2024 American Control Conference (ACC)
July 8-12, 2024. Toronto, Canada

Observer-based Consensus Strategy for Linear Multi-Agent Systems under Double Event-Triggering Conditions

Haochen Ding and Ming Xin

Abstract— **In this paper, an observer-based event-triggered consensus control (OETCC) strategy is proposed for linear multiagent systems (MASs) under strongly connected network. Two event-triggering conditions (ETCs) are designed to trigger information transmitting and control update separately so that continuously applying these tasks is avoided. In addition, the triggering times for both tasks are unnecessarily to be the same. Since the triggering time for control input update is predictable, continuously monitoring the related ETC can be avoided. It is proved that under the proposed strategy, consensus can be achieved exponentially. Its effectiveness is also verified by a numerical example.**

I. INTRODUCTION

In the past three decades, cooperative control of MASs has drawn more and more attention [1]. Since cooperative work of multiple simple vehicles has more advantages over the work by a single complicated vehicle [2], it has a wide range of applications such as monitoring forest fires, hazardous material handling, and so force [1,3,4]. Information consensus plays an important role during cooperative teamwork. Under distributed consensus algorithms, the agent in the system communicates with local neighbor(s) so that a common decision of the global system can be made.

The consensus algorithms require each agent to sample its own state, access to the state of its neighbor(s) and update its control input. In practice, these control tasks are implemented on a digital platform at discrete times [5]. The time instants can be determined in a periodic or event-triggered way. The former strategy is conservative since the constant period has to guarantee consensus in the worst-case scenario, while the latter one is more natural and flexible, which presets triggering conditions that depend on agents' real-time behaviors, and the control task is implemented once the related conditions are satisfied. Thus, under the event-triggered control law, more computation and energy resources as well as communication bandwidth can be saved. Recent results on event-triggered consensus algorithms are reviewed in [6,7].

Due to physical and cost constraints, agents may not be able to measure full states. In this case, an observer can be designed to estimate unmeasurable states via output feedback. The observer-based event-triggered linear consensus problem is investigated in [8,9]. In addition, fully distributed consensus strategy is proposed in [10] so that global network information is not required for each agent; Lipschitz and Lur'e nonlinear terms are added to the linear dynamics in [11,12] to represent more general physical systems; Consensus under bounded control inputs is investigated in [13]; The time delay and external disturbance are considered in [14] and [10,15], respectively, where the reset observer is introduced in [15] to improve the estimation performance; Consensus is guaranteed under Denial-of-Service Attacks in [16,17]. However, the foregoing results are based on the OETCC strategy with some limitations. In [8–10,12,15,17], although the event-triggered strategy avoids continuous communication, continuous control input update is required. This drawback is overcome in [11,13,14,16]. But the distributed control algorithm requires control inputs to be updated once each agent receives information. These limitations may cost extra communication, energy as well as computation resources. Moreover, in [8–17], the control input update and information transmission are triggered at the same time, which is not flexible.

Motivated by the above discussions, an OETCC algorithm is proposed for linear consensus under strongly connected network. Two ETCs are designed to trigger information transmission and control input update separately. The main advantages are three-fold: First, under the proposed event-triggering strategy, both continuous control input update and information transmission are not required. Second, the triggering times for both tasks are unnecessarily to be the same. In addition, when an agent receives information, neither of the control tasks is necessarily to be triggered immediately. Finally, the triggering time for control input update can be predictable so that continuously monitoring the related ETC can be avoided.

II. PRELIMINARIES AND PROBLEM STATEMENT

A. Notation and graph theory

Let the network topology be represented by a directed graph (digraph) $G=(V,E)$, where $V=\{v_i\}_{i=1}^N$ is the node set, and $E\subseteq V\times V$ is the edge set. $(v_i,v_j)\in E$ denotes that there is a directed edge from v_i to v_j, but not necessarily vice versa. In this case, v_i is called an in-neighbor of v_j, and v_j is an out-neighbor of v_i. We assume $(v_i,v_i)\notin E$. Let $\mathcal{N}_i=\{v_j\in V:(v_j,v_i)\in E\}$, and let $I_{\mathcal{N}_i}=\{1\le j\le N: v_j\in\mathcal{N}_i\}$. The adjacency matrix of the graph G is defined by $A_G=[a_{ij}]$, where $a_{ij}=1$ if $(v_j,v_i)\in E$ and $a_{ij}=0$, otherwise. The Laplacian matrix is defined by $L=[l_{ij}]$, where $l_{ii}=\sum_{k=1}^N a_{ik}$ and $l_{ij}=-a_{ij}$ if $j\neq i$. G is strongly connected if for each pair (v_i,v_j), there is a directed path from v_i to v_j.

Definition 1 [18] Given a strongly connected digraph G with the Laplacian matrix $L\in\mathbf{R}^{N\times N}$. Let $\boldsymbol{r}=\mathrm{col}(r_1,\ldots,r_N)$, $r_i>0$,

H. Ding (e-mail: hdnp7@umsystem.edu) and M. Xin are with the Department of Mechanical and Aerospace Engineering, University of Missouri, Columbia, MO 65211, USA (e-mail: xin@missouri.edu).

$1 \le i \le N$, satisfying $\boldsymbol{r}^{\mathrm{T}} L = \mathbf{0}_{1\times N}$ and $\boldsymbol{r}^{\mathrm{T}}\mathbf{1}_N = 1$. The general algebraic connectivity is defined by

$$a(G) = \min_{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{r}=0,\boldsymbol{x}\neq\mathbf{0}} \left(\boldsymbol{x}^{\mathrm{T}}\hat{L}\boldsymbol{x}\right)\Big/\left(\boldsymbol{x}^{\mathrm{T}}R\boldsymbol{x}\right)$$

where $\hat{L} = \left(RL + L^{\mathrm{T}}R\right)\big/2$ and $R = diag\left(r_1,\ldots,r_N\right)$.

B. Problem Statement

Consider a system with N agents. The dynamics of agent i, $1 \le i \le N$, is described by

$$\begin{cases} \dot{\boldsymbol{x}}_i(t) = A\boldsymbol{x}_i(t) + B\boldsymbol{u}_i(t) \\ \boldsymbol{y}_i(t) = C\boldsymbol{x}_i(t) \end{cases} \tag{1}$$

where $\boldsymbol{x}_i(t) \in \mathbf{R}^n$, $\boldsymbol{u}_i(t) \in \mathbf{R}^m$ and $\boldsymbol{y}_i(t) \in \mathbf{R}^r$ represent the state, control input, and output of agent *i,* respectively, and $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, and $C \in \mathbf{R}^{r\times n}$ are constant matrices. Assume that each agent can only sample its own output data. This paper aims to design a distributed OETCC law such that consensus can be achieved.

The following assumptions and lemmas are needed.

Assumption 1 (A,B,C) is stabilizable and detectable.

Assumption 2 The digraph G is strongly connected.

Lemma 1 [19] If the matrix pair (A,B) is stabilizable, for any number $\eta_1,\ \eta_2 > 0$, and matrix $Q > 0$, the algebraic Riccati equation (ARE) $PA + A^{\mathrm{T}}P - \eta_1 PBB^{\mathrm{T}}P = -\eta_2 Q$ has a unique solution $P > 0$.

Lemma 2 [18] Under Assumption 2, $a(G) > 0$. In addition, $a(G)$ is the maximum of β, which satisfies $S^{\mathrm{T}}\left(\hat{L} - \beta R\right)S \ge 0$, where $R = diag\left(r_1,\ldots,r_N\right)$ and $\hat{L}$ is defined in Definition 1, and $S = \begin{bmatrix} I_{N-1} & -\hat{\boldsymbol{r}}^{\mathrm{T}}/r_N \end{bmatrix}^{\mathrm{T}}$ with $\hat{\boldsymbol{r}} = \left[r_1,\ldots,r_{N-1}\right]$.

III. Main Results

In this section, an OETCC strategy is proposed to solve the consensus problem stated in Sec.II.B such that the Zeno behavior is excluded.

