DIAMETER ESTIMATES IN KÄHLER GEOMETRY

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ABSTRACT. Diameter estimates for Kähler metrics are established which require only an entropy bound and no lower bound on the Ricci curvature. The proof builds on recent PDE techniques for L^∞ estimates for the Monge-Ampère equation, with a key improvement allowing degeneracies of the volume form of codimension strictly greater than one. As a consequence, diameter bounds are obtained for long-time solutions of the Kähler-Ricci flow and finite-time solutions when the limiting class is big, as well as for special fibrations of Calabi-Yau manifolds.

1. Introduction

The diameter is one of the most important geometric invariants defined by a metric. Bounds for the diameter are for example essential in the study of convergence of manifolds, which is of particular interest in moduli problems and geometric flows, where one hopes to arrive at a canonical model by taking limits. Unfortunately, in Riemannian geometry, there are very few tools for estimating the diameter, besides comparison theorems and the Bonnet-Myers theorem which require that the Ricci curvature be strictly positive. The situation did not seem markedly different in Kähler geometry, although we can exploit there the fact that the potential can be viewed as the solution of a complex Monge-Ampère equation with right hand side given by its volume form (see e.g. [11, 37, 23, 17]). However, there has been considerable progress recently in PDE methods for L^∞ estimates for fully non-linear equations [15, 12, 13]. These new methods turn out to be particularly amenable to geometric estimates, and have been shown to imply some promising estimates for non-collapse [18] and for the Green's function [14].

The main goal of the present paper is to develop a general theory of diameter estimates in Kähler geometry. We shall be particularly interested in estimates which require only an upper bound for the entropy of the volume form, but not a lower bound for the Ricci curvature. For geometric applications, it is also important that

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the diameter estimates be uniform with respect to suitable subsets which can approach the boundary of the Kähler cone. To obtain such estimates, we build on the PDE methods mentioned above [15, 18, 14], but with an essential improvement. These methods originally applied to fully non-linear elliptic equations which satisfy a specific structural condition, corresponding to a condition of nowhere vanishing of the volume form in the case of Monge-Ampère. It has been recently shown by Harvey and Lawson [19] that this condition is natural and applies to very broad classes of equations. Nevertheless, for many applications which we shall consider in this paper, notably the Kähler-Ricci flow and the analytic Minimal Model Program, it is necessary to allow the volume form to be arbitrarily close to vanishing along subsets which are suitably small, such as a proper complex subvariety. It turns out that such a generalization is indeed possible, and it plays a major role in this whole paper.

We now state our general diameter estimates more precisely. Let (X, ω_X) be an n-dimensional compact Kähler manifold equipped with a Kähler metric ω_X. Let $\mathcal{K}(X)$ be the space of Kähler metrics on X. We define the p-Nash entropy of a Kähler metric $\omega \in \mathcal{K}(X)$ associated to (X, ω_X) by

$$\mathcal{N}_{X,\omega_X,p}(\omega) = \frac{1}{V_\omega}\int_X \left|\log\left((V_\omega)^{-1}\frac{\omega^n}{\omega_X^n}\right)\right|^p \omega^n,\ V_\omega = \int_X \omega^n = [\omega]^n, \tag{1.1}$$

for $p > 0$. If we write $e^F = \frac{1}{V_\omega}\frac{\omega^n}{\omega_X^n}$, then

$$\mathcal{N}_{X,\omega_X,p}(\omega) = \int_X e^F |F|^p \omega_X^n = \left\|e^F\right\|_{L^1 \log L^p(X,\omega_X)}$$

and

$$\mathcal{N}_{X,\omega_X,p}(\omega) \le \int_{F\ge 0} e^F F^p \omega_X^n + C$$

for some $C = C(X, \omega_X, p) > 0$.

We introduce the following set of admissible functions for given parameters $A, B, K > 0$, $p > n$,

$$\mathcal{V}(X, \omega_X, n, A, p, K) = \left\{\omega \in \mathcal{K}(X) : [\omega]\cdot[\omega_X]^{n-1} \le A,\ \mathcal{N}_{X,\omega_X,p}(\omega) \le K\right\}. \tag{1.2}$$

Let γ be a non-negative continuous function. We further define a subset of $\mathcal{V}(X, \omega_X, n, A, p, K)$ by

$$\mathcal{W}(X, \omega_X, n, A, p, K; \gamma) = \left\{\omega \in \mathcal{V}(X, \omega_X, n, A, p, K) :\ (V_\omega)^{-1}\frac{\omega^n}{\omega_X^n} \ge \gamma\right\}. \tag{1.3}$$

We also define the Green's function $G(x, y)$ associated to a Riemannian manifold (X, g) by (see e.g. [26])

$$\Delta_g G(x, \cdot) = -\delta_x(\cdot) + (\mathrm{Vol}_g(X))^{-1},$$

where Δ_g is the Laplace operator associated to g. The following is the main theorem of our paper:

Theorem 1.1. *Let X be an n-dimensional connected Kähler manifold equipped with a Kähler metric ω_X and let γ be a nonnegative continuous function on X satisfying*

$$\dim_{\mathcal{H}}\{\gamma = 0\} < 2n - 1, \ \gamma \geq 0, \tag{1.4}$$

where $\dim_{\mathcal{H}}$ is the Hausdorff dimension. Then for any $A, K > 0$ and $p > n$, there exist $C = C(X, \omega_X, n, A, p, K, \gamma) > 0$, $c = c(X, \omega_X, n, A, p, K, \gamma) > 0$ and $\alpha = \alpha(n, p) > 0$ such that for any $\omega \in \mathcal{W}(X, \omega_X, n, A, p, K; \gamma)$, we have the following bounds for

(a) *The Green's function:*

$$\int_X |G(x, \cdot)|\omega^n + \int_X |\nabla G(x, \cdot)|\omega^n + \left(-\inf_{y \in X} G(x, y)\right) \mathrm{Vol}_\omega(X) \leq C$$

for any $x \in X$;

(b) *The diameter:*

$$\mathrm{diam}(X, \omega) \leq C;$$

(c) *The volume element: for any $x \in X$ and any $R \in (0, 1]$,*

$$\frac{\mathrm{Vol}_\omega(B_\omega(x, R))}{\mathrm{Vol}_\omega(X)} \geq cR^\alpha.$$

This is the first general result on uniform diameter bounds and volume non-collapsing estimates for Kähler manifolds without any curvature assumption. We note that the assumption on the Hausdorff dimension of the set $\{\gamma = 0\}$ can be replaced by the weaker assumption that this set have small measure together with the connectedness of $\{\gamma > 0\}$ (c.f. Proposition 5.1 and Proposition 6.1), and that this is in fact how the theorem is proved. In practice, the theorem is often applied with $\{\gamma = 0\}$ supported on a proper analytic subvariety of X. It may be instructive to compare it with the situation in Riemannian geometry. There the sharp result is the theorem of S.Y. Cheng and P. Li [6], where a lower bound for the Green's function requires a lower bound on the Ricci curvature. Since $\mathrm{Ric}(\omega) = -i\partial\bar{\partial} \log \omega^n$ in Kähler geometry, and we allow the lower bound γ for the volume form ω^n to vanish, we see that Theorem 1.1 can give lower bounds for the Green's function even when no lower bound for the Ricci curvature is available. As we shall see, this flexibility is particularly important in the study of the Kähler-Ricci flow and of fibrations of Calabi-Yau manifolds.

As a consequence of Theorem 1.1, we obtain the following theorem, which can be viewed as a Kähler analogue of Gromov's precompactness theorem for metric spaces:

Theorem 1.2. *Let X be an n-dimensional connected Kähler manifold equipped with a Kähler metric ω_X and let γ be a nonnegative continuous functionon X with*

$$\dim_{\mathcal{H}}\{\gamma = 0\} < 2n - 1.$$

Then for any $A, K > 0$, $p > n$ and any sequence $\{\omega_j\}_{j=1}^{\infty} \subset \mathcal{W}(X, \omega_X, n, A, p, K; \gamma)$, after passing to a subsequence, (X, ω_j) converges in Gromov-Hausdorff topology to a compact metric space (X_∞, d_∞).

Geometric compactness is fundamental for understanding degeneration and moduli problems for complex Riemannian manifolds. Curvature bounds are usually necessary such as in the general theory of Cheeger-Colding [4]. Theorem 1.2 bypasses the curvature requirement to provide boundedness for families of Kähler manifolds. It might also be combined with techniques from [5] to explore formation of singularities and achieve stronger geometric regularity for the limiting metric spaces.

2. Applications to the Kähler-Ricci flow and cscK

We describe now the applications of Theorem 1.1 to the Kähler-Ricci flow and constant scalar curvature Kähler metrics. We shall be particularly interested in the analytic Minimal Model Program introduced in [33] in relation to the formation of both finite time and long time singularities in the Kähler-Ricci flow.

We first consider the following unnormalized Kähler-Ricci flow on a Kähler manifold X with an initial Kähler metric g_0

$$\begin{cases} \dfrac{\partial g}{\partial t} = -\mathrm{Ric}(g), \\ \\ g|_{t=0} = g_0. \end{cases} \tag{2.1}$$

Let

$$T = \sup\{t > 0 \mid [g_0] + t[K_X] > 0\} \in \mathbb{R} \cup \{\infty\}. \tag{2.2}$$

It is shown in [41, 39] that the Kähler-Ricci flow has a maximal solution $g(t)$ for $t \in [0, T)$.

2.1. **The case of finite-time singularities.** If $T < \infty$, the flow (2.1) must develop singularities at $t = T$. In this case, $\alpha_T = [g_0] + T[K_X]$ is a nef class on X. If $[g_0] \in H^2(X, \mathbb{Q})$, α_T is semi-ample by Kawamata's base point free theorem and the numerical dimension of α_T coincides with its Kodaira dimension. In general, it is unclear if there exists a smooth semi-positive closed $(1,1)$-form in α_T. The most interesting case is when α_T is big, i.e., there exists a Kähler current in α_T or equivalently, $(\alpha_T)^n > 0$. Such a bigness condition can be also interpreted by the total volume along the Kähler-Ricci flow as

$$(\alpha_T)^n = \lim_{t\to T} \mathrm{Vol}_{g(t)}(X) > 0. \tag{2.3}$$

This bigness assumption for the limiting class is in fact generic for finite-time singularities.

It is conjectured in [33] as part of the analytic minimal model program that when α_T is big, $(X, g(t))$ should converge to a compact Kähler variety and the flow will extend uniquely through the singular time through a canonical metric surgery. Such a surgery corresponds to either a divisorial contraction or a flip in birational geometry. After suitable blow-ups, the singularity model is expected to be a transition from a shrinking soliton to an expanding soliton (c.f. [28]). This is confirmed in the case of Kähler surfaces by [37, 38], where it is shown that the Kähler-Ricci flow contracts finitely many holomorphic S^2 of (-1) self-intersection in Gromov-Hausdorff topology. The following diameter bound is the first step to understand formation of finite time singularities of the Kähler-Ricci flow in general dimension:

Theorem 2.1. *Let (X, g_0) be a Kähler manifold equipped with a Kähler metric g_0. If $g(t)$ is the maximal solution of the Kähler-Ricci flow (2.1) for $t \in [0, T)$ for some $T \in \mathbb{R}^+$ and if the limiting class $[g_0]+T[K_X]$ is big, then there exist $C = C(X, g_0) > 0$, $c = c(X, g_0) > 0$ and $\alpha = \alpha(X, g_0) > 0$ such that for any $t \in [0, T)$,*

$$\mathrm{diam}(X, g(t)) \le C,$$

$$\int_X |G_t(x,\cdot)| dV_{g(t)} + \int_X |\nabla G_t(x,\cdot)| dV_{g(t)} + \left(-\inf_{y\in X} G_t(x,y)\right) \mathrm{Vol}_{g(t)}(X) \le C,$$

$$\frac{\mathrm{Vol}_{g(t)}(B_{g(t)}(x,R))}{\mathrm{Vol}_{g(t)}(X)} \ge cR^\alpha,$$

for any $x \in X$ and $R \in (0, 1]$, where G_t is the Green's function for $(X, g(t))$.

We stress that Theorem 2.1 holds for general Kähler manifolds, and no projectiveness assumption is needed. We prove it in section 8 by establishing a uniform upper bound for the p-Nash entropy for any $p > 0$ and a lower bound for the volume form along the flow.

2.2. **The case of long-time solutions.** It is well-known that the Kähler-Ricci flow has a long-time solution if and only if the canonical bundle K_X is nef. The underflying manifold X is then called a minimal model. The Kodaira dimension of X is defined by

$$\mathrm{Kod}(X) = \lim_{m\to\infty} \frac{\log h^0(X, mK_X)}{\log m}$$

if $h^0(X, mK_X) \neq 0$ for some $m \in \mathbb{Z}^+$. The Kodaira dimension of X is always no greater than n and is nonnegative as long as there exists one holomorphic pluricanonical section. The abundance conjecture predicts that if X is minimal, K_X must be semi-ample and hence the Kodaira dimension is always nonnegative. We would like now to obtain a uniform diameter bound for long time solutions of the Kähler-Ricci flow.

We consider the following normalized Kähler-Ricci flow with initial metric g_0 if the Kodaira dimension of X is nonnegative.

$$\begin{cases} \dfrac{\partial g}{\partial t} = -\mathrm{Ric}(g) - g, \\ \\ g|_{t=0} = g_0. \end{cases} \tag{2.4}$$

Obviously, the flow (2.4) exists for $t \in [0, \infty)$ since K_X is nef.

Theorem 2.2. *Let (X, g_0) be an n-dimensional Kähler manifold with nef K_X and non-negative Kodaira dimension. Let $g(t)$ be the solution of the normalized Kähler-Ricci flow (2.4). Then there exist $C = C(X, g_0) > 0$, $c = c(X, g_0) > 0$ and $\alpha = \alpha(X, g_0) > 0$ such that for any $t \geq 0$,*

$$\mathrm{diam}(X, g(t)) \leq C,$$
$$\int_X |G_t(x, \cdot)| dV_{g(t)} + \int_X |\nabla G_t(x, \cdot)| dV_{g(t)} + \left(-\inf_{y\in X} G_t(x, y)\right) \mathrm{Vol}_{g(t)}(X) \leq C,$$
$$\frac{\mathrm{Vol}_{g(t)}(B_{g(t)}(x, R))}{\mathrm{Vol}_{g(t)}(X)} \geq cR^\alpha,$$

for any $x \in X$ and $R \in (0, 1]$, where G_t is the Green's function for $(X, g(t))$.

The diameter is optimal for X with positive Kodaira dimension. When $c_1(X) = 0$ and hence $\kappa(X) = 0$, the diameter of $(X, g(t))$ decays at the exact rate $e^{-t/2}$. The uniform diameter bound in Theorem 2.2 is proved in [21] in the case when K_X is semi-ample, and the proof is built on works of [35, 1], relying on the uniform scalar curvature bound obtained in [34, 46] (see also [42] in the case of general type). Our proof in the present case is based rather on Theorem 1.1. This has many advantages, since we do not need then any assumption on the scalar curvature (for which bounds are not available in the nef case), nor on the projectiveness of X, nor on the abundance conjecture.

Next, we discuss the behavior of the flow near a singular fiber in the case of collapse. If K_X is semi-ample, the pluricanonical system of X induces a unique holomorphic fibration

$$\pi : X \to X_{can},$$

where X_{can} is the unique canonical model of X [1]. The Kodaira dimension of X coincides with the complex dimension of X_{can}. The general fibre of π is a smooth Calabi-Yau manifold. It is proved in [31, 32] that the normalized Kähler-Ricci flow converges weakly to a twisted possibly singular Kähler-Einstein metric g_∞ on X_{can} satisfying

$$\mathrm{Ric}(g_\infty) = -g_\infty + g_{WP},$$

where g_{WP} is the Weil-Petersson metric for the Calabi-Yau fibration $\pi : X \to X_{can}$, while the fibre metrics collapses along the normalized Kähler-Ricci flow. In particular, if $X_y = \pi^{-1}(Y)$ is a smooth fibre, it is shown in [40] that $e^t g(t)|_{X_y}$ converges to the unique Ricci-flat Kähler metric in $[g(0)|_{X_y}]$. The following theorem describe the asymptotic behavior near the singular fibre with at worst canonical singularities.

Theorem 2.3. *Let (X, g_0) be an n-dimensional projective manifold with semi-ample K_X and $\mathrm{Kod}(X) = 1$. Let $g(t)$ be the solution of the normalized Kähler-Ricci flow (2.4). If every fibre of $\pi : X \to X_{can}$ has at worst canonical singularities, then there exists $C = C(X, g_0)$ such that for all $t \geq 0$ and $y \in X_{can}^\circ$, we have*

$$\mathrm{diam}(X_y, g(t)|_{X_y}) \leq Ce^{-\frac{t}{2}}. \tag{2.5}$$

The fibre diameter estimate (2.5) is intrinsic as the diameter is achieved by a minimal geodesic in the fibre. It immediately implies that the extrinsic fibre diameter estimate holds uniformly for all fibres of $\pi : X \to X_{can}$ since X_{can}° is an open dense subset of X_{can}. Theorem 2.3 can be compared with the diameter estimates in Li [23, 24] for fibres of collapsing Ricci-flat Kähler metrics on a projective Calabi-Yau manifold. In fact, the proof of Theorem 2.3 based on Theorem 1.1 can serve as an alternative proof and improvement of the diameter estimates in [23, 24] (c.f. Theorem 11.1). One can further derive uniform bounds for the Green's function and volume non-collapsing on each smooth fibre with respect to the rescaled metric $e^t g(t)$ (c.f. Theorem 11.2).

[1] More generally, the abundance conjecture predicts that K_X is nef if and only if it is semi-ample when X is projective.

2.3. **The case of constant scalar curvature Kähler metrics.** Finally, we would like to extend our results to families of cscK metrics with bounded p-Nash entropy for $p \leq n$. We will apply Theorem 1.1 to cscK metrics on smooth minimal models.

