Distribution of Chores with Information Asymmetry

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Abstract. A well-regarded fairness notion when dividing indivisible chores is envy-freeness up to one item (EF1), which requires that pairwise envy can be eliminated by the removal of a single item. While an EF1 and Pareto optimal (PO) allocation of goods can always be found via well-known algorithms, even the existence of such solutions for chores remains open, to date. We take an *epistemic* approach utilizing information asymmetry by introducing *dubious chores*–items that inflict no cost on receiving agents but are perceived costly by others. On a technical level, dubious chores provide a more fine-grained approximation of envy-freeness than EF1. We show that finding allocations with minimal number of dubious chores is computationally hard. Nonetheless, we prove the existence of envy-free and fractional PO allocations for n agents with only $2n-2$ dubious chores and strengthen it to $n-1$ dubious chores in four special classes of valuations. Our experimental analysis demonstrates that often only a few dubious chores are needed to achieve envy-freeness.

1 Introduction

The fair allocation of resources and tasks is a fundamental role of any economy. It has garnered attention of diverse communities spanning computer science, artificial intelligence, political science and economics due to its broad applicability in healthcare, charitable donations, waste management, and task allocation [5, 23, 24, 41, 53, 54]. Traditionally, this field is concerned with allocating *indivisible* items that are positively-valued by agents (i.e., *goods*). However, many practical decisions distribute indivisible negatively-valued tasks (i.e., *chores*) too.

The allocation of chores is fundamentally different from its goods counterpart since chores must be fully allocated, whereas goods may be disposed of at no cost. Moreover, algorithmic techniques and axiomatic approaches developed for fair allocation of goods do not immediately translate to this setting. Thus, in recent years a large body of work has focused on investigating fairness axioms and algorithmic techniques specifically for allocation of chores [7, 19, 30, 37, 46]. The canonical fairness notion in the literature is envy-freeness (EF), an intrapersonal property which requires that each agent weakly prefers their own bundle to any other [36]. However, EF allocations may not exist and determining their existence is computationally intractable, motivating a number of relaxations.

One well-studied relaxation, *envy-freeness up to one item* (EF1), requires that any pairwise envy between the agents can be eliminated by the counterfactual removal of a single item (a good from the envied agent or a chore from the envious agent) [49, 23]. An EF1 allocation of goods always exists and can be computed efficiently along with economic efficiency notions such as Pareto optimality (PO) [12, 26]. In contrast, for chores, not only computing an EF1 and PO allocation is unknown, but even the existence of such allocations remains open to date. (EF1 allocations without PO can be computed in polynomial time [9, 17].) This has led several works to study EF1 and PO allocations under restricted domains [9, 35, 40, 48].

We take a different, *epistemic*, approach that utilizes information asymmetry. Rather than require counterfactual reasoning to achieve fairness, as with EF1, epistemic fair division considers allocations that are partially hidden [8] or for which information is withheld about the goods [45, 18]. To this end, we differentiate between the allocation of chores that agents are informed about and the allocation they actually receive. The difference is an over-representation about which tasks agents complete, above-and-beyond those they are actually assigned. That is, agents must only complete a subset of their assigned chores, which may contain duplicates.

We represent this technically through the introduction and allotment of *dubious chores*, copies of the original chores that bear no cost for agents that receive them, but are seen as costly by others. This models settings where agents do not have direct means of communication and cannot verify the exact costs incurred by each agent, such as with distributed computing of high-complexity problems or decentralized training of large language models. Formally, we propose a fairness notion of *envy-freeness up to k dubious chores* (DEF-k). Consider n agents, m chores, and an allocation $A = (A_1, \dots, A_n)$. We introduce up to k *dubious* copies of the original chores, representing additional tasks agents appear to be completing, that are distributed via the *dubious allocation* $A^D = (A_1^D, \dots, A_n^D)$. If every agent i prefers its own allocation A_i to the perceived allocation $A_h \cup A_h^D$ (combined real and dubious) of each other agent $h \neq i$, then we say that allocation A is DEF-k.

This approach differs from related work about EF1, duplicating chores, and chores with subsidies. First, agents in our approach measure whether an allocation is fair based on their information, which is subjective and may differ across agents. This contrasts EF1 which requires agents to counterfactually reason about other agents' bundles. Moreover, DEF-k offers a more fine-grained approximation of EF than EF1. Whereas the latter recognizes any allocation *close to* an envy-free allocation, DEF-k precisely identifies a trade-off between fairness and transparency: a DEF-0 allocation is necessarily EF, while at most k units of transparency must be compromised to make a DEF-k allocation envy-free for $k > 0$.