For $1 \le i \le N$, the dynamics of the observer state of agent i is described by

$$\begin{cases} \dot{\tilde{\boldsymbol{x}}}_i(t) = A\tilde{\boldsymbol{x}}_i(t) + B\boldsymbol{u}_i(t) + K_1\left[\tilde{\boldsymbol{y}}_i(t) - \boldsymbol{y}_i(t)\right] \\ \tilde{\boldsymbol{y}}_i(t) = C\tilde{\boldsymbol{x}}_i(t) \end{cases} \tag{2}$$

where $\tilde{\boldsymbol{x}}_i(t) \in \mathbf{R}^n$ is the observer state, $\tilde{\boldsymbol{y}}_i(t) \in \mathbf{R}^n$ is the output of the observer, $K_1 \in \mathbf{R}^{n\times r}$ is the observer gain to be determined.

The distributed consensus control law is based on the disagreement among agent i and its in-neighbor(s) defined by

$$\boldsymbol{\delta}_i(t) = \sum_{j\in I_{\mathcal{N}_i}} \left\lfloor \boldsymbol{x}_i(t) - \boldsymbol{x}_j(t) \right\rfloor \tag{3}$$

Note that consensus is achieved if for any $1 \le i \le N$, $\boldsymbol{\delta}_i(t) \to 0$ as $t \to \infty$ [20]. Let $\left\{t_{k_{1i}}^{1i} : k_{1i} = 0,1,\ldots\right\}$ and $\left\{t_{k_{2i}}^{2i} : k_{2i} = 0,1,\ldots\right\}$ be the sequences of triggering times corresponding to information transmission and control input update of agent i, respectively. For $t \in [t_{k_{1i}}^{1i}, t_{k_{1i}+1}^{1i})$, $k_{1i} = 0,1,\ldots$, define $\tilde{\boldsymbol{z}}_i(t)$ satisfying

$$\dot{\tilde{\boldsymbol{z}}}_i(t) = A\tilde{\boldsymbol{z}}_i(t) \tag{4}$$

with initial $\tilde{\boldsymbol{z}}_i\left(t_{k_{1i}}^{1i}\right) = \tilde{\boldsymbol{x}}_i\left(t_{k_{1i}}^{1i}\right)$. Let the disagreement vector based on the estimated state be

$$\tilde{\boldsymbol{\delta}}_i(t) = \sum_{j\in I_{\mathcal{N}_i}} \left[\tilde{\boldsymbol{z}}_i(t) - \tilde{\boldsymbol{z}}_j(t)\right] \tag{5}$$

The OETCC law is designed as:

$$\boldsymbol{u}_i(t) = K_2\hat{\boldsymbol{\delta}}_i(t) \tag{6}$$

where $K_2 \in \mathbf{R}^{n\times m}$ is the control gain to be determined and $\hat{\boldsymbol{\delta}}_i(t) = \tilde{\boldsymbol{\delta}}_i\left(t_{k_{2i}}^{2i}\right)$, $t \in [t_{k_{2i}}^{2i}, t_{k_{2i}+1}^{2i})$, $k_{2i} = 0,1,\ldots$. Define the measurement errors by

$$\begin{cases} \boldsymbol{e}_{1i}(t) = \tilde{\boldsymbol{x}}_i(t) - \tilde{\boldsymbol{z}}_i(t) \\ \boldsymbol{e}_{2i}(t) = \hat{\boldsymbol{\delta}}_i(t) - \tilde{\boldsymbol{\delta}}_i(t) \end{cases} \tag{7}$$

Let $\tilde{t}_{k_{1i}}^{1i}$ and $\tilde{t}_{k_{2i}}^{2i}$ be two latest triggering times of agent i. The next triggering time instant of agent i is determined by

$$\begin{cases} \tilde{t}_{k_{1i}+1}^{1i} = \inf\left\{t > \tilde{t}_{k_{1i}}^{1i} : \left\|\boldsymbol{e}_{1i}(t)\right\| \ge f_{1i}(t)\right\} \\ \tilde{t}_{k_{2i}+1}^{2i} = \inf\left\{t > \tilde{t}_{k_{2i}}^{2i} : \left\|\boldsymbol{e}_{2i}(t)\right\| \ge f_{2i}(t)\right\} \end{cases} \tag{8}$$

where $f_{1i}(t)$ and $f_{2i}(t)$ are the threshold functions to be determined. This means the triggering time is determined when the ETC $\left\|\boldsymbol{e}_{1i}(t)\right\| \ge f_{1i}(t)$ or $\left\|\boldsymbol{e}_{2i}(t)\right\| \ge f_{2i}(t)$ is satisfied. Let $\left\{\hat{t}_k : \hat{t}_k = k\Delta t, k = 1,2,\ldots\right\}$ be the sequence of time instants when each agent monitors ETCs and samples output. By choosing small enough Δt, it can be considered that the agent monitors ETCs and samples data continuously. The event-triggering mechanism for agent $i, 1 \le i \le N$, is summarized as follows: At $t = t_0$, agent i samples $\boldsymbol{y}_i(t_0)$, sets $\tilde{\boldsymbol{x}}_i(t_0) = \tilde{\boldsymbol{x}}_{0i}$, computes $\tilde{\boldsymbol{y}}_i(t_0)$ by (2); Agent i sets $t_0^{1i} = t_0$, $\tilde{\boldsymbol{z}}_i\left(t_0^{1i}\right) = \tilde{\boldsymbol{x}}_{0i}$, and sends it to its out-neighbor(s); Agent i receives $\tilde{\boldsymbol{z}}_j\left(t_0^{1j}\right), j \in I_{\mathcal{N}_i}$, sets $t_0^{2i} = t_0$; computes $\hat{\boldsymbol{\delta}}_i(t) = \tilde{\boldsymbol{\delta}}_i\left(t_0^{2i}\right)$ and $\boldsymbol{u}_i(t)$ by (5) and (6), respectively. At $t = \hat{t}_1$, agent i computes $\tilde{\boldsymbol{x}}_i\left(\hat{t}_1\right)$, $\tilde{\boldsymbol{y}}_i\left(\hat{t}_1\right)$, $\tilde{\boldsymbol{z}}_j\left(\hat{t}_1\right)$ $j \in I_{\mathcal{N}_i} \bigcup \{i\}$, and $\tilde{\boldsymbol{\delta}}_i\left(\hat{t}_1\right)$ by (2), (4) and (5), respectively, then agent i checks ETCs: if $\left\|\boldsymbol{e}_{1i}(t)\right\| \ge f_{1i}(t)$, agent i sets $t_1^{1i} = \hat{t}_1$, $\tilde{\boldsymbol{z}}_i\left(t_1^{1i}\right) = \tilde{\boldsymbol{x}}_i\left(\hat{t}_1\right)$, and so $\boldsymbol{e}_{1i}\left(t_1^{1i}\right) = \mathbf{0}$, then sends $\tilde{\boldsymbol{z}}_i\left(t_1^{1i}\right)$ to out-neighbor(s); if $\left\|\boldsymbol{e}_{2i}(t)\right\| \ge f_{2i}(t)$, agent i sets $t_1^{2i} = \hat{t}_1$, $\hat{\boldsymbol{\delta}}_i(t) = \tilde{\boldsymbol{\delta}}_i\left(\hat{t}_1\right)$, and so $\boldsymbol{e}_{2i}\left(t_2^{2i}\right) = \mathbf{0}$, then updates the control input by (6); Agent i samples $\boldsymbol{y}_i\left(\hat{t}_1\right)$; At $t = \hat{t}_2$, agent i repeats the previous procedure, and so force.

