has the same hyperbolic Weyl chamber picture as G. This completes the proof of Lemma 45. □

6.2. **Estimates for elements in the same Weyl chamber.** We continue our analysis of the dynamics of G relative to the Weyl chamber picture.

Lemma 47. *Suppose $h \in G_0$ has the property that A_h and A_f lie in the same Weyl chamber. Then*

(1) there exists $c > 0$ such that for any $h-$invariant ergodic measure ν,

$$\lambda^{\max}(Dh|_{E^s_f}, \nu) < -c, \text{ and } \lambda^{\min}(Dh|_{E^u_f}, \nu) > c;$$

(2) for every i, Dh either uniformly contracts or uniformly expands E^i.

Proof. (1): If A_h is in the same Weyl chamber as A_f, then as in Lemma 45 we have that T_{A_h} uniformly contracts $\pi(\mathcal{W}^s_f)$ and uniformly expands $\pi(\mathcal{W}^u_f)$. By Lemma 46, for any $x \in \mathbb{T}^d$ and $y \in \mathcal{W}^s_f(x, loc)$,

$$\limsup_{n\to\infty} \frac{1}{n} \log d_{\mathcal{W}^s_f}(h^n(x), h^n(y)) \le \delta \cdot \lambda^{\max}(T_{A_h}|_{\pi(\mathcal{W}^s_f)}) < 0, \tag{11}$$

where δ is the Hölder exponent of π in Lemma 46. Then for any $h-$invariant ergodic measure ν, Lemma 25 and (11) together imply that for $\nu-$almost every $x \in \mathbb{T}^{n+1}$, the Pesin stable manifold passing through x is $\mathcal{W}^s_f(x, loc)$, and

$$\lambda^{\max}(Dh|_{E^s_f}, \nu) \le \delta\lambda^{\max}(T_{A_h}|_{\pi(\mathcal{W}^s_f)}) < 0.$$

Similarly, for any $h-$invariant ergodic measure ν, we have

$$\lambda^{\min}(Dh|_{E^u_f}, \nu) \ge \delta\lambda^{\min}(T_{A_h}|_{\pi(\mathcal{W}^u_f)}) > 0.$$

Setting $c := \min(|\delta\lambda^{\max}(T_{A_h}|_{\pi(\mathcal{W}^s_f)})|, \delta\lambda^{\min}(T_{A_h}|_{\pi(\mathcal{W}^u_f)}))$ completes the proof of (1).

(2): Since $Dh|_{E^s_f}, Dh|_{E^u_f}$ are continuous, item (1) of Lemma 47 implies that $Dh|_{E^s_f}$ and $Dh|_{E^u_f}$ satisfy the conditions of Lemma 10. Thus $Dh|_{E^u_f}$ and $Dh^{-1}|_{E^s_f}$ have uniform exponential growth, which implies (2). □

Thus we have shown that Proposition 44 holds in one case:

$$\begin{aligned} & h \text{ and } f \text{ lie in the same hyperbolic Weyl chamber} \\ \iff \quad & A_h \text{ and } A_f \text{ lie in the same Weyl chamber.} \end{aligned}$$

6.3. **Proof of Proposition 44.** In this section we will prove Proposition 44 for those h for which A_h and A_f lie in different Weyl chambers.

Recall that we have ordered the bundles E^i_f and $E^i_{f_0}$ in i by decreasing size of Lyapunov exponents. Write

$$E^u_f = E^1_f \oplus E^2_f \oplus \cdots \oplus E^k_f, \quad \text{and } E^s_f = E^{k+1}_f \oplus \cdots \oplus E^\ell_f,$$

and for $i \leq j$, let

$$E^{[i,j]}_f := E^i_f \oplus E^2_f \oplus \cdots \oplus E^j_f$$

An immediate application of the normally hyperbolic theory in [35] implies that for every $i \in [1,k]$, $E^{[i,k]} \oplus E^c$ is integrable, tangent to an f-invariant foliation $\mathcal{W}^{[i,k]c}_f$ that projects under π to the affine foliation $\mathcal{W}^{[i,k]}_{A_f}$ tangent to $E^i_{A_f} \oplus \cdots \oplus E^k_{A_f}$. Further application of [35] gives the following.

Lemma 48. *For every $i \subset [1,k]$, there is an f-invariant foliation $\mathcal{W}^{[i,k]}_f$ with the following properties*

(1) $E^{[i,k]}_f$ is uniquely integrable, tangent to $\mathcal{W}^{[i,k]}_f$, and
(2) $\mathcal{W}^{[i,k]}_f$ and $\mathcal{W}^c$ are jointly integrable, tangent to the foliation $\mathcal{W}^{[i,k]c}_f$; the restriction of π to $\mathcal{W}^{[i,k]}_f$ is a bi-Hölder homeomorphism onto $\mathcal{W}^{[i,k]}_{A_f}(\pi(x))$.

Proof. For fixed $i \in [1,k]$ we apply the graph transform argument for f in restriction to the disjoint union of the leaves of $\mathcal{W}^{[i,k]c}_f$: as the splitting $E^{[i,k]}_f \oplus E^c_f$ is dominated, and since $E^{[i,k]}_f$ is uniformly expanded, it is uniquely integrable, tangent to a foliation $\mathcal{W}^{[i,j]}_f$. By construction $\mathcal{W}^{[i,j]}_f$ and $\mathcal{W}^c_f$ are jointly integrable. □

For $j \in [1,k]$, the bundle $E^{[1,j]}_f$ is a strong unstable bundle for f and therefore uniquely integrable, tangent to a foliation $\mathcal{W}^{[1,j]}_f$. By intersecting foliations $\mathcal{W}^{[1,j]}_f$ and $\mathcal{W}^{[i,k]}_f$ we obtain f-invariant foliations $\mathcal{W}^{[i,j]}_f$ tangent to the uniquely integrable bundle $E^{[i,j]}_f$, for any interval $[i,j] \subset [1,k]$. We denote by $\mathcal{W}^m_f$ the foliation $\mathcal{W}^{[m,m]}_f$, which is tangent to E^m_f. Lemma 13 implies that the foliations $\mathcal{W}^{[i,j]}_f$ are G_0-invariant.

Note that except in the case $j = k$, we do not know that $\mathcal{W}^{[i,j]}_f$ projects to under π to $\mathcal{W}^{[i,j]}_{A_f}$ (and *a priori* for a single f this will not be the case). We will need to use the maximality of the G_0-action to establish this.

Set $h_0 = f$. Since G induces a maximal Anosov action on $\mathbb{T}^{d-1}$, Lemma 30 implies that there exists a Weyl chamber adjacent to that of A_f such that for any element $h_i \in G_0$ with A_{h_i} in this chamber, the signs of all the exponents of A_f and A_{h_i} are the same except one exponent corresponding to $E^i_{f_0}$. By this process we produce elements $h_1, \dots, h_k \in G_0$. We prove the following statement inductively, for $i = 1, \dots, k$.

Inductive hypotheses (i):

- (A_i) $\mathcal{W}^{[i,k]}_f$ is absolutely continuous, with C^2 leaves,
- (B_i) $\pi(\mathcal{W}^{[i,k]}_f) = \mathcal{W}^{[i,k]}_{A_f}$, and $\pi(\mathcal{W}^i_f) = \mathcal{W}^i_{A_f}$. The restriction of π to $\mathcal{W}^i_f$ leaves is a bi-Holder homeomorphism onto $\mathcal{W}^i_{A_f}-$ leaves with Hölder exponent $\delta \to 1$ as $d_{C^1}(f_0, f) \to 0$.
- (C_i) Dh_i uniformly contracts E^i_f and expands $E^{[i+1,k]}_f$.

The inductive hypothesis holds vacuously for $i = 0$. Assume then that the hypothesis holds for $i-1$, for some $i \in \{1, \dots, k\}$. We establish the hypothesis for i in several steps.

Step 1: Show that $\mathcal{W}^{[i,k]}_f$ is absolutely continuous, with C^2 leaves.

Step 2: Define a G_0-invariant topological foliation $\mathcal{W}^{\#}$, subfoliating $\mathcal{W}^{[i,k]}_f$, such that $\pi(\mathcal{W}^{\#}) = \mathcal{W}^i_{A_f}$.