Second, Akrami et al. [3] recently introduced real copies of chores

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(i.e., "*surplus*") into the original fair division instance. Our approach differs on a conceptual level in that duplicates introduce *real* additional cost on the receiving agents, reducing overall welfare, and more duplicated chores than dubious ones may be needed to eliminate envy. Third, our method differs from approaches of eliminating envy by subsidizing agents with positively-valued money [22, 25, 44]. Whereas these approaches change real allocations by introducing unit-valued goods to achieve EF fairness, agents in our approach perceive their allocation as fair according to their visible information.

1.1 Our Contributions

We propose a novel fairness notion, DEF-k, that utilizes information asymmetry with dubious chores, items that are perceived as costly but inflict no actual cost on the receiving agent. Conceptually, dubious chores provide a natural approach when no envy-free solution exists. Technically, DEF-k provides a more fine-grained approximation of envy-freeness than EF1 which enables progress towards addressing open problems in fair allocation of indivisible chores, such as the existence and computation of EF1 and PO. As such, we make the following technical contributions for agents with additive valuations:

- We show the hardness of finding the minimal k for which DEF-k exists and its constant approximation (Corollary 1). In contrast, we obtain that DEF-$(n-1)$ allocations always exist and can be computed efficiently (Theorem 3).
- We achieve both fairness and efficiency by showing that there always exists an allocation satisfying DEF-$(2n-2)$ and fractional Pareto optimality (fPO), a more demanding efficiency requirement than PO (Theorem 4). Furthermore, we strengthen this result showing that DEF-$(n-1)$ and fPO allocations can be efficiently computed in four special cases: when agents' valuations are identical (Theorem 5), binary (Theorem 6), or bivalued (Theorem 7), or upon restricting chores to be two-typed (Theorem 8). Under binary valuations our algorithm also guarantees *envy-freeness up to any chore* (EFX) [9, 27], a strengthening of EF1 under which each pairwise envy can be eliminated by the removal of *any* negatively-valued chore of the envious agent.
- Our empirical study demonstrates that `Round Robin` produces allocations requiring small numbers of dubious chores to become envy-free. This beats the theoretical existence guarantee of DEF-$(n-1)$ by half. DEF-k also appears to correlate well with PO, which optimally require few dubious chores to become envy-free.

2 Related Work

While the literature on fair division of indivisible items is vast (see e.g., Amanatidis et al. [6] for a survey), our work fits best in the contexts of algorithms guaranteeing fair and efficient allocations (often for restricted preference classes), fairness achieved with copies, and fairness achieved by imposing epistemic constraints. The following related works hold for agents with additive valuations; see Section 3 for formal definitions.

2.1 Algorithms Yielding Fairness and Efficiency

For goods instances and agents with general valuations, Murhekar and Garg [52] identified a pseudo-polynomial time algorithm for computing EF1 and fPO that extends to polynomial time by fixing the numbers of agents or different values that agents have over goods. This improves algorithms by Aziz et al. [9] and Barman et al. [12] which are polynomial time for two agents with general valuations and if valuations are bounded, respectively. Garg and Murhekar [38] identified a polynomial time algorithm for computing EFX, a strict generalization of EF1, and fPO, for bivalued goods. The polynomial time algorithm of Gorantla et al. [43] computes EFX for two-types of goods. Garg et al. [40]'s polynomial time algorithm identifies EF1+fPO for *two-type* agents, where each agent has one of two utility functions. Still, computing EF1+fPO in polynomial time for the general goods case remains an open question.

For chores instances, Garg et al. [39] and Ebadian et al. [35] independently identified polynomial time algorithms to compute EF1+fPO allocations when agents' valuations only take on one of two possible values. Similarly, Aziz et al. [10] provided an algorithm to compute EF1+fPO allocations when there are only two types of chores. Garg et al. [40] further identified polynomial time algorithms that compute an allocation that is EF1 and fPO for either three agents, two-type agents, or *personalized* bivalued chores instances—bivalued chores in which agents may have different values—with different enough valuations.