Theorem 2.4. *Let X be an n-dimensional smooth minimal model of general type. For any Kähler class $\mathcal{A}$ of X, there exist $\delta_0 = \delta_0(\mathcal{A}) > 0$, $C = C(\mathcal{A}, \delta_0) > 0$, $\alpha = \alpha(\mathcal{A}, \delta_0)$ and $c = c(\mathcal{A}, \delta_0) > 0$ such that for any $0 < \delta < \delta_0$, there exists a unique cscK metric ω_δ in $K_X + \delta\mathcal{A}$ satisfying*

$$\mathrm{diam}(X, \omega_\delta) \leq C,$$

$$\int_X |G_\delta(x, \cdot)|\omega_\delta^n + \int_X |\nabla G_\delta(x, \cdot)|\omega_\delta^n + \left(-\inf_{y\in X} G_\delta(x, y)\right) \mathrm{Vol}_{\omega_\delta}(X) \leq C,$$

$$\frac{\mathrm{Vol}_\omega(B_{\omega_\delta}(x, R))}{\mathrm{Vol}_{\omega_\delta}(X)} \geq cR^\alpha,$$

for any $x \in X$ and $R \in (0, 1)$, where $G_\delta(x, y)$ is the Green's function of (X, ω_δ).

In particular, if the canonical model X_{can} of X has only isolated singularities, then (X, ω_δ) converges to the Kähler-Einstein metric space (X_{can}, d_{KE}) in Gromov-Hausdorff topologgy as $\delta \to 0$, where (X_{can}, d_{KE}) is the metric completion of the unique smooth Kähler-Einstein metric on the regular part of X_{can} in [29].

The existence of cscK metrics in a Kähler class near $[K_X]$ on a minimal model is proved in [43, 36, 20, 27, 30]. When X is a minimal model of general type, there exists a unique Kähler-Einstein current with bounded potentials by the work of [22, 8, 45] and it is a smooth Kähler-Einstein metric g_{KE} on X°_{can}, the regular part of X_{can}. It is proved in [29] that the metric completion of (X°_{can}, g_{KE}) is a compact metric space homeomorphic to X_{can} itself.

It is conjectured in [20] that if X is a minimal model, then the cscK metric spaces near the canonical class $[K_X]$ converge geometrically to the twisted Kähler-Einstein space (X_{can}, d_{can}). Theorem 2.4 can be viewed as a partial confirmation of this conjecture.

3. Bounded sets in the Kähler cone

Notational convention: if $\omega = (g_{i\bar{j}})$ is a Kähler metric and $\theta = (\theta_{i\bar{j}})$ is a $(1,1)$-form, we denote $\mathrm{tr}_\omega(\theta) = g^{i\bar{j}}\theta_{i\bar{j}}$, where $(g^{i\bar{j}})$ is the inverse of $(g_{i\bar{j}})$. For a number $p \in (1, \infty)$, we denote by p^* the conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$.

Proposition 3.1. *Let (X, ω_X) be an n-dimensional compact Kähler manifold equipped with a Kähler metric ω_X in a Kähler class α. For any $k \geq 0$, and a cohomology class $\beta \in H^{1,1}(X, \mathbb{R})$, there exists a smooth representative $\theta \in \beta$ such that*

$$\|\theta\|_{C^k(X,\omega_X)} \leq C, \tag{3.1}$$

for some constant $C = C(X, \omega_X, k, |\beta \cdot \alpha^{n-1}|, |\beta^2 \cdot \alpha^{n-2}|) > 0$.

Proof. We will take θ to be the unique harmonic $(1,1)$-form in the class β, relative to the Kähler metric ω_X, i.e.

$$\Delta_{\bar\partial}\theta = 0.$$

By the standard Bochner-Kodaira-Lichnerowicz formula, we have

$$0 = -\Delta_{\bar\partial}\theta_{i\bar k} = \frac{1}{2}(\theta_{i\bar k, j\bar j} + \theta_{i\bar k, \bar j j}) + \theta_{m\bar j}R_{i\bar m j\bar k} - \frac{1}{2}\theta_{m\bar k}R_{i\bar m} - \frac{1}{2}\theta_{i\bar m}R_{m\bar k}, \tag{3.2}$$

where $R_{i\bar m j\bar k}$, $R_{i\bar m}$ denote the Riemann and Ricci curvatures of the fixed metric ω_X, and $\theta_{i\bar k, j\bar j}$, $\theta_{i\bar k, \bar j j}$ denote the covariant derivatives of $\theta_{i\bar k}$ with respect to the connection induced by ω_X. It is well-known that equation (3.2) is a linear elliptic equation of the $(1,1)$-form $\theta_{i\bar k}$.

Taking traces on both sides of (3.2), we obtain

$$\Delta_{\omega_X}(\mathrm{tr}_{\omega_X}\theta) = 0,$$

hence $\mathrm{tr}_{\omega_X}\theta$ must be a constant. Then we have

$$c_1 = \int_X \theta \wedge \omega_X^{n-1} = \frac{V}{n}\mathrm{tr}_{\omega_X}\theta,$$

where $V = \int_X \omega_X^n = [\omega_X]^n$. On the other hand, we also have

$$c_2 = \int_X \theta^2 \wedge (\omega_X)^{n-2} = C(n)\int_X ((\mathrm{tr}_{\omega_X}\theta)^2 - |\theta|^2_{\omega_X})\omega_X^n. \tag{3.3}$$

From this we see that

$$\int_X |\theta|^2_{\omega_X}\omega_X^n \tag{3.4}$$

is uniformly bounded depending only on c_1, c_2 and ω_X. Applying Moser iteration to the equation (3.2) we get an L^∞ bound for θ. The uniform $C^{k,\alpha}$ estimates of θ then follow from the standard elliptic estimates applied to the linear elliptic equation (3.2). □

Note that when β is Kähler, c_2 in (3.3) is positive and so (3.4) is uniformly bounded depending only on c_1 and ω_X. The following corollary is then an immediate consequence of Proposition 3.1.

Corollary 3.1. *Let (X, ω_X) be an n-dimensional Kähler manifold equipped with a Kähler metric ω_X. Then the following hold.*

(1) *For any bounded set U in the Kähler cone of X, there exists $C = C(U) > 0$ such that for any Kähler class $\beta \in U$,*

$$\beta \cdot [\omega_X]^{n-1} < C.$$

(2) *For any $A > 0$ and a Kähler class β with*

$$\beta \cdot [\omega_X]^{n-1} < A,$$

there exists $C = C(A) > 0$ such that

$$C[\omega_X] - \beta$$

is a Kähler class.

Corollary 3.2. *Let (X, ω_X) be an n-dimensional Kähler manifold equipped with a Kähler metric ω_X. For any bounded set U in the Kähler cone of X, there exists a smooth closed $(1,1)$-form $\theta \in \beta$ for any $\beta \in U$ with the following uniform properties.*

(1) *There exists $C = C(U) > 0$ such that $\|\theta\|_{C^3(X,\omega_X)} \leq C$,*

(2) *There exist $\alpha = \alpha(U) > 0$ and $C = C(U, \alpha) > 0$ such that for any $\varphi \in \mathrm{PSH}(X, \theta)$,*

$$\int_X e^{-\alpha(\varphi - \sup_X \varphi)} (\omega_X)^n \leq C,$$

Proof. It suffices to prove (2). By Proposition 3.1, there exists $B = B(U) > 0$ such that $\theta < B\omega_X$. Then for any $\varphi \in \mathrm{PSH}(X, \theta)$, we have $\varphi \in \mathrm{PSH}(X, B\omega_X)$. The corollary then follows by applying the α-invariant of $B\omega_X$. □

4. L^∞-estimates for complex Monge-Ampère equations

Proposition 4.1. *Let (X, ω_X) be an n-dimensional compact Kähler manifold equipped with a Kähler metric ω_X. For any $K > 0$, $p > n$, there exist $C = C(X, \omega_X, n, p, K) > 0$ such that if θ a smooth closed $(1,1)$-form with $\theta \leq \omega_X$ and if $\omega = \theta + \sqrt{-1}\partial\overline{\partial}\varphi$ is a Kähler form satisfying*

$$\mathcal{N}_{X,\omega_X,p}(\omega) \leq K,$$

then

$$\|\varphi - \sup_X \varphi - \mathcal{V}_\theta\|_{L^\infty(X)} \leq C,$$

where $\mathcal{V}_\theta = \sup\{u : u \in \mathrm{PSH}(X, \theta),\ u \leq 0\}$ is the envelope of non-positive θ-psh functions.

Proof. By the assumption on θ, we have $\mathrm{PSH}(X,\theta) \subseteq \mathrm{PSH}(X,\omega_X)$. Therefore there exists $\alpha > 0$ and $C_\alpha > 0$ such that for any $\varphi \in \mathrm{PSH}(X,\theta)$,

$$\int_X e^{-\alpha(\varphi - \sup_X \varphi)} \omega_X^n \le C_\alpha.$$

Then the proposition follows from the uniform L^∞-estimates from [8, 7, 2, 11, 16] which generalize Kolodziej's L^∞ estimates in the case of a fixed background metric [22]. □

Corollary 4.1. *Let (X,ω_X) be an n-dimensional compact Kähler manifold equipped with a Kähler metric ω_X. For any $A, K > 0$ and $p > n$, if two Kähler forms ω_1 and ω_2 belong to the same Kähler class and*

$$\omega_1, \omega_2 \in \mathcal{V}(X, \omega_X, n, A, p, K),$$

then there exist $C = C(X, \omega_X, n, A, p, K) > 0$ and $\varphi \in C^\infty(X)$ such that

$$\omega_2 = \omega_1 + \sqrt{-1}\partial\overline{\partial}\varphi, \ \|\varphi - \sup_X \varphi\|_{L^\infty(X)} \le C. \tag{4.1}$$

Proof. Let γ be the Kähler class of ω_1 and ω_2. Then $\gamma \cdot [\omega_X]^{n-1} < A$ by our assumption. By Proposition 3.1, we can choose a smooth closed $(1,1)$-form $\theta \in \gamma$ (not necessarily positive) such that

$$\|\theta\|_{C^3(X,\omega_X)}$$

is uniformly bounded by a constant that only depends on A. Furthermore, We define $\varphi_i \in C^\infty(X)$ by

$$\omega_i = \theta + \sqrt{-1}\partial\overline{\partial}\varphi_i, \ \sup_X \varphi_i = 0, \ i = 1, 2.$$

Then φ_i satisfies the following complex Monge-Ampère equation

$$\frac{(\theta + \sqrt{-1}\partial\overline{\partial}\varphi_i)^n}{[\theta]^n} = e^{F_i}(\omega_X)^n, \ \sup \varphi_i = 0. \tag{4.2}$$

Since $\omega_i \in \mathcal{V}(X, \omega_X, n, A, p, K)$,

$$\|e^{F_i}\|_{L^1 \log L^p(X,\omega_X)} \le K, \ i = 1, 2.$$

Let $\mathcal{V}_\theta = \sup\{u : u \in \mathrm{PSH}(X,\theta), \ u \le 0\}$. Proposition 4.1 implies that there exists $C = C(X, \omega_X, n, A, p, K) > 0$ such that

$$\|\varphi_i - \mathcal{V}_\theta\|_{L^\infty(X)} \le C,$$

and so

$$\|\varphi_1 - \varphi_2\|_{L^\infty(X)} \le 2C.$$

The corollary is proved, with $\varphi = \varphi_1 - \varphi_2$. □

5. Estimates for the Green's functions

Throughout this section, we fix the n-dimensional Kähler manifold (X, ω_X) and constants $A, K > 0$ and $p > n$.

The following lemma is a natural extension of Lemma 2 in [14].

Lemma 5.1. *Suppose $\omega \in \mathcal{V}(X, \omega_X, n, A, p, K)$. Let $v \in L^1(X, \omega^n)$ be a function that satisfies $\int_X v\omega^n = 0$ and*

$$v \in C^2(\overline{\Omega_0}), \quad \Delta_\omega v \geq -a \text{ in } \Omega_0 \tag{5.1}$$

for some $a > 0$ and $\Omega_s = \{v > s\}$ is the super-level set of v. Then there is a uniform constant $C = C(X, \omega_X, A, p, K) > 0$ such that

$$\sup_X v \leq C(a + \frac{1}{[\omega]^n} \int_X |v|\omega^n). \tag{5.2}$$

For the convenience of the readers, we sketch the proof of Lemma 5.1.

Proof. We follow closely the arguments in [14].

First, we observe that it suffices to prove the lemma in the case $a = 1$. This is because both the equation (5.1) and the desired inequality (5.2) are homogenous under a simultaneous rescaling of a and v, $a \to 1$, $v \to \frac{v}{a}$.

Next, we may assume $\|v\|_{L^1(X,\omega^n)} \leq [\omega]^n$, otherwise, replace v by $\hat{v} := v \cdot [\omega]^n / \|v\|_{L^1(X,\omega^n)}$ which still satisfies (5.1) with the same $a = 1$ and $\|\hat{v}\|_{L^1(X,\omega^n)} = [\omega]^n$. It thus suffices to show $\sup_X v \leq C$ for some $C > 0$ with the dependence as stated in the lemma. By Proposition 3.1, we can choose a smooth closed $(1,1)$-form $\theta \in [\omega]$ with $\|\theta\|_{C^3(X,\omega_X)}$ uniformly bounded. We let $\omega = \theta + \sqrt{-1}\partial\overline{\partial}\varphi$ for $\varphi \in \mathrm{PSH}(X, \theta)$ and $\sup_X \varphi = 0$.

(1). We fix a sequence of positive smooth functions $\eta_k : \mathbb{R} \to \mathbb{R}_+$ such that $\eta_k(x)$ converges uniformly and monotonically decreasingly to the function $x \cdot \chi_{\mathbb{R}_+}(x)$, as $k \to \infty$. We may choose $\eta_k(x) \equiv 1/k$ for any $x \leq -1/2$. As in [15] we make use of auxiliary Monge-Ampère equations. More precisely, for each $s \geq 0$ and large k, we consider the following specific complex Monge-Ampère equations [14]

$$(\theta + \sqrt{-1}\partial\overline{\partial}\psi_{s,k})^n = [\omega]^n \frac{\eta_k(v-s)}{A_{s,k}} e^F \omega_X^n, \quad \sup_X \psi_{s,k} = 0, \tag{5.3}$$

where

$$A_{s,k} = \int_X \eta_k(v-s) e^F \omega_X^n \to \int_{\Omega_s} (v-s) e^F \omega_X^n =: A_s \text{ as } k \to \infty. \tag{5.4}$$

We have assumed that the open set $\Omega_s \neq \emptyset$ so $A_s > 0$. The assumption that $\|v\|_{L^1(X,\omega^n)}/[\omega]^n \leq 1$ implies that $A_s \leq 1$, hence $A_{s,k} \leq 2$ for large k.

(2). Recall that we have assumed that $a = 1$. We denote $\Lambda = C + 1$ where $C > 0$ is the constant in Corollary 4.1. Consider the test function

$$\Phi := -\varepsilon(-\psi_{s,k} + \varphi + \Lambda)^{\frac{n}{n+1}} + (v - s),$$

where $\varepsilon > 0$ is chosen such that

$$\varepsilon^{n+1} = \Big(\frac{n+1}{n^2}\Big)^n (1 + \varepsilon n)^n A_{s,k}. \tag{5.5}$$

It follows easily from $A_{s,k} \le 2$ and equation (5.5) that

$$\varepsilon \le C(n) A_{s,k}^{1/(n+1)}, \tag{5.6}$$

for some $C(n) > 0$ depending only on n. The function Φ is a C^2 function on Ω_0 and $-\psi_{t,k} + \varphi_t + \Lambda \ge 1$. As shown in [14], it follows from the maximum principle, the choice of ε in (5.5) and the equations of $\psi_{s,k}$ and φ that $\Phi \le 0$ on X.

(3). From $\Phi \le 0$ and (5.6), we have $(v - s)A_{s,k}^{-1/(n+1)} \le C_1(-\psi_{s,k} + \varphi + \Lambda)^{n/(n+1)}$ on X, for some $C_1 > 0$ depending only on n. This together with the α-invariant and Hölder-Young inequality (see [14] for more details) implies that for some uniform constant $C_2 > 0$

$$r\phi(s + r) \le C_2 \phi(s)^{1+\delta_0}, \quad \text{for } \forall s \ge 0 \text{ and } \forall r > 0, \tag{5.7}$$

where we denote $\delta_0 = \frac{p-n}{np} > 0$ and $\phi(s) = \int_{\Omega_s} e^F \omega_X^n$.

The assumption that $\|v\|_{L^1(X,\omega^n)} \le [\omega]^n$ implies that $\int_{\Omega_0} v e^F \omega_X^n \le 1$. Hence for any $s > 0$ we have

$$\phi(s) = \int_{\Omega_s} e^F \omega_X^n \le \frac{1}{s}\int_{\Omega_0} v e^F \omega_X^n \le \frac{1}{s}. \tag{5.8}$$

We can pick $s_0 = (2C_2)^{1/\delta_0}$ to ensure that $\phi(s_0)^{\delta_0} \le 1/(2C_2)$. Given (5.7), we can apply the De Giorgi type iteration argument of Kolodziej [22] to conclude that $\phi(s) = 0$ for any $s > S_\infty$ with

$$S_\infty = s_0 + \frac{1}{1 - 2^{-\delta_0}} = (2C_2)^{1/\delta_0} + \frac{1}{1 - 2^{-\delta_0}}.$$

This means that $\sup_X v \le S_\infty$ and the lemma is proved. □

The following corollary is an immediate consequence of Lemma 5.1.

Corollary 5.1. *Suppose $\omega \in \mathcal{V}(X, \omega_X, n, A, p, K)$. If $v \in C^2(X)$ satisfies*

$$|\Delta_\omega v| \le 1 \ \textit{and} \ \int_X v\omega^n = 0,$$

then there is a uniform constant $C = C(X, \omega_X, n, A, p, K) > 0$ such that

$$\sup_X |v| \le C(1 + \frac{1}{[\omega]^n} \int_X |v| \omega^n).$$

In order to bound $\frac{1}{[\omega]^n} \int_X |v|\omega^n$, we will have to impose a uniform lower bound for the normalized volume form

$$([\omega]^n)^{-1} \frac{\omega^n}{\omega_X^n}.$$

In particular, we will consider $\omega \in \mathcal{W}(X, \omega_X, n, A, p, K, \gamma)$ for some nonnegative continuous function γ.