Step 3: Show that for any h_i-invariant ergodic measure ν, $\mathcal{W}^{\#}$ coincides with the Pesin stable manifold of $h_i|\mathcal{W}^{[i,k]}_f$, $\nu-$almost everywhere. In particular $T\mathcal{W}^{\#}$ is well-defined $\nu-$almost everywhere, and $\mathcal{W}^{\#}$ is absolutely continuous in $\mathcal{W}^{[m,k]}_f$.

Step 4: Show that $T\mathcal{W}^{\#} = E^i_f, \mathrm{vol} - a.e.$. Consequently, there is a full volume set K such that for any $x \in K$, $\mathcal{W}^{\#}(x)$ is a C^1 manifold tangent to E^i_f everywhere.

Step 5: Using an approximation argument, show that $\mathcal{W}^{\#} = \mathcal{W}^i_f$ and $\pi(\mathcal{W}^i_f) = \mathcal{W}^i_{A_f}$. Obtain integrability of $E^i_f \oplus E^c$ and the fact that $\pi(\mathcal{W}^{ic}) = \mathcal{W}^{ic}_{A_f}$. Conclude that the restriction of π to $\mathcal{W}^i_f$ leaves is δ bi-Hölder, with δ close to 1.

Step 6: Using Lemma 25 , show that Dh_i uniformly contracts E^i_f and uniformly expands $E^{[i+1,k]}_f$.

Step 1. By the inductive hypothesis A_{i-1}, the foliation $\mathcal{W}^{[i-1,k]}_f$ is absolutely continuous, with C^2 leaves. Hypothesis C_{i-1} implies that the derivative Dh_{i-1} uniformly contracts E^{i-1}_f and expands $E^{[i,k]}_f$. It follows that $\mathcal{W}^{[i,k]}_f$ is an absolutely continuous subfoliation of $\mathcal{W}^{[i-1,k]}_f$ with C^2 leaves, and thus $\mathcal{W}^{[i,k]}_f$ is absolutely continuous. (Note that for $i = 1$, the statement holds, because $\mathcal{W}^{[1,k]}_f = \mathcal{W}^u_f$).

Step 2. We define the G_0-invariant topological foliation $\mathcal{W}^{\#}$ to be the lift of $\mathcal{W}^i_{A_f} = \mathcal{W}^s_{A_{h_i}} \cap \mathcal{W}^u_{A_f}$ by π^{-1} on $\mathcal{W}^{[i,k]}_f-$leaves.

Henceforth to simplify notation, we write $h = h_i$, $\mathcal{W} = \mathcal{W}_f^{[i,k]}$, $E = E_f^i$, $E' = E_f^{[i+1,k]}$, *and* $F = E_f^{[i,k]} = E \oplus E' = T\mathcal{W}$. By construction, the topological foliation $\mathcal{W}^\#$ subfoliates the absolutely continuous foliation $\mathcal{W}$, whose leaves are C^2.

Step 3. Since π is bi-Hölder when restricted to $\mathcal{W}_f^u$ leaves, the map h contracts distance in the leaves of $\mathcal{W}^\#$ exponentially fast; i.e. for any $x \in \mathbb{T}^d$ and $y \in \mathcal{W}^\#(x, loc)$,

$$\limsup_{n\to\infty} \frac{1}{n} \log d_{\mathcal{W}_f^u}(h^n(x), h^n(y)) \leq \lambda_0 < 0, \tag{12}$$

where $\mathcal{W}^\#(x, loc)$ is defined to be the lift of $\mathcal{W}^s_{A_h} \cap \mathcal{W}^{[i,k]}_{A_f}(\pi(x), loc)$ to $\mathcal{W}^\#(x)$.

Proposition 24 implies that for any G_0-invariant ergodic measure ν, the leaf $\mathcal{W}^\#(x)$ coincides with the (global) Pesin stable manifold $\mathcal{W}^{Pe}_{h|\mathcal{W}}(x, gl)$ of x (for the restricted dynamics $h|\mathcal{W}$), for $\nu-$almost every x, since globally h contracts $\mathcal{W}^\#$ exponentially fast. Therefore $\mathcal{W}^\#(x, loc)$ is tangent to $E^s_{h,\nu} \cap F(x)$ at x for $\nu-$almost every x, where $E^s_{h,\nu}$ is the Oseledec stable space of (Dh, ν).

Step 4. Restricting to the case $\nu = \mathrm{vol}_{\mathbb{T}^d}$ in Step 3, we then have

Proposition 49. *The measurable distribution* $E^s_{h,\mathrm{vol}} \cap F$ *coincides with* E, vol$-$*a.e.*

Proof of Proposition 49. We split the proof into two cases.

Case 1: $\dim\left(E^s_{h,\mathrm{vol}} \cap F\right) (= \dim E) = 1$. The following lemma is easy to show.

Lemma 50. *There exists* E_f^j *such that* $E^s_{h,\mathrm{vol}} \cap F \subset E_f^j$, vol$-$*almost everywhere.*

Proof of Lemma 50. Evidently $E^s_{h,\mathrm{vol}} \cap F$ is a vol$-$a.e. defined, one dimensional, $Df-$invariant distribution within F, in particular we have Lyapunov exponents defined for (v, x) for vol$-$a.e. x and $v \in E^s_{h,\mathrm{vol}} \cap F$.

For any regular point x and any $v \in T_x\mathbb{T}^d - \{0\}$, the forward and the backward Lyapunov exponents for v exist and coincide. Then v is contained in some Oseledec subspace, and hence is contained in some E^j_x. Therefore $\mathbb{T}^d$ decomposes as a finite union of measurable sets $\cup_j X_j$ such that for all m, $f(X_j) = X_j$, and

$$E^s_{h,\mathrm{vol}} \cap F \subset E^j(x),$$

for all $x \in X_j$. By ergodicity of f we have that one of the X_j has full volume. □

Let $V := E_f^j$, where E_f^j is the subbundle obtained from Lemma 50.

Lemma 51. *For a full volume set of* $x \in \mathbb{T}^d$, *the leaf* $\mathcal{W}^\#(x)$ *is a* C^1 *submanifold tangent to* V *everywhere.*

Proof of Lemma 51. The absolute continuity of the foliation $\mathcal{W}$ from Step 1 implies that any set of full volume meets almost every leaf of $\mathcal{W}$ in a set of full leaf volume. Hence

there is a full volume, f-invariant set $P \subset M$ of Pesin regular points for (f, vol) in M such that for every $p \in P$, the leaf $\mathcal{W}$ meets P in a set of full leafwise volume.

Let $N := \bigsqcup_{p\in M} \mathcal{W}(p)$ be the disjoint union of unstable manifolds: it is a non-compact C^2 Riemannian manifold. The maps induced by f and h on N are C^2, with uniform bounds on the derivatives. Applying the arguments in [66] to the (Pesin regular) points in $P_N := \bigsqcup_{p\in P} P \cap \mathcal{W}(p)$, we obtain that the Pesin local stable manifolds

$$\mathcal{P}_{loc} := \{\mathcal{P}_{loc}(x) := \mathcal{W}^{Pe}_{h|\mathcal{W}}(x, loc) : x \in P\}$$

of $h|N$ form an absolutely continuous family of disks. In particular, for every $p \in P$, a set $B \subset \mathcal{W}(p)$ has $\mathrm{vol}_{\mathcal{W}}$-measure 0 in $\mathcal{W}(p)$ if and only if it has $\mathrm{vol}_{\mathcal{P}_{loc}(p)}$-measure 0 in $\mathcal{P}_{loc}(z)$, for almost every $z \in \mathcal{W}$.

This implies in particular that for vol$-$a.e. $x \in \mathbb{T}^d$, there is a dense subset of $y \in \mathcal{P}_{loc}(x)$ such that y belongs to P. For such y, the smooth disk $\mathcal{P}_{loc}(x)$ is tangent to $E(y)$. Thus for vol$-$a.e. $x \in \mathbb{T}^d$, the submanifold $\mathcal{P}_{loc}(x)$ is tangent to E on a dense subset, and hence by continuity of E, $\mathcal{P}_{loc}(x)$ is tangent to V everywhere and is therefore a C^1 submanifold.