Remark 1 Note that agent i cannot predict $\tilde{\boldsymbol{x}}_j(t), j \in I_{\mathcal{N}_i}$, by (2). Thus, if $\tilde{\boldsymbol{\delta}}_i(t)$ is defined by $\tilde{\boldsymbol{\delta}}_i(t) = \sum_{j\in I_{\mathcal{N}_i}} \left[\tilde{\boldsymbol{x}}_i(t) - \tilde{\boldsymbol{x}}_j(t)\right]$, agent i needs to receive $\tilde{\boldsymbol{x}}_j(t)$ continuously to monitor the ETC. To overcome this drawback, for $j \in I_{\mathcal{N}_i} \bigcup \{i\}$, $\tilde{\boldsymbol{x}}_j(t)$ is replaced by $\tilde{\boldsymbol{z}}_j(t)$ that can be predicted by (4), and one more ETC is introduced to restrict the difference between $\tilde{\boldsymbol{x}}_j(t)$ and

$\tilde{z}_j(t)$. When the difference between $\tilde{x}_j(t)$ and $\tilde{z}_j(t)$ is large, the related ETC is satisfied so that agent j sends the latest $\tilde{x}_j(t)$ to agent i, and agent i uses this information as the initial value to predict $\tilde{z}_j(t)$ by (4).

The main result is presented as follows. For ease of notation, some index t is omitted.

Theorem 1 Consider the linear MASs with dynamics described by (1) and OETCC law (6) under Assumptions 1 and 2. Let the observer gain in (2) be $K_1 = -\nu_1 P_1 C^{\mathrm{T}}$, where $\nu_1 > 0$, $P_1 > 0$ is the solution of the ARE

$$P_1 A^{\mathrm{T}} + A P_1 - 2\nu_1 P_1 C^{\mathrm{T}} C P_1 = -\nu_2 I_n \tag{9}$$

with $\nu_2 > 0$. Let the control gain in (6) be $K_2 = -\nu_3 B^{\mathrm{T}} P_2$, where $\nu_3 > 0$ and $P_2 > 0$ is the solution of the ARE

$$P_2 A + A^{\mathrm{T}} P_2 - \frac{1}{2} a(G) \nu_3 P_2 B B^{\mathrm{T}} P_2 = -\nu_4 I_n \tag{10}$$

with $\nu_4 > 0$. Let the threshold functions in (8) be $f_{ji}(t) = \sqrt{\frac{\mu_j}{4 g_j}\left\|\hat{\boldsymbol{\delta}}_i\right\|^2 + c_{ji}^2 e^{-2\alpha_{ji}(t-t_0)}}$, $j = 1,2$, where μ_1, $\mu_2 > 0$ and $\mu_1 + \mu_2 \le 1$, c_{ji}, $\alpha_{ji} > 0$, $j = 1,2$, $g_1 = 5\frac{\beta_1}{\breve{r}} + 4\frac{\nu_3}{\nu_4}\frac{\beta_2}{a(G)\breve{r}}$ with $\beta_1 = \left\|L^{\mathrm{T}} R L\right\|$, $\beta_2 = \left\|\left(L^{\mathrm{T}}\right)^2 R L^2 \otimes P_2 B B^{\mathrm{T}} P_2\right\|$ and $\breve{r} = \min\{r_i : 1 \le i \le N\}$, r_i is defined in Definition 1, and $g_2 = 5 + 4\frac{\nu_3}{\nu_4}\frac{\beta_3}{a(G)\breve{r}}$ with $\beta_3 = \left\|L^{\mathrm{T}} R L \otimes P_2 B B^{\mathrm{T}} P_2\right\|$. Then consensus can be achieved exponentially, and no agent will exhibit Zeno behavior.

Proof: For $1 \le i \le N$, define the observer error as

$$\bar{\boldsymbol{e}}_i = \tilde{\boldsymbol{x}}_i - \boldsymbol{x}_i \tag{11}$$

We first prove that there exists γ_1 and θ_1 such that $\sum_{i=1}^{N}\left\|\bar{\boldsymbol{e}}_i\right\|^2 \le \gamma_1 e^{-\theta_1(t-t_0)}$. This means the observer state can approach the real state exponentially. According to (1) and (2), $\dot{\bar{\boldsymbol{e}}}_i = \dot{\tilde{\boldsymbol{x}}}_i - \dot{\boldsymbol{x}}_i = (A + K_1 C)\bar{\boldsymbol{e}}_i$. Let $V_1 = \sum_{i=1}^{N} \bar{\boldsymbol{e}}_i^{\mathrm{T}} P_1^{-1} \bar{\boldsymbol{e}}_i$ be a Lyapunov function. It follows from Rayleigh quotient that

$$\lambda_{\min}\left(P_1^{-1}\right)\sum_{i=1}^{N}\left\|\bar{\boldsymbol{e}}_i\right\|^2 \le V_1 \le \lambda_{\max}\left(P_1^{-1}\right)\sum_{i=1}^{N}\left\|\bar{\boldsymbol{e}}_i\right\|^2 \tag{12}$$

Taking the time derivative of V_1 along the trajectory of $\bar{\boldsymbol{e}}_i$ and using $K_1 = -\nu_1 P_1 C^{\mathrm{T}}$ yields $\dot{V}_1 = 2\sum_{i=1}^{N} \bar{\boldsymbol{e}}_i^{\mathrm{T}} P_1^{-1} \dot{\bar{\boldsymbol{e}}}_i = 2\sum_{i=1}^{N} \bar{\boldsymbol{e}}_i^{\mathrm{T}} P_1^{-1}(A + K_1 C)\bar{\boldsymbol{e}}_i = \sum_{i=1}^{N} \bar{\boldsymbol{e}}_i^{\mathrm{T}}\left(P_1^{-1} A + A^{\mathrm{T}} P_1^{-1} - 2\nu_1 C^{\mathrm{T}} C\right)\bar{\boldsymbol{e}}_i$. Since (A, C) is detectable, $\left(A^{\mathrm{T}}, C^{\mathrm{T}}\right)$ is stabilizable. Let P_1 be the solution of (9), then $\dot{V}_1 = -\nu_2 \sum_{i=1}^{N} \bar{\boldsymbol{e}}_i^{\mathrm{T}}\left(P_1^{-1}\right)^2 \bar{\boldsymbol{e}}_i \le -\nu_2 \lambda_{\min}\left[\left(P_1^{-1}\right)^2\right]\sum_{i=1}^{N}\left\|\bar{\boldsymbol{e}}_i\right\|^2$. Combining this inequality and (12) yields $\dot{V}_1 \le -\theta_1 V_1$, where $\theta_1 = \left\{\lambda_{\min}\left[\left(P_1^{-1}\right)^2\right] / \lambda_{\max}\left(P_1^{-1}\right)\right\}\nu_2$. By comparison principle, $V_1 \le e^{-\theta_1(t-t_0)} V_1(t_0)$. Combining this inequality and (12) yields

$$\sum_{i=1}^{N}\left\|\bar{\boldsymbol{e}}_i\right\|^2 \le \gamma_1 e^{-\theta_1(t-t_0)} \tag{13}$$

where $\gamma_1 = V_1(t_0) / \left[\lambda_{\min}\left(P_1^{-1}\right)\right]$. Next, we prove for $1 \le i \le N$, $\boldsymbol{\delta}_i$ will go to zero exponentially. According to (1), (3) and (6), $\dot{\boldsymbol{\delta}}_i = A\boldsymbol{\delta}_i + BK_2 \sum_{j \in I_{\mathcal{N}_i}}\left(\hat{\boldsymbol{\delta}}_i - \hat{\boldsymbol{\delta}}_j\right)$. Let $\boldsymbol{\delta} = \mathrm{col}(\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_N)$ and $\hat{\boldsymbol{\delta}} = \mathrm{col}\left(\hat{\boldsymbol{\delta}}_1, \ldots, \hat{\boldsymbol{\delta}}_N\right)$. Then

$$\dot{\boldsymbol{\delta}} = (I_N \otimes A)\boldsymbol{\delta} + (L \otimes BK_2)\hat{\boldsymbol{\delta}} \tag{14}$$

Let $\bar{\boldsymbol{\varepsilon}}_i = \sum_{j \in I_{\mathcal{N}_i}}\left(\bar{\boldsymbol{e}}_i - \bar{\boldsymbol{e}}_j\right)$ and $\boldsymbol{\varepsilon}_{1i} = \sum_{j \in I_{\mathcal{N}_i}}\left(\boldsymbol{e}_{1i} - \boldsymbol{e}_{1j}\right)$. According to (3), (5), (7) and (11), one has

$$\hat{\boldsymbol{\delta}}_i = \boldsymbol{\delta}_i + \bar{\boldsymbol{\varepsilon}}_i - \boldsymbol{\varepsilon}_{1i} + \boldsymbol{e}_{2i} \tag{15}$$