Lemma 5.2. *Suppose $\gamma \ge 0$ is a continuous function on X with $\{\gamma > 0\}$ being connected. Then for any open subset $V \subset\subset \{\gamma > 0\}$, there exists a connected open subset U of X with*

$$V \subset\subset U \subset\subset \{\gamma > 0\}.$$

Proof. Obviously, $\{\gamma > 0\}$ is path connected since it is open and connected. We choose a fixed base point $p \in V$. For any $q \in \overline{V}$, there exists a continuous path $\mathcal{C}$ joining p and q in $\{\gamma > 0\}$. We can find an open tubular neighborhood U_q of $\mathcal{C}$ such that $U_q \subset\subset \{\gamma > 0\}$. Then

$$\overline{V} \subset \cup_{q \in \overline{V}} U_q$$

and we can find finitely many $q_1, q_2, ..., q_N \subset \overline{V}$ such that

$$\overline{V} \subset U = \cup_{j=1}^N U_{q_j}.$$

Then $U \subset\subset \{\gamma > 0\}$ is open and connected since every U_{q_j} is path connected with a common point p. The lemma is then proved. □

Lemma 5.3. *There exists $\varepsilon = \varepsilon(X, \omega_X, n, A, p, K) > 0$ such that if*

(1) *$\gamma \ge 0$ is a continuous function on X such that*

$$|\{\gamma = 0\}|_{\omega_X} \le \epsilon, \ \{\gamma > 0\} \ \textit{ is connected,}$$

(2) *$\omega \in \mathcal{W}(X, \omega_X, n, A, p, K; \gamma)$,*

(3) *$v \in C^2(X)$ satisfies*

$$|\Delta_\omega v| \le 1 \ \textit{ and } \int_X v\omega^n = 0,$$

then there exists $C = C(X, \omega_X, n, A, p, K, \varepsilon, \gamma) > 0$ such that

$$\frac{1}{[\omega]^n} \int_X |v| \omega^n \le C.$$

Proof. The proof is by contradiction. Suppose Lemma 5.3 fails. Then there exist a sequence of $\omega_j \in \mathcal{W}(X, \omega_X, n, A, p, K; \gamma)$ and $v_j \in C^2(X)$ satisfying

$$\Delta_{\omega_j} v_j = h_j, \quad \int_X v_j \omega_j^n = 0,$$

for some $|h_j| \leq 1$, and as $j \to \infty$

$$\frac{1}{[\omega_j]^n} \int_X |v_j| \omega_j^n =: N_j \to \infty. \tag{5.9}$$

We define F_j by

$$e^{F_j} = ([\omega_j]^n)^{-1} \frac{\omega_j^n}{\omega_X^n}$$

and immediately we have

$$e^{F_j} \geq \gamma.$$

We now consider $\tilde{v}_j$ defined by $\tilde{v}_j = v_j / N_j$. Clearly we have

$$|\Delta_{\omega_j} \tilde{v}_j| = |h_j|/N_j \to 0, \ \frac{1}{[\omega_j]^n} \int_X |\tilde{v}_j| \omega_j^n = 1.$$

Applying Lemma 5.1 to $\tilde{v}_j$, there exists a uniform $C > 0$ such that for all $j \geq 1$,

$$\sup_X |\tilde{v}_j| \leq C. \tag{5.10}$$

From the equation of $\tilde{v}_j$ and integration by parts, we see that

$$\int_X |\nabla \tilde{v}_j|^2_{\omega_j} e^{F_j} \omega_X^n = \frac{1}{[\omega_j]^n} \int_X |\nabla \tilde{v}_j|^2_{\omega_j} \omega_j^n \to 0. \tag{5.11}$$

Let $U_0 = \{\gamma > 0\}$. Suppose $|\{\gamma = 0\}|_{\omega_X} < \varepsilon$ for some sufficiently small $\varepsilon > 0$ to be determined. Then for any $\varepsilon > 0$, we can pick open connected subsets $U_{3\varepsilon} \subset\subset U_{2\varepsilon} \subset\subset \{\gamma > 0\}$ as in Lemma 5.2 such that

$$|X \setminus U_{2\varepsilon}|_{\omega_X} < 2\varepsilon, \ |X \setminus U_{3\varepsilon}|_{\omega_X} < 3\varepsilon.$$

Without loss of generality, we can assume both $U_{2\varepsilon}$ and $U_{3\varepsilon}$ have smooth boundaries. Let

$$\delta_\varepsilon = \inf_{U_{2\varepsilon}} \gamma. \tag{5.12}$$

Then we have $\inf_{U_{2\varepsilon}} e^{F_j} \geq \delta^{-1}$ and

$$\begin{aligned}
\int_{U_{2\varepsilon}} |\nabla \tilde{v}_j|_{\omega_X} \omega_X^n &\leq \Big(\int_{U_{2\varepsilon}} |\nabla \tilde{v}_j|^2_{\omega_j} e^{F_j} \omega_X^n \Big)^{1/2} \Big(\int_{U_{2\varepsilon}} e^{-F_j} \mathrm{tr}_{\omega_X}(\omega_j) \omega_X^n \Big)^{1/2} \\
&\leq \delta_\varepsilon^{-1/2} \Big(\int_X |\nabla \tilde{v}_j|^2_{\omega_j} e^{F_j} \omega_X^n \Big)^{1/2} \Big(\int_X \omega_X^{n-1} \wedge \omega_j \Big)^{1/2} \\
&\leq (A\delta_\varepsilon)^{-1/2} \Big(\int_X |\nabla \tilde{v}_j|^2_{\omega_j} e^{F_j} \omega_X^n \Big)^{1/2} \to 0
\end{aligned}$$

by (5.11) as $j \to \infty$. Therefore $\tilde{v}_j$ is uniformly bounded in $W^{1,1}(U_{2\varepsilon}, \omega_X)$ by the above estimate and (5.10). By the Sobolev embedding theorem, after passing to a subsequence, we can assume that $\tilde{v}_j$ converges to $\tilde{v}_\infty$ in $L^1(\overline{U_{3\varepsilon}}, \omega_X)$. Furthermore, since $\tilde{v}_j$ is uniformly bounded in $L^\infty(\overline{U_{3\varepsilon}})$ and converges almost everywhere to $\tilde{v}_\infty$ in $\overline{U_{3\varepsilon}}$, $\tilde{v}_\infty$ is also bounded in $L^\infty(\overline{U_{3\varepsilon}}, \omega_X)$.

Since $\lim_{j\to\infty} \int_{U_{2\varepsilon}} |\nabla \tilde{v}_j|_{\omega_X} \omega_X^n = 0$, we have for any test function $f \in C_0^\infty(U_{2\varepsilon})$

$$\begin{aligned}
\left| \int_X \tilde{v}_j (\Delta f) \omega_X^n \right| &= \left| \int_X \langle \nabla \tilde{v}_j, \nabla f \rangle_{\omega_X} \omega_X^n \right| \\
&\leq (\sup_X |\nabla f|_{\omega_X}) \int_{U_{2\varepsilon}} |\nabla \tilde{v}_j|_{\omega_X} \omega_X^n \to 0
\end{aligned}$$

as $j \to \infty$, where Δ is the Laplace operator with respect to ω_X. Therefore by Weyl's lemma for the Laplace equation, $\tilde{v}_\infty$ solves the Laplace equation $\Delta \tilde{v}_\infty = 0$ on $U_{3\varepsilon}$ and $\tilde{v}_\infty \in C^\infty(U_{3\varepsilon})$. For any smooth vector field $Y \in C_0^\infty(U_{3\varepsilon})$, we have

$$\begin{aligned}
\int_X \langle \nabla \tilde{v}_\infty, Y \rangle_{\omega_X} \omega_X^n &= -\int_X \tilde{v}_\infty \cdot \mathrm{div}_{\omega_X} Y \omega_X^n \\
&= -\lim_{j\to\infty} \int_X \tilde{v}_j \cdot \mathrm{div}_{\omega_X} Y \omega_X^n \\
&= \lim_{j\to\infty} \int_X \langle \nabla \tilde{v}_j, Y \rangle_{\omega_X} \omega_X^n = 0.
\end{aligned}$$

Here the first and third lines follow from the divergence theorem, and the second equality holds since $\tilde{v}_j$ converge in $L^1(\overline{U_{3\varepsilon}})$ to $\tilde{v}_\infty$ and the function $\mathrm{div}_{\omega_X} Y$ is compactly supported in $U_{3\varepsilon}$. By taking $Y = \eta \nabla \tilde{v}_\infty$ for any nonnegative function $\eta \in C_0^\infty(U_{2\delta})$, we see immediately that $\nabla \tilde{v}_\infty \equiv 0$ on $U_{3\varepsilon}$, and hence $\tilde{v}_\infty$ is constant on $U_{3\varepsilon}$ since $U_{3\varepsilon}$ is connected.

We first derive a uniform positive lower bound for $\int_{U_{3\varepsilon}} |\tilde{v}_j| \omega_X^n$. In fact, by the Hölder-Young inequality, there exists $C > 0$ such that for any $\delta > 0$ and smooth functions u and F, we have

$$|u| e^F = |\delta^{-1} u| e^{F + \log \delta} \leq \delta e^F (1 + |F| + |\log \delta|) + C \delta^{-1} |u| e^{\delta^{-1}|u|}. \tag{5.13}$$

Applying (5.13), we have

$$\begin{aligned}
1 &= \frac{1}{[\omega_j]^n} \int_X |\tilde{v}_j| \omega_j^n = \int_X |\tilde{v}_j| e^{F_j} \omega_X^n \\
&\leq \delta \int_X e^{F_j} (1 + |F_j| + |\log \delta|) \omega_X^n + C \delta^{-1} \int_X |\tilde{v}_j| e^{\delta^{-1} |\tilde{v}_j|} \omega_X^n \\
&\leq \delta \int_X e^{F_j} (1 + |F_j| + |\log \delta|) \omega_X^n + C \delta^{-1} \int_{U_{3\varepsilon}} |\tilde{v}_j| e^{\delta^{-1} |\tilde{v}_j|} \omega_X^n + C e^{C \delta^{-1}} \int_{X \setminus U_{3\varepsilon}} \omega_X^n
\end{aligned}$$

$$\begin{aligned}
&\le \delta \int_X e^{F_j}(1+|F_j|+|\log \delta|)\omega_X^n + C\delta^{-1}\int_{U_{3\varepsilon}} |\tilde{v}_j| e^{\delta^{-1}|\tilde{v}_j|}\omega_X^n + 2\varepsilon C e^{C\delta^{-1}}\\
&\le \frac{1}{2} + C_\delta([\omega_X]^n)^{-1}\int_{U_{3\varepsilon}} |\tilde{v}_j|\omega_X^n,
\end{aligned}$$

for all j and some uniform constants $C = C(A,p,K)$ and $C_\delta = C_\delta(A,p,K) > 0$, if we choose $\delta = \delta(A,p,K) > 0$ sufficiently small such that

$$\delta \int_X e^{F_j}(1+|F_j|+|\log\delta|)\omega_X^n < \frac{1}{4},$$

and then choose

$$\varepsilon < \varepsilon_1 = \frac{e^{-C\delta^{-1}}}{8C}. \tag{5.14}$$

Immediately we have

$$([\omega_X]^n)^{-1}\int_{U_{3\varepsilon}} |\tilde{v}_j|\omega_X^n \ge (2C_\delta)^{-1} \tag{5.15}$$

for sufficiently large j.

On the other hand, we can extend $\tilde{v}_\infty$ to a constant function on X. By applying the Hölder-Young inequality again and by the fact that on $U_{2\varepsilon}$ is connected, for any $\epsilon' > 0$, there exists $C_1 > 0$ and $C_2 = C_2(A,p,K) > 0$ such that for sufficiently large j, we have

$$\begin{aligned}
|\tilde{v}_\infty| &= \left|\frac{1}{[\omega_j]^n}\int_X (\tilde{v}_\infty - \tilde{v}_j)\,\omega_j^n\right|\\
&\le \frac{1}{[\omega_j]^n}\int_{U_{3\varepsilon}} |\tilde{v}_\infty - \tilde{v}_j|\,\omega_j^n + \frac{1}{[\omega_j]^n}\int_{X\setminus U_{3\varepsilon}} |v_\infty - \tilde{v}_j|\omega_j^n\\
&= \int_{U_{3\varepsilon}} |\tilde{v}_\infty - \tilde{v}_j|\, e^{F_j}\omega_X^n + \int_{X\setminus U_{3\varepsilon}} |v_\infty - \tilde{v}_j| e^{F_j}\omega_X^n\\
&\le \epsilon' \int_X e^{F_j}(1+|F_j|+|\log\epsilon'|)\omega_X^n + C_1(\epsilon')^{-1}e^{(\epsilon')^{-1}\sup_X|\tilde{v}_j-\tilde{v}_\infty|}\int_{U_{3\varepsilon}} |\tilde{v}_j - \tilde{v}_\infty|\omega_X^n\\
&\quad + C_1(\epsilon')^{-1}e^{(\epsilon')^{-1}\sup_X(|v_\infty-\tilde{v}_j|)}\int_{X\setminus U_{3\varepsilon}} |v_\infty - \tilde{v}_j|\omega_X^n\\
&\le C_2(\epsilon')^{1/2} + e^{C_2(\epsilon')^{-1}}\varepsilon\\
&< (4C_\delta)^{-1},
\end{aligned}$$

if we choose ϵ' and ε with

$$\epsilon' < (8C_2C_\delta)^{-2}, \text{ and } \varepsilon < \varepsilon_2 = \left(8e^{C_2(\epsilon')^{-1}}C_\delta\right)^{-1}, \tag{5.16}$$

Since $\tilde{v}_j$ converges to $\tilde{v}_\infty$ in $L^1(\overline{U_{3\varepsilon}})$, for sufficiently large j, we have

$$([\omega_X]^n)^{-1}\int_{U_{3\varepsilon}} |\tilde{v}_j|\omega_X^n < ([\omega_X]^n)^{-1}\int_{U_{3\varepsilon}} |\tilde{v}_\infty|\omega_X^n + (4C_\delta)^{-1} < (2C_\delta)^{-1}.$$

This contradicts the lower bound (5.15). From now on, we will fix the choice for

$$\varepsilon = \frac{\min(\varepsilon_1, \varepsilon_2)}{2}$$

from (5.14) and (5.16) for the parameter ε in the assumption of the lemma.

We have now completed the proof of the lemma. □

Let ω be a Kähler metric on X. We let $G(x,\cdot)$ be the Green's function of (X,ω) with base point x, for any $x \in X$.

Lemma 5.4. *There exists $\varepsilon = \varepsilon(X, \omega_X, n, A, p, K) > 0$ such that if*

(1) $\gamma \geq 0$ *is a continuous function on X such that*

$$|\{\gamma = 0\}|_{\omega_X} \leq \varepsilon, \ \{\gamma > 0\} \text{ is connected},$$

(2) $\omega \in \mathcal{W}(X, \omega_X, n, A, p, K; \gamma)$,

then there exists $C = C(X, \omega_X, n, A, p, K, \varepsilon, \gamma) > 0$ such that for any $x \in X$

$$\int_X |G(x,\cdot)|\omega^n \leq C, \ \text{ and } \inf_{y\in X} G(x,y) \geq -\frac{C}{[\omega]^n},$$

where $G(x,\cdot)$ is the Green's function of (X,ω).

Proof. We now fix $\omega \in \mathcal{W}(X, \omega_X, n, A, p, K; \gamma)$ satisfying the assumption and $x \in X$. Take a sequence of smooth functions h_k which are uniformly bounded and converge in $L^q(X,\omega)$ for some fixed sufficiently large $q > 0$, to $-\chi_{\{G(x,\cdot)\leq 0\}} + \frac{1}{[\omega]^n}\int_{\{G(x,\cdot)\leq 0\}}\omega^n$, where we denote χ_E to be the characteristic function of a Borel set E. We can also choose h_k to satisfy

$$\sup_X |h_k| \leq 2, \quad \text{and} \quad \frac{1}{[\omega]^n}\int_X h_k\omega^n = 0.$$

Immediately, there exists a unique smooth solution solving the linear equation

$$\Delta_\omega v_k = h_k, \quad \frac{1}{[\omega]^n}\int_X v_k\omega^n = 0.$$

By Lemma 5.3, there exists $C > 0$ independent of k such that

$$\sup_X |v_k| \leq C.$$

Applying the Green's formula, we have by the dominated convergence theorem

$$v_k(x) = \int_X G(x,y)(-h_k(y))\omega^n(y) \to \int_{\{G(x,\cdot)\leq 0\}} G(x,\cdot)\omega^n$$

as $k \to \infty$. Combining this with the fact that

$$\int_{\{G(x,\cdot)\geq 0\}} G(x,\cdot)\omega^n = -\int_{\{G(x,\cdot)\leq 0\}} G(x,\cdot)\omega^n.$$

we easily find that $\int_X |G(x,\cdot)|\omega^n \leq C$.

For the lower bound of the Green's function, we apply Lemma 5.1 to the function $v := -[\omega]^n \cdot G(x,\cdot)$ and $a = 1$. It then follows that

$$-[\omega]^n \cdot \inf_X G(x,\cdot) \leq C([\omega]^n + \int_X |G(x,\cdot)|\omega^n) \leq C.$$

This completes the proof of the lemma. □

We observe that Lemma 5.4 implies a lower bound of the first nonzero eigenvalue of the Laplacian operator Δ_ω. To see this, suppose $\lambda_1 > 0$ is such an eigenvalue and $f \in C^\infty(X)$ is an associated eigenfunction normalized by $\int_X f^2\omega^n = [\omega]^n$. Then $\Delta_\omega f = -\lambda_1 f$. If we let $x_0 \in X$ be a maximum point of $|f|$, by the Green's formula we have

$$0 \neq f(x_0) = \frac{1}{[\omega]^n}\int_X f\omega^n - \int_X G(x_0,\cdot)\Delta_\omega f\omega_t^n = \lambda_1 \int_X G(x_0,\cdot) f\omega^n.$$

Hence

$$|f(x_0)| \leq \lambda_1 |f(x_0)| \int_X |G(x_0,\cdot)|\omega^n \leq C|f(x_0)|\lambda_1,$$

by Lemma 5.4. This immediately gives the uniform positive lower bound of λ_1.