Fix a positive volume, compact Pesin block Λ for h. By Pesin theory, for $y \in \Lambda$, the size of $\mathcal{P}_{loc}(y)$ is at least $r_0 > 0$. Let x be an f-regular point in Λ. Then there exist infinitely many n such that $f^{-n}(x), n \geq 0$ intersects Λ infinitely many times (this property holds for vol$-$almost every x, by Poincaré recurrence). Then the submanifold $f^n(\mathcal{P}_{loc}(f^{-n}(x)))$

- is contained in $\mathcal{W}^{\#}(x)$
- is tangent to V everywhere.
- has length $\geq r_0 e^{\lambda n}$ for some $\lambda > 0$, for all n with $f^{-n}(x) \in \Lambda$.

As n tends to infinity, we obtain that $\mathcal{W}^{\#}(x) = \cup_{n\geq 0} f^n(\mathcal{P}_{loc}(f^{-n}(x)))$ is a C^1 submanifold tangent to V everywhere. Since the h-Pesin blocks exhaust the volume, we conclude that for vol$-$a.e. x, $\mathcal{W}^{\#}(x)$ is a C^1 submanifold tangent to V. Let K be the set of such x. Then K is dense since it has full volume, completing the proof of Lemma 51. □

Now we claim that $j = i$, and thus $V = E$. Suppose $j \neq i$. By Hölder continuity of π, there exist positive constants ϵ_1, C_1 such that for any x and any $y \in \mathcal{W}^{\#}(x)$ with $d_{\mathcal{W}}(x, y) \leq \epsilon_1$, we have

$$d_{\mathbb{T}^{d-1}}(\pi(x), \pi(y)) \leq C_1\, \epsilon_1. \tag{13}$$

Now we pick an arbitrary $x \in K$ and consider the C^1 submanifold $\mathcal{W}^{\#}(x)$. Since, by Lemma 51, $\mathcal{W}^{\#}(x)$ is everywhere tangent to $V = E^j_f$, f expands $\mathcal{W}^{\#}$ at a rate slower than $e^{\lambda^j(f_0)+\eta}$ for some $\eta \ll \lambda^{j-1}(f_0) - \lambda^j(f_0)$ (by smallness of $d_{C^1}(f, f_0)$). Choose $y \in \mathcal{W}^{\#}(x, loc)$ such that $d_{\mathcal{W}^{\#}}(x, y) \leq \epsilon_1$. For n large, $\pi(f^n(x)), \pi(f^n(y))$ can be connected by a $\mathcal{W}^i_{A_f}-$path with length less than $O(C_1\, \epsilon_1\, e^{n(\lambda^j+\eta)})$. But since T_{A_f} expands $\mathcal{W}^i_{A_f}$ leaves at a constant rate $e^{\lambda^i(f_0)}$, the points $\pi(f^n(x)), \pi(f^n(y))$ cannot be linked by a $\mathcal{W}^i_{A_f}$ path with length $o(e^{n\lambda^j(f_0)})$, a contradiction. Therefore j must be i. This completes the proof of Proposition 49 in Case 1.

Case 2: $\dim\left(E^s_{h,\mathrm{vol}} \cap F\right) (= \dim E) > 1$. Suppose that $E^s_{h,\mathrm{vol}} \cap F$ does not coincide vol−a.e. with E. Then we have

Lemma 52. *The measurable distribution $E^s_{h,\mathrm{vol}} \cap F$ has non-trivial intersection (over a positive volume set) with $E' = E_f^{[i+1,k]}$.*

Proof of Lemma 52. Suppose that $E^s_{h,\mathrm{vol}} \cap F$ has trivial intersection with E', vol-almost everywhere. Since $E^s_{h,\mathrm{vol}} \cap F$ does not coincide vol−a.e. with E, Lusin's theorem implies that there is a compact set K_2 with positive volume and a positive constant δ_2 such that for any $x \in K_2$,

$$\angle(E^s_{h,\mathrm{vol}} \cap F(x), E(x)) > \delta_2. \tag{14}$$

Therefore for any $n \geq 1$ such that $f^n(x) \in K_2$,

$$\angle(E^s_{h,\mathrm{vol}} \cap F(f^n(x)), E(f^n(x))) > \delta_2. \tag{15}$$

On the other hand since $F = E \oplus E'$ is a dominated splitting, and $E^s_{h,\mathrm{vol}} \cap F(x)$ is Df−invariant and has trivial intersection with E', we have that

$$\lim_{n\to\infty} \angle(E^s_{h,\mathrm{vol}} \cap F(f^n(x)), E(f^n(x))) = \angle(Df^n(E^s_{h,\mathrm{vol}} \cap F(x)), Df^n(E(x))) = 0.$$

If $x \in K_2$ is recurrent, then this contradicts (15). Since almost every $x \in K_2$ is recurrent, this gives a contradiction, completing the proof of Lemma 52. □

As in the proof of Lemma 51, we thus obtain that $\mathcal{W}^\#$ is absolutely continuous and that there is a full volume set $K \subset \mathbb{T}^d$ such that for any $x \in K$, $\mathcal{W}^\#(x)$ is a C^1 submanifold, and $T\mathcal{W}^\#(x)$ has non-trivial intersection with E' everywhere. This uses Lemma 52, and the continuity of E' and $T\mathcal{W}^\#$ along the leaves of $\mathcal{W}^\#$.

Moreover, by the Cauchy-Peano existence theorem, for $x \in K$, there exists a C^1 path $\gamma : I \to \mathcal{W}^\#(x)$ such that for any $t \in I$, $\gamma'(t) \in E' \cap T\mathcal{W}^\#$. Let $z_0 = \gamma(0)$ and $z_1 = \gamma(1)$. As in Case 1, $f^n(z_0)$ and $f^n(z_1)$ can be linked by a C^1 path $f^n(\gamma)$ in $\mathcal{W}^\#(x)$ of length $O(e^{n(\lambda^{i+1}(f_0)+\eta')})$ for some $\eta' \ll \lambda^i(f_0) - \lambda^{i+1}(f_0)$; this implies that $\pi(f^n(z_0))$ and $\pi(f^n(z_1))$ can be linked by a $\mathcal{W}^i_{A_f}$−path of length $O(e^{n(\lambda^{i+1}(f_0)+\eta')})$ (since (13) holds here). On the other hand, since $\pi(f^n(z_i)) = T^n_{A_f}(\pi(z_i)), i = 0, 1$, it follows that $\pi(f^n(z_0))$ and $\pi(f^n(z_1))$ cannot be connected by a $\mathcal{W}^i_{A_f}$−path of length $o(e^{n\lambda^1(f_0)})$, which is a contradiction. This completes the proof of Proposition 49. □

Combining Proposition 49 from Step 3 with Lemma 51, we obtain a full volume subset $K \subset \mathbb{T}^d$ such that for $x \in K$, $\mathcal{W}^\#(x)$ is a C^1 manifold tangent to E everywhere and coinciding with a global Pesin stable manifold. By the absolute continuity of the family Pesin disks tangent to E and Fubini's theorem, it follows that $\mathcal{W}^\#$ is absolutely continuous.

Step 5. If a topological foliation has almost every leaf coinciding with another topological foliations, the two foliations must coincide. It follows that $\mathcal{W}^\# = \mathcal{W}^i_f$. and in particular $\pi(\mathcal{W}^i_f) = \mathcal{W}^i_{A_f}$. Observe that the leaves of $\pi^{-1}\left(\mathcal{W}^i_{A_f}\right)$ are jointly subfoliated

by $\mathcal{W}^c_f$ and $\mathcal{W}^i_f$, both of which have C^1 leaves. Therefore $E \oplus E^c$ is integrable and tangent to $\mathcal{W}^{ic}$. Integrability of $E' \oplus E^c$ follows from normal hyperbolicity.

