POTENTIAL THEORY AND QUASISYMMETRIC MAPS BETWEEN COMPACT AHLFORS REGULAR METRIC MEASURE SPACES VIA BESOV FUNCTIONS: PRELIMINARY

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Abstract. We study Besov capacities in a compact Ahlfors regular metric measure space by means of hyperbolic fillings of the space. This approach is applicable even if the space does not support any Poincaré inequalities. As an application of the Besov capacity estimates we show that if a homeomorphism between two Ahlfors regular metric measure spaces preserves, under some additional assumptions, certain Besov classes, then the homeomorphism is necessarily a quasisymmetric map.

Dedicated to Professor Vladimir Maz'ya
for his ground-breaking contributions to potential theory.

1. Introduction

The study of potential theory is usually directed towards Sobolev spaces of functions on Riemannian manifolds, and more recently, Newton-Sobolev spaces of functions on complete doubling metric measure spaces supporting a Poincaré inequality. These Sobolev-type spaces of functions are associated with a gradient structure, with weak (distributional) derivatives in the Riemannian case and minimal weak upper gradients in the metric measure space case. Such gradients have the property that if f is a function in the Sobolev-type class and f is constant on an open subset of the metric space, then the norm of the weak derivative (in the Riemannian setting) and the minimal weak upper gradient (in the metric setting) are zero almost everywhere in that open set. In the language of Dirichlet forms and Markov processes, this property is called *strongly local* property of the energy associated with the Sobolev classes. Tools used to study potential theory related

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to Sobolev spaces include locality together with the doubling property of the measure and the Poincaré inequality. The books [1, 41], and especially [41, Sections 10.4.1, 13.1.2], have an excellent sampling of results in potential theory in the Euclidean setting.

However, there are many compact doubling metric measure spaces that do not have sufficient number of non-constant rectifiable curves in order to support a Poincaré inequality. Examples of such spaces include the standard (thin) Sierpiński carpet and the Sierpiński gasket, Rickman rug, as well as the von Koch snowflake curve [14, Proposition 4.5]. In such metric spaces a more suitable replacement for Sobolev spaces might be Besov spaces. Unfortunately (or fortunately, depending on the perspective) the energy associated with the Besov spaces are not local. In this paper we use the tools of hyperbolic filling and the associated lifting of measures as developed in [9] to study potential theory associated with Besov function spaces on compact metric measure spaces. We establish Besov capacitary estimates for various configurations of pairs of subsets of the metric space under the assumption that the measure is Ahlfors Q-regular; see Subsection 2.2 for the definition. A discussion regarding recent developments connecting Besov spaces of functions in Euclidean spaces and Sobolev spaces can also be found in [41, Sections 10.3, 10.5].

The results in this note are motivated by the study in [36, 37]. The results of [36] use a characterization of Besov spaces via scaled Hajłasz-type gradients from [28]. Our motivation is two-fold; first, to provide an alternate proof of the potential theoretic results in [36, Lemma 3.3 and Lemma 3.4] using the new perspective of hyperbolic filling that enable us to avoid the scaled Hajłasz-type method and directly handle the Besov norm as in (2.2), and second, to extend these capacitary estimates to spaces where the measure is Ahlfors regular but may not support any Poincaré inequality. As an application of the capacitary estimates, we extend at the end of this note the discussion relating Besov space preservation property and qusiconformal maps, given in [36] for Ahlfors regular spaces supporting a Poincaré inequality, to a more general class of Ahlfors regular compact metric measure spaces that may not support a Poincaré inequality but are linearly locally path connected. However, since we do not assume that the metric spaces support a Poincaré inequality, we assume a stronger condition on the homeomorphism, namely that it is quasisymmetric. We show that homeomorphisms between two Ahlfors regular (but not necessarily of the same dimension) compact metric measure spaces that are linearly locally connected, have the property that if they preserve certain Besov classes under composition, with control over the Besov norms, then the mapping is necessarily quasisymmetric. This is the content of Theorem 4.3.

At present we do not know whether every quasisymmetric map between two compact Ahlfors regular linearly locally path connected metric measure spaces with the same regularity dimension preserves certain Besov spaces, though this is known to be true under some additional conditions, including a Poincaré inequality, see [36].

2. Preliminaries

This section is devoted to describing the background notions used in this note, with the setting considered here delineated in Subsection 2.5.

2.1. **Newton-Sobolev spaces.** Let $1 \le p < \infty$. When Ω is an n-dimensional Euclidean (or a Riemannian) domain and $f \in L^p(\Omega)$, we say that f is in the Sobolev class $W^{1,p}(\Omega)$ if f has a weak derivative $\nabla f \in L^p(\Omega : \mathbb{R}^n)$. Note that if f is of class $C^1(\Omega)$, then for each compact rectifiable curve γ in Ω we have

$$|f(y) - f(x)| \le \int_\gamma g\, ds, \tag{2.1}$$

with $g = |\nabla f|$, where x and y denote the two end points of γ. However, if f is not of class $C^1(\Omega)$, a weaker analog of this holds, see [48], namely, there is a family Γ_f of compact rectifiable curves in Ω such that whenever γ is a compact rectifiable curve in Ω that *does not* belong to Γ_f, then (2.1) holds. Moreover, the family Γ_f is of p-modulus zero, that is, there is a non-negative Borel measurable function $\rho \in L^p(\Omega)$ such that $\int_\gamma \rho\, ds = \infty$ for each $\gamma \in \Gamma_f$.

This is the starting point for the theory of Newton-Sobolev functions on metric measure spaces where weak derivatives do not make sense. Let Y be a metric space equipped with a Radon measure μ, and let f be a function on Y. We say that a non-negative Borel measurable function g is a *p-weak upper gradient* of f if there is a family Γ of non-constant compact rectifiable curves in Y (possibly empty) such that there is a non-negative Borel measurable function $\rho \in L^p(Y)$ satisfying $\int_\gamma \rho\, ds = \infty$ for each $\gamma \in \Gamma_f$, and for each non-constant compact rectifiable curve γ in Y with $\gamma \notin \Gamma$, the pair f and g satisfies (2.1). We set $\widetilde{N}^{1,p}(Y)$ to be the collection of all functions f such that $\int_Y |f|^p\, d\mu < \infty$ and f has a p-weak upper gradient $g \in L^p(Y)$. Note that we do not ask that $f \in L^p(Y)$ as elements of $L^p(Y)$ are equivalence classes of functions that agree outside measure-null sets, but the existence of a weak upper gradient from $L^p(Y)$ may fail if we modify f on a set of measure zero. The Newton-Sobolev space $N^{1,p}(Y)$ is set to be the collection $\widetilde{N}^{1,p}(Y)/\sim$ of equivalence classes, where two functions $f_1, f_2 \in \widetilde{N}^{1,p}(Y)$ are equivalent, $f_1 \sim f_2$, if $\|f_1 - f_2\|_{N^{1,p}(Y)} = 0$. Here

$$\|f\|^p_{N^{1,p}(Y)} := \int_X |f|^p\, d\mu + \inf_g \int_X g^p\, d\mu,$$

with the infimum taken over all p-weak upper gradients g of f. For $1 \le p < \infty$, for each $f \in N^{1,p}(Y)$ there is a minimal p-weak upper gradient $g_f \in L^p(Y)$, that has the smallest L^p-norm of all p-weak upper gradients of f. We refer the interested reader to [32] and [5] for more on Newton-Sobolev spaces.