For convenience of notation, we write

$$\mathcal{G}(x,\cdot) = G(x,\cdot) - \inf_{x,y\in X} G(x,y) + 1 > 0. \tag{5.17}$$

It is clear that $\int_X \mathcal{G}(x,\cdot)\omega^n \leq C$.

Lemma 5.5. *There exist $\varepsilon = \varepsilon(X,\omega_X,n,A,p,K) > 0$ and $\varepsilon' = \varepsilon'(n,p) > 0$ such that if*

(1) $\gamma \geq 0$ *is a continuous function on X such that*

$$|\{\gamma = 0\}|_{\omega_X} \leq \varepsilon,\ \{\gamma > 0\} \text{ is connected,}$$

(2) $\omega \in \mathcal{W}(X,\omega_X,n,A,p,K;\gamma)$,

then there exists $C = C(A,p,K,\varepsilon,\gamma,\varepsilon') > 0$ such that for any $x \in X$, we have

$$\int_X \mathcal{G}(x,\cdot)^{1+\varepsilon'}\omega^n \leq C([\omega]^n)^{-\varepsilon'}.$$

Proof. We fix $x \in X$ and a small constant $\varepsilon' > 0$ to be determined. Fix a large $k \gg 1$ and consider a smooth positive function H_k which is a smoothing of $\min\{\mathcal{G}(x,\cdot), k\}$. Without loss of generality, we can assume that H_k converges increasingly to $\mathcal{G}(x,\cdot)$ as $k \to \infty$. In particular, there exists $C = C(A, p, K, \varepsilon, \gamma) > 0$ such that for any k

$$0 < \int_X H_k e^F \omega_X^n \leq \frac{1}{[\omega]^n} \int_X \mathcal{G}(x,\cdot)\omega^n \leq \frac{C}{[\omega]^n},$$

where $F = ([\omega]^n)^{-1} \frac{\omega^n}{\omega_X^n}$.

We now consider the following linear equation on X

$$\left\{ \begin{array}{l} \Delta_\omega u_k = -H_k^{\varepsilon'} + \frac{1}{[\omega]^n} \int_X H_k^{\varepsilon'} \omega^n, \\ \frac{1}{[\omega]^n} \int_X u_k \omega^n = 0. \end{array} \right. \tag{5.18}$$

Equation (5.18) admits a unique smooth solution since the smooth function on the right-hand side of the first equation has integral 0. We cannot apply the maximum principle to u_k directly since the term $-H_k^{\varepsilon'}$ on the right-hand side of 5.18 is unbounded.

We will let $\chi \in [\omega]$ be a smooth closed $(1,1)$-form such that $\|\chi\|_{C^3(X,\omega_3)}$ is uniformly bounded some constant that only depends on A. We let

$$\omega = \chi + \sqrt{-1}\partial\overline{\partial}\varphi, \ \sup_X \varphi = 0,$$

and let

$$\hat{H}_k := [\omega]^n \cdot H_k.$$

Then we consider the following auxiliary complex Monge-Ampère equation which admits a smooth solution by [44]

$$\frac{1}{[\omega]^n}(\chi + \sqrt{-1}\partial\overline{\partial}\psi_k)^n = \frac{(\hat{H}_k)^{n\varepsilon'} + 1}{\int_X \Big((\hat{H}_k)^{n\varepsilon'} + 1\Big)\omega^n}\omega^n = \frac{(\hat{H}_k)^{n\varepsilon'} + 1}{B_k} e^F \omega_X^n, \tag{5.19}$$

with

$$\sup_X \psi_k = 0, \ B_k = \int_X ((\hat{H}_k)^{n\varepsilon'} + 1) e^F \omega_X^n.$$

By the Hölder inequality, there exists $C = C(A, p, K, \varepsilon', \gamma) > 0$ for sufficiently small $0 < \varepsilon' < n^{-1}$ such that for all k, we have

$$C^{-1} \leq \int_X \gamma \omega_X^n \leq B_k \leq \int_X e^F \omega_X^n + \Big(\int_X e^F \omega_X^n\Big)^{1-n\varepsilon'} \Big(\int_X \hat{H}_k e^F \omega_X^n\Big)^{n\varepsilon'} \leq C. \tag{5.20}$$

For fixed $p' \in (n, p)$, the p'-th entropy of the function $((\hat{H}_k)^{n\varepsilon'} + 1)e^F/B_k$ on the right-hand side of (5.19) satisfies

$$\frac{1}{B_k} \int_X ((\hat{H}_k)^{n\varepsilon'} + 1) \Big| - \log B_k + F + \log(1 + (\hat{H}_k)^{n\varepsilon'}) \Big|^{p'} e^F \omega_X^n \tag{5.21}$$

$$\le \frac{|\log B_k|^{p'}}{B_k}\int_X ((\hat H_k)^{n\varepsilon'}+1)e^F\omega_X^n + \frac{1}{B_k}\int_X ((\hat H_k)^{n\varepsilon'}+1)\left(\log((\hat H_k)^{n\varepsilon'}+1)\right)^{p'} e^F\omega_X^n$$
$$+\frac{1}{B_k}\int_X ((\hat H_k)^{n\varepsilon'}+1)|F|^{p'}e^F\omega_X^n.$$

The first integral on the right hand side in (5.21) is bounded due to the estimate of the constant B_k in 5.20, the Hölder inequality and the uniform L^1-bound of

$$\int_X \hat H_k e^F\omega_X^n \le \int_X \mathcal{G}(x,\cdot)\omega^n$$

by Lemma 5.4.

The second integral on the right hand side in (5.21) is also uniformly bounded by a similar argument since

$$\frac{1}{B_k}\int_X (\hat H_k^{n\varepsilon'}+1)[\log(\hat H_k^{n\varepsilon'}+1)]^{p'}e^F\omega_X^n \le C\int_X \hat H_k^{\varepsilon'}(\hat H_k^{n\varepsilon'}+1)e^F\omega_X^n \le C,$$

by the Hölder inequality and the calculus inequality $(\log(1+x))^{p'} \le Cx^{\varepsilon'/n}$ for any $x>0$. We have also chosen $\varepsilon'>0$ small so that $(n+1)\varepsilon'<1$.

To deal with the last integral in (5.21), we observe that by Young's inequality

$$((\hat H_k)^{n\varepsilon'}+1)|F|^{p'} \le \frac{|F|^p}{p/p'} + \frac{(\hat H_k^{n\varepsilon'}+1)^{(p/p')^*}}{(p/p')^*},$$

where $(p/p')^*>1$ is the conjugate exponent of $p/p'>1$. Hence the last term in (5.21) satisfies

$$\begin{aligned} &\frac{1}{B_k}\int_X ((\hat H_k)^{n\varepsilon'}+1)|F|^{p'}e^F\omega_X^n\\ \le\ & C\int_X |F|^p e^F\omega_X^n + C\int_X ((\hat H_k)^{n\varepsilon'(p/p')^*}+1)e^F\omega_X^n\\ \le\ & C, \end{aligned}$$

if we choose ε' small so that $n\varepsilon'(p/p')^*<1$.

From now on we fix a small $\varepsilon'>0$ that meets the requirements above and so the p'-th entropy of the function on the right-hand side of (5.19) is uniformly bounded. We apply Corollary 4.1 to conclude that

$$\sup_X |\psi_k-\varphi| \le C, \tag{5.22}$$

for some uniform constant $C=C(A,p,K,\varepsilon,\gamma,\varepsilon')>0$.

We now consider the function

$$v_k := (\psi_k-\varphi) - \frac{1}{[\omega]^n}\int_X (\psi_k-\varphi)\omega^n + \varepsilon'' u_k, \tag{5.23}$$

where $\varepsilon'' > 0$ is a suitable constant to be chosen later. It follows from the definition that $\frac{1}{[\omega]^n}\int_X v_k\omega^n = 0$ and v_k is a smooth function.

Let $\omega_{\psi_k} = \chi + \sqrt{-1}\partial\overline{\partial}\psi_k$. We then calculate the Laplacian of v in (5.23) and there exists $C > 0$ such that

$$\begin{aligned}
\Delta_\omega v_k &= \mathrm{tr}_\omega(\omega_{\psi_k}) - n + \varepsilon''\Delta_\omega u_k \\
&\geq n\Big(\frac{\omega_{\psi_k}^n}{\omega^n}\Big)^{1/n} - n - \varepsilon'' H_k^{\varepsilon'} + \frac{\varepsilon''}{[\omega]^n}\int_X H_k^{\varepsilon'}\omega^n \\
&= nB_k^{-1/n}(\hat{H}_k^{n\varepsilon'} + 1)^{1/n} - n - \varepsilon'' H_k^{\varepsilon'} + \frac{\varepsilon''}{([\omega]^n)}\int_X H_k^{\varepsilon'}\omega^n \\
&\geq nC^{-1}([\omega]^n)^{\varepsilon'} H_k^{\varepsilon'} - n - \varepsilon'' H_k^{\varepsilon'} \geq -n,
\end{aligned}$$

if we choose $\varepsilon'' = nC^{-1}([\omega]^n)^{\varepsilon'}$. We apply the Green's formula to the function v_k at x

$$\begin{aligned}
v_k(x) &= \frac{1}{[\omega]^n}\int_X v_k\omega^n + \int_X G(x,\cdot)(-\Delta_\omega v_k)\omega^n = \int_X \mathcal{G}(x,\cdot)(-\Delta_\omega v_k)\omega^n \\
&\leq n\int_X \mathcal{G}(x,\cdot)\omega^n \leq C,
\end{aligned}$$

where the last inequality follows from the uniform $L^1(X,\omega^n)$-bound of $\mathcal{G}(x,\cdot)$. It then follows from (5.22) that

$$u_k(x) \leq C([\omega]^n)^{-\varepsilon'}$$

for a uniform constant $C > 0$. We now apply the Green's formula to u_k at $x \in X$

$$\begin{aligned}
u_k(x) &= \frac{1}{[\omega]^n}\int_X u_k\omega^n + \int_X \mathcal{G}(x,\cdot)(-\Delta_\omega u_k)\omega^n \\
&= \int_X \mathcal{G}(x,\cdot)\Big((H_k)^{\varepsilon'} - \frac{1}{[\omega]^n}\int_X (H_k)^{\varepsilon'}\omega^n\Big)\omega^n.
\end{aligned}$$

It then follows that

$$\begin{aligned}
\int_X \mathcal{G}(x,\cdot)(H_k)^{\varepsilon'}\omega^n &\leq u_k(x) + C\frac{1}{[\omega]^n}\int_X (H_k)^{\varepsilon'}\omega^n \\
&\leq C([\omega]^n)^{-\varepsilon'} + C\Big(\frac{1}{[\omega]^n}\int_X H_k\omega^n\Big)^{\varepsilon'} \\
&\leq 2C([\omega]^n)^{-\varepsilon'},
\end{aligned}$$

for some uniform constant $C > 0$. Letting $k \to \infty$ and applying the monotone convergence theorem, we can conclude that

$$\int_X \mathcal{G}(x,\cdot)^{1+\varepsilon'}\omega^n \leq C([\omega]^n)^{-\varepsilon'},$$

for some uniform constant $C > 0$. The proof of the lemma is now complete. □

We observe the following elementary estimate which follows easily from the Green's formula.

Lemma 5.6. *Under the same assumptions of Lemma 5.5, for any $\beta > 0$ we have*

$$\sup_{x\in X}\int_X \frac{|\nabla_y \mathcal{G}(x,y)|^2_{\omega(y)}}{\mathcal{G}(x,y)^{1+\beta}}\omega^n(y) \le \frac{([\omega]^n)^\beta}{\beta}. \tag{5.24}$$

Proof. Fix $\beta > 0$ and a point $x \in X$. The function $u(y) := \mathcal{G}(x,y)^{-\beta}$ is a continuous function with $u(x) = 0$ and $u \in C^\infty(X\backslash\{x\})$. By the definition of $\mathcal{G}$ in (5.17), for any $y \in X$, we have

$$0 \le u(y) \le ([\omega]^n)^\beta. \tag{5.25}$$

Applying the Green's formula, we have

$$\begin{aligned} 0 = u(x) &= \frac{1}{[\omega]^n}\int_X u\omega^n + \int_X \mathcal{G}(x,\cdot)(-\Delta_\omega u)\omega^n \\ &= \frac{1}{[\omega]^n}\int_X u\omega^n - \beta\int_X \frac{|\nabla \mathcal{G}(x,\cdot)|^2_\omega}{\mathcal{G}(x,\cdot)^{1+\beta}}\omega^n. \end{aligned}$$

In the last inequality, we apply the integration by parts using the asymptotic behavior of $\mathcal{G}(x,y)$ near $y = x$. The lemma then follows easily from 5.25. ☐

Finally we are ready to derive the uniform $L^1(X,\omega^n)$ bound for the gradient of $G(x,\cdot)$.

Lemma 5.7. *Under the same assumptions of Lemma 5.5, for any $s \in [1, \frac{2+2\varepsilon'}{2+\varepsilon'})$ there is a uniform constant $C = C(s) > 0$ such that for any $x \in X$, we have*

$$\int_X |\nabla G(x,\cdot)|^s_\omega \omega^n \le \frac{C}{([\omega]^n)^{s-1}}. \tag{5.26}$$

Proof. It suffices to prove the same estimate for $\mathcal{G}(x,\cdot)$. For fixed $x \in X$, we regard $\mathcal{G}(y) := \mathcal{G}(x,y)$ as a function of y. By fixing $s \in [1, \frac{2+2\varepsilon'}{2+\varepsilon'})$ and applying Hölder inequality, we have

$$\begin{aligned} \int_X |\nabla G(x,\cdot)|^s_\omega \omega^n &\le \Big(\int_X \frac{|\nabla \mathcal{G}|^2_\omega}{\mathcal{G}^{1+\beta}}\omega^n\Big)^{s/2}\Big(\int_X \mathcal{G}^{1+\varepsilon'}\omega^n\Big)^{(2-s)/2} \\ &\le C([\omega]^n)^{\beta s/2}([\omega]^n)^{-\varepsilon'(2-s)/2} \\ &= C([\omega]^n)^{1-s}, \end{aligned} \tag{5.27}$$

where $\beta > 0$ is chosen by $1+\beta = (1+\varepsilon')\frac{2-s}{s}$. We apply the estimates in Lemmas 5.5 and 5.6 for the second inequality in (5.27). ☐

Combining the estimates above, we have established the following main result of this section.

Proposition 5.1. *For any $A, K > 0$ and $p > n$, there exist $\varepsilon = \varepsilon(X, \omega_X, n, A, p, K) > 0$ and $\varepsilon' = \varepsilon'(n, p) > 0$ such that if*

(1) *$\gamma \geq 0$ is a continuous function on X such that*

$$|\{\gamma = 0\}|_{\omega_X} \leq \varepsilon, \ \{\gamma > 0\} \ \text{ is connected,}$$

(2) *$\omega \in \mathcal{W}(X, \omega_X, n, A, p, K; \gamma)$,*

(3) *$s \in [1, \frac{2+2\varepsilon'}{2+\varepsilon'})$,*

there exist $C_1 = C(X, \omega_X, n, A, p, K, \gamma, \varepsilon) > 0$, $C_2 = C_2(X, \omega_X, n, A, p, K, \gamma, \varepsilon, \varepsilon') > 0$ and $C_3 = C_3(X, \omega_X, n, A, p, K, \gamma, \varepsilon, \varepsilon', s) > 0$ such that

$$\inf_{y \in X} G(x, y) \geq -C_1 \left([\omega]^n\right)^{-1},$$
$$\int_X |G(x, \cdot)|^{1+\varepsilon'} \omega^n \leq C_2,$$
$$\int_X |\nabla G(x, \cdot)|^{1+s} \omega^n \leq C_3,$$

for any $x \in X$.

6. Diameter and volume estimates

In this section, we will establish the following diameter and volume estimates by applying Proposition 5.1.

Proposition 6.1. *For any $A, K > 0$ and $p > n$, there exist $\varepsilon = \varepsilon(X, \omega_X, n, A, p, K) > 0$ and $\varepsilon' = \varepsilon'(n, p) > 0$ such that if*

(1) *$\gamma \geq 0$ is a continuous function on X such that*

$$|\{\gamma = 0\}|_{\omega_X} \leq \varepsilon, \ \{\gamma > 0\} \ \text{ is connected,}$$

(2) *$\omega \in \mathcal{W}(X, \omega_X, n, A, p, K; \gamma)$,*

there exist $\alpha = \alpha(n, p)$, $C = C(X, \omega_X, n, A, p, K, \gamma, \varepsilon) > 0$ and $c = c(X, \omega_X, n, A, p, K, \gamma, \varepsilon, \alpha) > 0$ such that

$$\operatorname{diam}(X, \omega) \leq C, \tag{6.1}$$
$$\frac{\operatorname{Vol}_\omega(B_\omega(x, R))}{[\omega]^n} \geq cR^\alpha, \tag{6.2}$$

for any $x \in X$ and $R \in (0, 1]$.

Proof. We first prove the diameter bound. Since (X, ω) is compact and complete, there exist a pair of points $x_0, y_0 \in X$ such that $d_\omega(x_0, y_0) = \mathrm{diam}(X, \omega)$. We define the 1-Lipschitz function $d(\cdot)$ on X by

$$d(y) = d_\omega(x_0, y).$$

Apply the Green's formula to d at a point $x \in X$. We obtain

$$d(x) = \frac{1}{[\omega]^n}\int_X d(y)\omega^n(y) + \int_X \langle \nabla_y G(x,y), \nabla d(y)\rangle_{\omega(y)}\omega^n(y). \tag{6.3}$$

By letting $x = x_0$, we have $d(x_0) = 0$ and

$$\begin{aligned} \frac{1}{[\omega]^n}\int_X d(y)\omega^n(y) &= -\int_X \langle \nabla_y G(x_0,y), \nabla d(y)\rangle_{\omega(y)}\omega^n(y) \\ &\leq \int_X |\nabla_y G(x_0,y)|_{\omega(y)}\omega^n(y). \end{aligned}$$

Finally we establish the uniform diameter by applying 6.3 to $x = z_0$ with

$$\begin{aligned} \mathrm{diam}(X,\omega) &= d(y_0) \\ &= \frac{1}{[\omega]^n}\int_X d(y)\omega^n(y) + \int_X \langle \nabla_y G(y_0,y), \nabla d(y)\rangle_{\omega(y)}\omega^n(y) \\ &\leq \int_X |\nabla_y G(x_0,y)|_{\omega(y)}\omega^n(y) + \int_X |\nabla_y G(y_0,y)|_{\omega(y)}\omega^n(y) \\ &\leq C, \end{aligned}$$

for some uniform constant $C > 0$ by Proposition 5.1.