Note that any leaf conjugacy from $(f, \mathcal{W}^c_f)$ to $(f_0, \mathcal{W}^c_{f_0})$ to close to the identity must map $\mathcal{W}^{ic}_{f_0}$ to $\mathcal{W}^{ic}_f$. Lemma 23 implies there is a leaf conjugacy h^c to $(f_0, \mathcal{W}^c_{f_0})$ that is δ-bi-Hölder along $\mathcal{W}^i_{f_0}$ leaves, where $\delta \to 1$ as $d_{C^1}(f, f_0) \to 0$. Then $\pi = P \circ h^c$, where $P\colon \mathbb{T}^d \to \mathbb{T}^{d-1}$ is the coordinate projection. It follows that that the restriction of π to $\mathcal{W}^i_f$ leaves has bi-Hölder exponent δ as well. This completes Step 5.

Step 6. Since $T^n_{A_h}$ exponentially contracts distances in $\mathcal{W}^i_{A_f}$ leaves and π is bi-Hölder between $\mathcal{W}^i_f$ and $\mathcal{W}^i_{A_f}$ leaves, we have that h^n exponentially contracts distances in $\mathcal{W}^i_f$ leaves. Lemma 25 implies that Dh uniformly contracts $T\mathcal{W}^i_f = E^i_f = E$.

Similarly, since $T^{-n}_{A_h}$ exponentially contracts distances in $\mathcal{W}^{[i+1,k]}_{A_f}$ leaves, h^{-n} exponentially contracts distances in $\mathcal{W}^{[i+1,k]}_f$ leaves. Again, Lemma 25 implies that Dh uniformly expands $T\mathcal{W}^{[i+1,k]}_f = E^{[i+1,k]}_f = E'$. Note that he same argument holds for any $h \in G_0$ such that A_h is not in a Weyl chamber wall.

This completes the induction, and so Proposition 44 holds for the elements $h_1, \dots h_k$ and the bundles $E^1_f, \dots, E^k_f$. Working in the stable bundle $\mathcal{W}^s_f$ with f^{-1} we obtain elements $h_{k+1} \dots, h_\ell$ satisfying the conclusions of Proposition 44 for f^{-1} and the bundles $E^{k+1}_f, \dots, E^\ell_f$. The remark above shows that that the conclusions hold for any $h \in G_0$ such that A_h is not in a Weyl chamber wall. This completes the proof of Proposition 44.

6.4. **Existence of partially hyperbolic elements and topological rigidity.** We now return to the proof of Proposition 43. The next step is to show that there is a partially hyperbolic element in every hyperbolic Weyl chamber.

Proposition 53. *Suppose f satisfies the hypotheses of Proposition 43. Then in each hyperbolic Weyl chamber of G_0, there exists a partially hyperbolic element h.*

Proof of Proposition 53. Fix an $h \in G_0$ such that A_h is not in any Weyl chamber wall.

Lemma 54. *For any $\epsilon > 0$, there exists $n \in \mathbb{Z}^+$ such that for any j, $\log \mathrm{Jac}|Dh^n|_{E^j_f}|$ lies in the interval*

$$\begin{cases} (\dim(E^j_f) \cdot (\delta^{-1}\lambda^j(A_h) - \epsilon) \cdot n),\ \dim(E^j_f) \cdot (\delta\lambda^j(A_h) + \epsilon) \cdot n),\ \text{if } \lambda^j(A_h) < 0; \\ (\dim(E^j_f) \cdot (\delta\lambda^j(A_h) - \epsilon) \cdot n),\ \dim(E^j_f) \cdot (\delta^{-1}\lambda^j(A_h) + \epsilon) \cdot n),\ \text{if } \lambda^j(A_h) > 0, \end{cases} \tag{16}$$

where $\lambda^j(A_h)$ is the Lyapunov exponent of $A_h|_{\mathcal{W}^j_{A_f}}$, $\delta \approx 1$ is the Hölder exponent given by Proposition 43, and $\mathrm{Jac}(\cdot|_{E^j_f})$ is the leafwise Jacobian for the map restricted on $\mathcal{W}^j_f$, .

Proof. Without loss of generality, assume that $\lambda^j(A_h) < 0$. Lifting the action of T_{A_h} to h and using the δ-Hölder continuity of π restricted to $\mathcal{W}^j_f$, we obtain that that for each

$\epsilon > 0$, there exist $\eta > 0$ and $N \in \mathbb{N}$ such that for all $x \in M$, and $y \in \mathcal{W}_f^j(x)$:

$$d(x,y) < \eta \implies e^{(\delta^{-1}\lambda^j(A_h)-\epsilon)n} \leq d(f^n(x), f^n(y)) \leq e^{(\delta^{-1}\lambda^j(A_h)+\epsilon)n},$$

for all $n \geq N$. The conclusion follows easily from Lemma 25, completing the proof of Lemma 54. □

Lemma 11 implies that h is volume preserving. Since E_f^i, E_f^c are all continuous distributions in $T\mathbb{T}^d$, there exists $C_0 \geq 1$, depending only on the angles between E_f^j, E_f^c, such that for any $k \in \mathbb{Z}$,

$$C_0^{-1} \leq (\prod_j \mathrm{Jac}(Dh^k|_{E_f^j})) \cdot \|Dh^k|_{E_f^c}\| \leq C_0; \tag{17}$$

since A_h has determinant 1, we also have

$$\sum_j \dim(E^j)\lambda^j(A_h) = 0. \tag{18}$$

Therefore by (17), (18) and Lemma 54, we have that for n large enough,

$$\|Dh^n|_{E_f^c}\| \in [e^{-\gamma n}, e^{\gamma n}], \tag{19}$$

where γ is small if δ is sufficiently close to 1 and ϵ in (16) is small.

Comparing (19) with (16), for $|\lambda^j(A_h)| \gg \gamma$ (which holds for any f which is sufficiently C^1 close to f_0 and any h that is not close to the Weyl chamber wall), we get h is in fact a partially hyperbolic diffeomorphism, with $E_h^s \oplus E_h^u = \oplus_j E_f^j$, and $E_h^c = E_f^c$, completing the proof of Proposition 53. □

From the existence of partially hyperbolic elements in every chamber, we obtain topological rigidity of the action.

Proposition 55. *If G induces a maximal Anosov action on $\mathbb{T}^{d-1}$, then*

1. *there exists a G_0-invariant continuous metric on E_f^c; and*
2. *f is Hölder conjugate to $T_{A_f} \times R_\theta$ for some $\theta \notin \mathbb{Q}/\mathbb{Z}$. Similarly, any $h \in G_0$ is Hölder conjugate (by the **same** Hölder conjugacy) to a product of T_{A_h} with a circle rotation.*

Proof. Proposition 53 implies that the G_0 action is partially hyperbolic, with hyperbolic subbundle $E^H := \oplus_j E_f^j$. The rigidity of such actions is studied in [24, 25]; in particular, the proofs of Proposition 8.1 in [25] and Proposition 5.1 in [24] imply that E^H is tangent to a C^1 foliation $\mathcal{W}^H$.

Denote by $\mathcal{W}_f^c(x_0)$ a G_0-fixed center leaf. Since E^H is integrable, there is no open accessibility class for f. Proposition 18 implies that $\mathbb{T}^d$ has a *product structure*, i.e. $\mathbb{T}^d$ is topologically the product of $\mathcal{W}_f^c(x_0)$ and $\mathbb{T}^{d-1}/\mathcal{W}^c$. By Hölder continuity of π and $\mathcal{W}_f^c$ this product structure is Hölder continuous as well.

Consider the projection Pr^c from $\mathbb{T}^d$ to $\mathcal{W}^c_f(x_0)$ along $\mathcal{W}^H$. Since $\mathcal{W}^H$ is a C^1 foliation, Pr^c is C^1 as well. Therefore $\mathrm{Pr}^c_*(\mathrm{vol}_{\mathbb{T}^d})$ is an $f-$invariant volume on $\mathcal{W}^c_f(x_0)$ with continuous density function, and $f|_{\mathcal{W}^c_f(x_0)}$ is C^1 conjugate to a circle rotation R_θ. By ergodicity of f, the rotation number θ must be irrational.

The continuous density function mentioned above gives an $f-$invariant continuous metric on $T\mathcal{W}^c_f(x_0)$, and this pulls back via $D\mathrm{Pr}^c|_{E^c}$ to an f-invariant metric on E^c. Since the construction of this $f-$invariant continuous metric on E^c only depends on the product structure and the volume form on $\mathbb{T}^d$, it must be G_0-invariant. This proves (1).