Let $\bar{\boldsymbol{\varepsilon}} = \mathrm{col}(\bar{\boldsymbol{\varepsilon}}_1, \ldots, \bar{\boldsymbol{\varepsilon}}_N)$, $\boldsymbol{\varepsilon}_1 = \mathrm{col}(\boldsymbol{\varepsilon}_{11}, \ldots, \boldsymbol{\varepsilon}_{1N})$, and $\boldsymbol{e}_2 = \mathrm{col}(\boldsymbol{e}_{21}, \ldots, \boldsymbol{e}_{2N})$. Then $\hat{\boldsymbol{\delta}} = \boldsymbol{\delta} + \bar{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}_1 + \boldsymbol{e}_2$. It follows from (14) that

$$\dot{\boldsymbol{\delta}} = (I_N \otimes A)\boldsymbol{\delta} + (L \otimes BK_2)(\boldsymbol{\delta} + \bar{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}_1 + \boldsymbol{e}_2) \tag{16}$$

Let $V_2 = \sum_{i=1}^{N} r_i \boldsymbol{\delta}_i^{\mathrm{T}} P_2 \boldsymbol{\delta}_i$ be a Lyapunov function. It follows that

$$\lambda_{\min}(P_2)\sum_{i=1}^{N} r_i\left\|\boldsymbol{\delta}_i\right\|^2 \le V_2 \le \lambda_{\max}(P_2)\sum_{i=1}^{N} r_i\left\|\boldsymbol{\delta}_i\right\|^2 \tag{17}$$

Note that V_2 can also be written as $V_2 = \boldsymbol{\delta}^{\mathrm{T}}(R \otimes P_2)\boldsymbol{\delta}$. Taking the time derivative of V_2 along the trajectory of (16), and using $K_2 = -\nu_3 B^{\mathrm{T}} P_2$ and Definition 1 yields

$$\begin{aligned}
\dot{V}_2 &= 2\boldsymbol{\delta}^{\mathrm{T}}(R \otimes P_2)\dot{\boldsymbol{\delta}} \\
&= 2\boldsymbol{\delta}^{\mathrm{T}}(R \otimes P_2 A)\boldsymbol{\delta} - 2\boldsymbol{\delta}^{\mathrm{T}}\left(RL \otimes \nu_3 P_2 B B^{\mathrm{T}} P_2\right)(\boldsymbol{\delta} + \bar{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}_1 + \boldsymbol{e}_2) \\
&\le \boldsymbol{\delta}^{\mathrm{T}}\left\{R \otimes \left[P_2 A + A^{\mathrm{T}} P_2 - 2\nu_3 a(G) P_2 B B^{\mathrm{T}} P_2\right]\right\}\boldsymbol{\delta} \\
&\quad -2\boldsymbol{\delta}^{\mathrm{T}}\left(RL \otimes \nu_3 P_2 B B^{\mathrm{T}} P_2\right)\bar{\boldsymbol{\varepsilon}} + 2\boldsymbol{\delta}^{\mathrm{T}}\left(RL \otimes \nu_3 P_2 B B^{\mathrm{T}} P_2\right)\boldsymbol{\varepsilon}_1 \\
&\quad -2\boldsymbol{\delta}^{\mathrm{T}}\left(RL \otimes \nu_3 P_2 B B^{\mathrm{T}} P_2\right)\boldsymbol{e}_2
\end{aligned} \tag{18}$$

Let $\sqrt{R} = diag\left(\sqrt{r_1}, \ldots, \sqrt{r_N}\right)$. By Young's Inequality

$$\begin{aligned}
&-2\boldsymbol{\delta}^{\mathrm{T}}\left(RL \otimes \nu_3 P_2 B B^{\mathrm{T}} P_2\right)\bar{\boldsymbol{\varepsilon}} \\
&= -2\nu_3\left[\left(\sqrt{R} \otimes B^{\mathrm{T}} P_2\right)\boldsymbol{\delta}\right]^{\mathrm{T}}\left[\left(\sqrt{R} L \otimes B^{\mathrm{T}} P_2\right)\bar{\boldsymbol{\varepsilon}}\right] \\
&\le \nu_3\left[\varphi_1 \boldsymbol{\delta}^{\mathrm{T}}\left(R \otimes P_2 B B^{\mathrm{T}} P_2\right)\boldsymbol{\delta} + 1/\varphi_1 \bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}\left(L^{\mathrm{T}} R L \otimes P_2 B B^{\mathrm{T}} P_2\right)\bar{\boldsymbol{\varepsilon}}\right]
\end{aligned}$$

where $\varphi_1 = \sigma_1 2a(G)$ with $\sigma_1 \in (0,1)$ to be determined later. Let $\bar{\boldsymbol{e}} = \mathrm{col}(\bar{\boldsymbol{e}}_1, \ldots, \bar{\boldsymbol{e}}_N)$. Then $\bar{\boldsymbol{\varepsilon}} = (L \otimes I_n)\bar{\boldsymbol{e}}$, and so $\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}\left(L^{\mathrm{T}} R L \otimes P_2 B B^{\mathrm{T}} P_2\right)\bar{\boldsymbol{\varepsilon}} = \bar{\boldsymbol{e}}^{\mathrm{T}}\left[\left(L^{\mathrm{T}}\right)^2 R L^2 \otimes P_2 B B^{\mathrm{T}} P_2\right]\bar{\boldsymbol{e}} \le \beta_2\left\|\bar{\boldsymbol{e}}\right\|^2$. Thus,

$$\begin{aligned}
&-2\boldsymbol{\delta}^{\mathrm{T}}\left(RL \otimes \nu_3 P_2 B B^{\mathrm{T}} P_2\right)\bar{\boldsymbol{\varepsilon}} \\
&\le \boldsymbol{\delta}^{\mathrm{T}}\left(R \otimes \sigma_1 2\nu_3 a(G) P_2 B B^{\mathrm{T}} P_2\right)\boldsymbol{\delta} + \frac{\nu_3 \beta_2}{2\sigma_1 a(G)}\left\|\bar{\boldsymbol{e}}\right\|^2
\end{aligned} \tag{19}$$

Let $\boldsymbol{e}_1 = \mathrm{col}(\boldsymbol{e}_{11}, \ldots, \boldsymbol{e}_{1N})$. Following similar procedure yields

$$\begin{aligned}
&2\boldsymbol{\delta}^{\mathrm{T}}\left(RL \otimes \nu_3 P_2 B B^{\mathrm{T}} P_2\right)\boldsymbol{\varepsilon}_1 \\
&\le \boldsymbol{\delta}^{\mathrm{T}}\left(R \otimes \sigma_2 2\nu_3 a(G) P_2 B B^{\mathrm{T}} P_2\right)\boldsymbol{\delta} + \frac{\nu_3 \beta_2}{2\sigma_2 a(G)}\left\|\boldsymbol{e}_1\right\|^2
\end{aligned} \tag{20}$$

$$-2\boldsymbol{\delta}^{\mathrm{T}}\left(RL \otimes \nu_3 P_2 B B^{\mathrm{T}} P_2\right)\boldsymbol{e}_2$$

$$\leq \boldsymbol{\delta}^{\mathrm{T}}\left(R\otimes\sigma_3 2\nu_3 a(G)P_2BB^{\mathrm{T}}P_2\right)\boldsymbol{\delta}+\frac{\nu_3\beta_3}{2\sigma_3 a(G)}\|\boldsymbol{e}_2\|^2 \quad (21)$$