We now turn to the non-collapsing estimate for metric balls in (X, ω). Fix a point $x \in X$ and a number $R \in (0, 1]$. Let $B(x, R) \subset X$ be the geodesic ball in (X, ω) with center x and radius $R > 0$. We choose a smooth cut-off function η with support in $B(x, R)$ satisfying

$$\eta \equiv 1, \text{ on } B\Big(x, \frac{R}{2}\Big), \ \sup_X |\nabla\eta|_\omega \leq \frac{4}{R}.$$

We let $d(y) = d_\omega(x, y)$ be the geodesic distance from x to $y \in X$. Applying the Green's formula to the Lipschitz function $d \cdot \eta$, we have for any $z \in X$

$$d(z)\eta(z) = \frac{1}{[\omega]^n}\int_X d(y)\eta(y)\omega^n(y) + \int_X \langle \nabla_y G(z,y), \eta(y)\nabla d(y) + d(y)\nabla\eta(y)\rangle_{\omega(y)}\omega^n(y). \tag{6.4}$$

Take $s = \frac{2+1.5\varepsilon'}{2+\varepsilon'} > 1$ for ε' from the assumption in Proposition 5.1. We apply (6.4) to a point $\tilde{z} \in X \backslash \overline{B(x, R)}$. Then $d(\tilde{z})\eta(\tilde{z}) = 0$ and by Lemma 5.7, we have

$$\frac{1}{[\omega]^n}\int_X d(y)\eta(y)\omega^n(y)$$

$$\begin{aligned}
&\leq \int_X |\nabla_y G(\tilde z, y)|_{\omega(y)} \Big(\eta(y) + d(y)|\nabla\eta(y)|_{_{\omega(y)}}\Big)\, \omega^n(y) \\
&\leq 5\Big(\int_X |\nabla_y G(\tilde z, y)|^s_{\omega(y)}\omega^n(y)\Big)^{1/s} \cdot \Big(\mathrm{Vol}_\omega(B(x,R))\Big)^{1/s^*} \\
&\leq C([\omega]^n)^{-\frac{s-1}{s}}\Big(\mathrm{Vol}_\omega(B(x,R))\Big)^{1/s^*} \\
&= C\Big(\frac{\mathrm{Vol}_\omega(B(x,R))}{[\omega]^n}\Big)^{1/s^*},
\end{aligned}$$

where $s^* = \frac{s}{s-1}$ is the conjugate exponent of s. Next we apply (6.4) to a point $\hat z \in \partial B(x, R/2)$ where $d(\hat z)\eta(\hat z) = R/2$. Applying the above estimate along with the same argument, we have

$$\begin{aligned}
\frac{R}{2} &\leq \frac{1}{[\omega]^n}\int_X d(y)\eta(y)\omega^n(y) + \int_X |\nabla_y G(\hat z, y)|_{\omega(y)}\Big(\eta(y) + d(y)|\nabla\eta(y)|_{_{\omega(y)}}\Big)\,\omega^n(y) \\
&\leq C\left(\frac{\mathrm{Vol}_\omega(B(x,R))}{[\omega]^n}\right)^{1/s^*},
\end{aligned}$$

for some uniform constant $C > 0$. This immediately gives a lower bound of the volume of $B(x,R)$,

$$\frac{\mathrm{Vol}_\omega(B(x,R))}{[\omega]^n} \geq cR^\alpha,$$

for some uniform constants $\alpha = s^*(n,p) > 0$ and $c = c(A, p, K, \gamma, \varepsilon, \varepsilon', \alpha) > 0$. □

Remark. We briefly explain an application of the noncollapsing estimate (c) of Theorem 1.1 to the pre-compactness in Gromov-Hausdorff (GH) topology. Let (X, ω_j) be a sequence of Kähler metrics satisfying the assumptions in Theorem 1.1. By Gromov's precompactness theorem, it suffices to verify the following:

for any $\epsilon > 0$, there exists an $N(\epsilon) > 0$ which is independent of j such that there exists an ϵ-dense set $\{x_j^a\}_{a=1}^{M_j}$ in the metric space (X, ω_j) with $M_j \leq N(\epsilon)$.

In fact, suppose $\{x_j^a\}_{a=1}^{M_j}$ is an ϵ-dense set in the metric space (X, ω_j), by which we mean a maximal collection of points where any two of them have distance at least ϵ. By definition, the geodesic balls $\{B_j(x_j^a, \epsilon/2)\}_a$ are pairwise disjoint, hence by (c) of Theorem 1.1,

$$c(\epsilon/2)^\alpha([\omega]^n)M_j \leq \sum_{a=1}^{M_j}\mathrm{Vol}_{\omega_j}(B_{\omega_j}(x_j^a, \epsilon/2)) \leq ([\omega]^n) = \mathrm{Vol}(X, \omega_t^n),$$

so $M_j \leq c^{-1}(\epsilon/2)^{-\alpha} =: N(\epsilon)$.

This shows that up to a subsequence the metric spaces (X, ω_j) converge in GH topology to a *compact* metric space (Z, d_Z).

Now we can complete the proof of Theorem 1.1.

Proof of Theorem 1.1. It suffices to show that if S is a closed subset of X with $\dim_{\mathcal{M}} S < 2n-1$, then $X \setminus S$ is connected. Since the Cech cohomological dimension is always no greater than the topological dimension, which is no greater than Minkowski dimension, we have

$$\check{H}^{2n-1}(S) = \check{H}^{2n}(S) = 0$$

and so by Poincare-Alexander-Lefschetz duality $H_1(X, X \setminus S) = \check{H}^{2n-1}(S) = 0$ and $H_0(X, X\setminus S) = \check{H}^{2n}(S) = 0$ (c.f. Theorem 8.3, Chapter VI, in [3]). The exact sequence for reduced coholomology gives

$$0 = H_1(X, X \setminus S) \to \tilde{H}_0(X \setminus S) \to \tilde{H}_0(X) \to H_0(X, X\setminus S) = 0.$$

Therefore $\tilde{H}_0(X \setminus S) = \tilde{H}_0(X) = \mathbb{Z}$ and so $X \setminus S$ is connected. Then Theorem 1.1 is a direct consequence of Proposition 5.1 and Proposition 6.1. □

7. A uniform Sobolev inequality

In this section, we will prove a special Sobolev-type inequality for Kähler metrics satisfying the assumption in Proposition 5.1. The main feature of this inequality is the uniformity of the constants.

We first improve Lemma 5.3 in the following lemma.

Lemma 7.1. *For any $A, K > 0$ and $p > n$, there exist $\varepsilon = \varepsilon(X, \omega_X, n, A, p, K) > 0$ and $\varepsilon' = \varepsilon'(n, p) > 0$ such that if*

(1) *$\gamma \geq 0$ is a continuous function on X such that*

$$|\{\gamma = 0\}|_{\omega_X} \leq \varepsilon, \ \{\gamma > 0\} \text{ is connected},$$

(2) *$\omega \in \mathcal{W}(X, \omega_X, n, A, p, K; \gamma)$,*

then there exists $C = C(A, p, K, \varepsilon, \gamma, \varepsilon') > 0$ such that for any $v \in C^\infty(X)$ satisfying

$$\frac{1}{[\omega]^n} \int_X |\Delta v|^{(1+\varepsilon')^*} \omega^n \leq 1, \ \int_X v\omega^n = 0,$$

we have

$$\sup_X |v| \leq C.$$

Proof. Let $p = 1 + \varepsilon'$ and $q = p^*$. Then applying Proposition 5.1, there exists $C = C(A, p, K, \varepsilon, \gamma, \varepsilon') > 0$ such that

$$v(x) \;\; = \;\; -\int_X G(x, y)\Delta v(y)\omega^n(y)$$

$$\begin{aligned}
&\leq \left(\int_X |G(x,\cdot)|^p \omega^n\right)^{1/p} \left(\int_X |\Delta v|^q \omega^n\right)^{1/q} \\
&\leq C\left([\omega]^n\right)^{-\varepsilon'/p} \left(\int_X |\Delta v|^q \omega^n\right)^{1/q} \\
&\leq C\left([\omega]^n\right)^{-\varepsilon'/p+1/q} \\
&\leq C,
\end{aligned}$$

since $-\frac{\varepsilon'}{p} + \frac{1}{q} = 1 - \frac{\varepsilon'+1}{p} = 0$. □

We can now apply Lemma 7.1 to derive a Sobolev-type inequality with large exponents.

Lemma 7.2. *For any $A, K > 0$ and $p > n$, there exist $\varepsilon = \varepsilon(X, \omega_X, n, A, p, K) > 0$ and $\varepsilon' = \varepsilon'(n,p) > 0$ such that if*

(1) *$\gamma \geq 0$ is a continuous function on X such that*

$$|\{\gamma = 0\}|_{\omega_X} \leq \varepsilon, \ \{\gamma > 0\} \text{ is connected,}$$

(2) *$\omega \in \mathcal{W}(X, \omega_X, n, A, p, K; \gamma)$,*

(3) *$s \in (1, \frac{2+2\varepsilon'}{2+\varepsilon'})$,*

then there exists $C = C(A, p, K, \varepsilon, \gamma, s) > 0$ such that for any $u \in C^\infty(X)$ satisfying $\int_X u\omega^n = 0$,

$$\|u\|_{L^\infty(X)} \leq C\left(\frac{1}{[\omega]^n}\int_X |\nabla u|^{\frac{s}{s-1}}\omega^n\right)^{\frac{s-1}{s}}.$$

Proof. By the Green's formula and integration by parts, we have

$$\begin{aligned}
|u(x)| &= \left|\int_{X\setminus\{x\}} \langle \nabla G(x,\cdot), \nabla u(\cdot)\rangle \omega^n\right| \\
&\leq \left(\int_X |\nabla G(x,\cdot)|^s \omega^n\right)^{\frac{1}{s}} \left(\int_X |\nabla u|^{\frac{s}{s-1}}\omega^n\right)^{\frac{s-1}{s}} \\
&\leq C([\omega]^n)^{-\frac{s-1}{s}} \left(\int_X |\nabla u|^{s/(s-1)}\omega^n\right)^{\frac{s-1}{s}} \\
&= C\left(\frac{1}{[\omega]^n}\int_X |\nabla u|^{s/(s-1)}\omega^n\right)^{\frac{s-1}{s}},
\end{aligned}$$

for some uniform constant $C = C(A, p, K, \varepsilon, \gamma, s) > 0$, after applying Proposition 5.1. □

We remark that Proposition 6.1 can also be proved by directly applying Lemma 7.2.

8. Finite time solutions of the Kähler-Ricci flow

We will prove Theorem 2.1 in this section by applying Theorem 1.1. The key is to bound the p-Nash entropy from above and the volume form from below along the Kähler-Ricci flow.

We consider the unnormalized Kähler-Ricci flow (2.1) on a Kähler manifold X with an initial Kähler metric g_0. Suppose the flow develops finite time singularity. Without loss of generality by rescaling, we can assume the singular time is given by

$$T = \sup\{t > 0 \mid [\omega_0] + t[K_X] > 0\} = 1.$$

By choosing a smooth closed $(1,1)$-form $\chi \in K_X$, the Kähler-Ricci flow (2.1) is equivalent to the following parabolic complex Monge-Ampère flow.

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} = \log \dfrac{\left(\omega_0 + t\chi + \sqrt{-1}\partial\overline{\partial}\varphi\right)^n}{\Omega}, \\ \\ \varphi|_{t=0} = 0, \end{cases} \tag{8.1}$$

where Ω is a smooth volume form on X satisfying

$$\sqrt{-1}\partial\overline{\partial}\log\Omega = \chi \in [K_X].$$

We let $\omega_t = \omega_0 + t\chi$ and $\omega = \omega(t) = \omega_t + \sqrt{-1}\partial\overline{\partial}\varphi$.

Lemma 8.1. *There exists $C > 0$ such that*

$$\varphi \le C,\ \frac{\partial \varphi}{\partial t} \le C$$

on $X \times [0,1)$.

Proof. The upper bound for φ follows directly from the maximum principle. Let

$$u = t\frac{\partial\varphi}{\partial t} - \varphi - nt.$$

Then u satisfies

$$\left(\frac{\partial}{\partial t} - \Delta\right) u = -tr_\omega(\omega_0) \le 0.$$

By the maximum principle,

$$\sup_{X\times[0,1)} u \le \sup_X u(\cdot, 0) = 0$$

and so $t\frac{\partial\varphi}{\partial t}$ is also uniformly bounded from above. The lemma immediately follows by considering $t \in [1/2, 1)$ since $\frac{\partial\varphi}{\partial t}$ is uniformly bounded for $t \in [0, 1/2]$. $\square$

We can now view the Monge-Ampère flow as a family of complex Monge-Ampère equations

$$(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{\frac{\partial\varphi}{\partial t}}\Omega$$

for $t \in [0,1)$. If $[\omega_0] + [K_X]$ is big,

$$\lim_{t\to 1}[\omega_t]^n = ([\omega_0] + [K_X])^n > 0.$$

Lemma 8.2. *There exists $\psi \in \mathrm{PSH}(X,\chi)$ such that $\omega_0 + \chi + \sqrt{-1}\partial\overline{\partial}\psi$ is a Kähler current on X. Furthermore, ψ has analytic singularities and is smooth outside the locus of its singularities.*

Proof. The lemma is a consequence of [9] (The regularization theorem 3.2). □

We can assume that $\omega_0 + \chi + \sqrt{-1}\partial\overline{\partial}\psi > \epsilon\omega_0$ for some $\epsilon > 0$.

Lemma 8.3. *There exists $C > 0$ such that on $X \times [0,1)$, we have*

$$\varphi \geq \psi - C.$$

Proof. Let $u = \varphi - \psi$. Then u is bounded below and tends to ∞ near the singular locus of ψ for each $t \in [0,1)$. u satisfies the evolution equation

$$\frac{\partial u}{\partial t} = \log \frac{\left(\omega_0 + \chi + \sqrt{-1}\partial\overline{\partial}\psi - (1-t)\chi + \sqrt{-1}\partial\overline{\partial}u\right)^n}{\Omega}.$$

Let

$$t' = \inf\{0 < t < 1 \mid \epsilon\omega_0 > 2(1-s)\chi, \text{ for all s} \in \text{(t, 1)}\}.$$

Obviously, $t' < 1$. Suppose $\inf_{X\times[\max(1/2,t'),t_0)} u = u(z_0,t_0)$. Then ψ is smooth at z_0. By applying the maximum principle, we have at (z_0,t_0)

$$\begin{aligned}
\frac{\partial u}{\partial t} &\geq \log \frac{\left(\omega_0 + \chi + \sqrt{-1}\partial\overline{\partial}\psi - (1-t_0)\chi\right)^n}{\Omega} \\
&\geq \log \frac{(\epsilon\omega_0 - (1-t_0)\chi)^n}{\Omega} \\
&\geq \log \frac{\omega_0^n}{\Omega} - C,
\end{aligned}$$

for some uniform $C > 0$. Therefore u is uniformly bounded below for $t \in [0,1)$. principle. The lemma then immediately follows. □

Lemma 8.4. *There exist $A, C > 0$ such that*

$$\frac{\partial\varphi}{\partial t} \geq A\psi - C.$$

Proof. Let $u = \frac{\partial\varphi}{\partial t} + 2A(\varphi - \psi)$ for some fixed $A > 2\epsilon^{-1} > 0$. Then the evolution for u is given by

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) u &= 2A\frac{\partial\varphi}{\partial t} + tr_\omega(2A\omega_t + 2A\sqrt{-1}\partial\overline{\partial}\psi + \chi) - 2nA \\ &= 2A\frac{\partial\varphi}{\partial t} + tr_\omega(2A(\omega_1 + \sqrt{-1}\partial\overline{\partial}\psi) + (1 - 2A(1-t))\chi) - 2nA \\ &\geq 2A\frac{\partial\varphi}{\partial t} + tr_\omega\left(2A\epsilon\omega_0 + (1 - 2A(1-t))\chi\right) - 2nA \\ &\geq 2A\frac{\partial\varphi}{\partial t} + 2A\epsilon tr_\omega(\omega_0) - 2nA \\ &\geq 2A\frac{\partial\varphi}{\partial t} + A\epsilon\left(\frac{\omega_0^n}{\omega^n}\right)^{1/n} - 2nA \\ &\geq 2A\frac{\partial\varphi}{\partial t} + e^{-n^{-1}\frac{\partial\varphi}{\partial t}} - 2nA.\end{aligned}$$

by choosing $A >> \epsilon^{-1}$ for $t > 1 - A^{-1}$. Let p be the minimum point of u at $t > 1 - A^{-1}$. Then p does not lie in the singular locus of ψ and by applying the maximum principle,we have

$$\frac{\partial\varphi}{\partial t}(p) \geq -C$$

for some uniform constant $C > 0$. Hence $u(p)$ is uniformly bounded below by applying Lemma 8.3. The lemma then immediately follows. □

Corollary 8.1. *For any $p > n$, there exists $C > 0$ such that for all $t \in [0, 1)$*

$$\mathcal{N}_{X,\omega_0,p}(\omega(t)) \leq C.$$

Furthermore, for any $p > n$, there exist $A, B, K > 0$ such that for all $t \geq 0$,

$$\omega(t) \in \mathcal{W}\left(X, \omega_0, n, A, p, K; e^{B\psi - B}\right),$$

where ψ is defined in Lemma 8.2.

Proof. By combining the previous lemmas, there exist $C_1, C_2 > 0$ such that on $X \times [0, 1)$, we have

$$C_2^{-1} e^{C_1\psi}(\omega_0)^n \leq \omega^n \leq C_2(\omega_0)^n.$$

The corollary immediately follows.