For (2), we know that the action induced by f

- on $\mathbb{T}^d/\mathcal{W}^c$ is Hölder conjugate to T_{A_f} on $\mathbb{T}^{d-1}$; and
- on $\mathbb{T}^d/\mathcal{W}^H$ is C^1-conjugate to R_θ.

Using the product structure of f, we obtain that f is Hölder conjugate to the product of T_{A_f} on $\mathbb{T}^{d-1}$ with an irrational rotation R_θ. The same proof also works for any $h \in G_0$ (although if h is not ergodic, the rotation number might not be irrational). Therefore, by the same conjugacy, h is Hölder conjugate to the product of T_{A_h} with a circle rotation. □

6.5. **Absolute continuity of $\mathcal{W}^c_f$: volume and equilibrium states.** The following proposition is a partially hyperbolic version of Theorem 20.4.1. in [44].

Proposition 56. *Let $f\colon M \to M$ be a C^{1+}, volume preserving partially hyperbolic diffeomorphism. Suppose that for any $f-$invariant ergodic measure ν, the central Lyapunov exponents of f with respect to ν are all zero. Then the volume vol_M is an equilibrium state of the potential $\varphi := -\log J^u(f) := -\log|\det Df|_{E^u}|$.*

Proof. The proof is basically contained in [36]. By the Pesin entropy formula [63] and the vanishing of the central Lyapunov exponents, we have

$$h_{\mathrm{vol}}(f) = \int_M \log J^u(f)(x) d\mathrm{vol}(x).$$

Therefore $P_{\mathrm{vol}}(\varphi) = 0$, where

$$P_{\mathrm{vol}}(\varphi) = h_{\mathrm{vol}}(f) + \int \varphi \, d\mathrm{vol}.$$

is the free energy of φ with respect to vol. We need only show that that the pressure of φ vanishes:

$$P(\varphi) := \sup_{\mu: f_*\mu=\mu} \left(h_\mu(f) + \int \varphi \, d\mu \right) = 0.$$

In [36], the authors introduce the concept of *unstable pressure* $P^u(f,\psi) = P^u(\psi)$ for any continuous ψ and C^1 partially hyperbolic diffeomorphism f. Corollary A.2 and the paragraph right after the statement of Corollary A.2 in [36] implies that $P^u(\psi) \le P(\psi)$ for any continuous ψ. Moreover if f is C^{1+} and there is no positive Lyapunov exponent in the center direction with respect to any $f-$invariant ergodic measure ν, then equality holds. Corollary C.1 in [36] implies that $P^u(\varphi) = 0$. for the potential $\varphi = -\log J^u(f)$.

The assumptions of Proposition 56 imply that for any $\psi \in C(M, \mathbb{R})$, $P^u(\psi) = P(\psi)$, and it follows that $P(\varphi) = 0$. This completes the proof of Proposition 56. □

6.6. **Absolute continuity of $\mathcal{W}^c_f$: $\mathcal{W}^H$-leafwise cocycle rigidity of higher rank partially hyperbolic actions.** Proposition 55, implies that the action of G_0 on $\mathbb{T}^d$ is Hölder conjugate to an *irrational rotation extension* α over $\bar{\alpha}$, where $\bar{\alpha}$ is the maximal linear Anosov $\mathbb{Z}^{d-2}$ action on $\mathbb{T}^d$. Here α is a $\mathbb{Z}^{d-2}-$action on $\mathbb{T}^d$ defined by $\alpha(\mathbf{a}) = \bar{\alpha}(\mathbf{a}) \times R_{\theta(\mathbf{a})}$ for $\mathbf{a} \in \mathbb{Z}^{d-2}$, where $\mathbf{a} \mapsto R_{\theta(\mathbf{a})}$ is an action by circle rotations with at least one $\theta(\mathbf{a})$ irrational.

Recall that a continuous function $\beta : \mathbb{Z}^{d-2} \times \mathbb{T}^d \to \mathbb{R}$ is an *(additive) cocycle* over α if $\beta(a + b, x) = \beta(a, \alpha(b) \cdot x) + \beta(b, x)$ holds for all $a, b \in \mathbb{Z}^{d-2}$ and $x \in \mathbb{Z}^d$. A cocycle β_1 is *cohomologous to* another cocycle β_2 if there exists a continuous function (called the *transfer function*) $\Psi : \mathbb{T}^d \to \mathbb{R}$ such that $\beta_1(a, x) = \beta_2(a, x) + \Psi(\alpha(a) \cdot x) - \Psi(x)$.

It is well known that for a maximal $\mathbb{Z}^{d-2}-$Anosov action $\bar{\alpha}$ on $\mathbb{T}^{d-1}$ (see Lemma 32), any Hölder continuous cocycle over $\bar{\alpha}$ is cohomologous to a constant cocycle.

We obtain here a corresponding result for the irrational rotation extension α over $\bar{\alpha}$. A cocycle β on $\mathbb{T}^{d-1} \times \mathbb{T}$ is *constant on* $\mathbb{T}^{d-1}$ if $\beta(a, x) = \beta(a, y)$ whenever x, y have the same $\mathbb{T}$-component, i.e. they lie on the same leaf of the *horizontal* $\mathbb{T}^{d-1}$*-foliation* $\{\mathbb{T}^{d-1} \times \{t\} : t \in \mathbb{T}\}$.

Proposition 57. *Let α be an irrational rotation extension over a maximal, linear Anosov $\mathbb{Z}^{d-2}$-action $\bar{\alpha}$ on $\mathbb{T}^{d-1}$. Then any Hölder continuous cocycle over α is cohomologous to a cocycle that is constant on $\mathbb{T}^{d-1}$.*

This proposition is a direct corollary of the following more general result on partially hyperbolic actions:

Proposition 58. *Let α be a partially hyperbolic $\mathbb{Z}^k$ action with coarse Lyapunov distributions E^j and corresponding coarse Lyapunov foliations $\mathcal{F}^j$, $j = 1, \dots, r$. Assume that:*

(1) $\bigoplus_{j=1}^r E^j$ integrates to a Hölder foliation $\mathcal{W}^H$ with compact smooth leaves.
(2) For any two $i, j \in \{1, \dots, r\}$ there exists a Weyl chamber $\mathcal{C}$ and an action element $a \in \mathcal{C}$ such that $\alpha(a)$ is partially hyperbolic and uniformly contracts both E^i and E^j.

Then any Hölder continuous cocycle over α is cohomologous to a cocycle that is constant along the leaves of $\mathcal{W}^H$.

Proof. The proof is an application of the periodic cycle functionals argument for higher rank actions developed in [20, 46] (cf. [87] for the rank-1 case). The main idea is that within each accessibility class of the action one can define a transfer map for the cocycle along *Lyapunov paths* (these are broken paths with pieces completely contained in leaves of foliations $\mathcal{F}^1, \dots, \mathcal{F}^1$), see [20, Definition 4]. Such a transfer map gives rise to a well-defined global Hölder map provided that its values along any two broken paths with same endpoints are the same [20, Definition 5]. In other words, the value of the linear functional

thus defined (called the periodic cycle functional [20, Proposition 2]) should be trivial on a closed Lyapunov path. This holds as in [20, Section 3.3] if the system of foliations $\mathcal{F}^1, \dots, \mathcal{F}^1$ satisfies the condition (2), which is also known as the *totally non-symplectic* (TNS) condition. The actions considered in [20] are assumed to be accessible, so the whole manifold is one accessibility class and the periodic cycle functionals argument implies in the case of actions in [20] that any Hölder cocycle is cohomologous to an everywhere constant cocycle.

In the situation we have here the exact same argument applies along leaves of the $\mathcal{W}^H$ foliation, since within each leaf we have accessibility of the coarse Lyapunov foliations and property (2). By the same argument as in [20, Section 3.3], this implies that any Hölder cocycle over α is cohomologous to a cocycle which is constant along the leaves of the foliation $\mathcal{W}^H$.