where $\sigma_2,\ \sigma_3\in(0,1)$ are parameters to be determined. Substituting (19)–(21) to (18), one has $\dot{V}_2\leq\boldsymbol{\delta}^{\mathrm{T}}\{R\otimes[P_2A+A^{\mathrm{T}}P_2-2(1-\bar{\sigma}_1)\nu_3 a(G)P_2BB^{\mathrm{T}}P_2]\}\boldsymbol{\delta}+\frac{\nu_3}{2a(G)}\left(\frac{\beta_2}{\sigma_1}\|\bar{\boldsymbol{e}}\|^2+\frac{\beta_2}{\sigma_2}\|\boldsymbol{e}_1\|^2+\frac{\beta_3}{\sigma_3}\|\boldsymbol{e}_2\|^2\right)$, where $\bar{\sigma}_1=\sum_{i=1}^{3}\sigma_i$. Take $\sigma_i=1/4,\ 1\leq i\leq 3$, then $\bar{\sigma}_1=3/4$. Under Assumption 1, according to Lemma 1, one can let P_2 be the solution of (10). It follows that $\dot{V}_2\leq-\nu_4\sum_{i=1}^{N}r_i\|\boldsymbol{\delta}_i\|^2+\frac{2\nu_3}{a(G)}\left(\beta_2\|\boldsymbol{e}_1\|^2+\beta_3\|\boldsymbol{e}_2\|^2\right)+\frac{2\nu_3\beta_2}{a(G)}\|\bar{\boldsymbol{e}}\|^2$ Combining this inequality and (17) yields $\dot{V}_2\leq-\theta_2V_2-\bar{\sigma}_2\nu_4\sum_{i=1}^{N}r_i\|\boldsymbol{\delta}_i\|^2+\frac{2\nu_3}{a(G)}\left(\beta_2\|\boldsymbol{e}_1\|^2+\beta_3\|\boldsymbol{e}_2\|^2\right)+\frac{2\nu_3\beta_2}{a(G)}\|\bar{\boldsymbol{e}}\|^2$, where $\bar{\sigma}_2\in(0,1)$ and $\theta_2=\nu_4(1-\bar{\sigma}_2)/\lambda_{\max}(P_2)$. Choosing $\bar{\sigma}_2=1/2$ and using $\breve{r}=\min\{r_i:1\leq i\leq N\}$, one has

$$\begin{aligned}\dot{V}_2&\leq-\theta_2V_2-\frac{1}{2}\nu_4\sum_{i=1}^{N}r_i\|\boldsymbol{\delta}_i\|^2+\frac{2\nu_3}{a(G)}\left(\beta_2\sum_{i=1}^{N}\|\boldsymbol{e}_{1i}\|^2\right.\\&\quad\left.+\beta_3\sum_{i=1}^{N}\|\boldsymbol{e}_{2i}\|^2\right)\Bigg]+\frac{2\nu_3\beta_2}{a(G)}\|\bar{\boldsymbol{e}}\|^2\\&\leq-\theta_2V_2+\frac{1}{2}\nu_4\sum_{i=1}^{N}r_i\Big[-\|\boldsymbol{\delta}_i\|^2+\frac{4\nu_3}{a(G)\nu_4\breve{r}}\Big(\beta_2\|\boldsymbol{e}_{1i}\|^2\\&\quad+\beta_3\|\boldsymbol{e}_{2i}\|^2\Big)\Big]+\frac{2\nu_3\beta_2}{a(G)}\|\bar{\boldsymbol{e}}\|^2\end{aligned} \quad (22)$$

According to (15),

$$\begin{aligned}&\sum_{i=1}^{N}r_i\left(-\|\boldsymbol{\delta}_i\|^2\right)\\&=\sum_{i=1}^{N}r_i\Big[-\hat{\boldsymbol{\delta}}_i^{\mathrm{T}}\hat{\boldsymbol{\delta}}_i-\bar{\boldsymbol{\varepsilon}}_i^{\mathrm{T}}\bar{\boldsymbol{\varepsilon}}_i-\boldsymbol{\varepsilon}_{1i}^{\mathrm{T}}\boldsymbol{\varepsilon}_{1i}-\boldsymbol{e}_{2i}^{\mathrm{T}}\boldsymbol{e}_{2i}+2\hat{\boldsymbol{\delta}}_i^{\mathrm{T}}\bar{\boldsymbol{\varepsilon}}_i\\&\quad-2\hat{\boldsymbol{\delta}}_i^{\mathrm{T}}\boldsymbol{\varepsilon}_{1i}+2\hat{\boldsymbol{\delta}}_i^{\mathrm{T}}\boldsymbol{e}_{2i}+2\bar{\boldsymbol{\varepsilon}}_i^{\mathrm{T}}\boldsymbol{\varepsilon}_{1i}-2\bar{\boldsymbol{\varepsilon}}_i^{\mathrm{T}}\boldsymbol{e}_{2i}+2\boldsymbol{\varepsilon}_{1i}^{\mathrm{T}}\boldsymbol{e}_{2i}\Big]\end{aligned} \quad (23)$$

where $2\hat{\boldsymbol{\delta}}_i^{\mathrm{T}}\bar{\boldsymbol{\varepsilon}}_i\leq\frac{1}{4}\hat{\boldsymbol{\delta}}_i^{\mathrm{T}}\hat{\boldsymbol{\delta}}_i+4\bar{\boldsymbol{\varepsilon}}_i^{\mathrm{T}}\bar{\boldsymbol{\varepsilon}}_i$, $-2\hat{\boldsymbol{\delta}}_i^{\mathrm{T}}\boldsymbol{\varepsilon}_{1i}\leq\frac{1}{4}\hat{\boldsymbol{\delta}}_i^{\mathrm{T}}\hat{\boldsymbol{\delta}}_i+4\boldsymbol{\varepsilon}_{i1}^{\mathrm{T}}\boldsymbol{\varepsilon}_{1i}$, $2\hat{\boldsymbol{\delta}}_i^{\mathrm{T}}\boldsymbol{e}_{2i}\leq\frac{1}{4}\hat{\boldsymbol{\delta}}_i^{\mathrm{T}}\hat{\boldsymbol{\delta}}_i+4\boldsymbol{e}_{2i}^{\mathrm{T}}\boldsymbol{e}_{2i}$, $2\bar{\boldsymbol{\varepsilon}}_i^{\mathrm{T}}\boldsymbol{\varepsilon}_{1i}\leq\bar{\boldsymbol{\varepsilon}}_i^{\mathrm{T}}\bar{\boldsymbol{\varepsilon}}_i+\boldsymbol{\varepsilon}_{1i}^{\mathrm{T}}\boldsymbol{\varepsilon}_{1i}$, $-2\bar{\boldsymbol{\varepsilon}}_i^{\mathrm{T}}\boldsymbol{e}_{2i}\leq\bar{\boldsymbol{\varepsilon}}_i^{\mathrm{T}}\bar{\boldsymbol{\varepsilon}}_i+\boldsymbol{e}_{2i}^{\mathrm{T}}\boldsymbol{e}_{2i}$, and $2\boldsymbol{\varepsilon}_{1i}^{\mathrm{T}}\boldsymbol{e}_{2i}\leq\boldsymbol{\varepsilon}_{i1}^{\mathrm{T}}\boldsymbol{\varepsilon}_{1i}+\boldsymbol{e}_{2i}^{\mathrm{T}}\boldsymbol{e}_{2i}$. Applying these inequalities to (23) yields $\sum_{i=1}^{N}r_i\left(-\|\boldsymbol{\delta}_i\|^2\right)\leq\sum_{i=1}^{N}r_i\left(-\frac{1}{4}\|\hat{\boldsymbol{\delta}}_i\|^2+5\|\bar{\boldsymbol{\varepsilon}}_i\|^2+5\|\boldsymbol{\varepsilon}_{1i}\|^2+5\|\boldsymbol{e}_{2i}\|^2\right)$. Recall $\bar{\boldsymbol{\varepsilon}}=(L\otimes I_n)\bar{\boldsymbol{e}}$. Then apply $\beta_1=\|L^{\mathrm{T}}RL\|$, one has $\sum_{i=1}^{N}r_i\|\bar{\boldsymbol{\varepsilon}}_i\|^2=\sum_{i=1}^{N}r_i\bar{\boldsymbol{\varepsilon}}_i^{\mathrm{T}}\bar{\boldsymbol{\varepsilon}}_i=\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}(R\otimes I_n)\bar{\boldsymbol{\varepsilon}}=\bar{\boldsymbol{e}}^{\mathrm{T}}(L^{\mathrm{T}}RL\otimes I_n)\bar{\boldsymbol{e}}\leq\beta_1\|\bar{\boldsymbol{e}}\|^2$. For the same reason, applying $\boldsymbol{\varepsilon}_1=(L\otimes I_n)\boldsymbol{e}_1$ and $\breve{r}=\min\{r_i:1\leq i\leq N\}$ yields $\sum_{i=1}^{N}r_i\|\boldsymbol{\varepsilon}_{1i}\|^2\leq\beta_1\|\boldsymbol{e}_1\|^2\leq\sum_{i=1}^{N}r_i\frac{\beta_1}{\breve{r}}\|\boldsymbol{e}_{1i}\|^2$. Therefore,