□

Proof of Theorem 2.1. The assumption of Theorem 1.1 is satisfied due to Corollary 8.1. Theorem 2.1 immediately follows. □

9. Long time solutions of the Kähler-Ricci flow

We will prove Theorem 2.2 in this section as an application of Theorem 1.1. As in the previous section, we will bound the p-Nash entropy from above and the volume form from below along the Kähler-Ricci flow.

Let X be a Kähler manifold with nef K_X and nonnegative Kodaira dimension. For any smooth closed $(1,1)$-form $\chi \in K_X$, we can find a smooth volume form Ω such that

$$\chi = \sqrt{-1}\partial\overline{\partial}\log\Omega, \ \int_X \Omega = 1.$$

We let $\omega_t = (1-e^{-t})\chi + e^{-t}\omega_0$. The numerical dimension of K_X is defined by

$$\kappa = \kappa(X) = \max\{k \geq 0 : [K_X]^k \neq 0 \text{ in } H^{k,k}(X,\mathbb{R})\}.$$

We will assume $\mathrm{Kod}(X) \geq 0$, the numerical dimension $\kappa(X) \geq \mathrm{Kod}(X) \geq 0$. The normalized Kähler-Ricci flow (2.4) is equivalent to the following parabolic complex Monge-Ampère equation

$$\begin{cases} \dfrac{\partial\varphi}{\partial t} = \log \dfrac{e^{(n-\kappa)t}\left(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi\right)^n}{\Omega} - \varphi, \\ \\ \varphi|_{t=0} = 0. \end{cases} \tag{9.1}$$

We let $\omega = \omega(\cdot,t) = \omega_t + \sqrt{-1}\partial\overline{\partial}\varphi$ solving the Monge-Ampère flow (9.1).

We first derive the volume growth for $(X, g(t))$.

Lemma 9.1. *There exists $C > 0$ such that for all $t \geq 0$, we have*

$$C^{-1}e^{-(n-\kappa)t} \leq [\omega_t]^n \leq Ce^{-(n-\kappa)t}.$$

Proof. Let $\alpha = [\omega_0] - K_X$. Then by the definition of κ, we have

$$(K_X)^\kappa \cdot \alpha^{n-\kappa} > 0$$

and

$$\begin{aligned} e^{(n-\kappa)t}[\omega_t]^n &= \sum_{l=0}^{n} C_n^l e^{(l-\kappa)t}(K_X)^l \cdot \alpha^{n-l} \\ &= \sum_{l=0}^{\kappa} C_n^l e^{(l-\kappa)t}(K_X)^l \cdot \alpha^{n-l} \\ &= C_n^\kappa (K_X)^\kappa \cdot \alpha^{n-\kappa} + O(e^{-t}). \end{aligned}$$

This proves the lemma. ∎

Lemma 9.2. *There exists $C > 0$ such that for all $t \in [0, \infty)$,*

$$-C \leq \sup_X \varphi(\cdot, t) \leq C, \ \sup_X \left(\frac{\partial \varphi}{\partial t} + \varphi \right)(\cdot, t) \leq C.$$

Proof. By Jensen inequality and Lemma 9.1,

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\int_X \varphi(\cdot, t)\Omega\right) &= \int_X \left(\log \frac{e^{(n-\kappa)t}\omega^n}{\Omega}\right)\Omega - \int_X \varphi(\cdot, t)\Omega \\
&\leq \log\left(\int_X e^{(n-\kappa)t}\omega^n\right) - \int_X \varphi(\cdot, t)\Omega \\
&\leq \log\left(e^{(n-\kappa)t}[\omega_t]^n\right) - \int_X \varphi(\cdot, t)\Omega \\
&\leq C - \int_X \varphi(\cdot, t)\Omega.
\end{aligned}$$

Hence $\int_X \varphi(\cdot, t)\Omega$ is uniformly bounded above. Since $\varphi \in \mathrm{PSH}(X, \omega_t) \subset \mathrm{PSH}(X, A\omega_0)$ for some fixed sufficiently large $A > 0$, by the mean value theorem for plurisubharmonic functions, there exists $C > 0$ such that for any $t \in [0, \infty)$ and $x \in X$,

$$\varphi(x, t) \leq \int_X \varphi(\cdot, t)\Omega + C.$$

This proves the uniform upper bound for φ.

Now we let $u = \frac{\partial \varphi}{\partial t} - e^{-t}\varphi$. Then the evolution for u is given by

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u - u - e^{-t}\frac{\partial \varphi}{\partial t} - e^{-t} tr_\omega(\omega_t + \omega_0 - \chi) + n - \kappa + ne^{-t} \\
&= \Delta u - u + e^{-t}\log\frac{\Omega}{\omega^n} - e^{-t} tr_\omega(\omega_0 + e^{-t}(\omega_0 - \chi)) \\
&\quad + e^{-t}\frac{\partial \varphi}{\partial t} + n - \kappa + e^{-t}(n - (n-\kappa)t).
\end{aligned}$$

Then there exist $C_1, C_2 > 0$ such that for all $t \geq 0$,

$$\begin{aligned}
\frac{\partial u}{\partial t} &\leq \Delta u - u - \frac{1}{2}e^{-t}\left(tr_\omega(\omega_0) - \log\frac{\omega_0^n}{\omega^n}\right) + C_1 \\
&\leq \Delta u - u - C_2.
\end{aligned}$$

The uniform upper bound for u immediately follows from the maximum principle. Therefore.

$$\frac{\partial \varphi}{\partial t} + \varphi = u + (1 + e^{-t})\varphi$$

is uniformly bounded above.

We now prove the lower bound for $\sup_X \varphi(\cdot,t)$. By the upper bound of $\frac{\partial\varphi}{\partial t} - e^{-t}\varphi$ and by Lemma 9.1, there exist $c_1, c_2 > 0$ such that for all $t \geq 0$, we have

$$\begin{aligned} e^{(1+e^{-t})\sup_X \varphi(\cdot,t)} &\geq c_1 e^{(1+e^{-t})\sup_X \varphi + \sup_X \left(\frac{\partial\varphi}{\partial t} - e^{-t}\varphi\right)} \\ &\geq \int_X e^{\frac{\partial\varphi}{\partial t}+\varphi}\Omega \\ &= \int_X e^{(n-\kappa)t}\omega^n \\ &\geq c_2. \end{aligned}$$

This completes the proof of the theorem. □

Let

$$\mathcal{V}_t(x) = \sup\{\phi(x) \mid \omega_t + \sqrt{-1}\partial\overline{\partial}\phi \geq 0,\ \phi \leq 0\} \tag{9.2}$$

be the extremal function associated to ω_t for any $t \in [0,\infty)$. We let V_∞ be the extremal function associated to χ. Since we assume $\kappa > 0$, there exists a holomorphic section $\sigma \in |mK_X|$ for some sufficiently large m. Let h be the hermitian metric on mK_X with $\mathrm{Ric}(h) = m\chi$ and $\sup_X |\sigma|_h^2 = 1$. Then

$$\mathcal{V}_\infty \geq \frac{1}{m}\log|\sigma|_h^2 \tag{9.3}$$

because

$$\chi + \frac{1}{m}\sqrt{-1}\partial\overline{\partial}\log|\sigma|_h^2 = [\sigma] \geq 0.$$

The following lemma is obvious by definition.

Lemma 9.3. *For any $t_1 \leq t_2$,*

$$\mathcal{V}_\infty \leq \mathcal{V}_{t_2} \leq \mathcal{V}_{t_1}.$$

Lemma 9.4. *There exists $C > 0$ such that on $X \times [0,\infty)$, we have*

$$\varphi(\cdot,t) \geq \mathcal{V}_t - C \geq \mathcal{V}_\infty - C. \tag{9.4}$$

Proof. By Lemma 9.1 and Lemma 9.2, There exists $C > 0$ such that on $X \times [0,\infty)$, the normalized volume measure is uniformly bounded above by a smooth volume form

$$\frac{\omega^n}{[\omega]^n} \leq C\Omega.$$

The lemma then follows from Proposition 4.1 and the uniform lower bound of $\sup_X \varphi(\cdot,t)$. □

Lemma 9.5. *There exists $C > 0$ such that on $X \times [0,\infty)$, we have*

$$\frac{\partial^2\varphi}{\partial t^2} + \frac{\partial\varphi}{\partial t} \leq C.$$

Proof. Let $u = \frac{\partial \varphi}{\partial t} + \varphi$. Let R be the scalar curvature of ω. Since the scalar curvature of the Kähler-Ricci flow is uniformly bounded below, there exists $C > 0$ such that on $X \times [0, \infty)$, we have

$$\Delta u = \Delta \left(\log \frac{\omega^n}{\Omega} \right) = -\mathrm{R} - tr_\omega(\chi) \leq C - tr_\omega(\chi).$$

On the other hand,

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 \varphi}{\partial t^2} + \frac{\partial \varphi}{\partial t} \\ &\leq \Delta u + tr_\omega(\chi) - \kappa \\ &\leq C - \kappa. \end{aligned}$$

This proves the lemma. □

Lemma 9.6. *Suppose $f = f(x)$ is a smooth function for $x \geq 0$ and it satisfies the differential inequality*

$$f'' + f' \leq 1, \ |f| \leq A$$

for all $x \geq 0$ and for some fixed $A > 1$. Then for any $x \geq 0$, we have,

$$f' \geq -5A.$$

Proof. Suppose the lemma fails. Then there exists $x_0 \geq 0$, such that

$$f'(x_0) < -5A.$$

Since $|f| \leq A$, there must exist an $a \in (0, 1)$ such that

$$f'(x_0 + a) = -2A.$$

Otherwise, $f(x) < f(x_0) - 2A \leq -A$ for $x \in (x_0, x_0 + 1)$, contradicting the assumption.

By our assumption, $(f' + f - x)' \leq 0$, and so

$$f'(x_0 + a) + f(x_0 + a) - (x_0 + a) \leq f'(x_0) + f(x_0) - x_0.$$

This implies that

$$-2A = f'(x_0 + a) \leq f'(x_0) + f(x_0) - f(x_0 + a) + a \leq -5A + 2A + 1 = -3A + 1.$$

This leads to contradiction as $A > 1$. □

Lemma 9.7. *There exists $C > 0$ such that on $X \times [0, \infty)$, we have*

$$\frac{\partial \varphi}{\partial t} \geq 5\mathcal{V}_\infty - C.$$

Proof. By Lemma 9.6 and Lemma 9.4, there exists $C > 0$ such that for any $x \in X$ and $t \geq 0$,we have

$$\frac{\partial^2 \varphi}{\partial t^2} + \frac{\partial \varphi}{\partial t} \leq C, \ -\mathcal{V}_\infty - C \leq \varphi \leq C$$

The lemma is then proved by directly applying Lemma 9.6. □

We then immediately have the following corollary by combining Lemma 9.2, Lemma 9.7 and (9.3).

Corollary 9.1. *There exists $C = C(X, g_0) > 0$ such that on $X \times [0, \infty)$, we have*

$$C^{-1} \exp\left(\frac{6}{m} \log |\sigma|_h^2\right) \leq \frac{1}{[\omega]^n} \frac{\omega^n}{\Omega} \leq C.$$

Proof. By Lemma 9.1, there exists $C > 0$ such that

$$C^{-1} \exp\left(\frac{\partial \varphi}{\partial t} + \varphi\right) \leq \frac{1}{[\omega]^n} \frac{\omega^n}{\Omega} \leq C \exp\left(\frac{\partial \varphi}{\partial t} + \varphi\right).$$

The corollary is then a direct consequence of Lemma 9.7, Lemma 9.4 and (9.3). □

Corollary 9.1 implies the bound for p-Nash entropy and a pointwise lower bound for $\omega(t)$.

Corollary 9.2. *For any $p > n$, there exists $C > 0$ such that for all $t \geq 0$*

$$\mathcal{N}_{X, \omega_0, p}(\omega(t)) \leq C.$$

Furthermore, for any $p > n$, there exist $A, B, K > 0$ such that for all $t \geq 0$,

$$\omega(t) \in \mathcal{W}(X, \omega_0, n, A, p, K; B^{-1} |\sigma|_h^{2B}),$$

where ψ is defined in Lemma 8.2.

Proof of Theorem 2.2. The assumption of Theorem 1.1 is satisfied due to Corollary 9.2. Theorem 2.2 immediately follows. □

10. Family of projective manifolds

In this section, we will extend Theorem 1.1 to a projective family with not only varying Kähler classes but also complex structures. Such extensions will allow us to obtain fibre diameter estimates for degeneration of canonical Kähler metrics on special fibrations.

Let

$$\pi : \mathcal{X} \subset \mathbb{CP}^N \times \mathbb{D} \to \mathbb{D} \tag{10.1}$$

be a projective family over a unit disk $\mathbb{D}$ with $\mathcal{X}_t = \pi^{-1}(t)$ being a smooth n-dimensional projective manifold for each $t \in \mathbb{D}^*$. We further assume that π is proper and flat, and the central fibre $\mathcal{X}_0 = \pi^{-1}(0)$ is reduced and irreducible. Let θ_t be the restriction of the Fubini-Study metric θ of $\mathbb{CP}^N$ to $\mathcal{X}_t$.

Theorem 10.1. *Let $\pi : \mathcal{X} \subset \mathbb{CP}^N \times \mathbb{D} \to \mathbb{D}$ be a projective family defined as above in (10.1). Let γ be a nonnegative continuous function on $\mathbb{CP}^N \times \mathbb{D}$ such that $\{\gamma = 0\}$ is a proper subvariety of $\mathbb{CP}^N \times \mathbb{D}$ and $\{\gamma = 0\}$ does not contain $\mathcal{X}_t$ for any $t \in \mathbb{D}$. Then for any $A > 0$, $p > n$ and $K > 0$, there exists $C = C(A, p, K, \gamma) > 0$ such that for any $t \in \frac{1}{2}\mathbb{D}^*$ and any Kähler metric ω on $\mathcal{X}_t$, if*

$$\mathcal{N}_{\mathcal{X}_t,\theta_t,p}(\omega) \leq K, \ [\omega] \leq A[\theta_t], \ \frac{1}{\mathrm{Vol}_\omega(\mathcal{X}_t)} \frac{\omega^n}{\theta_t^n} \geq \gamma, \tag{10.2}$$

then

$$\mathrm{diam}(\mathcal{X}_t, \omega) \leq C,$$

and

$$\int_{\mathcal{X}_t} \big(|G(x, \cdot)| + |\nabla G(x, \cdot)|\big)\omega^n + \left(-\inf_{y \in \mathcal{X}_t} G(x, y)\right) \mathrm{Vol}_\omega(\mathcal{X}_t) \leq C,$$

for any $x \in \mathcal{X}_t$, where G is the Green's function of $(\mathcal{X}_t, \omega)$. Moreover, there exist a uniform constant $c = c(A, p, K, \gamma) > 0$ and $\alpha = \alpha(n, p) > 0$ such that for any $x \in \mathcal{X}_t$ and $R \in (0, 1]$,

$$\frac{\mathrm{Vol}_\omega(B_\omega(x, R))}{\mathrm{Vol}_\omega(\mathcal{X}_t)} \geq cR^\alpha.$$

The proof of Theorem 10.1 is almost identical to the proof of Theorem 1.1 except for a few technical differences as presented below.

Lemma 10.1. *Let $\pi : \mathcal{X} \subset \mathbb{CP}^N \times \mathbb{D} \to \mathbb{D}$ be a projective family defined as above in (10.1) and let S be a subvariety of $\mathbb{CP}^N \times \mathbb{D}$. If S does not contain $\mathcal{X}_t$ for any $t \in \mathbb{D}$ and $\mathrm{Sing}(\mathcal{X}_0) \times \{0\} \subset S$, then for any $\epsilon > 0$ and $\mathcal{K} \subset\subset \mathcal{X} \setminus S$, there exists $\rho_\epsilon \in C^\infty\big(\big(\mathbb{CP}^N \times \mathbb{D}\big) \setminus S\big)$ such that*

1. *$0 \leq \rho_\epsilon \leq 1$,*
2. *$\mathrm{Supp}\rho_\epsilon \subset\subset \big(\mathbb{CP}^N \times \mathbb{D}\big) \setminus S$,*
3. *$\rho_\epsilon = 1$ on $\mathcal{K}$,*
4. *$\int_{\mathcal{X}_t} |\nabla\rho_\epsilon|^2\theta_t^n < \epsilon$, for any $t \in \frac{1}{2}\mathbb{D}$.*

Proof. We can assume S is a union of smooth divisors after replacing $\mathbb{CP}^N \times \mathbb{D}$ by its blow-ups $\mathcal{Y}$. Let σ be the defining section for S and h be a smooth hermitian metric on the line bundle associated to S. Without loss of generality, we can assume that $|\sigma|_h^2 \leq 1$. We can always pick a Kähler metric τ on $\mathcal{Y}$ such that $\tau > \mathrm{Ric}(h) = -i\partial\bar\partial \log h$ and $\tau_{\mathbb{CP}^N} \geq \theta$ after slightly shrinking $\mathbb{D}$. For simplicity, we identify θ with its pullback from $\mathbb{CP}^N \times \mathbb{D}$.