□

6.7. **Absolute continuity of $\mathcal{W}^c_f$: uniqueness of the measure of maximal entropy.** Consider the diffeomorphism $T_A \times R_\theta : \mathbb{T}^{d-1} \times \mathbb{T} \to \mathbb{T}^{d-1} \times \mathbb{T}$ where R_θ is an irrational rotation on circle and $A \in \mathrm{SL}(d-1, \mathbb{Z})$ is hyperbolic.

Lemma 59. *The volume* $\mathrm{vol}_{\mathbb{T}^d}$ *on* $\mathbb{T}^{d-1} \times \mathbb{T}$ *is the unique measure of maximal entropy of* $T_A \times R_\theta$.

Proof. This lemma is probably well-known; we sketch the proof. The projection of the measure of maximal entropy ν for $T_A \times R_\theta$ to $\mathbb{T}^{d-1}$ is the measure of maximal entropy for T_A, which is volume. On the other hand, the projection of ν to the circle $\mathbb{T}$ is R_θ-invariant, and hence is Lebesgue measure. Therefore ν must be $\mathrm{vol}_{\mathbb{T}^d}$, since zero entropy systems are disjoint from Bernoulli systems (cf. [31]). □

The following proposition is a corollary of Proposition 57.

Proposition 60. *Let f satisfy the hypotheses of Theorem 6, and let $G_0, G \subset \mathrm{Diff}(\mathbb{T}^d)$ be the finitely generated abelian groups defined in Section 5.1. If G defines a maximal linear Anosov action, then the volume* $\mathrm{vol}_{\mathbb{T}^d}$ *is the unique measure of maximal entropy of f.*

Proof. By the discussion in Section 6.5 and Proposition 55 we know that $\mathrm{vol}_{\mathbb{T}^d}$ is an equilibrium state of the potential $\varphi := -\log J^u(f)$ for f. We define the cocycle $\beta := -\log J^u$ over the action of G_0 as follows. For $f_1 \in G_0, x \in \mathbb{T}^d$, we set

$$\beta(f_1, x) := -\log |\det Df_1|_{E^u_f(x)}|.$$

Clearly β is a cocycle over the action of G_0, and $\beta(f, x) = \varphi(x)$, for all x.

The action of G_0 is Hölder conjugate to the algebraic action α defined in Section 6.6. By Proposition 57 we know any Hölder continuous cocycle over α is cohomologous to a cocycle that is constant on $\mathbb{T}^{d-1}$. Therefore β must be cohomologous to a cocycle that is constant on each horizontal $\mathcal{W}^H-$leaf. In particular, there exist continuous functions $\psi, \Psi : \mathbb{T}^d \to \mathbb{R}$, such that

$$\varphi = \psi + \Psi \circ f - \Psi, \tag{20}$$

and $\psi(x) = \psi(y)$ whenever x, y lie in the same $\mathcal{W}^H-$leaf.

As in the proof of Proposition 55, we denote by $\mathcal{W}^c_f(x_0)$ a G_0-fixed center leaf, and let $\Pr^c : \mathbb{T}^d \to \mathcal{W}^c_f(x_0)$ be the projection along the horizontal foliation $\mathcal{W}^H$. Then ψ defined in (20) induces a well-defined continuous function ψ^c on $\mathcal{W}^c_f(x_0)$ such that

$$\psi = \psi^c \circ \Pr^c. \tag{21}$$

Now we claim that for any $f-$invariant measure μ, $\int_{\mathbb{T}^d} \varphi \, d\mu$ is independent of μ. Indeed

$$\begin{aligned}\int_{\mathbb{T}^d} \varphi \, d\mu &= \int_{\mathbb{T}^d} (\psi + \Psi \circ f - \Psi) \, d\mu \text{ (by (20))} = \int_{\mathbb{T}^d} \psi \, d\mu \text{ (since } \mu \text{ is } f-\text{invariant)} \\ &= \int_{\mathcal{W}^c_f(\bar{x_0})} \psi^c d\Pr^c_*(\mu) \text{ (since } \psi \text{ is constant along each horizontal leaf).}\end{aligned}$$

But $f|_{\mathcal{W}^c_f(x_0)}$ is uniquely ergodic, and $\Pr^c_*(\mu)$ is $f-$invariant on $\mathcal{W}^c_f(x_0)$. Then the integral $\int_{\mathcal{W}^c_f(x_0)} \psi^c \, d\Pr^c_*(\mu)$ (and hence $\int_{\mathbb{T}^d} \varphi \, d\mu$) is independent of μ. Write $s(\varphi)$ for the value $\int_{\mathbb{T}^d} \varphi \, d\mu$ of this integral.

Since $\mathrm{vol}_{\mathbb{T}^d}$ is an equilibrium state of the potential φ, we have that

$$\begin{aligned}P_{\mathrm{vol}}(\varphi) &= \sup_{\mu \text{ is } f-inv} h_\mu(f) + \int_\mu \varphi \\ &= \sup_{\mu \text{ is } f-inv} h_\mu(f) + s(\varphi) \text{ (since } \int_\mu \varphi = s(\varphi), \text{ which is independent of } \mu).\end{aligned}$$

But $P_{\mathrm{vol}}(\varphi) = h_{\mathrm{vol}}(f) + \int_{\mathrm{vol}} \varphi = h_{\mathrm{vol}}(f) + s(\varphi)$. Therefore $h_{\mathrm{vol}}(f) = \sup_{\mu \text{ is } f-inv} h_\mu(f)$, which implies $\mathrm{vol}_{\mathbb{T}^d}$ is a measure of maximal entropy of f. But by Proposition 55 we know f is conjugate to $T_{A_f} \times R_\theta$, for some $\theta \notin \mathbb{Q}$, therefore by Lemma 59, $\mathrm{vol}_{\mathbb{T}^d}$ is the unique measure of maximal entropy of f. □

As a corollary, the conjugacy between f and $T_{A_f} \times R_\theta$ identifies the measure of maximal entropy $\mathrm{vol}_{\mathbb{T}^d}$ of $T_{A_f} \times R_\theta$ with the measure of maximal entropy $\mathrm{vol}_{\mathbb{T}^d}$ of f. Recall that $\mathrm{vol}_{\mathbb{T}^d}$, the measure of maximal entropy of $T_{A_f} \times R_\theta$ is the product of $\Pr^{\mathbb{T}}_*(\mathrm{vol}_{\mathbb{T}^d})$ and $\Pr^{\mathbb{T}^{d-1}}_*(\mathrm{vol}_{\mathbb{T}^d})$. Therefore $\mathrm{vol}_{\mathbb{T}^d}$, the measure of maximal entropy of f, is the product of $\Pr^c_*(\mathrm{vol}_{\mathbb{T}^d})$ and $\Pr^H_*(\mathrm{vol}_{\mathbb{T}^d})$, where $\Pr^H$ is the projection from $\mathbb{T}^d$ to $\mathbb{T}^d/\mathcal{W}^c_f$ along $\mathcal{W}^c_f$.

In particular, since $\Pr^c_*(\mathrm{vol}_{\mathbb{T}^d})$ is absolutely continuous with respect to the Lebesgue measure on $\mathcal{W}^c_f(\bar{x_0})$ (since $\Pr^c$ is C^1!), it follows that $\mathrm{vol}_{\mathbb{T}^d}$ has Lebesgue disintegration along $\mathcal{W}^c_f$. This completes the proof of Proposition 43, which implies Theorem 6.

7. Proof of Theorem 4

Let f_0 be as in Theorem 4. Let $f \in \mathrm{Diff}^\infty_{\mathrm{vol}}(\mathbb{T}^d)$ be a C^1-small ergodic perturbation of f_0. Denote by $\lambda^i(f_0)$ the distinct Lyapunov exponents of f_0 (ordered in i by decreasing size) and by $T\mathbb{T}^d = \oplus E^i_{f_0} \oplus E^c_{f_0}$ the corresponding Df_0-invariant Lyapunov splitting. Let $T\mathbb{T}^d = \oplus E^i_f \oplus E^c_f$ be the corresponding $Df-$invariant dominated splitting.