$$\begin{aligned}&\sum_{i=1}^{N}r_i\left(-\|\boldsymbol{\delta}_i\|^2\right)\\&\leq\sum_{i=1}^{N}r_i\left(-\frac{1}{4}\|\hat{\boldsymbol{\delta}}_i\|^2+5\frac{\beta_1}{\breve{r}}\|\boldsymbol{e}_{1i}\|^2+5\|\boldsymbol{e}_{2i}\|^2\right)+5\beta_1\|\bar{\boldsymbol{e}}\|^2\end{aligned} \quad (24)$$

Applying (13), (24) to (22), and using $g_1=5\frac{\beta_1}{\breve{r}}+4\frac{\nu_3}{\nu_4}\frac{\beta_2}{a(G)\breve{r}}$, $g_2=5+4\frac{\nu_3}{\nu_4}\frac{\beta_3}{a(G)\breve{r}}$ yield $\dot{V}_2\leq-\theta_2V_2+\frac{1}{2}\nu_4\sum_{i=1}^{N}r_i\left(-\frac{1}{4}\|\hat{\boldsymbol{\delta}}_i\|^2+g_1\|\boldsymbol{e}_{1i}\|^2+g_2\|\boldsymbol{e}_{2i}\|^2\right)+\left(\frac{5}{2}\nu_4\beta_1+\frac{2\nu_3\beta_2}{a(G)}\right)\gamma_1e^{-\theta_1(t-t_0)}$. By event-triggering mechanism and (8), before the next triggering time, $\|\boldsymbol{e}_{ji}(t)\|\leq f_{ji}(t)$, $j=1,2$. For $j=1,2$, let $f_{ji}(t)=h_{ji}(t)+c_{ji}e^{-\alpha_{ji}(t-t_0)}$, where $c_{ji},\ \alpha_{ji}>0$, and $h_{ji}(t)$ is to be determined. Then $f_{ji}^2=h_{ji}^2+2c_{ji}e^{-\alpha_{ji}(t-t_0)}h_{ji}+c_{ji}^2e^{-2\alpha_{ji}(t-t_0)}$, and so $\dot{V}_2\leq-\theta_2V_2+\frac{1}{2}\nu_4\sum_{i=1}^{N}\sum_{j=1}^{2}r_i\left(g_jh_{ji}^2+2c_{ji}g_je^{-\alpha_{ji}(t-t_0)}h_{ji}-\frac{1}{4}\mu_j\|\hat{\boldsymbol{\delta}}_i\|^2\right)+\frac{1}{2}\nu_4\sum_{i=1}^{N}\sum_{j=1}^{2}r_ig_jc_{ji}^2e^{-2\alpha_{ji}(t-t_0)}+\left(\frac{5}{2}\nu_4\beta_1+\frac{2\nu_3\beta_2}{a(G)}\right)\gamma_1e^{-\theta_1(t-t_0)}$. Let $h_{ji}=\frac{\sqrt{\Delta_{ji}}-2c_{ji}e^{-\alpha_{ji}(t-t_0)}g_j}{2g_j}$, where $\Delta_{ji}=\mu_jg_j\|\hat{\boldsymbol{\delta}}_i\|^2+4c_{ji}^2g_j^2e^{-2\alpha_{ji}(t-t_0)}$, then $g_jh_{ji}^2+2c_{ji}g_je^{-\alpha_{ji}(t-t_0)}h_{ji}-\frac{1}{4}\mu_j\|\hat{\boldsymbol{\delta}}_i\|^2=0$ and $f_{ji}=\frac{\sqrt{\Delta_{ji}}}{2g_j}=\sqrt{\frac{\mu_j}{4g_j}\|\hat{\boldsymbol{\delta}}_i\|^2+c_{ji}^2e^{-2\alpha_{ji}(t-t_0)}}$. Let $\widehat{c}=\max\{c_{ji}:1\leq i\leq N,\ j=1,2\}$ and $\breve{\alpha}_1=\min\{\alpha_{ji}:1\leq i\leq N,\ j=1,2\}$. Then $\sum_{i=1}^{N}\sum_{j=1}^{2}r_ig_jc_{ji}^2e^{-2\alpha_{ji}(t-t_0)}\leq\sum_{i=1}^{N}\sum_{j=1}^{2}r_ig_j\widehat{c}^2e^{-2\breve{\alpha}_1(t-t_0)}$. Let $\chi=\frac{1}{2}\nu_4\sum_{i=1}^{N}\sum_{j=1}^{2}r_ig_j\widehat{c}^2+\left(\frac{5}{2}\nu_4\beta_1+\frac{2\nu_3\beta_2}{a(G)}\right)\gamma_1$ and $\breve{\alpha}_2=\min\{2\breve{\alpha}_1,\theta_1\}$. Then $\frac{1}{2}\nu_4\sum_{i=1}^{N}\sum_{j=1}^{2}r_ig_jc_{ji}^2e^{-2\alpha_{ji}(t-t_0)}+\left(\frac{5}{2}\nu_4\beta_1+\frac{2\nu_3\beta_2}{a(G)}\right)\gamma_1e^{-\theta_1(t-t_0)}\leq\chi e^{-\breve{\alpha}_2(t-t_0)}$. Therefore, one has

$$\dot{V}_2\leq-\theta_2V_2+\chi e^{-\breve{\alpha}_2(t-t_0)} \quad (25)$$

Note that this inequality holds at the time when $\|\boldsymbol{e}_{ji}\|\leq f_{ji}$ for all $1\leq i\leq N$, $j=1,2$. However, at some time instants, $\|\boldsymbol{e}_{ji}\|$ may be greater than the threshold function for some $1\leq i\leq N$, $j=1,2$. According to (4), $\tilde{z}_i$ is discontinuous at $t_{k_{1i}}^{1i}$, $k_{1i}=1,2,\ldots$. It follows from (5) that $\tilde{\boldsymbol{\delta}}_i$ in (7) is also discontinuous at these times. By (8), this may lead to $\|\boldsymbol{e}_{2i}\|>f_{2i}$ at some $t_{k_{1i}}^{1i}$. On the other hand, according to (6), $\hat{\boldsymbol{\delta}}_i$ is discontinuous at $t_{k_{2i}}^{2i}$, $k_{2i}=1,2,\ldots$. Since f_{1i} depends on $\hat{\boldsymbol{\delta}}_i$, it follows from (8) that $\|\boldsymbol{e}_{1i}\|>f_{1i}$ may occur at some $t_{k_{2i}}^{2i}$. In addition, when agent i receives $\tilde{\boldsymbol{z}}_j$ from agent j, the discontinuity of $\tilde{\boldsymbol{z}}_j$ leads to discontinuity $\tilde{\boldsymbol{\delta}}_i$. This may further lead to $\|\boldsymbol{e}_{2i}\| > f_{2i}$. Since it can be considered that each agent monitors ETCs continuously, $\boldsymbol{e}_{ji}$ becomes zero once $\|\boldsymbol{e}_{ji}\| > f_{ji}$. Hence, $\|\boldsymbol{e}_{ji}\| > f_{ji}$ only happens at some discrete time instants. Let $\tilde{t}_1, \tilde{t}_2, \ldots$ be the times when $\|\boldsymbol{e}_{ji}\| > f_{ji}$, then $\left\|\boldsymbol{e}_{ji}\left(\tilde{t}_k^+\right)\right\| = 0$, $k = 1, 2, \ldots$, where $\left\|\boldsymbol{e}_{ji}\left(\tilde{t}_k^+\right)\right\| = \lim_{t \to \tilde{t}_k^+} \|\boldsymbol{e}_{ji}(t)\|$. According to (25), $\dot{V}_2 \le -\theta_2 V_2 + \chi e^{-\breve{\alpha}_2(t-t_0)}$ on $[t_0, \tilde{t}_1) \cup \bigcup_{k \in \mathbf{Z}^+} (\tilde{t}_k, \tilde{t}_{k+1})$. Let $W(t)$ be the solution of