2.2. **Poincaré inequalities and doubling measures.** For $1 \le p < \infty$, we say that the metric measure space (Y, d, μ) supports a p-Poincaré inequality if there is a constant $C > 0$ such that

$$\fint_B |f - f_B| \, d\mu \le C \, \mathrm{rad}(B) \left(\fint_B g^p \, d\mu \right)^{1/p}$$

whenever B is a ball in Y, f is integrable in Y, and g is a p-weak upper gradient of f in Y. Here we use the notation

$$u_B = \fint_B u \, d\mu = \mu(B)^{-1} \int_B u \, d\mu$$

for the mean-value integral over B. The validity of a p-Poincaré inequality immediately implies that Y is connected. If μ is in addition doubling, and Y is locally compact, then Y is quasiconvex, that is, for each $x, z \in Y$ there is a curve γ in Y with end points x, y and length $\ell(\gamma) \le C \, d(x, z)$ with C independent of x, z. This quasiconvexity property was first proved in [21] in the context of complete metric measure spaces, but see [23, Theorem 3.1] for the corresponding proof for locally compact metric measure spaces.

Recall that a Radon measure μ is doubling if there is a constant $C_d \ge 1$ such that whenever $y \in Y$ and $r > 0$, we have

$$0 < \mu(B(x, 2r)) \le C_d \, \mu(B(x, r)) < \infty.$$

We say that μ is Ahlfors Q-regular for some $Q > 0$ if there is a constant $C > 0$ such that whenever $x \in Y$ and $0 < r < 2 \, \mathrm{diam}(Y)$,

$$\frac{r^Q}{C} \le \mu(B(x, r)) \le C \, r^Q.$$

Note that Ahlfors Q-regular measures are comparable to the Q-dimensional Hausdorff measure $\mathcal{H}^Q$ (see Subsection 2.6).

See [32] and [5] for more details on analysis on doubling metric measure spaces supporting Poincaré inequalities.

2.3. **Besov spaces.** The primary focus of our note is on the Besov spaces. These were initially formulated by O. V. Besov in order to study Sobolev extension and restriction theorems for somewhat smooth Euclidean domains, see for instance [3, 41, 43]. Let $1 \le p < \infty$ and $0 < \theta < 1$. A function $u \in L^p(Y)$ is said to be in the Besov space $B^\theta_{p,p}(Y)$ if its *Besov energy*

$$\|u\|^p_{B^\theta_{p,p}(Y)} := \int_X \int_X \frac{|u(x) - u(z)|^p}{d(x, z)^{\theta p} \mu(B(x, d(x, z)))} \, d\mu(z) \, d\mu(x) \tag{2.2}$$

is finite. Unlike the Newton-Sobolev functions, an arbitrary perturbation of a Besov function on a set of measure zero gives an equivalent Besov function.

If (Y, μ) is a doubling metric measure space supporting a p-Poincaré inequality, then $B^\theta_{p,p}(Y)$ is obtained via a real interpolation of $N^{1,p}(X)$ with $L^p(X)$, see [2] for the Euclidean setting and [27] for more on this in the metric setting. However, in this note we are not interested in the interpolation properties connecting Newton-Sobolev spaces to Besov spaces, but in the trace properties. Jonsson and Wallin studied the trace relationship between Sobolev classes on Euclidean spaces and Ahlfors regular compact subsets of the Euclidean spaces, see [33, 34].

If Y is a non-complete, locally compact metric measure space, we set $\partial Y := \overline{Y} \setminus Y$. Here $\overline{Y}$ is the metric completion of Y, obtained by considering equivalence classes of Cauchy sequences in Y; hence ∂Y consists of equivalence classes of Cauchy sequences in Y that do not converge in Y. Observe that if Y is locally compact, then necessarily Y is an open subset of $\overline{Y}$. Suppose that ν is a Borel measure on ∂Y and that ∂Y is proper, that is, all closed and bounded sets in ∂Y are compact. We say that $B^\theta_{p,p}(\partial Y)$ is the trace space of $N^{1,p}(Y)$ if there is a bounded operator

$$T : N^{1,p}(Y) \to B^\theta_{p,p}(\partial Y)$$

and a bounded linear extension operator

$$E : B^\theta_{p,p}(\partial Y) \to N^{1,p}(Y)$$

such that

(1) $Tu = \overline{u}|_{\partial Y}$ whenever u is a Lipschitz function on Y; here $\overline{u}$ is the unique continuous extension of u to ∂Y,
(2) $T \circ E$ is the identity map on $B^\theta_{p,p}(\partial Y)$.

The subject of traces of Sobolev functions in Euclidean domains dates back to the work of Besov, Gagliardo, Jonsson, and Wallin [3, 4, 25, 45, 33, 34]. The canonical textbook of Maz'ya [41] contains a nice discussion on traces in Chapter 11, while [42, 43] contain results linking traces of Sobolev spaces to Besov-type spaces in certain Euclidean domains. See also the text [44] for a general treatment of boundary values of Sobolev functions on "bad" Euclidean domains; these are merely a few papers on the topic from a vast literature on traces, as we cannot hope to list all papers on the topic of traces here. We refer interested readers to [27, 39, 40, 8] for more on Besov spaces as traces of Newton-Sobolev spaces in the metric setting, and to [47] for connections to other expressions of Besov spaces.

2.4. **Hyperbolic fillings and uniformization.** Throughout this note, the triple (Z, d_Z, ν) denotes a compact metric space (Z, d_Z) together with a doubling measure ν. For brevity, we say in the rest of the paper that (Z, d_Z, ν) is a compact doubling metric measure space, and without loss of generality we may assume that $0 < \operatorname{diam}(Z) < 1$. For $\alpha > 1$ and $\tau > 1$, we construct a Gromov hyperbolic space X from Z as a graph. For each non-negative integer n we set A_n to be a maximal α^{-n} -separated subset of Z such that $A_n \subset A_{n+1}$ for each $n \in \mathbb{N}_0$. The vertex set of the graph X is the set $\bigcup_{n\in\mathbb{N}}\{n\} \times A_n$. Two vertices $v = (n, x)$ and $w = (m, y)$ are neighbors if $v \neq w$, $|n - m| \leq 1$, and $B(x, \tau\alpha^{-n}) \cap B(y, \tau\alpha^{-m})$ is non-empty if $n = m$ and $B(x, \alpha^{-n}) \cap B(y, \alpha^{-m})$ is non-empty if $n \neq m$. We consider each pair of neighbors to be connected with an edge that is an interval of unit length. There is only one vertex p_0 corresponding to the level $n = 0$, that is, $\{0\} \times A_0 = \{p_0\}$.

Variants of hyperbolic fillings have been constructed in [19, 15, 13, 11, 9, 12], but the one described above is from [9] where it was also shown that X is a Gromov hyperbolic space and that with $\varepsilon = \log(\alpha)$, the uniformization X_ε of X as in [10] yields a uniform space such that Z is biLipschtiz equivalent to ∂X_ε. Here, the uniformization is accomplished via the modified metric d_ε given by

$$d_\varepsilon(x, y) = \inf_\gamma \int_\gamma e^{-\varepsilon d(\gamma(t),p_0)}\, ds(t)$$

with the infimum over all rectifiable curves γ in X with end points x and y. Since the notions of doubling property of the measure ν on Z and quasisymmetric maps on Z are invariant under compositions with biLipschitz maps (though the associated constants might change—tracking of which is not of concern in this note), and as Besov energies are quasi-preserved by biLipschitz change in the metric, from now on we consider Z to be equipped with the extension of the metric d_ε as described above. Recall that X_ε is a uniform space if there is a constant $A \ge 1$ such that for each pair of points $x, y \in X_\varepsilon$ there is a curve γ in X_ε with end points x and y such that $\ell_\varepsilon(\gamma) \le A\, d_\varepsilon(x, y)$ and for each $z \in \gamma$,

$$\min\{\ell(\gamma_{x,z}), \ell(\gamma_{z,y})\} \le A\, \delta_\varepsilon(z).$$

Here, $\gamma_{x,z}$ and $\gamma_{z,y}$ denote subcurves of γ with end points x, z and z, y respectively, and

$$\delta_\varepsilon(z) := \operatorname{dist}_{d_\varepsilon}(z, \partial X_\varepsilon) := \inf_{w\in\partial X_\varepsilon} d_\varepsilon(z, w).$$

When Z is equipped with a measure ν, we can lift up this measure to a measure μ_+ on X by setting balls of radius 1 centered at vertices $v = (n, x)$ to have measure equal to $\nu(B(x, \alpha^{-n}))$. For each $\beta > 0$ we can uniformize this measure to obtain a measure μ_β on X_ε by setting $d\mu_\beta(v) = e^{-\beta d(v,p_0)}\, d\mu_+(v)$. This gives us a one-parameter family of lifted measures on X_ε, first constructed in [9].