Let F be the standard smooth cut-off function on $[0, \infty)$ with $F = 1$ on $[0, 1/2]$ and $F = 0$ on $[1, \infty)$. We then let

$$\eta_\epsilon = \max(\log |\sigma|_h^2, \log \epsilon).$$

For $\epsilon < 1$, we have $\log \epsilon \leq \eta_\epsilon \leq 0$. Then obviously, $\eta_\epsilon \in \mathrm{PSH}(\mathcal{Y}, \tau) \cap C^0(\mathcal{Y})$. Now we let

$$\rho_\epsilon = F\left(\frac{\eta_\epsilon}{\log \epsilon}\right).$$

Then $\rho_\epsilon = 1$ on $\mathcal{K}$ if ϵ is sufficiently small. Let $\mathcal{X}_t'$ be the proper transform of $\mathcal{X}_t$ by the blow-up and $\rho_{\epsilon,t} = \rho_\epsilon|_{\mathcal{X}_t}$, $\eta_{\epsilon,t} = \eta_\epsilon|_{\mathcal{X}_t}$, $\tau_t = \tau|_{\mathcal{X}_t}$. For simplicity, we identify ρ_ϵ and θ_t with their pullbacks from $\mathcal{X}_t$ to $\mathcal{X}_t'$. Straightforward calculations give

$$\begin{aligned}
& \int_{\mathcal{X}_t'} \sqrt{-1}\partial\rho_{\epsilon,t} \wedge \overline{\partial}\rho_{\epsilon,t} \wedge \theta_t^{n-1} \\
= \;& (\log \epsilon)^{-2} \int_{\mathcal{X}_t'} (F')^2 \sqrt{-1}\partial\eta_{\epsilon,t} \wedge \overline{\partial}\eta_{\epsilon,t} \wedge \theta_t^{n-1} \\
\leq \;& C(\log \epsilon)^{-2} \int_{\mathcal{X}_t'} (-\eta_{\epsilon,t}) \sqrt{-1}\partial\overline{\partial}\eta_{\epsilon,t} \wedge \theta_t^{n-1} \\
= \;& C(\log \epsilon)^{-2} \int_{\mathcal{X}_t'} (-\eta_{\epsilon,t})(\tau_t + \sqrt{-1}\partial\overline{\partial}\eta_{\epsilon,t}) \wedge \theta_t^{n-1} + C(\log \epsilon)^{-2} \int_{\mathcal{X}_t'} \eta_{\epsilon,t}\; \tau_t \wedge \theta_t^{n-1} \\
\leq \;& C(-\log \epsilon)^{-1} \int_{\mathcal{X}_t'} (\tau_t + \sqrt{-1}\partial\overline{\partial}\eta_{\epsilon,t}) \wedge \theta_t^{n-1} \\
\leq \;& C(-\log \epsilon)^{-1} [\tau]^n \cdot \mathcal{X}_t' \\
\leq \;& C(-\log \epsilon)^{-1} \to 0
\end{aligned}$$

as $\epsilon \to 0$, where the constant C is independent of $t \in \frac{1}{2}\mathbb{D}$. Therefore we obtain $\rho_\epsilon \in C^0(\mathcal{X}_t')$ satisfying the conditions in the lemma. The lemma is then proved by smoothing ρ_ϵ on $(\mathrm{Supp}\ \rho_\epsilon) \setminus \mathcal{K}$. $\square$

Lemma 10.2. *Let $\pi : \mathcal{X} \subset \mathbb{CP}^N \times \mathbb{D} \to \mathbb{D}$ be a projective family defined as above. Then there exist $\alpha > 0$ and $C > 0$ such that for any $t \in \frac{1}{2}\mathbb{D}^*$ and $\varphi \in \mathrm{PSH}(\mathcal{X}_t, \theta_t)$, we have*

$$\int_{\mathcal{X}_t} e^{-\alpha\left(\varphi - \sup_{\mathcal{X}_t} \varphi\right)} \theta_t^n \leq C. \tag{10.3}$$

Proof. The lemma is an immediate consequence of the results in [10] (c.f. Theorem 3.4 in [10]). ☐

Lemma 10.3. *Let $\pi : \mathcal{X} \subset \mathbb{CP}^N \times \mathbb{D} \to \mathbb{D}$ be a projective family defined as above. For any $A > 0$, $p > n$, $K > 0$, there exists $C = C(A, p, K) > 0$ such that for any $t \in \frac{1}{2}\mathbb{D}^*$, if η is a smooth closed $(1,1)$-form on $\mathcal{X}_t$ with $\eta \leq A\theta_t$ and if $\omega = \eta + \sqrt{-1}\partial\overline{\partial}\varphi$ is a Kähler form on $\mathcal{X}_t$ satisfying*

$$\mathcal{N}_{\mathcal{X}_t,\theta_t,p}(\omega) \leq K, \tag{10.4}$$

then

$$\|\varphi - \sup_X \varphi - \mathcal{V}_\eta\|_{L^\infty(\mathcal{X}_t)} \leq C,$$

where $\mathcal{V}_\eta = \sup\{u \in \mathrm{PSH}(\mathcal{X}_t, \eta) : \ u \leq 0\}$ is the envelope of non-positive η-PSH functions.

Proof. By Lemma 10.2, the α-invariant for $\mathrm{PSH}(\mathcal{X}_t, A\theta_t)$ is uniformly bounded for all $t \in \frac{1}{2}\mathbb{D}^*$. Since $[\omega] \leq A[\theta_t]$, we can choose a smooth closed $(1,1)$-form $\eta \in [\omega]$ with $\eta \leq A\theta_t$ and then $\mathrm{PSH}(\mathcal{X}_t, \eta) \subset \mathrm{PSH}(\mathcal{X}_t, A\theta_t)$. We can consider the following Monge-Ampère equation

$$(\eta + \sqrt{-1}\partial\overline{\partial}\varphi)^n = \omega^n, \ \sup_X \varphi = 0.$$

The right hand side has uniformly bounded p-Nash entropy with respect to θ_t for some $p > n$. The lemma follows the work of [8, 7, 2, 11, 16] as generalizations of Kolodziej's work [22] (c.f. Proposition 4.1). ☐

Lemma 10.4. *Let $\pi : \mathcal{X} \subset \mathbb{CP}^N \times \mathbb{D} \to \mathbb{D}$ be a projective family defined as above. Then for any $A > 0$, $p > n$, $K > 0$, there is a uniform constant $C = C(A, p, K) > 0$ such that for any $t \in \frac{1}{2}\mathbb{D}^*$, any Kähler metric ω on $\mathcal{X}_t$ satisfying*

$$\mathcal{N}_{\mathcal{X}_t,\theta_t,p}(\omega) \leq K, \ [\omega] \leq A[\theta_t], \tag{10.5}$$

and $v \in C^2(\mathcal{X}_t)$ satisfying

$$|\Delta_\omega v| \leq 1 \ \text{ and } \int_{\mathcal{X}_t} v\omega^n = 0,$$

we have

$$\sup_{\mathcal{X}_t} |v| \leq C\left(1 + \frac{1}{[\omega]^n}\int_{\mathcal{X}_t} |v|\omega^n\right).$$

Proof. The lemma can be proved by the same argument for Lemma 5.1 and Corollary 5.1 with the estimate established in Lemma 10.3. ☐

Lemma 10.5. *Let $\pi : \mathcal{X} \subset \mathbb{CP}^N \times \mathbb{D} \to \mathbb{D}$ be a projective family defined as above. Let γ be a nonnegative smooth function on $\mathbb{CP}^N \times \mathbb{D}$ such that $\{\gamma = 0\}$ is a subvariety of $\mathbb{CP}^N \times \mathbb{D}$ and $\{\gamma = 0\}$ does not contain $\mathcal{X}_t$ for any $t \in \mathbb{D}$. Then for any $A > 0$, $p > n$, $K > 0$, there exists $C = C(A, p, K, \gamma) > 0$ such that for any $t \in \frac{1}{2}\mathbb{D}^*$, any Kähler metric ω on $\mathcal{X}_t$ satisfying*

$$\mathcal{N}_{\mathcal{X}_t,\theta_t,p}(\omega) \le K,\ [\omega] \le A[\theta_t],\ \frac{\omega^n}{\theta_t^n} \ge \gamma, \tag{10.6}$$

and $v \in C^2(\mathcal{X}_t)$ satisfying

$$|\Delta_\omega v| \le 1 \ \text{ and } \int_{\mathcal{X}_t} v\omega^n = 0,$$

we have

$$\frac{1}{[\omega]^n}\int_{\mathcal{X}_t} |v|\omega^n \le C.$$

Proof. . We will follow the same proof of Lemma 5.3 by the argument of contradiction. Suppose Lemma 10.5 fails. Then there exist a sequence of $t_j \in \frac{1}{2}\mathbb{D}$, Kähler metric ω_j of $\mathcal{X}_{t_j}$ and $v_j \in C^2(\mathcal{X}_{t_j})$ satisfying

$$\mathcal{N}_{\mathcal{X}_{t_j},\theta_{t_j},p}(\omega_j) \le K,\ [\omega_j] \le A[\theta_{t_j}],\ \frac{\omega_j^n}{\theta_{t_j}^n} \ge \gamma,$$

and

$$\Delta_{\omega_j} v_j = h_j, \quad \int_X v_j\omega_j^n = 0,$$

for some $|h_j| \le 1$, and as $j \to \infty$

$$\frac{1}{V_j}\int_X |v_j|\omega_j^n =: N_j \to \infty, \tag{10.7}$$

where $V_j = [\omega_j]^n = \int_{\mathcal{X}_{t_j}} \omega_j^n$. We consider $\tilde{v}_j$ defined by $\tilde{v}_j = v_j/N_j$, which clearly satisfies

$$|\Delta_{\omega_j}\tilde{v}_j| = |h_j|/N_j \to 0, \ \text{ and } \frac{1}{V_j}\int_X |\tilde{v}_j|\omega_j^n = 1.$$

We define F_j on $\mathcal{X}_{t_j}$ by

$$\omega_j^n = [\omega_j]^n e^{F_j}\theta_{t_j}^n.$$

Applying Lemma 10.4 to $\tilde{v}_j$, we get $\sup_X |\tilde{v}_j| \le C$ for some uniform $C > 0$. From the equation of $\tilde{v}_j$ and integration by parts, we see that

$$\frac{1}{V_j}\int_X |\nabla\tilde{v}_j|^2_{\omega_j} e^{F_j}\theta_{t_j}^n = \frac{1}{V_j}\int_{\mathcal{X}_{t_j}} |\nabla\tilde{v}_j|^2_{\omega_{t_j}}\omega_j^n \to 0.$$

Let $\mathcal{S} = \{\gamma = 0\} \cup (\mathrm{Sing}\mathcal{X}_0 \times \{0\})$. For each $k > 0$, we pick a sequence of a pair open subsets $U_k \subset\subset V_k \subset\subset (\mathcal{X} \times \mathbb{D}) \setminus \mathcal{S}$ and cut-off functions ρ_k in Lemma 10.1 satisfying the following conditions.

(1) $V_k \subset V_{k+1}$, $U_k \subset U_{k+1}$,

(2) $\lim_{k\to\infty} U_k = (\mathcal{X}\times\mathbb{D})\setminus\mathcal{S}$ and $\mathrm{Vol}_{\theta_t}(\mathcal{X}_t\setminus U_k) < \frac{1}{k}$ for each $t\in\frac{1}{2}\mathbb{D}$,

(3) $\rho_k = 1$ on U_k and $\rho_k = 0$ on $(\mathcal{X}\times\mathbb{D})\setminus V_k$,

(4) $\int_{\mathcal{X}_t} |\nabla\rho_k|^2_{\theta_t}\theta_t^n < \frac{1}{k^2}$, for each $t\in\mathbb{D}$.

Let

$$\mathcal{C}_k = \sup_{t\in\frac{1}{2}\mathbb{D}}\ \sup_{\mathcal{X}_t\cap V_k} e^{-\gamma}.$$

Then for any j and $t\in\frac{1}{2}\mathbb{D}$, we have

$$\inf_{V_k\cap\mathcal{X}_t} e^{F_j} \geq (\mathcal{C}_k)^{-1}.$$

After passing to a subsequence and relabelling j, we can assume that

$$\int_{\mathcal{X}_{t_j}} |\nabla\tilde{v}_j|^2_{\omega_j}\omega_j^n \leq j^{-2}\mathcal{C}_j^{-1}.$$

We now define

$$u_j = \rho_j\tilde{v}_j.$$

Then there exists $C>0$ such that for all j, we have

$$\begin{aligned}
&\int_{\mathcal{X}_{t_j}} |\nabla u_j|_{\theta_{t_j}}\theta_{t_j}^n\\
\leq\ & \int_{\mathcal{X}_{t_j}} |\tilde{v}_j||\nabla\rho_j|_{\theta_{t_j}}\theta_{t_j}^n + \int_X \rho_k j|\nabla\tilde{v}_j|_{\theta_{t_j}}\theta_{t_j}^n\\
\leq\ & Cj^{-1} + \Big(\int_{\mathcal{X}_{t_j}} |\nabla\tilde{v}_j|^2_{\omega_j}e^{F_j}\theta_{t_j}^n\Big)^{1/2}\Big(\int_{\mathcal{X}_{t_j}} \rho_j^2 e^{-F_j}\mathrm{tr}_{\theta_{t_j}}(\omega_j)\theta_{t_j}^n\Big)^{1/2}\\
\leq\ & Cj^{-1} + j^{-1}\Big(\int_{\mathcal{X}_{t_j}} \omega_j\wedge\theta_{t_j}^{n-1}\Big)^{1/2}\\
\leq\ & 2Cj^{-1}.
\end{aligned}$$

Without loss of generality, we can assume that $t_j\to 0$. For any $\mathcal{K}\subset\subset(\mathcal{X}\times\mathbb{D})\setminus\mathcal{S}$, we can pick a sufficiently small $0\in\Delta\Subset\mathbb{D}$ such that $\pi^{-1}(\Delta)\cap K\subset\cup_i(W_i\times\Delta)$, with holomorphic local coordinates $(z_1,...,z_n,t)$ with $t\in\Delta$. Since u_j is uniformly bounded in $W^{1,1}(\mathcal{X}_{t_j},\theta_{t_j})$, by Sobolev embedding theorem, after passing to a subsequence, we can assume that u_j converges to u_∞ in $L^1_{\mathrm{loc}}(\mathcal{X}_0\setminus\mathcal{S},\theta_0)$. Furthermore, u_∞ and u_j are uniformly bounded in $L^\infty(\mathcal{X}_0)$ and $L^\infty(\mathcal{X}_{t_j})$.

Since $\lim_{j\to\infty}\int_{\mathcal{X}_{t_j}}|\nabla u_j|_{\theta_{t_j}}\theta_{t_j}^n = 0$, for any test function $f\in C^\infty(\mathcal{X}_0\setminus\mathcal{S})$ with compact support in $\mathcal{X}_0\setminus\mathcal{S}$, we can extend f to $C^\infty((\mathcal{X}\times\mathbb{D})\setminus\mathcal{S})$ with compact support in

$(\mathcal{X} \times \mathbb{D}) \setminus \mathcal{S}$ and

$$\begin{aligned} \left| \int_{\mathcal{X}_{t_j}} u_j (\Delta_{\theta_{t_j}} f) \theta_{t_j}^n \right| &= \int_{\mathcal{X}_{t_j}} |\nabla u_j|_{\theta_{t_j}} |\nabla f|_{\theta_{t_j}} \theta_{t_j}^n \\ &\leq \left(\sup_{t \in \mathbb{D}, \mathcal{X}_{t_j}} |\nabla f|_{\theta_t} \right) \int_{\mathcal{X}_{t_j}} |\nabla u_j|_{\theta_{t_j}} \theta_{t_j}^n \\ &\to 0 \end{aligned}$$

as $j \to \infty$. Therefore

$$\int_{\mathcal{X}_0 \cap (W_i \times \Delta)} u_\infty \left(\Delta_{\theta_0} f \right) \theta_0^n = 0$$

and by Weyl's lemma for the Laplace equation, u_∞ solves the Laplace equation $\Delta u_\infty = 0$ on $\mathcal{X}_0 \setminus \mathcal{S}$ and $u_\infty \in C^\infty(\mathcal{X}_0 \setminus \mathcal{S})$. This immediately implies that

$$\int_{\mathcal{X}_0} \left(\rho_{j,0} \right)^2 u_\infty \left(\Delta_{\theta_0} u_\infty \right) \theta_0^n = 0$$

for any j, where $\rho_{j,0}$ is the restriction of ρ_j to $\mathcal{X}_0$. Then

$$\int_{\mathcal{X}_0} (\rho_{j,0})^2 |\nabla u_\infty|_{\theta_0}^2 \theta_0^n \leq 8 \sup_{\mathcal{X}_0} |u_\infty|^2 \left(\int_{\mathcal{X}_0} |\nabla \rho_{j,0}|_{\theta_0}^2 \theta_0^n \right) \to 0$$

as $j \to \infty$ by the choice of ρ_j. Therefore u_∞ is a constant on $\mathcal{X}_0$.

Following the same argument in the proof of Lemma 5.3, we can show that u_∞ cannot be 0 by the assumption

$$\frac{1}{V_j} \int_{\mathcal{X}_{t_j}} |u_j| \omega_j^n = 1.$$

On the other hand, $\mathcal{X}_0 \setminus \mathcal{S}$ is connected because $\mathcal{X}_0$ is irreducible and $\mathcal{X}_0 \cap \mathcal{S}$ is a subvariety of $\mathcal{X}$. By the same argument in the proof of Lemma 5.3, we can also show that u_∞ must vanish everywhere in $\mathcal{X}_0 \setminus \mathcal{S}$. This leads to contradiction and we have completed the proof for the lemma. □

With Lemma 10.5, we can complete the proof of Theorem 10.1 by the same argument for Theorem 1.1.

11. Calabi-Yau fibrations

In this section, we will apply Theorem 10.1 to collapsing Kähler metrics on a fibration of Calabi-Yau manifolds. Our first result is to extend the results of [24] for collapsing Ricci-flat Kähler metrics on a fibred Calabi-Yau manifold.

Theorem 11.1. *Let (X, ω_X) be an n-dimensional projective Calabi-Yau manifold and $\pi : X \to Y$ be a holomorphic fibration over a Rieman surface Y. Suppose each fibre $X_y = \pi^{-1}(y)$ is normal and has at worst canonical singularities. Let ω_Y be a fixed Kähler metric on Y and let ω_ϵ be the unique Ricci-flat Kähler metric in $\in [\omega_X + \epsilon^{-1}\omega_Y]$ for each $\epsilon \in (0,1)$. Then there exist $\alpha > 0$ and $C > 0$ such that for any $\epsilon > 0$, any smooth fibre X_y of π and any $x \in X_y$,*

$$\mathrm{diam}(X_y, \omega_\epsilon|_{X_y}) \le C,$$

$$\int_{X_y} \left(|G_{y,\epsilon}(x,\cdot)| + |\nabla G_{y,\epsilon}(x,\cdot)|\right) \left(\omega_\epsilon|_{X_y}\right)^n + \left(-\inf_{z\in X_y} G_{y,\epsilon}(x,z)\right) \mathrm{Vol}_{\omega_\epsilon|_{X_y}}(X_y) \le C,$$

$$\frac{\mathrm{Vol}_{\omega_\epsilon|_{X_y}}(B_{\omega_\epsilon|_{X_y}}(x,R))}{\mathrm{Vol}_{\omega_\epsilon|_{X_y}}(X)} \ge C^{-1}R^\alpha,$$

where $G_{y,\epsilon}$ is the Green's function for $(X_y, \omega_\epsilon|_{X_y})$ and $B_{\omega_\epsilon|_{X_y}}(x,R))$ is the geodesic ball in $(X_y, \omega_\epsilon|_{X_y})$.