2.2 Epistemic Fairness and Copies

Our work aligns closely with *epistemic* fair division which describes agents with limited information about allocations. Aziz et al. [8], and Caragiannis et al. [28] define epistemic EF and EFX as allocations in which each agent, based on information about their own bundle only, believes it is possible to allocate the remaining items to other agents such that the overall allocation satisfies EF or EFX. Thus, each agent's view may be substantially different from each other or the real underlying allocation. The notion of hidden envy-freeness of Hosseini et al. [45] supposes that each agent agrees on the visible information, but their belief about what's hidden may differ. Other related works include a Bayesian approach to incomplete information [32], a generalization of maximin share [29], and where agents only compare their values to their neighbors on a social network [1, 13, 16, 21, 33]. Bliznets et al. [18] combine these approaches to study EF for agents embedded in a social network who can hide at most one good in their allocation, and at most k goods are hidden.

The concept of "copies" of items appears in several works, but in very different settings to ours. For example, Akrami et al. [3] considered actual copies of chores that are added to allocations to obtain fair solutions. In this case, each actual copy of a chore creates additional disutility for the agents. Therefore, their approach for chores resembles the literature on goods allocation with charity, in which a small number of goods may be left unallocated in order to obtain a fair solution. Literature on EFX with charity was introduced by Caragiannis et al. [26] and followed-up with many relaxations [31, 15, 2, 14, 50]. On a technical level, Akrami et al. [3] develops an algorithm that yields an EF1 and fPO allocation with $n-1$ surplus, whereas our algorithms guarantees EF and fPO upon adding $2n-2$ dubious chores to the original instances.

Separately, Aleksandrov [4] considered modified definitions of EF1 and EFX in which a counterfactual "copy" of an item is added to a bundle to eliminate pairwise envy. Gorantla et al. [43] studied the setting of goods divided into several types such that each agent values goods of the same type identically. Hence, goods of each type can be seen as "copies," but they are given exogenously, in contrast to our setting. Likewise, Aziz et al. [10] focused on the setting with two types of chores.

3 Preliminaries

Problem instance. An *instance* $\mathcal{I} = \langle \mathcal{N}, \mathcal{M}, \mathcal{V} \rangle$ of the fair division problem is defined by a set of n *agents* $\mathcal{N}$, a set of m *chores* $\mathcal{M}$, and a *valuation profile* $\mathcal{V} = (v_i)_{i \in \mathcal{N}}$ that specifies the preferences of every agent $i \in \mathcal{N}$ over each subset of the chores in $\mathcal{M}$ via a *valuation function* $v_i : 2^{\mathcal{M}} \to \mathbb{R}$. If $v_i(S) \geq 0\ \forall i \in \mathcal{N}, S \subseteq \mathcal{M}$, then we call the items *goods*; likewise, if $v_i(S) \leq 0$ the items are *chores*. We assume that the valuation functions are *additive*, i.e., for any $i \in \mathcal{N}$ and $S \subseteq \mathcal{M}$, $v_i(S) = \sum_{c \in S} v_i(\{c\})$, where $v_i(\emptyset) = 0$. We write $v_i(c)$ instead of $v_i(\{c\})$ for a single chore $c \in \mathcal{M}$.

Allocation. A real *allocation* $A = (A_1, \dots, A_n)$ refers to an n-partition of the set of chores $\mathcal{M}$, where $A_i \subseteq \mathcal{M}$ is the *bundle* allocated to agent $i \in \mathcal{N}$ and no chore of $\mathcal{M}$ is unallocated. The utility of agent i for the bundle A_i is given by $v_i(A_i) = \sum_{c \in A_i} v_i(c)$. An allocation is *fractional* if chores may be shared between agents and *integral* otherwise; each chore is nevertheless fully allocated across the agents.

Restricted valuations. We consider four special cases of agent valuation functions. The instance $\mathcal{I}$ has *identical valuation* if for any two agents $i, j \in \mathcal{N}$ and chore $c \in \mathcal{M}$ it holds that $v_i(c) = v_j(c)$. It has *binary valuations* if $v_i(c) \in \{-1, 0\}$ for every agent $i \in \mathcal{N}$ and chore $c \in \mathcal{M}$. The instance has *bivalued valuations* if there exist two real numbers $x < y < 0$ such that $v_i(c) \in \{x, y\}$ for every $i \in \mathcal{N}$ and $c \in \mathcal{M}$. Finally, the instance has *two types of chores* if $\mathcal{M}$ can be partitioned into two sets X and Y such that, for every agent $i \in \mathcal{N}$ and two chores c, c' from the same set, it holds that $v_i(c) = v_i(c')$.