Recently Clark Butler extended the construction of hyperbolic fillings from compact doubling metric measure spaces to complete doubling metric spaces that are unbounded, see [16, 17, 18]. It was shown in [16] that trace and extension theorems similar to the ones in [9] hold even for the unbounded setting. In this note we focus on compact spaces Z, but point out that with minimal effort the results here can be extended to unbounded complete doubling metric measure spaces as well by using the tools of [16, 17].

2.5. **Uniformized measure μ_β and connection to ν.** For $\beta > 0$ let μ_β be the uniformized lift of ν to X_ε as constructed in [9] and described in Subsection 2.4. From the results in [9] we know that the metric measure space $(X_\varepsilon, d_\varepsilon, \mu_\beta)$ is a doubling metric measure space supporting the best possible Poincaré inequality, namely the 1-Poincaré inequality.

There is a relationship between ν and μ_β; whenever $z \in Z$ and $0 < r \leq \operatorname{diam}(Z)$, we have, by [9, Theorem 10.3] and the doubling property of μ_β, that

$$\mu_\beta(B(z,r)) \simeq r^{\beta/\varepsilon}\nu(B(z,r)). \tag{2.3}$$

We treat ν as a measure on $\overline{X}$, obtained by extending ν from $Z = \partial X_\varepsilon$ to X by zero.

Proposition 2.1 ([9, Theorem 1.1 and Theorem 10.2])**.** *With the choice of α and ε as above, the uniformized space X_ε, equipped with the metric d_ε and the measure μ_β, is doubling and supports a 1-Poincaré inequality. Moreover, for the choice $\theta = 1 - \beta/(\varepsilon p)$, the Besov space $B^\theta_{p,p}(Z)$ is the trace space of $N^{1,p}(X_\varepsilon)$.*

The above proposition is a key tool for us in this note. We will exploit this identification of $B^\theta_{p,p}(Z)$ with the trace of $N^{1,p}(X_\varepsilon)$ frequently. The fine properties of functions in $N^{1,p}(X_\varepsilon, \mu_\beta)$ follow from the results of [32, 5] thanks to the doubling property of μ_β and the support of the 1-Poincaré inequality. While $N^{1,p}(X_\varepsilon)$ also depends on the choice of β in defining the measure on X_ε, we will suppress this dependance in our notation as we fix θ and p, and hence β in this note.

Also the following construction of the extension Eu of $u \in B^\theta_{p,p}(Z)$ to X_ε will be important. In [9, Theorem 12.1], the extension is constructed by first defining $Eu((n,z))$, $z \in A_n$, by

$$Eu((n,z)) = \fint_{B(z,\alpha^{-n})} u\, d\nu, \tag{2.4}$$

and then extending Eu linearly (with respect to the uniformized metric d_ε) to the edges that make up the graph X_ε. It is shown in [9, Theorem 12.1] that $Eu \in N^{1,p}(X_\varepsilon, \mu_\beta)$ when $\theta = 1 - \beta/(p\varepsilon)$, with $TEu = u$ ν-a.e. in Z, and moreover

$$\int_{\overline{X}_\epsilon} |Eu|^p d\mu_\beta \lesssim \int_Z |u|^p\, d\nu$$

and

$$\int_{\overline{X}_\epsilon} g_{Eu}^p d\mu_\beta \lesssim \|u\|^p_{B^\theta_{p,p}(Z)}. \tag{2.5}$$

2.6. **Capacities and Hausdorff content.** As mentioned in Subsection 2.1, functions in $N^{1,p}(Y)$ cannot be arbitrarily modified on general sets of measure zero. Therefore, to study fine properties of such functions, we need a finer notion than null measure, and this is one purpose of the notion of capacity.

Let (Y, d, μ) be a metric measure space with μ a Radon measure. Given a set $E \subset Y$ and $1 \leq p < \infty$, we set the Newton-Sobolev p-capacity of E to be the number

$$\operatorname{Cap}_{N^{1,p}(Y)}(E) := \inf_u \|u\|^p_{N^{1,p}(Y)},$$

where the infimum is over all functions $u \in \widetilde{N}^{1,p}(Y)$ satisfying $u \geq 1$ on E. It follows from the results of [46, 32] that Newton-Sobolev functions can be arbitrarily perturbed only on sets of capacity zero.

On the other hand, Besov functions can be perturbed arbitrarily on sets of measure zero. For this reason the Besov capacity of E is set to be

$$\mathrm{Cap}_{B^\theta_{p,p}(Y)}(E) := \inf_u \int_Y |u|^p \, d\mu + \|u\|^p_{B^\theta_{p,p}(Y)}$$

with infimum over all $u \in B^\theta_{p,p}(Y)$ such that $u \geq 1$ on a *neighborhood of* E.

Related to the above two capacities there is a notion of *relative capacity* of a condenser $(E, F; Y)$. If $E, F \subset Y$, then

$$\mathrm{cap}_{N^{1,p}(Y)}(E, F) := \inf_u \int_Y g_u^p \, d\mu$$

where the infimum is over all $u \in N^{1,p}(Y)$ satisfying $u \geq 1$ in E and $u \leq 0$ in F, and g_u is the minimal p-weak upper gradient of u as described at the end of Subsection 2.1. Similarly,

$$\mathrm{cap}_{B^\theta_{p,p}(Y)}(E, F) := \inf_u \|u\|^p_{B^\theta_{p,p}(Y)},$$

where the infimum is over all $u \in B^\theta_{p,p}(Y)$ satisfying $u \geq 1$ in a neighborhood of E and $u \leq 0$ in a neighborhood of F. Note that if $F \subset F_1$ and $E \subset E_1$, then

$$\mathrm{cap}_{B^\theta_{p,p}(Y)}(E, F) \leq \mathrm{cap}_{B^\theta_{p,p}(Y)}(E_1, F_1).$$

Returning to our setting, it was shown in [9] that $N^{1,p}(X_\varepsilon) = N^{1,p}(\overline{X}_\varepsilon)$ and that when $E \subset Z$,

$$\mathrm{Cap}_{N^{1,p}(\overline{X}_\varepsilon)}(E) \simeq \mathrm{Cap}_{B^\theta_{p,p}(Z)}(E).$$

It was shown there moreover that if $\mathrm{Cap}_{N^{1,p}(\overline{X}_\varepsilon)}(E) = 0$ then necessarily $\nu(E) = 0$. Note here that the statement holds regardless of the value of $\beta > 0$ that generated the measure μ_β on X_ε, provided that β is chosen so that $\theta = 1 - \frac{\beta}{\varepsilon p}$.