Proof. Let η be a non-where vanishing holomorphic volume form on X and let η_y be the relative holomorphic volume form defined by $\eta = \eta_y \wedge dy$ locally on $\mathbb{D} \subset Y$. Then ω_ϵ satisfies

$$\omega_\epsilon^n = A_\epsilon \eta \wedge \overline{\eta},$$

where $A_\epsilon = \frac{[\omega_\epsilon]^n}{\int_X \eta\wedge\overline{\eta}} = O\left(\epsilon^{-1}\right)$.

Let $\theta_y = \omega_X|_{X_y}$ be the restriction of ω_X to X_y. It is proved in [24] (c.f. Proposition 2.1 and Proposition 2.3 in [24]) that there exist $p > 1$ and $C > 0$ such that for any $y \in Y$ and $\epsilon \in (0,1)$, we have

$$\int_{X_y} \left|\frac{\eta_y \wedge \overline{\eta_y}}{(\theta_y)^{n-1}}\right|^p (\theta_y)^{n-1} \le C, \tag{11.8}$$

$$\sup_X \mathrm{tr}_{\omega_\epsilon}(\epsilon^{-1}\omega_Y) \le C. \tag{11.9}$$

(11.9) implies that on each X_y, we have on X_y

$$\frac{\omega_\epsilon^{n-1} \wedge \omega_Y}{\omega_X^{n-1} \wedge \omega_Y} = \frac{\omega_\epsilon^{n-1} \wedge \omega_Y}{\omega_\epsilon^n} \frac{\omega_\epsilon^n}{\omega_X^{n-1} \wedge \omega_Y} \le C \frac{\eta \wedge \overline{\eta}}{\omega_X^{n-1} \wedge \omega_Y} \le C \frac{\eta_y \wedge \overline{\eta_y}}{\theta_y^{n-1}}.$$

Then (11.8) immediately gives the uniform upper bound for the q-Nash entropy $\mathcal{N}_{X_y,\theta_y,q}(\omega_\epsilon|_{X_y})$ for any $\epsilon \in (0,1)$ and fixed $q > 0$.

By Theorem 2.5 of [24] as a refined Schwarz lemma, there exists $C > 0$ such that

$$\omega_\epsilon \ge C^{-1}\omega_X,$$

and so

$$\left.\frac{\omega_\epsilon^{n-1}}{(\theta_y)^{n-1}}\right|_{X_y}$$

is uniformly bounded away from 0.

Therefore we can directly apply Theorem 10.1 to complete the proof of Theorem 11.1. □

In the setting of Theorem 11.1, it is proved in [24] that the extrinsic diameter of a smooth fibre X_y is uniformly bounded, i.e., any two points on a smooth fibre X_y can be joined by a path in X with uniformly bounded arc length with respect to ω_ϵ for all $\epsilon > 0$. The stronger intrinsic diameter bound for X_y is achieved in Theorem 11.1.

We will also apply Theorem 10.1 to the long time collapsing solutions of the Kähler-Ricci flow. Let X be an n-dimensional projective manifold of nef K_X. Suppose K_X is semi-ample and the Kodaira dimension of X is one. The pluricanonical map

$$\pi : X \to X_{can}$$

is a holomorphic fibration over the canonical model X_{can}. The general fibre of π is a smooth Calabi-Yau manifold of dimension $n-1$. We consider the Kähler-Ricci flow

$$\frac{\partial g}{\partial t} = -\mathrm{Ric}(g),\ g(0) = g_0 \tag{11.10}$$

for a given Kähler metric g_0. Then (11.10) admits a smooth solution $g(t)$ for all $t \geq 0$. We would like to investigate the geometric behavior of $g(t)$ near a singular fibre with mild singularities.

Theorem 11.2. *Let X be an n-dimensional projective manifold with semi-ample K_X and $\mathrm{Kod}(X) \leq 1$. Let $g(t)$ be the solution of the Kähler-Ricci flow (11.10). Suppose each fibre of $\pi : X \to X_{can}$ has at worst canonical singularities. Then there exists $C > 0$ such that for any $t > 0$, any smooth fibre X_y of π and any $x \in X_y$,*

$$\mathrm{diam}(X_y, g(t)|_{X_y}) \leq C,$$

$$\int_{X_y} \big(|G_{t,y}(x,\cdot)| + |\nabla G_{t,y}(x,\cdot)|\big) dV_{g(t)|_{X_y}} + \left(-\inf_{z\in X_y} G_{t,y}(x,z)\right) \mathrm{Vol}_{g(t)|_{X_y}}(X_y) \leq C,$$

$$\frac{\mathrm{Vol}_{g(t)|_{X_y}}(B_{g(t)|_{X_y}}(x,R))}{\mathrm{Vol}_{g(t)|_{X_y}}(X)} \geq C^{-1} R^\alpha,$$

where $G_{t,y}$ is the Green's function for $(X_y, g(t)|_{X_y})$ and $B_{g(t)|_{X_y}}(x,R))$ is the geodesic ball in $(X_y, \omega_\epsilon|_{X_y})$. .

Proof. When the Kodaira dimension is 0, $c_1(X) = 0$ and the flow (11.10 converges smoothly to a Ricci-flat Kähler metric in the initial Kähler class. Therefore it suffices to prove the theorem in the case of Kodaira dimension equal to one.

Let X°_{can} be the regular values of $\pi : X \to X_{can}$ and $X_{can} \setminus X^\circ_{can}$ is a finite set of points since X_{can} is a smooth Riemann surface. We follow the same argument in the proof of Theorem 11.1. The estimate (11.8) still holds due to the fibration structure of $\pi : X \to X_{can}$. Since X_{can} is smooth, we can choose a smooth Kähler metric $\hat{g}$ on X_{can} such that $\pi^*\hat{g} \in [K_X]$. Let χ, ω_0 and $\omega(t)$ be the smooth closed $(1,1)$-forms corresponding to $\pi^*\hat{g}$, g_0 and $g(t)$. We let $\omega_t = \omega_0 + t\chi$ and (11.10) can be reduced to a complex Monge-Ampère flow

$$\frac{\partial \varphi}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi)^n}{\Omega},\ \varphi(0) = 0,$$

where Ω is the volume form on X with $\sqrt{-1}\partial\overline{\partial}\log\Omega = \chi$.

Since K_X is semi-ample, the Schwarz lemma analogous to (11.9) also holds from the general estimate in [31, 32], i.e. there exists $C > 0$ such that for all $t \geq 0$,

$$\mathrm{tr}_{g(t)}(t\chi) \leq C.$$

By the same argument in the proof of Theorem 11.1, for any fixed $q > 0$, there exists $C = C(g_0, q) > 0$ such that for all $t \geq 0$ and $y \in X_{can}$, we have

$$\mathcal{N}_{X_y, g_{0,y}, q}\left(g(t)|_{X_y}\right) \leq C,$$

where $g_{0,y} = g_0|_{X_y}$. Since X_y is normal, the uniform Nash entropy bound (for $q > n$) above implies that there exists $C > 0$ such that for all $t \geq 0$ and $y \in X_{can}$,

$$\|\varphi(t)|_{X_y} - \sup_{X_y} \varphi(t)|_{X_y}\| \leq C. \tag{11.11}$$

If we let $\hat{\varphi}(t) = \frac{1}{V_{g_{0,y}}(X_y)} \int_{X_y} \varphi dV_{g_{0,y}}$ be the average of φ along the fibre, then $(\varphi(t) - \hat{\varphi}(t))$ is uniformly bounded due to (11.11).

Let

$$H = \log \mathrm{tr}_{g(t)}(tg_0) - 2A\Big(\varphi - \frac{1}{V_{g_{0,y}}(X_y)} \int_{X_y} \varphi dV_{g_{0,y}}\Big)$$

for some $A \geq 1$ to be determined. Since $V_{g_{0,y}}(X_y)$ is independent of $y \in X_{can}$ and $\|\frac{\partial \varphi}{\partial t}\|_{L^\infty(X)}$ is uniformly bounded for all $t \geq 0$ by [34], we can apply the similar second order calculations in [31] to the evolution of H by

$$\begin{aligned}
\Box_t H &\leq C\mathrm{tr}_{g(t)}(g_0) - 2A\mathrm{tr}_{g(t)}(g_t) - \frac{2A}{V_{g_{0,y}}(X_y)}\mathrm{tr}_{g(t)}\Big(\sqrt{-1}\partial\overline{\partial}\int_{X_y} \varphi\omega_0^{n-1}\Big) + CA \\
&\leq C\mathrm{tr}_{g(t)}(g_0) - 2A\mathrm{tr}_{g(t)}(g_t) - \frac{2A}{V_{g_{0,y}}(X_y)}\mathrm{tr}_{g(t)}\Big(\int_{X_y} (\omega - \omega_t) \wedge \omega_0^{n-1}\Big) + CA
\end{aligned}$$

$$\leq \quad C\mathrm{tr}_{g(t)}(g_0) - 2A\mathrm{tr}_{g(t)}(g_t) - \frac{2A}{V_{g_{0,y}}(X_y)}\mathrm{tr}_{g(t)}\Big(\int_{X_y}\omega_0^n\Big) + CA,$$

for some uniform constant $C > 0$ independent of A. We define a closed $(1,1)$ form as the push-forward of ω_0^n to X_{can} by

$$\hat{\chi} = \frac{\int_{X_y}\omega_0^n}{\mathrm{Vol}_{g_{0,y}}(X_y)}.$$

It is proved in [31, 32] that

$$f = \frac{\hat{\chi}}{\chi}$$

is smooth away from the singular fibres and f is $L^q(X_{can}, \chi)$ integrable for some $q > 1$. There exists a unique solution $\psi \in L^\infty(X_{can})$ to the following linear equation

$$\Delta_\chi \psi = f - \bar{f},\ \ \bar{f} = \frac{\int_{X_{can}} f\chi}{\int_{X_{can}}\chi} = \frac{\int_{X_{can}}\hat{\chi}}{\int_{X_{can}}\chi},$$

since f is L^q-integrable for some $q > 1$. Furthermore, both f and ψ are smooth on X_{can}°, and

$$\sqrt{-1}\partial\bar{\partial}\psi = \hat{\chi} - \bar{f}\chi.$$

Let σ be a defining section of points corresponding to $X_{can} \setminus X_{can}^\circ$ and let h be a smooth hermitian metric on the line bundle associated to $[\sigma]$. Then we let

$$H_\epsilon = \log|\sigma|_h^{2\epsilon}\mathrm{tr}_{g(t)}(tg_0) - 2A\psi - 2A\Big(\varphi - \frac{1}{V_{g_{0,y}}(X_y)}\int_{X_y}\varphi dV_{g_{0,y}|_{X_y}}\Big),$$

for any sufficiently small $\epsilon > 0$. Then there exists $C > 0$ such that for all $t \geq 0$ and $\epsilon \in (0,1)$,

$$\Box_t H_\epsilon \quad \leq \quad -A\mathrm{tr}_{g(t)}(g_t) + 2CA$$

for a fixed sufficiently large $A \geq 0$. Applying the maximum principle, $H_\epsilon \leq C$ for a uniform constant $C > 0$ independent of $\epsilon \in (0,1)$. Letting $\epsilon \to 0$, $\mathrm{tr}_{g(t)}(g_t)$ is uniformly bounded above, or equivalently, there exists $c > 0$ such that for all $t \geq 0$,

$$g \geq cg_t.$$

Therefore we have derived a uniform positive lower bound of

$$\frac{(\omega(t))^{n-1}}{\omega_0^{n-1}}$$

on each fibre X_y.

Combining the above estimates, we can apply Theorem 1.1 to complete the proof. $\Box$

Theorem 2.3 immediately follows from Theorem 11.2 by parabolic scaling.

12. Constant scalar curvature Kähler metrics with bounded Nash entropy

In this section, we will obtain geometric estimates for cscK metrics near the canonical class of a smooth minimal model of general type.

Let X be an n-dimensional Kähler manifold with big and nef K_X, i.e. a minimal model of general type. It is well-known in birational geometry that K_X is semi-ample and the pluricanonical system of X induces a unique surjective birational morphism $\pi : X \to X_{can}$ from X to its unique canonical model X_{can}. In particular, π is isomorphic between X°_{can} and X°, where X°_{can} is the set of regular values of π and $X^\circ = \pi^{-1}(X^\circ_{can})$. We choose Ω to be a smooth volume form on X on X such that $\chi = \sqrt{-1}\partial\overline{\partial}\log\Omega \in [K_X]$ is a semi-positive $(1,1)$-form. It is proved in [8] that there exists a unique $\varphi_{KE} \in \mathrm{PSH}(X,\chi) \cap L^\infty(X)$ solving the complex Monge-Ampère equation

$$(\chi + \sqrt{-1}\partial\overline{\partial}\varphi_{KE})^n = e^{\varphi_{KE}}\Omega,\ \sup_X \varphi_{KE} = 0$$

on X. If we let $\omega_{KE} = \chi + \sqrt{-1}\partial\overline{\partial}\varphi_{KE}$, then ω_{KE} descends to X_{can} as a Kähler-Einstein current since $K_X = \pi^* K_{X_{can}}$. Furthermore, ω_{KE} is smooth on X°_{can}. If we let g_{KE} be the smooth Kähler-Einstein metric on X°_{can} associated to the Kähler-Einstein current ω_{KE}, then it is proved in [29] that the metric completion of (X°_{can}, g_{KE}) is a compact metric space homeomorphic to X_{can}.

The following result of [20] shows that there always exists a unique cscK metric in a Kähler class near the canonical class.

Lemma 12.1. *For any Kähler class $\mathcal{A}$ of X, there exists $\delta_0 = \delta_0(\mathcal{A}) > 0$ such that for any $0 < \delta < \delta_0$, there exists a unique cscK metric $g_\delta \in [K_X + \delta\mathcal{A}]$.*

Let ω_δ be the Kähler form associated to g_δ in Lemma 12.1 and $\omega_{\mathcal{A}} \in [\mathcal{A}]$ be a fixed Kähler form. Then we can write

$$\omega_\delta = \chi + \delta\omega_{\mathcal{A}} + \sqrt{-1}\partial\overline{\partial}\varphi_\delta$$

for a unique $\varphi_\delta \in C^\infty(X)$ with $\sup_X \varphi_\delta = 0$.

The following estimates are proved in [25].

Lemma 12.2. *Let $\mathcal{A}$ be a Kähler class of X and $\delta_0 = \delta_0(\mathcal{A}) > 0$ as in Lemma 12.1. Then there exists $C > 0$ such that for any $0 < \delta < \delta_0$,*

$$C^{-1} \le \inf_X \frac{\omega_\delta^n}{\omega_{\mathcal{A}}^n} \le \sup_X \frac{\omega_\delta^n}{\omega_{\mathcal{A}}^n} \le C,\ \|\varphi_\delta\|_{L^\infty(X)} \le C.$$

Furthermore, φ_δ converges to φ_{KE} in $C^\infty(X^\circ)$ as $\delta \to 0$.

Lemma 12.2 implies that the cscK metrics g_δ converge to the Kähler-Einstein metrics g_{KE} on X° smoothly.

Proof of Theorem 2.4. The uniform estimates for the diameter, Green's function and volume non-collapsing follows immediately by applying Lemma 12.2 to Theorem 1.1.

It remains to prove that (X, g_δ) converges to (X_{can}, d_{KE}) as $\delta \to 0$, if X_{can} has only isolated singularities. Suppose not. We can choose a sequence $\delta_j \to 0$ such that (X, g_{δ_j}) converges to a compact metric space (Y, d_Y) not isomorphic to (X_{can}, d_{KE}). Since g_{δ_j} converges smoothly to g_{KE} on X°_{can}, (X°_{can}, g_{KE}) can be embedded in (Y, d_Y) with local isomorphisms. Then there must be a point $p \in Y$ such that

$$d_Y(p, X^\circ_{can}) > 0.$$

Otherwise, (Y, d_Y) will be isomorphic to (X_{can}, d_{KE}) because Y is compact and (X_{can}, d_{KE}) is the compactification of (X°_{can}, g_{KE}) by adding finitely many points from $X_{can} \setminus X^\circ_{can}$. We can pick a sequence of points $p_j \in (X, g_{\delta_j})$ such that p_j converges to $p \in Y$ in Gromov-Hausdorff distance. After possibly passing to a sequence, p_j must also converge to a singular point p_∞ of X_{can} with respect to any fixed metric on X_{can} (for example, the Fubini-Study metric from a projective embedding of X_{can}).

We choose an exhaustion of X°_{can} with a sequence of open sets $U_1 \subset\subset U_2 \subset\subset .... \subset\subset U_k \subset\subset ... \subset X^\circ_{can}$ such that

$$\lim_{k\to\infty} U_k = X^\circ_{can}, \ \lim_{k\to\infty} \mathrm{Vol}_{g_{KE}}(U_k) = \mathrm{Vol}_{g_{KE}}(X^\circ_{can}) = [K_X]^n.$$

From the above assumption, there exists $\epsilon > 0$ such that for any $k > 0$, there exists $j > 0$ such that

$$d_{g_{\delta_j}}(p_j, U_k) > \epsilon,$$

and so

$$B_{g_{\delta_j}}(p_j, \epsilon) \subset X \setminus U_k.$$

By the uniform volume non-collapsing, there exists $\delta > 0$ such that for any $k > 0$, there exists $j > 0$ such that

$$B_{g_{\delta_j}}(p_j, \epsilon) > \delta$$

and

$$\mathrm{Vol}_{g_{\delta_j}}(U_k) < [K_X]^n - \delta.$$

This leads to contradiction by choosing sufficiently large k. □

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