Lemma 61. *If $d_{C^1}(f, f_0)$ is sufficiently small then the cocycles $Df^{-1}|_{E^u_f}, Df|_{E^s_f}$ satisfy the narrow band condition defined in Section 3.6.*

Proof. It is clear that the cocycles $Df_0^{-1}|_{E^u_{f_0}}, Df_0|_{E^s_{f_0}}$ have point Mather spectrums. If $d_{C^1}(f, f_0)$ is small then E^i_f is close to $E^i_{f_0}$ and therefore the Mather spectrum of $Df|_{E^i_f}$ for each i is contained in an arbitrarily small narrow band, which implies Lemma 61. □

Since f is leaf conjugate to f_0, there is an $f-$fixed center leaf $W^c_f(x_0)$. As in the proof of Proposition 40, for any $s \geq 1$, $\mathcal{Z}_s(f)$ is virtually G_0, where

$$G_0 := \{h \in \mathcal{Z}_s(f) : h \text{ preserves the orientation of } \mathcal{W}^c_f, \text{ and } h(\mathcal{W}^c_f(x_0)) = \mathcal{W}^c_f(x_0)\}.$$

By Proposition 18 there is a Hölder continuous fiber bundle $\pi : \mathbb{T}^d \to \mathbb{T}^{d-1}$ such that $\pi \circ f = T_{A_f} \circ \pi$, and the fibers of π are leaves of $\mathcal{W}^c_f$. For any $h \in G_0$, h preserves the fiber bundle structure, and there is an automorphism $T_{A_h} \colon \mathbb{T}^{d-1} \to \mathbb{T}^{d-1}$ such that $\pi \circ h = T_{A_h} \circ \pi$. As in the proof of Proposition 40, we consider the group $\mathcal{Z}^c$ of center-fixing elements in G_0 and we let $G = \{A_h : h \in G_0\}$. Then G_0 is a group extension of G by $\mathcal{Z}^c$.

In the case that $\mathcal{W}^c_f$ is a smooth foliation, the volume has a smooth disintegration along $\mathcal{W}^c_f$, and f is smoothly conjugate to an ergodic smooth isometric extension g_ρ of g, where g is the map on the base $\mathbb{T}^{d-1}$ and ρ is a function on the base (see the definition of isometric extension in Section 2.1) , such that ρ is homotopic to identity. We have the following lemma for g and ρ:

Lemma 62. *Let r be as in Theorem 4. If $d_{C^1}(f, f_0)$ is sufficiently small and $\mathcal{W}^c_f$ is a smooth foliation, then one of the following holds.*

1. *$\mathcal{Z}_s(f)$ is virtually $\mathbb{Z} \times \mathbb{T}$ for every $s \geq r$. In this case, either g is not C^∞ conjugate to T_{A_f}, or ρ is not C^∞ cohomologous to a constant.*
2. *$\mathcal{Z}_s(f)$ is virtually $\mathbb{Z}^{\ell_0(A_f)} \times \mathbb{T}$ for every $s \geq 1$, and g_ρ is C^∞ conjugate to $T_{A_f} \times R_\theta$.*

Proof. Fix $s \geq r$. Let $G(g_\rho), \mathcal{Z}^c(g_\rho)$, and $G_0(g_\rho)$ be the groups defined in Section 5.1 for g_ρ.

The proof of Lemma 41 and ergodicity of g_ρ imply that any h commuting with g_ρ is an isometric extension, $\mathcal{Z}_s(g_\rho)$ is virtually $G_0(g_\rho)$, and $G_0(g_\rho)$ is a group extension of $G(g_\rho)$ by $\mathcal{Z}^c(g_\rho)$. Moreover, by the proof of Proposition 40, $\mathcal{Z}^c(g_\rho) = \{\mathrm{id} \times R_\theta, \theta \in \mathbb{T}\}$, and $G_0(g_\rho)$ is virtually $\mathbb{Z}^\ell \times \mathcal{Z}^c$, where ℓ is the rank of the finitely generated abelian group $G(g_\rho)$; to see this, note that in any short exact sequence of abelian groups: $0 \to H \to G_0 \to G \to 0$, with G finitely generated, the group G_0 is virtually the product of H with the torsion free part of G.

It is not hard too see that $Dg_\rho|_{E^s_{g_\rho}}$, $Dg_\rho{}^{-1}|_{E^u_{g_\rho}}$ have narrow band spectrum if and only if $Dg|_{E^s_g}$ and $Dg^{-1}|_{E^u_g}$ do. Since f has narrow band spectrum, and f is smoothly conjugate to g_ρ, both g and g_ρ have narrow band spectrum. By Corollary 33, $\mathcal{Z}_s(g)$ is either virtually trivial or g is smoothly conjugate to T_A. The former implies that $G(g_\rho)$ is virtually trivial, hence $\ell = 1$ and item (1) of Lemma 62 holds.

If g is smoothly conjugate to T_{A_f}, then without loss of generality we may assume that $g = T_{A_f}$. If ρ is smoothly cohomologous to a constant θ, then by ergodicity $\theta \notin \mathbb{Q}/\mathbb{Z}$, and item (2) of Lemma 62 holds.

We claim now that if $g = T_{A_f}$, and ρ is not C^∞ cohomologous to a constant then $\ell = 1$ for any $s \geq 1$. Suppose $\ell > 1$ for some $s \geq 1$. By taking a finite iterate if necessary, we can assume that there is an isometric extension $(T_B)_{\rho_B}$ (a priori C^s) such that $(T_B)_{\rho_B}$ commutes with $g_\rho = (T_{A_f})_\rho$, and the group generated by A_f, B is not virtually trivial. Using commutativity, by considering the induced action of $(T_B)_{\rho_B}, (T_{A_f})_\rho$ on $\pi_1(\mathbb{T}^d)$, we get that ρ_B is cohomologous to a constant, which can be viewed as a function on $\mathbb{T}^{d-1}$. By Lemma 31, the group generated by T_{A_f}, T_B on $\mathbb{T}^n$ is a higher rank action, therefore by Lemma 32, ρ, ρ_B are (simultaneously) cohomologous to constants. By Livšic's theorem the conjugacy is smooth, i.e. ρ is C^∞ cohomologous to a constant, which is a contradiction. □

Proof of Theorem 4. If the disintegration of volume along $\mathcal{W}^c_f$ leaves is not Lebesgue, then Theorem 4 is a corollary of Theorem 6. Assume that $\mathrm{vol}_{\mathbb{T}^d}$ has Lebesgue disintegration along $\mathcal{W}^c_f$. Proposition 18 implies that one of the following cases holds:

Case 1: f is accessible, and the disintegration of $\mathrm{vol}_{\mathbb{T}^d}$ has a continuous density function on the leaves of W^c_f. By [2, Theorem E], there is a volume-preserving flow φ_t tangent to and C^∞ along the leaves of $\mathcal{W}^c_f$, commuting with f and satisfying $\varphi_1 = \mathrm{id}$. Lemma 41 implies that $h = \varphi_{\rho(h)}$ for any $h \in \mathcal{Z}^c$, i.e. $\mathcal{Z}^c \subset \{\varphi_t\}_{t\in\mathbb{T}}$.

Let $D := \{t \in \mathbb{T} : \varphi_t \in \mathcal{Z}^c\}$. There are two possibilities:

(1) $D < \mathbb{T}$ *is discrete.* Then $\mathcal{Z}^c$ is finite. By Lemma 15, the group G is abelian with rank $\ell \leq \ell_0$.
   (a) $\ell < \ell_0$ or $\ell = \ell_0 = 1$. Since G_0 is abelian group extension of G by $\mathcal{Z}^c$, by finiteness of $\mathcal{Z}^c$ we can construct a finite index subgroup of G_0 isomorphic to the torsion free part of G, which is $\mathbb{Z}^\ell$. By the same proof as in Proposition 40, we have that $\mathcal{Z}_s(f)$ is virtually G_0, therefore Theorem 4 holds in this case.
   (b) $\ell = \ell_0 > 1$. As in the proof of Theorem 6, we can construct partially hyperbolic elements in all the Weyl chambers of the action of $\mathcal{Z}_0$, which implies that $E^u_f \oplus E^s_f$ is jointly integrable, contradicting the accessiblity of f.