Sobolev capacity is associated with Hausdorff content, as seen for example in [41, Section 1.1.18] in the Euclidean setting and [31, Theorem 2.26] in Euclidean domains equipped with admissible weights. Given a set $E \subset Y$, $0 < \alpha < \infty$, and $0 < \tau \leq \infty$, the α-dimensional Hausdorff content of E at scale τ is the number

$$\mathcal{H}^\alpha_\tau(E) := \inf_{(B_i)_{i \in I \subset \mathbb{N}}} \sum_{i \in I} \mathrm{diam}(B_i)^\alpha,$$

where the infimum is over all countable covers $(B_i)_{i \in I \subset \mathbb{N}}$ of the set E, by balls B_i, such that for each $i \in I$ we have $\mathrm{diam}(B_i) < \tau$. The α-dimensional Hausdorff measure of E is then given by

$$\mathcal{H}^\alpha(E) := \lim_{\tau \to 0^+} \mathcal{H}^\alpha_\tau(E).$$

Hausdorff measures are a natural metric tool to use in an Ahlfors Q-regular space Y to analyze Sobolev capacities. For instance, if we assume in addition that Y is complete, unbounded and supports a p-Poincaré inequality, with $1 < p \le Q$, then it follows from the results in [22] that if $\mathrm{Cap}_{N^{1,p}(Y)}(E) = 0$, then $\mathcal{H}^s_\infty(E) = 0$ for every $s > Q - p$, and conversely, if $\mathcal{H}^{Q-p}_\infty(E) = 0$ (or even $\mathcal{H}^{Q-p}_\infty(E) < \infty$, when $1 < p < Q$), then $\mathrm{Cap}_{N^{1,p}(Y)}(E) = 0$. We refer the interested reader to [24, Section 4.7.2] for the Euclidean setting. Additional information can be found in [41, pages 28, 760]. In more general doubling metric measure spaces co-dimensional Haudorff measures are more useful in controlling Sobolev capacites, see for instance [26, Proposition 3.11, Section 8], and relative capacities, see e.g. [38, Propositions 4.1 and 4.3].

3. Besov capacitary estimates

In studying quasisymmetric mappings between metric spaces, there are two types of configurations that play a key role. The first type of configuration is that of an annulus $B(x,R) \setminus B(x,r)$, and the associated condenser is the triplet $(\overline{B}(x,r), X \setminus B(x,R), X)$ for $0 < r < R \le \mathrm{diam}(X)/2$. Here by $\overline{B}(x,r)$ we mean the closed ball $\{w \in X : d_X(x,w) \le r\}$. The second type of configuration arises from considering two compact continua E, F contained in a ball $B(x,R)$ with $\min\{\mathrm{diam}(E), \mathrm{diam}(F)\} \ge R/C$, and the associated condenser is (E, F, X). We consider these two configurations in the two subsections of this section.

We assume throughout this section that the measure ν on Z is Ahlfors Q-regular for some $Q > 0$. The results of this section are modeled after [36, Lemma 2.4 and Lemma 2.3] and [30].

3.1. Relative Besov capacitary estimates for annular rings.

In this subsection we consider annular rings in Z, namely sets of the form $E = \overline{B}(x_0, r)$ and $F = Z \setminus B(x_0, R)$ for $x_0 \in Z$ and $0 < r < R$. An analog of Case 2 of the following theorem for relative Newton-Sobolev capacity $\mathrm{cap}_{N^{1,Q}(Z)}$ can be found in [30, Lemma 3.14].

Theorem 3.1. *Assume that Z is a compact metric space and that ν is an Ahlfors Q-regular measure on Z, for some $Q > 0$. Let $1 < p < \infty$ and $0 < \theta < 1$, and suppose that $0 < r < R/2$ and $x_0 \in Z$. Then*

$$\mathrm{cap}_{B^\theta_{p,p}(Z)}(\overline{B}(x_0,r), Z \setminus B(x_0,R)) \le \xi(R)\,\Xi(r)\,\Psi(R/r),$$

where

(1) *if $p\theta > Q$, then $\xi(R) \simeq R^{Q-\theta p}$, $\Xi(r) = 1$ and $\Psi(R/r) = 1$.*
(2) *if $p\theta = Q$, then $\xi(R) \simeq \Xi(r) \simeq 1$ and $\Psi(R/r) = \log(R/r)^{1-p}$.*
(3) *if $p\theta < Q$, then $\xi(R) = 1$, $\Xi(r) \simeq r^{Q-\theta p}$, and $\Psi(R/r) = 1$.*

Therefore, when $p\theta = Q$ or when $p\theta < Q$ and $R \le 1$, we have

$$\mathrm{cap}_{B^\theta_{p,p}(Z)}(\overline{B}(x_0,r), Z \setminus B(x_0,R)) \lesssim \log(R/r)^{1-p}.$$

Proof. We give to this result an alternate proof from the one in [36], by utilizing the hyperbolic filling. We fix θ with $0<\theta<1$ and choose $\beta>0$ such that $\theta=1-\beta/(\varepsilon p)$, and consider the space $(X_\varepsilon, d_\varepsilon, \mu_\beta)$ as described in Subsection 2.4. Then $B^\theta_{p,p}(Z)$ is the trace space of $N^{1,p}(X_\varepsilon,\mu_\beta)$, as explained in Proposition 2.1.

In Case 1, that is, when $p\theta>Q$, we consider the test function u given by

$$u(x)=\left(1-\frac{2\operatorname{dist}(x,B(x_0,R/2))}{R}\right)_+.$$

Then $u=1$ on $B(x_0,r)\subset B(x_0,R/2)$, $u=0$ on $X_\varepsilon\setminus B(x_0,R)$, and u is $2/R$-Lipschitz continuous. Therefore

$$\int_{X_\varepsilon} g_u^p\,d\mu_\beta \lesssim \mu_\beta(B(x_0,R)\setminus B(x_0,R/2))\left(\frac{2}{R}\right)^p.$$

Note that the balls considered in the above estimate are all centered at points in $Z=\partial X_\varepsilon$, and so we are in the realm of (2.3). Using the facts that $\mu_\beta(B(x_0,R)\setminus B(x_0,R/2))\lesssim R^{Q+\beta/\varepsilon}$ and $\theta=1-\beta/(p\varepsilon)$, we obtain

$$\int_{X_\varepsilon} g_u^p\,d\mu_\beta \lesssim R^{Q-p\theta}.$$

In Case 3, that is, when $p\theta<Q$, we instead consider the function u given by

$$u(x)=\left(1-\frac{\operatorname{dist}(x,B(x_0,r))}{r}\right)_+$$

and note that $u=1$ on $B(x_0,r)$ and $u=0$ on $X_\varepsilon\setminus B(x_0,2r)$. Thus we see that

$$\int_{X_\varepsilon} g_u^p\,d\mu_\beta \lesssim r^{Q-\theta p}.$$

In both of these cases, by [9, Theorem 11.1(11.2)], with u also denoting the trace of u to Z (and as u is Lipschitz continuous, this is a pointwise identification), we have the desired upper bound for the Besov capacity as well.

Finally, in Case 2 ($p\theta=Q$) we define the function u on $\overline{X}_\varepsilon$ by

$$u(x):=\min\left\{\left(\frac{\log(R/d(x,x_0))}{\log(R/r)}\right)_+,1\right\}.$$

Then by the chain rule for upper gradients (see [32, (6.3.19)] or [5, Theorem 2.16]) and by the fact that 1 is an upper gradient of the distance function, we see that

$$g_u(x)\le \frac{1}{\log(R/r)}\,\frac{1}{d(x,x_0)}\,\chi_{B(x_0,R)\setminus B(x_0,r)}(x). \tag{3.1}$$