$$\dot{W} = -\theta_2 W + \chi e^{-\breve{\alpha}_3(t-t_0)} \tag{26}$$

with initial $W(t_0) = V_2(t_0)$, where $\breve{\alpha}_3 = \breve{\alpha}_2$ if $\breve{\alpha}_2 < \theta_2$, and $\breve{\alpha}_3 < \theta_2$ otherwise. By the comparison Lemma, $V_2(t) \le W(t)$ on $[t_0, \tilde{t}_1)$. Since $V_2(t)$ and $W(t)$ are continuous at $t = \tilde{t}_1$, $V_2(\tilde{t}_1) = \lim_{t \to \tilde{t}_1^-} V_2(t) \le \lim_{t \to \tilde{t}_1^-} W(t) = W(\tilde{t}_1)$. Thus, applying the comparison lemma on $[\tilde{t}_1, \tilde{t}_2)$ yields $V_2(t) \le W(t)$ on $[\tilde{t}_1, \tilde{t}_2)$. It follows that $V_2(t) \le W(t)$ on $[t_0, \tilde{t}_2)$. Repeating this procedure gives $V_2(t) \le W(t)$, $t \ge t_0$. Using (26) and $\breve{\alpha}_3 < \theta_2$ yields $W(t) = e^{-\theta_2(t-t_0)} V_2(t_0) + \dfrac{\chi}{\theta_2 - \breve{\alpha}_3} e^{-\breve{\alpha}_3(t-t_0)} \left[1 - e^{-(\theta_2 - \breve{\alpha}_3)(t-t_0)}\right]$. It follows that $V_2(t) \le \left[V_2(t_0) + \dfrac{\chi}{\theta_2 - \breve{\alpha}_3}\right] e^{-\breve{\alpha}_3(t-t_0)}$. Using (17) to this inequality yields $\sum_{i=1}^{N} r_i \|\boldsymbol{\delta}_i\|^2 \le \gamma_2 e^{-\breve{\alpha}_3(t-t_0)}$, where $\gamma_2 = \dfrac{1}{\lambda_{\min}(P_2)} \left[V_2(t_0) + \dfrac{\chi}{\theta_2 - \breve{\alpha}_3}\right]$. This implies that consensus can be achieved exponentially. By following a similar procedure as in [10], it can be proved that no agent will exhibit Zeno behavior. ■

Remark 2 According to the event-triggering mechanism and (8), continuous information transmission and the control input update are avoided, and they are unnecessary to be triggered at the same time. Different from [11,13,14,16], (6) implies that the latter is unnecessary to be triggered once an agent receives information. However, the analysis in the proof of Theorem 1 shows that when one task is triggered, the other one may also be triggered. In addition, when an agent receives information the control task may be triggered as well.

Remark 3 By (5), (7), (8) along with the threshold function f_{2i} determined by Theorem 1, for $1 \le i \le N$, at the latest triggering time $\tilde{t}_{k_{2i}}^{2i}$, agent i can predict the next triggering time, and continuously monitoring this ETC can be avoided.

IV. Simulation Example

In this section, an example is given to illustrate the effectiveness of the proposed OETCC law.

Consider a consensus problem of 6 agents. The dynamic of each agent is described by (1), where A=[0 -1 0; 1 0 0; 0 1 0], B=[1; 0; 0], and C=[0 0 1]. It can be verified that Assumption 1 is satisfied. The network topology is described in Fig. 1, which is strongly connected. The corresponding

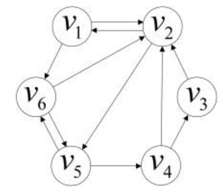


Fig. 1 Network topology

Laplacian matrix is L=[1 -1 0 0 0 0; -1 4 -1 -1 0 -1; 0 0 1 -1 0 0; 0 0 0 1 -1 0; 0 -1 0 0 2 -1; -1 0 0 0 -1 2].

We first compute general algebraic connectivity: Under Assumption 2, by Lemma 2, one can obtain $a(G) = 0.7132$. Next, determine the observer gain K_1 and control gain K_2: Choose $\nu_1 = 0.5$, $\nu_2 = 1$, $\nu_3 = 0.25$ and $\nu_4 = 1$, and solve ARE (9) and (10) for P_1 and P_2, respectively. Then according to Theorem 1, one can obtain $K_1 = [0.2480\ \text{-}0.6622\ \text{-}0.9551]^{\mathrm{T}}$ and $K_2 = [\text{-}2.1202\ \text{-}0.6764\ \text{-}0.8373]$. Finally, we determine the parameters in the threshold functions f_{1i} and f_{2i}: Choose $\mu_1 = 0.5$ and $\mu_2 = 0.5$, and for $1 \le i \le 6$, choose $c_{1i} = 0.085$, $\alpha_{1i} = 0.025$, $c_{2i} = 0.275$ and $\alpha_{2i} = 0.025$.

Assume $t_0 = 0$. Let each agent monitors ETCs at $k\Delta t$ with $\Delta t = 0.01$ and $k = 1, 2, \ldots$. Write $\boldsymbol{x}_i$ as $\boldsymbol{x}_i = \mathrm{col}\left(x_1^i, x_2^i, x_3^i\right)$. The initial state of agent i, $1 \le i \le 6$, is assumed to be $\boldsymbol{x}_i(0) = \mathrm{col}(x_{10}^i, x_{20}^i, x_{30}^i)$, where $x_{10}^i = 2i - 7$, $x_{20}^i = 2i - 6$ and $x_{30}^i = i - 3$. The initial observer state is set as $\tilde{\boldsymbol{x}}(0) = \mathbf{0}$. Write the observer error as $\bar{\boldsymbol{e}} = \mathrm{col}\left(\bar{e}_{11}, \bar{e}_{12}, \bar{e}_{13}, \ldots, \bar{e}_{61}, \bar{e}_{62}, \bar{e}_{63}\right)$. For $1 \le j \le 3$, let $\hat{\boldsymbol{x}}_j = \mathrm{col}\left(x_j^1, \ldots, x_j^6\right)$. Note that consensus is achieved if for any $1 \le j \le 3$, and $1 \le p, q \le 6$, $x_j^p - x_j^q = 0$. The performance of the MAS is simulated during [0, 60]. The histories of the components of $\bar{\boldsymbol{e}}$, $\hat{\boldsymbol{x}}_j$ and control input are presented in Fig. 2, 3 and 4, respectively. We focus on the histories during [0, 25]. Fig. 2 indicates that the observer error almost becomes zero before $t = 20$. Fig. 3 shows that consensus is almost achieved before $t = 25$. In Fig 4, the magnitude of control input decreases when the disagreement among agents becomes smaller. The histories of the triggering times related to information transmission and control input update during [0, 60] are presented in Fig 5a and Fig 5b, respectively. For most of the agents, both tasks are triggered at a high frequency before $t = 10$, and the frequency decreases significantly after this time. Note that in Fig 2 and 3, the observer error and the disagreement among agents are also reduced significantly after $t = 10$. The numbers of triggering times during [0, 60] are listed in Table 1, where N_{trig}^{1i} and N_{trig}^{2i} denote the number of triggering times of agent i related to information transmission and control input update, respectively. Since each ETC is monitored 6,000 times during this period, both tasks are not triggered frequently.