Definition 1 (**Dubious chores**). *A* dubious *chore c' allocated to agent $i \in \mathcal{N}$ is a* copy *of the chore $c \in \mathcal{M}$ that upholds the same perceived costs for all agents but no incurred cost for i. That is, $v_h(c') = v_h(c)\ \forall h \in \mathcal{N} \setminus \{i\}$ and $v_i(c') = 0$. Given an instance $\mathcal{I}$ and real allocation A, a* dubious multiset *D refers to a multiset containing dubious chores copied from $\mathcal{M}$. A* dubious allocation *$A^D = (A_1^D, \dots, A_n^D)$ is an n-partition of the multiset D.*

We define the augmented allocation *$A^* = A \cup A^D$ such that for each $i \in \mathcal{N}$, $A_i^* = A_i \cup A_i^D$. The utility of agent i for its own augmented bundle is $v_i(A_i^*) = v_i(A_i)$ while for other agents $h \in \mathcal{N}$ is $v_h(A_i^*) = v_h(A_i) + v_h(A_i^D)$.*

Definition 2 (**Envy-freeness**). *An allocation A (real, dubious, or augmented) is* envy-free *(*EF*) if for every pair of agents $i, h \in \mathcal{N}$, $v_i(A_i) \geq v_i(A_h)$. The allocation A is* envy-free up to one chore *(*EF1*) if for every pair of agents $i, h \in \mathcal{N}$ such that $A_i \neq \emptyset$, there exists some chore $c \in A_i$ such that $v_i(A_i \setminus \{c\}) \geq v_i(A_h)$. Furthermore, A is* envy-free up to any chore *(*EFX*) if for every pair of agents $i, h \in \mathcal{N}$ and chore $c \in A_i$ such that $v_i(c) < 0$, it holds that $v_i(A_i \setminus \{c\}) \geq v_i(A_h)$.*

Definition 3 (**Envy-freeness with dubious chores**). *A real allocation A is said to be* envy-free up to k dubious chores *(*DEF-k*) if there exists a dubious multiset D and dubious allocation A^D such that: (i) D consists of up to k dubious chores copied from $\mathcal{M}$, and (ii) the augmented allocation $A^* = A \cup A^D$ is envy-free.*

Remark. *It follows from the above definitions that an allocation is* EF *if and only if it is* DEF-0*. Moreover, for any $k \geq 0$, a* DEF-k *allocation is also* DEF-$(k+1)$*, but the converse may not hold.*

In the full version of the paper [47], we also consider two stronger notions of DEF-k: *singly* and *personalized*. There, we introduce restrictions on how dubious chores may be allocated that may be well motivated in some of the potential applications.

agent	c_1	c_2	$\dots$	c_{n-1}	c_n	c'_n	c'_n	$\dots$	c'_n
1	(-1)	-1	$\dots$	-1	$-n$	(0)	$-n$	$\dots$	$-n$
2	-1	(-1)	$\dots$	-1	$-n$	$-n$	(0)	$\dots$	$-n$
$\vdots$		$\vdots$	$\ddots$		$\vdots$	$\vdots$		$\ddots$	$\vdots$
$n-1$	-1	-1	$\dots$	(-1)	$-n$	$-n$	$-n$	$\dots$	(0)
n	-1	-1	$\dots$	-1	(-n)	$-n$	$-n$	$\dots$	$-n$

Table 1. An example allocation and its dubious augmentation.

Pareto optimality. An allocation A is *Pareto dominated* by allocation B if $v_i(B_i) \geq v_i(A_i)\ \forall i \in \mathcal{N}$, with at least one of the inequalities being strict. A *Pareto optimal* (PO) allocation is one that is not Pareto dominated by any other allocation. A *fractional Pareto optimal* (fPO) allocation is one that is not Pareto dominated by any other fractional allocation.

Note that an fPO allocation is also PO, but the converse may not necessarily hold. PO does imply fPO for special cases such as bivalued preferences [35].

Example 1. *Consider the instance $\mathcal{I}$ in Table 1 with n agents and $m = n$ chores defined with agents' valuations $v_i(c)$ for $i \in \mathcal{N}, c \in \mathcal{M}$. A real allocation A is identified by the circled valuations such that $A_i = \{c_i\}$, $\forall i \in \mathcal{N}$. Clearly A is* EF1 *and* PO *and agent n envies each other agent $i \neq n$.*

Consider the dubious multiset D containing $n-1$ dubious copies of c_n that is allocated as $A^D = (\{c'_n\}, \{c'_n\}, \dots, \{c'_n\}, \emptyset)$. Then, the augmented allocation $A^* = A \cup A^D$ is envy-free. This follows because for any pair of agents $i, h \in \mathcal{N} \setminus \{n\}$, we have that $v_n(A_i^*) = -n-1 < -n = v_n(A_n^*)$, $v_i(A_h^*) = -n-1 < -1 = v_i(A_i^*)$, and $v_i(A_n^*) = -n < -1 = v_i(A_i^*)$. Hence, A is DEF-$(n-1)$.