Again by [9, Theorem 11.1(11.2)], we have

$$\|u\|^p_{B^\theta_{p,p}(Z)}\lesssim \int_{X_\varepsilon} g_u^p\,d\mu_\beta.$$

Hence it suffices to obtain integral estimates for g_u. Note that $B(x_0,R)\setminus B(x_0,r)\subset\bigcup_{j=0}^{n_R}B(x_0,2^{j+1}r)\setminus B(x_0,2^jr)$ where n_R is the smallest positive integer such that $2^{n_R}r\ge R$. We have $n_R\simeq\log(R/r)$. Then by the bound on g_u in (3.1) and by (2.3),

$$\begin{aligned}\int_{X_\varepsilon} g_u^p\,d\mu_\beta &\le \log(R/r)^{-p}\sum_{j=0}^{n_R}\int_{B(x_0,2^{j+1}r)\setminus B(x_0,2^jr)}\frac{1}{d(x,x_0)^p}\,d\mu_\beta(x)\\ &\lesssim \log(R/r)^{-p}\sum_{j=0}^{n_R}\frac{\mu_\beta(B(x_0,2^jr))}{(2^jr)^p}\\ &\simeq \log(R/r)^{-p}\sum_{j=0}^{n_R}\frac{\nu(B(x_0,2^jr))}{(2^jr)^{p-\beta/\varepsilon}}\\ &\simeq \log(R/r)^{-p}\sum_{j=0}^{n_R}\frac{1}{(2^jr)^{p-Q-\beta/\varepsilon}} \quad = \quad \log(R/r)^{-p}\,n_R,\end{aligned}$$

where the last equality followed from the identity $\theta=1-\beta/(\varepsilon p)$ together with $p\theta=Q$, and the penultimate estimate came from the assumption that ν is Ahlfors Q-regular. Since $n_R\simeq\log(R/r)$, we have

$$\int_{X_\varepsilon} g_u^p\,d\mu_\beta\lesssim\log(R/r)^{-p}n_R\simeq\log(R/r)^{1-p},$$

verifying the claim in Case 2. □

Remark 3.2. Note that in Case 2, the capacity of the annulus tends to zero as $R/r\to\infty$. In Case 3, the capacity of the annulus tends to zero as $r\to 0$. This perspective plays a key role in the study of homeomorphisms that induce Besov space morphisms, and their relationship to local quasisymmetry and metric quasiconformality, see Section 4 below.

We record also the following converse of Theorem 3.1. These bounds are not needed in the later results, but they in particular show that the estimates in Theorem 3.1 are often optimal. We refer the interested reader to [32, Lemma 9.3.6] and [6, Sections 6 and 7] for the analogous estimates for Sobolev capacity in doubling metric measure spaces supporting a p-Poincaré inequality.

Theorem 3.3. *Assume that Z is a compact metric space and that ν is an Ahlfors Q-regular measure on Z, for some $Q>0$. Let $1<p<\infty$ and $0<\theta<1$, and suppose that $x_0\in Z$ and $0<r<R<\operatorname{diam}(Z)/4C_0$ for suitably large constant $C_0>2$. Then*

$$\operatorname{cap}_{B^\theta_{p,p}(Z)}(\overline{B}(x_0,r),Z\setminus B(x_0,R))\ge\xi(R)\,\Xi(r)\,\Psi(R/r),$$

where ξ,Ξ, and Ψ are as in Theorem 3.1, and in the case $p\theta=Q$ we assume in addition that $r\le R/2$.

Proof. Fix $0 < \theta < 1$ and choose $\beta > 0$ so that $\theta = 1 - \beta/(p\varepsilon)$. Let $u \in B^\theta_{p,p}(Z)$ be such that $u = 1$ in a neighborhood of $\overline{B}(x_0, r)$ and $u = 0$ in a neighborhood of $Z \setminus B(x_0, R)$.

Let Eu be the extension of u to the uniformization X_ε of the hyperbolic filling X of Z as explained in Section 2.5. Then $Eu \in N^{1,p}(\overline{X}_\epsilon, \mu_\beta)$, and by (2.5) we have

$$\int_{\overline{X}_\epsilon} g_{Eu}^p d\mu_\beta \lesssim \|u\|^p_{B^\theta_{p,p}(Z)}.$$

From the way the extension Eu is defined in [9, Theorem 12.1], see (2.4), it follows that $Eu = 1$ on $\overline{B}_\varepsilon(x_0, r/(\tau\alpha))$ and $Eu \le 1/2$ on $\overline{X_\varepsilon} \setminus B_\varepsilon(x_0, \tau\alpha R)$; here the parameters $\tau > 1$ and $\alpha > 1$ are as in Section 2.4 with τ is chosen to be sufficiently large, (see below) and the subscript ε in B_ε refers to the fact that these balls are with respect to $\overline{X_\varepsilon}$.

More precisely, from [9, Theorem 12.1], for a vertex $x \in \overline{X_\varepsilon} \setminus B_\varepsilon(x_0, \tau\alpha R)$ we know that

$$Eu(x) = \frac{1}{\nu(B_\varepsilon(z_x, r_x))} \int_{B_\varepsilon(z_x, r_x)} u \, d\nu,$$

where z_x is the nearest point to x in $Z = \partial X_\varepsilon$ and $r_x = \log(\alpha)\, d_\varepsilon(x, z_x)$. If $B_\varepsilon(z_x, r_x) \cap B_\varepsilon(x_0, R) = \emptyset$, then clearly $Eu(x) = 0 \le 1/2$. On the other hand, if the balls $B_\varepsilon(z_x, r_x)$ and $B_\varepsilon(x_0, R)$ intersect, then as $d(x, x_0) \ge \tau\alpha R$, we see that $r_x \ge (\tau\alpha - 1)R/2$. Since u is supported in $B(x_0, R)$, it follows from the Q-regularity of ν that

$$Eu(x) \le \frac{\nu(B_\varepsilon(x_0, R))}{\nu(B_\varepsilon(z_x, r_x))} \lesssim \frac{R^Q}{(\tau\alpha - 1)^Q\,(R/2)^Q} \le \frac{1}{2},$$

where the last inequality holds provided that $\tau > 1$ is chosen to be sufficiently large. We also ensure that $C_0 \ge 2\tau\alpha$.

It follows that the function $v = 2(Eu - \frac{1}{2})_+$ satisfies $v = 1$ on $\overline{B}_\varepsilon(x_0, r/(\tau\alpha))$ and $v = 0$ on on $\overline{X_\varepsilon} \setminus B_\varepsilon(x_0, \tau\alpha R)$. Moreover,

$$\int_{\overline{X}_\epsilon} g_v^p d\mu_\beta \le 2^p \int_{\overline{X}_\epsilon} g_{Eu}^p d\mu_\beta \lesssim \|u\|^p_{B^\theta_{p,p}(Z)}.$$

Write $r' = r/(\tau\alpha)$ and $R' = \tau\alpha R$. By our assumption on C_0, we know that $R' < \operatorname{diam}(Z)/4$. Then v is a test function for the capacity

$$\operatorname{cap}_{N^{1,p}(\overline{X_\varepsilon})}(B_\varepsilon(x_0, r'), \overline{X_\varepsilon} \setminus B_\varepsilon(x_0, R'))$$

and so

$$\int_{\overline{X_\varepsilon}} g_v^p \, d\mu_\beta \ge \operatorname{cap}_{B^\theta_{p,p}(Z)}(B_\varepsilon(x_0, r'), \overline{X_\varepsilon} \setminus B_\varepsilon(x_0, R')).$$

Recall from [9, Lemma 10.6] that as Z is Ahlfors Q-regular, we have a lower mass bound exponent for μ_β on $\overline{X_\varepsilon}$ given by $Q_\beta := \max\{1, Q + \frac{\beta}{\varepsilon}\}$. Also from Proposition 2.1 we know that $(\overline{X_\varepsilon}, d_\varepsilon, \mu_\beta)$ is doubling and supports a 1-Poincaré inequality; hence we are in a position to apply [32, Lemma 9.3.6] together with [35], to obtain that

$$\mathrm{cap}_{N^{1,p}(\overline{X_\varepsilon})}(B_\varepsilon(x_0,r'),\overline{X_\varepsilon}\setminus B_\varepsilon(x_0,R'))\geq C(R',r'),$$

where

(1) if $1<p<Q_\beta$, then $Q_\beta=Q+\beta/\varepsilon>1$ and thus

$$\begin{aligned}C(R',r')&\simeq\frac{\mu_\beta(B_\varepsilon(x_0,r'))^{1-\frac{p}{Q_\beta}}\,\mu_\beta(B_\varepsilon(x_0,R'))^{\frac{p}{Q_\beta}}}{(R')^p}\\&\simeq(r')^{(Q+\frac{\beta}{\varepsilon})(1-\frac{p}{Q_\beta})}\,(R')^{\frac{p}{Q_\beta}(Q+\frac{\beta}{\varepsilon})-p}\simeq r^{Q+\frac{\beta}{\varepsilon}-p}.\end{aligned}$$