V. Conclusion

In this paper, an observer-based event-triggered consensus problem has been considered for general linear MASs under strongly connected network. Two ETCs have been applied to

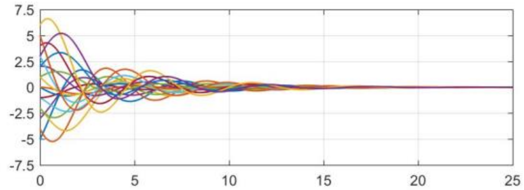


Fig. 2 Histories of observer error

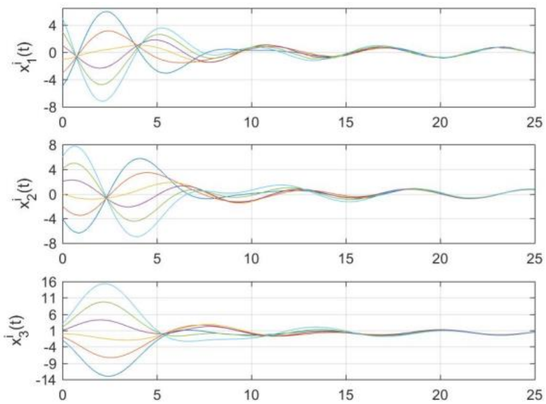


Fig. 3 Histories of state

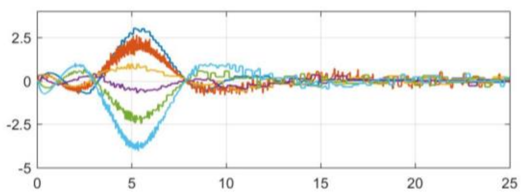


Fig. 4 Histories of control input

the consensus algorithm so that both continuous control input update and information transmission are not required. In addition, the triggering times for both tasks are unnecessarily to be the same. It has been verified that under the proposed algorithm, consensus can be achieved exponentially.

REFERENCES

[1] R. M. Murray, “Recent research in cooperative control of multivehicle systems,” *J. Dynam. Syst.,Meas., Control*, vol. 129, no. 5, pp. 571–583, 2007.

[2] Y. Cao, W. Yu, W. Ren, and G. Chen, “An overview of recent progress in the study of distributed multi-agent coordination,” *IEEE Trans. Ind. Informat*., vol. 9, no. 1, pp. 427–438, Feb. 2013.

[3] W. Ren, R. W. Beard, and E. M. Atkins, “Information consensus in multivehicle cooperative control,” *IEEE Control Syst. Mag.*, vol. 27, no. 2, pp. 71–82, Feb. 2007.

[4] W. Ren and R. W. Beard, *Distributed Consensus in Multi-vehicle Cooperative Control.* London, U.K.: Springer-Verlag, pp. 3, 28-32, 2008.

[5] W. Zhu, Z.-P. Jiang, and G. Feng, “Event-based consensus of multi-agent systems with general linear models,” *Automatica*, vol. 50, no. 2, pp. 552–558, 2014.

[6] L. Ding, Q.-L. Han, X. Ge, and X.-M. Zhang, “An overview of recent advances in event-triggered consensus of multiagent systems,” *IEEE Trans. Cybern.*, vol. 48, no. 4, pp. 1110–1123, Apr. 2018.

[7] X. Ge, Q.-L. Han, L. Ding, Y.-L. Wang and X.-M. Zhang, “Dynamic event-triggered distributed coordination control and its applications: A survey of trends and techniques”, *IEEE Trans. Syst. Man Cybern. Syst.*, vol. 50, no. 9, pp. 3112-3125, Sep. 2020.

[8] H. Zhang, G. Feng, H. Yan, and Q. Chen, “Observer-based output feedback event-triggered control for consensus of multi-agent systems,” *IEEE Trans. Ind. Electron*., vol. 61, no. 9, pp. 4885–4894, Sep. 2014.

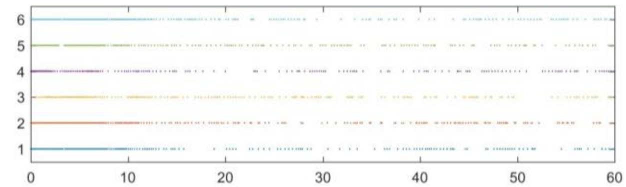
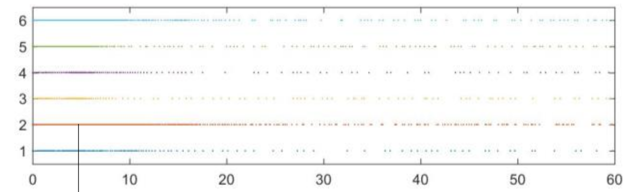


Fig. 5a Histories of time for information transmitting



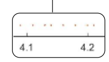


Fig. 5b Histories of time for control update

Table 1: Number of triggering times

i	1	2	3	4	5	6
N_{trig}^{1i}	344	286	189	184	298	406
N_{trig}^{2i}	131	560	117	126	199	402

[9] D. Yang, W. Ren, X. Liu, and W. Chen, “Decentralized event- triggered consensus for linear multi-agent systems under general directed graphs,” *Automatica*, vol. 69, pp. 242–249, Jul. 2016.

[10] Y.-Y. Qian, L. Liu, and G. Feng, “Distributed event-triggered adaptive control for consensus of linear multi-agent systems with external disturbances,” *IEEE Trans. Cybern*. vol. 50,no. 5,pp. 2197–2208, 2020.

[11] J. Lu and L. Li, “Leader-following consensus of nonlinear multi-agent systems using observer-based event-triggered control,” *Chinese Intelligent Systems Conference*, vol.593, pp.419, 2020.

[12] J. Sun, H. Liu, C. Du, X. Liu, Z. Chen, and P. Lu, “Distributed output feedback consensus control of multiple lur’e systems based on event-triggered mechanism,” 59th *IEEE Conference on Decision and Control* (*CDC*), pp. 2374–2379. Dec. 2020

[13] D. Ye et. al., “Observer-based event-triggered consensus of multi-agent systems with input saturation,” 33rd *Youth Academic Annual Conference of Chinese Association of Automation*, pp. 805–809. May 2018.

[14] W. Liu, C. Yang, Y. Sun and J. Qin, “Observer-based event-triggered control for consensus of multi-agent systems with time delay. In *2016 Chinese Control and Decision Conference (CCDC)*, *Yinchuan, China*. May 2016.

[15] G. Zhao, C. Hua, and X. Guan, “Reset observer-based zeno-free dynamic event-triggered control approach to consensus of multiagent systems with disturbances,” *IEEE Trans. Cybern*., vol. 52, no. 4, pp. 2329–2339, Apr. 2022.

[16] Q. Wang, W. Diao, L. Zino, X. Peng, and W. Zhong, “Observer-based secure event-triggered bipartite consensus control of linear multiagent systems subject to denial-of-service attacks,” *IEEE Trans. Circuits Syst. II: Exp. Briefs*, vol. 69, no. 12, pp. 5054–5058, Dec. 2022.

[17] T. Wang, J. Feng, J.-A. Wang, J. Zhang, and X. Wen, ‘‘Observer-based distributed event-triggered secure consensus of multi-agent system with DoS attack,’’ *IEEE Access*, vol. 11, pp. 34736–34745, Mar. 2023.

[18] W. Yu, G. Chen, M. Cao, and J. Kurths, “Second-order consensus for multiagent systems with directed topologies and nonlinear dynamics,” *IEEE Trans. Syst., Man, Cybern. B, Cybern*., vol. 40, no. 3, pp. 881–891, Jun. 2009.

[19] K. Vladimir, “A contribution to matrix quadratic equations,” *IEEE Trans. Autom. Control*, vol. 17, no. 3, pp. 344–347, Jan. 1972.

[20] W. Hu, C. Yang, T. Huang, and W. Gui, “A distributed dynamic event-triggered control approach to consensus of linear multiagent systems with directed networks,” *IEEE Trans. Cybern.*, vol. 50, no. 2, pp. 869–874, Feb. 2020.