(2) $D < \mathbb{T}$ *is dense.* Lemma 61 implies that the triple (f, φ_t, X) satisfies the hypotheses of Proposition 28; applying this result, we obtain that $D = \mathbb{R}$, X is a C^∞ vector field and so φ_t is a C^∞ flow. Therefore $\mathcal{W}^c_f$ is a smooth foliation, and f is smoothly conjugate to an isometric extension g_ρ. Then by Lemma 62 and accessibility of f, item (2) of Theorem 4 holds for f.

Case 2: f is topologically conjugate to $T_{A_f} \times R_\theta$ for some $\theta \notin \mathbb{Q}/\mathbb{Z}$. Then $E^u_f \oplus E^s_f$ is integrable and tangent to the horizontal foliation $\mathcal{W}^H$. By Lemma 21, $\mathcal{W}^H$ is a C^1 foliation.

For any $x \in \mathbb{T}^d$, we denote by $\Pr_x^c$ the projection from $\mathbb{T}^d$ to $\mathcal{W}_f^c(x)$ along $\mathcal{W}^H$ and let $\mu_x := \Pr_{x*}^c(\mathrm{vol}_{\mathbb{T}^d})$. Then the family $\{\mu_x, x \in \mathbb{T}^d\}$ is $f-$invariant, i.e.

$$(f|_{W_f^c(x)})_*\mu_x = \mu_{f(x)}. \tag{22}$$

The C^1-ness of $\mathcal{W}^H$ implies that the family of measures $\{\mu_x, x \in \mathbb{T}^d\}$ along $\mathcal{W}_f^c-$leaves have continuous density functions. Therefore f is center $r-$bunched, for all $r > 0$, which implies $\mathcal{W}^c$ has C^∞ leaves, and the stable and unstable holonomies between center leaves are uniformly smooth. Since $\mathcal{W}^u, \mathcal{W}^s$ have uniformly smooth leaves, Journé's lemma implies that $\mathcal{W}^H$ has uniformly smooth leaves as well. In summary, $\mathcal{W}^H$ is a smooth foliation.

Since $\mathcal{W}_f^c$ is absolutely continuous, [3, Theorem C (1)] implies that there exists a continuous, volume-preserving flow φ_t on $\mathbb{T}^d$ commuting with f whose generating vector field is tangent to the leaves of $\mathcal{W}_f^c$. Moreover, $\varphi_1 = \mathrm{id}$ and $\mathcal{Z}^c \subset \{\varphi_t\}_{t\in\mathbb{T}}$.

The rest of the proof for Case 2 is similar to that of Case 1. Again we take the set $D := \{t \in \mathbb{T}, \varphi_t \in \mathcal{Z}^c\}$, and consider the following cases.

(1) $D < \mathbb{T}$ *is discrete.* Then $\mathcal{Z}^c$ is finite. As in Case 1, we consider the abelian group G which is virtually $\mathbb{Z}^\ell, \ell \le \ell_0$.
   (a) $\ell < \ell_0$, or $\ell = \ell_0 = 1$. then by exactly the same proof as in Case 1 we can prove the conclusion of Theorem 4.
   (b) $\ell = \ell_0 > 1$. First we claim that the action of $\mathcal{Z}_s(f)$ on $\mathbb{T}^d$ is C^∞ (a priori it is only C^s). For any $g \in \mathcal{Z}_s(f)$, g preserves the smooth density on $\mathcal{W}^c$ (induced by $\{\mu_x, x \in \mathbb{T}^d\}$). Since $s \ge r > r_0(A) = \max(\frac{\lambda^s}{\mu^s}, \frac{\lambda^u}{\mu^u})$, Lemma 27 implies that if f is C^1-close to f_0, then f preserves a C^∞ normal form, and $r(f) < r \le s$. Theorem 9 then implies that g also preserves the smooth normal form on $\mathcal{W}_f^u$ and $\mathcal{W}_f^s$, which implies that g is uniformly smooth along $\mathcal{W}_f^s$ and $\mathcal{W}_f^u$. Therefore by Journé's lemma, g is uniformly smooth. So the action by G_0 is smooth and volume preserving on $\mathbb{T}^d$. Since G has rank $\ell_0 > 1$, following the proof of Theorem 6, we can construct partially hyperbolic elements in all the Weyl chambers of the action of G_0. Then the global rigidity result in [23] implies that the action of G_0 is rigid (see Proposition 63 in the the Appendix). Thus f is smoothly conjugate to $T_{A_f} \times R_\theta$ for some $\theta \notin \mathbb{Q}$.

(2) $D < \mathbb{T}$ *is dense.* By the same proof as in Case 1, we obtain that f is smoothly conjugate to an isometric extension. Then one of the two alternatives in Lemma 62 imply the alternatives (2) and (3) in Theorem 4.

This completes the proof of Theorem 4. □

Appendix A. Global rigidity of conservative partially hyperbolic abelian actions on the torus

We state here the main result in [23], which plays a crucial role in the proof of Theorem 4. The setting is as follows. Suppose $\alpha : \mathbb{Z}^k \to \mathrm{Diff}^\infty_{\mathrm{vol}}(\mathbb{T}^d)$ is a smooth, volume preserving ergodic abelian action. We assume that there exists at least one $a \in \mathbb{Z}^k$ such that $\alpha(a)$ is

a fibered partially hyperbolic diffeomorphism and all the partially hyperbolic elements of α preserve a common circle center foliation $\mathcal{W}^c$.

As explained in Section 3.7, the distribution $E^H := E^u_a \oplus E^s_a$ for a partially hyperbolic element $\alpha(a)$ is $\alpha-$invariant, and we consider the Lyapunov functionals χ_i and the hyperbolic Weyl chamber picture induced by the cocycle $D\alpha|_{E^H}$ with respect to $\mathrm{vol}_{\mathbb{T}^d}$.

Theorem 11. [23] *Assume that each hyperbolic Weyl chamber for α contains a partially hyperbolic element. Suppose that there is no pair of Lyapunov functionals χ_i, χ_j and $c \in (-\infty, \frac{1}{2}] \cup [2, \infty)$ such that $\chi_i = c\chi_j$. Then α is smoothly conjugate to the product of an affine Anosov action on $\mathbb{T}^{d-1}$ with an action by rotations on $\mathbb{T}^1$.*

We now verify that Theorem 11 applies to the action in the proof of Theorem 4.

Proposition 63. *The action of G_0 in (1)(b) of Case 2 in Section 7 satisfies all the conditions in Theorem 11. Therefore G_0 is smoothly conjugate to a product of a linear Anosov action on $\mathbb{T}^{d-1}$ and a rotation action on $\mathbb{T}^1$. In particular, in (1)(b) of Case 2 in Section 7, f is smoothly conjugate to $A_f \times R_\theta$ for some $\theta \notin \mathbb{Q}/\mathbb{Z}$.*

Proof. Recall that in (1)(b) of Case 2 in Section 7, we obtain that the action of G_0 on $\mathbb{T}^d$ is abelian, C^∞ and volume preserving. Every element h in G_0 preserves the common center foliation $\mathcal{W}^c$ and there is a Hölder continuous fiber bundle $\pi : \mathbb{T}^d \to \mathbb{T}^{d-1}$ such that for any $h \in G_0$, there is a linear automorphism $T_{A_h} : \mathbb{T}^{d-1} \to \mathbb{T}^{d-1}$ satisfying $\pi \circ h = T_{A_h} \circ \pi$.

Since $G = \{A_h, h \in G_0\}$ has rank $\ell_0 > 1$, by Lemma 30 G induces a maximal Anosov affine action on $\mathbb{T}^{n-1}$. Therefore, the action of G is TNS (i.e., there are no negatively proportional Lyapunov functionals) and conformal on each coarse Lyapunov foliation. By the discussion in Section 6.1, the Lyapunov functionals of the action of G_0 are close to that of G (see also Step 7. of Section 6.3); therefore the action of G_0 is TNS, and the Lyapunov functionals satisfy the $\frac{1}{2}-$pinching condition. Moreover, by following the proof of Theorem 6, we can construct partially hyperbolic elements in all the Weyl chambers of the action of G_0. Thus the action of G_0 in (2)(b) of Case 2 of Section 7 satisfies all the conditions in Theorem 11, and by Theorem 11 it is globally rigid. □

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