(2) if $1<p=Q_\beta$, then again $Q_\beta=Q+\beta/\varepsilon$ and so

$$\begin{aligned}C(R',r')&\simeq\frac{\mu_\beta(B_\varepsilon(x_0,R'))}{(R')^{Q_\beta}}\left(\log\left(\frac{C\,\mu_\beta(B_\varepsilon(x_0,R'))}{\mu_\beta(B_\varepsilon(x_0,r'))}\right)\right)^{1-Q_\beta}\\&\simeq(R')^{Q+\frac{\beta}{\varepsilon}-Q_\beta}\left(\log\left(C\tfrac{R'}{r'}\right)\right)^{1-Q_\beta}\simeq\left(\log\tfrac{R}{r}\right)^{1-p}.\end{aligned}$$

In the last step we need the assumption that $R/r\geq 2$.

(3) if $p>Q_\beta$, then

$$C(R',r')\simeq\frac{\mu_\beta(B_\varepsilon(x_0,R'))}{(R')^p}\simeq(R')^{Q+\frac{\beta}{\varepsilon}-p}\simeq R^{Q+\frac{\beta}{\varepsilon}-p}.$$

In the above cases we also used (2.3) and the fact that ν is Ahlfors Q-regular. Note that the balls $B_\varepsilon(x_0,R)$ and $B_\varepsilon(x_0,r)$ are balls centered at the point $x_0\in Z$. (Alternatively, similar estimates as above can be obtained by applying the capacity estimates given in [6, Sections 6 and 7].)

From the above estimates and (2.5) we conclude that

$$\|u\|^p_{B^\theta_{p,p}(Z)}\geq C(R,r),$$

where $C(R,r)$ has the desired forms as in the statement of Theorem 3.1 since $\theta p=p-\beta/\varepsilon$. The claim follows by taking the infimum over all such capacity test functions u. □

3.2. **Loewner-type bounds for Besov capacity.** Next we obtain an estimate for the Besov capacity associated to two compact continua E,F, given in terms of their Hausdorff contents. Recall the definition of Hausdorff content from Subsection 2.6.

Theorem 3.4. *Assume that Z is a compact metric space and that ν is an Ahlfors Q-regular measure on Z, for some $Q>0$. Let $x_0\in Z$, $R>0$ and $0<s<Q$. Suppose also that E,F are two disjoint compact sets such that $E,F\subset B(x_0,R)$. Then for each $p>\max\{1,Q-s\}$ and for each θ satisfying $\frac{Q-s}{p}<\theta<1$, we have*

$$\mathrm{cap}_{B^\theta_{p,p}(Z)}(E,F)\gtrsim\frac{\mathcal{H}^s_\infty(E)\wedge\mathcal{H}^s_\infty(F)}{R^{s-Q+\theta p}}.$$

The proof of the theorem, given next, is modeled after the corresponding result for Sobolev capacities found in [30].

Proof. Fix $p > 1$ such that $\frac{Q-s}{p} < 1$, and let $\theta > 0$ be such that $\frac{Q-s}{p} < \theta < 1$. Choose $\beta > 0$ in the hyperbolic filling construction given in Subsection 2.4 so that $\theta = 1 - \beta/(p\varepsilon)$. Then, because of the condition that $Q - s < \theta p$, necessarily $p + s - Q - \beta/\varepsilon > 0$. Let $u \in B^{\theta}_{p,p}(Z)$ such that $u = 1$ in a neighborhood of E and $u = 0$ in a neighborhood of F, and let Eu be the extension of u to the uniformization X_ε of the hyperbolic filling X of Z as explained in Section 2.5. Then $Eu \in N^{1,p}(\overline{X}_\epsilon, \mu_\beta)$, and by (2.5) we have

$$\int_{\overline{X}_\epsilon} g^p_{Eu} d\mu_\beta \lesssim \|u\|^p_{B^{\theta}_{p,p}(Z)}.$$

We now proceed essentially as in [30, Proof of Theorem 5.9]. We cannot apply the theorem from [30] directly because we do not have knowledge of the requisite lower mass bound property for μ_β on $\overline{X}_\varepsilon$. Nevertheless, their proof does apply here because we only need to apply the lower mass bound property on balls centered at points in $\partial X_\varepsilon = Z$, and for such balls we have the needed lower mass bound estimate from (2.3). For the convenience of the reader, we provide the complete proof here. See also [36] for a similar adaptation of [30].

We first show that

$$\frac{\mathcal{H}^s_\infty(E) \wedge \mathcal{H}^s_\infty(F)}{R^{s+p-(Q+\beta/\varepsilon)}} \lesssim \int_{B(x_0,4R)} g^p_{Eu} d\mu_\beta.$$

If there exist points $x \in E$ and $y \in F$ such that neither $|Eu(x) - (Eu)_{B_\varepsilon(x,R)}|$ nor $|Eu(y) - (Eu)_{B_\varepsilon(y,3R)}|$ exceeds $1/3$, then

$$1 \le |Eu(x) - Eu(y)| \le \frac{1}{3} + |Eu_{B_\varepsilon(x,R)} - Eu_{B_\varepsilon(y,3R)}| + \frac{1}{3},$$

and so from the 1-Poincaré inequality on $\overline{X}_\varepsilon$ together with Hölder's inequality, the above inequality implies that

$$\frac{1}{3} \le C \fint_{B_\varepsilon(y,3R)} |Eu - Eu_{B_\varepsilon(y,3R)}| d\mu_\beta \le CR\Big(\fint_{B_\varepsilon(y,3R)} g^p_{Eu} d\mu_\beta\Big)^{1/p}.$$

Hence from (2.3) we get

$$\frac{\nu(B_\varepsilon(y,R))}{R^{p-\beta/\varepsilon}} \lesssim \frac{\mu_\beta(B_\varepsilon(y,R))}{CR^p} \le \int_{B_\varepsilon(y,3R)} g^p_{Eu} d\mu_\beta. \tag{3.2}$$

Then, from the Ahlfors Q-regularity of ν, together with the estimates $\mathcal{H}^s_\infty(E) \lesssim R^s$ and $\mathcal{H}^s_\infty(F) \lesssim R^s$ and the identity $\theta p = p - \beta/\varepsilon$, it follows that

$$\frac{\mathcal{H}^s_\infty(E) \wedge \mathcal{H}^s_\infty(F)}{R^{s-Q+\theta p}} \lesssim \frac{R^s}{R^{s-Q+p-\beta/\varepsilon}} \lesssim \int_{B_\varepsilon(x_0,4R)} g^p_{Eu} d\mu_\beta$$

as desired.

Now suppose that the above assumption fails. Then either for each $x \in E$ we have $1/3 \leq |Eu(x) - Eu_{B_\varepsilon(x,R)}|$, or else for each $y \in F$ we have $1/3 \leq |Eu(y) - Eu_{B_\varepsilon(y,3R)}|$. Suppose now that for each $x \in E$ we have

$$\frac{1}{3} \leq |Eu(x) - Eu_{B_\varepsilon(x,R)}|.$$

Set $\tau := \frac{s+p-(Q+\beta/\varepsilon)}{p}$; note that $\tau > 0$. Then since x is a Lebesgue point of Eu, we have

$$\begin{aligned} C(\tau)\sum_{j=0}^{\infty} 2^{-i\tau} &\lesssim \sum_{j=0}^{\infty} |Eu_{B_j(x)} - Eu_{B_{j+1}(x)}| \lesssim \sum_{j=0}^{\infty} 2^{-j}R\left(\fint_{B_j(x)} g_{Eu}^p d\mu_\beta\right)^{1/p} \\ &\lesssim \sum_{j=0}^{\infty} (2^{-j}R)^{1-(Q+\beta/\varepsilon)/p}\left(\int_{B_j(x)} g_{Eu}^p d\mu_\beta\right)^{1/p}, \end{aligned}$$

where $B_j(x) := B(x, 2^{-j}R)$. Here we also used the fact that for balls $B_\varepsilon(x,\rho)$ with $x \in Z$ and $0 < \rho \leq \mathrm{diam}(Z)$ we have $\mu_\beta(B_\varepsilon(x,\rho) \simeq \rho^{Q+\beta/\varepsilon}$. Hence there exists $j_x \in \mathbb{N} \cup \{0\}$ such that

$$2^{-j_x\tau p} \lesssim (2^{-j_x}R)^{p-(Q+\beta/\varepsilon)} \int_{B_{j_x}(x)} g_{Eu}^p d\mu_\beta. \tag{3.3}$$

The above inequality, together with our choice of τ, gives

$$2^{-j_x s} \lesssim R^{p-(Q+\beta/\varepsilon)} \int_{B_{j_x}(x)} g_{Eu}^p d\mu_\beta.$$

By the 5-covering Lemma [29] there exists a countable pairwise disjoint family of balls $\{B(x_k, 2^{-j_{x_k}}R)\}_{k\in\mathbb{N}}$ such that

$$E \subseteq \bigcup_k B(x_k, 2^{-j_{x_k}}5R)$$

and

$$2^{-j_{x_k}s} \lesssim R^{p-(Q+\beta/\varepsilon)} \int_{B_{j_{x_k}}(x_k)} g_{Eu}^p d\mu_\beta. \tag{3.4}$$

Hence, by (3.4) and the pairwise disjointness property, we have

$$\mathcal{H}_\infty^s(E) \leq C\sum_{k=1}^{\infty}(2^{-j_{x_k}}R))^s \lesssim R^{s+p-(Q+\beta/\varepsilon)} \int_{B(x_0,4R)} g_{Eu}^p d\mu_\beta.$$

A similar argument shows that if for each $y \in F$ we have

$$\frac{1}{3} \leq |Eu(y) - Eu_{B(y,3R)}|,$$

then

$$\mathcal{H}_\infty^s(F) \lesssim R^{s+p-(Q+\beta/\varepsilon)} \int_{B(x_0,4R)} g_{Eu}^p d\mu_\beta.$$

Combining the two possibilities and applying the identity $\theta p = p - \beta/\varepsilon$, we see that

$$\frac{\mathcal{H}^s_\infty(E) \wedge \mathcal{H}^s_\infty(F)}{R^{s-Q+\theta p}} \lesssim \int_{B(x_0,4R)} g^p_{Eu} \, d\mu_\beta \tag{3.5}$$

as desired.

The proof is completed by first recalling from (2.5) that $\int_{X_\varepsilon} g^p_{Eu}\, d\mu_\beta \lesssim \|u\|_{B^\theta_{p,p}(Z)}$, and then taking the infimum over all capacity test functions u in the above two cases. ☐

If E and F are connected sets and $s = 1$, then $\mathcal{H}^s_\infty(E) \simeq \operatorname{diam}(E)$ and $\mathcal{H}^s_\infty(F) \simeq \operatorname{diam}(E)$. If they are not necessarily connected but $\nu(E) > 0$ and $\nu(F) > 0$, then for each $0 < s < Q$ we have that $\mathcal{H}^s_\infty(E) \geq \nu(E)\, R^{s-Q}$ and $\mathcal{H}^s_\infty(F) \geq \nu(F)\, R^{s-Q}$.

4. $B^\theta_{p,p}$-morphisms and quasisymmetric maps

From [20, Theorem 1.1] it is known that there is a correspondence between quasisymmetric mappings between two Ahlfors regular compact metric spaces and certain classes of weights on the hyperbolic fillings of either of the metric spaces. The perspective of [36, 37] is different in that unlike [20], they consider impact of quasisymmetric mappings on the relevant Besov classes of functions on the metric spaces themselves.

In this section, we extend the theory from [36] to Ahlfors regular spaces which do not support any Poincaré inequalities, see Theorem 4.3 below. We begin by recalling the definitions of quasisymmetry.

Definition 4.1. Let (Z, d_Z) and (W, d_W) be metric spaces.

(a) A homeomorphism $\varphi : Z \to W$ is a *quasisymmetric map* if there is a continuous monotone increasing function $\eta : [0, \infty) \to [0, \infty)$ with $\eta(0) = 0$ and $\eta(t) > 0$ when $t > 0$, such that for each triple of points $x, y, z \in Z$ we have

$$\frac{d_W(\varphi(x), \varphi(z))}{d_W(\varphi(x), \varphi(y))} \leq \eta\left(\frac{d_Z(x, z)}{d_Z(x, y)}\right).$$

(b) A homeomorphism $\varphi : Z \to W$ is *weakly quasisymmetric* if there is some $H > 0$ such that for each triple of points $x, y, z \in Z$ we have

$$\frac{d_W(\varphi(x), \varphi(z))}{d_W(\varphi(x), \varphi(y))} \leq H \quad \text{whenever} \quad \frac{d_Z(x, z)}{d_Z(x, y)} \leq 1.$$

In addition, φ is *uniformly locally weakly quasisymmetric* if there is some $\rho > 0$ such that the restriction of φ to balls in Z of radii at most ρ are weakly quasisymmetric with the same constant H.

Remark 4.2. Recall that we assume (Z, d_Z) is compact and is equipped with a doubling measure ν. From [29, Theorem 10.19] we know that if both Z and W are connected doubling metric spaces, then weak quasisymmetry is equivalent to quasisymmetry. Moreover, the proof given there works even if φ is only known to be uniformly locally weakly quasisymmetric; this is seen as follows.

From uniformly locally weak quasisymmetry, together with the connectendess property, we know that the homeomorphism is promoted to uniformly local quasisymmetry; that is, there is some $r_0 > 0$ such that whenever $x, y, z \in Z$ are three distinct points such that $\operatorname{diam}\{x, y, z\} \leq r_0$, we have

$$\frac{d_W(\varphi(x), \varphi(y))}{d_W(\varphi(x), \varphi(z))} \leq \eta\left(\frac{d_Z(x,y)}{d_Z(x,z)}\right).$$

Since Z (and hence also W) is compact and φ^{-1} is continuous, it follows that there is some $\kappa > 0$ such that for all $x, y \in Z$ we have that $d_W(\varphi(x), \varphi(y)) \geq \kappa$ whenever $d_Z(x, y) \geq r_0/4$. If $x, y, z \in Z$ are three distinct points such that $d_Z(x, z) \leq r_0/2$ and $d_Z(x, y) > r_0/2$, then by the connectedness property of Z we can find $w_0 \in Z$ such that $d_Z(x, y_0) = r_0/2$, and so by the monotonicity of the quasisymmetry gauge η,

$$\begin{aligned}
\frac{d_W(\varphi(x), \varphi(y))}{d_W(\varphi(x), \varphi(z))} &= \frac{d_W(\varphi(x), \varphi(y))}{d_W(\varphi(x), \varphi(y_0))}\,\frac{d_W(\varphi(x), \varphi(y_0))}{d_W(\varphi(x), \varphi(z))} \\
&\leq \frac{\operatorname{diam}(W)}{\kappa}\,\eta\left(\frac{d_Z(x,y_0)}{d_Z(x,z)}\right) \leq \frac{\operatorname{diam}(W)}{\kappa}\,\eta\left(\frac{d_Z(x,y)}{d_Z(x,z)}\right).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\frac{d_W(\varphi(x), \varphi(z))}{d_W(\varphi(x), \varphi(y))} &= \frac{d_W(\varphi(x), \varphi(z))}{d_W(\varphi(x), \varphi(y_0))}\,\frac{d_W(\varphi(x), \varphi(y_0))}{d_W(\varphi(x), \varphi(y))} \\
&\leq \frac{\operatorname{diam}(W)}{\kappa}\,\eta\left(\frac{d_Z(x,z)}{d_Z(x,y_0)}\right) \\
&\leq \frac{\operatorname{diam}(W)}{\kappa}\,\eta\left(\frac{d_Z(x,z)}{d_Z(x,y)}\,\frac{d_Z(x,y)}{d_Z(x,y_0)}\right) \\
&\leq \frac{\operatorname{diam}(W)}{\kappa}\,\eta\left(\frac{2\operatorname{diam}(Z)}{r_0}\,\frac{d_Z(x,z)}{d_Z(x,y)}\right).
\end{aligned}$$

Finally, if $d_Z(x, y) \geq r_0/2$ and $d_Z(x, z) \geq r_0/2$, then by the monotonicity of η again,

$$\frac{d_W(\varphi(x), \varphi(z))}{d_W(\varphi(x), \varphi(y))} \leq \frac{\operatorname{diam}(W)}{\kappa} \leq \frac{\operatorname{diam}(W)}{\kappa}\,\frac{\eta\left(\frac{d_Z(x,z)}{d_Z(x,y)}\right)}{\eta\left(\frac{r_0}{2\operatorname{diam}(Z)}\right)}.$$

It follows that φ is globally quasisymmetric as well, with quasisymmetry gauge $\widehat{\eta}$ given by

$$\widehat{\eta}(t) = \max\left\{\eta(t), \tfrac{\operatorname{diam}(W)}{\kappa}\eta(t), \tfrac{\operatorname{diam}(W)}{\kappa}\eta(\tfrac{2\operatorname{diam}(Z)}{r_0}t), \tfrac{\operatorname{diam}(W)}{\kappa\,\eta\left(\frac{r_0}{2\operatorname{diam}(Z)}\right)}\,\eta(t)\right\}.$$

Theorem 4.3. *Assume that (Z, d_Z, ν_Z) and (W, d_W, ν_W) are compact metric measure spaces, with ν_Z Ahlfors Q_Z-regular and ν_W Ahlfors Q_W-regular for some $Q_Z, Q_W > 0$. Suppose that a homeomorphism $\varphi : Z \to W$ induces* *a bounded linear operator $\varphi_\# : B^{\theta_W}_{p,p}(W) \to B^{\theta_Z}_{p,p}(Z)$, that is, there is a constant $C_\varphi > 0$ such that whenever $f \in B^{\theta_W}_{p,p}(W)$ we have that $f \circ \varphi \in B^{\theta_Z}_{p,p}(Z)$ with*

$$\|f \circ \varphi\|_{B^{\theta_Z}_{p,p}(Z)} \le C \, \|f\|_{B^{\theta_W}_{p,p}(W)},$$

where $\theta_Z = Q_Z/p$ and $\theta_W \le Q_W/p$. Suppose in addition that W is linearly locally path-connected, that is, there is a constant $C_L > 2$ such that given $w \in W$, $0 < r < \operatorname{diam}(W)$, and $w_1, w_2 \in B(w, r) \setminus B(w, r/2)$ there is a path γ in $B(w, C_L r) \setminus B(w, r/C_L)$ with end points w_1, w_2. Then φ is a quasisymmetric map.

Here we should be careful in stating what $f \circ \varphi$ is, as it may be the case that φ pulls back a set of ν_W-measure zero to a set of positive ν_Z-measure. Instead, we here require that we only consider the Besov quasicontinuous f in looking at $f \circ \varphi$. Such quasicontinuous representatives of functions in $B^{\theta_W}_{p,p}(W)$ (which are, strictly speaking, equivalence classes of functions) are guaranteed to exit, thanks to the results in [9].

The argument below is very similar to that of [30] where both the metric measure spaces are assumed to be connected and uniformly locally Ahlfors Q-regular, and to support a uniformly local Q-Poincaré inequality.

Proof of Theorem 4.3. Since W is connected, therefore Z is also connected, and so by Remark 4.2, it suffices to show that φ is uniformly locally weakly quasisymmetric.

Let φ be as in the statement of the theorem. Since φ is continuous on the compact space Z, it is uniformly continuous. Hence we can find $R_0 > 0$ such that whenever $x_1, x_2 \in Z$ with $d(x_1, x_2) \le R_0$ we have that $d(\varphi(x_1), \varphi(x_2)) < \operatorname{diam}(W)/10C_L^4$. By choosing R_0 small, we can also ensure that $R_0 \le \operatorname{diam}(Z)/10$.

We fix $x \in Z$ and consider $y, z \in Z$ such that $r := d(x, y) \le d(x, z) =: R < R_0$. We wish to find an upper bound for $d(\varphi(x), \varphi(y))/d(\varphi(x), \varphi(z))$. Set $L = d(\varphi(x), \varphi(y))$ and $l = d(\varphi(x), \varphi(z))$. If $L \le 4C_L^2 l$, then we have a bound in terms of $4C_L^2$. So suppose that $L > 4C_L^2 l$. By the choice of R_0 we can find $w \in W$ such that

$$d(\varphi(x), w) > \operatorname{diam}(W)/3 > \operatorname{diam}(W)/10C_L^4 > C_L^2 L.$$

Then $d(\varphi^{-1}(w), x) > R_0$.

Let $E, F \subset W$ such that E is a curve in $W \setminus B(\varphi(x), 2C_L l)$ with end points $w, \varphi(y)$ and F is a curve in $B(\varphi(x), C_L l)$ with end points $\varphi(x), \varphi(z)$; these curves are guaranteed by the linear local path-connectedness of W. Then by Theorem 3.1,

$$\begin{aligned} \operatorname{cap}_{B^{\theta_W}_{p,p}(W)}(E, F) &\le \operatorname{cap}_{B^{\theta_W}_{p,p}(W)}(B(\varphi(x), C_L l), W \setminus B(\varphi(x), L/C_L)) \\ &\le C \, \log(L/(C_L^2 l))^{\beta_p} \end{aligned}$$

where $\beta_p = 1 - p$. By the assumed morphism property of φ, it follows that

$$\operatorname{cap}_{B^{\theta_Z}_{p,p}(Z)}(E', F') \le C \, \log(L/(C_L^2 l))^{\beta_p},$$

where $E' = \varphi^{-1}(E)$ and $F' = \varphi^{-1}(F)$. On the other hand, both E' and F' are connected subsets of $\varphi^{-1}(B(\varphi(x), C_L l))$ and $\varphi^{-1}(W \setminus B(\varphi(x), L/C_L))$. Moreover, F' contains both x and z, while E' contains both y and w. It follows that

$$\min\{\mathcal{H}^1_\infty(E' \cap B(x, 2r)), \mathcal{H}^1_\infty(F' \cap B(x, 2r))\} \ge r,$$

and so by Theorem 3.4 and the assumption $\theta_Z p = Q_Z$, we have

$$\operatorname{cap}_{B^{\theta_Z}_{p,p}(Z)}(E', F') \ge 1/C.$$

It follows that $\log(L/l)^{p-1} \le C$, that is,

$$L \le e^{C^{-1/\beta_p}} \, l,$$

as desired. $\square$

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