Remark. *Every real allocation A is* DEF-$(m \cdot (n-1))$ *since it can be augmented with a dubious multiset D with $n-1$ dubious copies of each chore. Allocating these one per agent, except for the agent that receives the corresponding real chore in A, will make $A^* = A \cup A^D$ envy-free. However, this is a trivial allotment akin to only revealing each agent's own bundle to themselves, as in epistemic envy-freeness [8, 28]. In this work we provide computational limits and theoretical bounds on the minimum k for which* DEF-k *allocations exist.*

To conclude this section, we recall three techniques for allocating chores to agents that we use in our subsequent theorems.

Round Robin algorithm. Fix a permutation σ of the agents. The `Round Robin` algorithm cycles through the agents according to σ. In each round, an agent gets its favorite (i.e., least undesirable) chore from the pool of remaining chores.

Envy Graph algorithm [49]. The `Envy Graph` algorithm iterates through each chore c in rounds by first resolving cycles in the *top trading envy graph* T_A based on the partial allocation A. This graph is built over n vertices and composed of edges (i, h) if agent i's (weakly) most preferred bundle is h in A. The algorithm resolves cycles in T_A, by transferring to each agent i the bundle of the next agent in the cycle. Afterwards, one agent who does not envy any other agent is given c.

Fisher Markets. A *pricing vector* is a function $p : \mathcal{M} \to \mathbb{R}_{\geq 0}$. Intuitively, the price $p(c)$ of chore $c \in \mathcal{M}$ is a payment that an agent receives for doing the chore. For a subset $S \subseteq \mathcal{M}$, we have $p(S) = \sum_{c \in S} p(c)$. Given p, agent i's *minimum pain per buck* is the set of

agent	$d^1_{1,1}$	⋯	$d^4_{1,1}$	$d^1_{1,2}$	⋯	$d^4_{1,2}$	$\cdots$	$d^1_{1,n}$	⋯	$d^4_{1,n}$	$d^1_{2,1}$	⋯	$d^4_{2,1}$	$d^1_{2,2}$	$\cdots$	$d^4_{2,n}$	$d^1_{3,1}$	$\cdots$	$d^4_{n,n}$	c_1	c_2	$\cdots$	c_m
0	0	⋯	0	0	⋯	0	$\cdots$	0	⋯	0	0	⋯	0	0	$\cdots$	0	0	$\cdots$	0	$\textcircled{0}$	$\textcircled{0}$	$\cdots$	$\textcircled{0}$
1	$\textcircled{-1}$	⋯	$\textcircled{-1}$	-1	⋯	-1	$\cdots$	-1	⋯	-1	$\textcircled{0}$	⋯	$\textcircled{0}$	0	$\cdots$	0	$\textcircled{0}$	$\cdots$	0				
2	0	⋯	0	$\textcircled{0}$	⋯	$\textcircled{0}$	$\cdots$	0	⋯	0	-1	⋯	-1	$\textcircled{-1}$	$\cdots$	-1	0	$\cdots$	0				
3	0	⋯	0	0	⋯	0	$\cdots$	0	⋯	0	0	⋯	0	0	$\cdots$	0	-1	$\cdots$	0	S_1	S_2	$\cdots$	S_m
⋮		⋮			⋮				⋮			⋮			⋮			⋮					
n	0	⋯	0	0	⋯	0	$\cdots$	$\textcircled{0}$	⋯	$\textcircled{0}$	0	⋯	0	0	$\cdots$	$\textcircled{0}$	0	$\cdots$	$\textcircled{-1}$				

Table 2. An illustration to the proof of Theorem 2.

chores with the minimum absolute value of valuation to price ratio: $MPB_i = \arg\min_{c \in \mathcal{M}} |v_i(c)|/p(c)$; intuitively, the most profitable chores for i.

A pair of a real allocation and a pricing vector (A, p) is a *Fisher market equilibrium* if every agent $i \in \mathcal{N}$ receives only its minimal pain per buck chores, i.e., $A_i \subseteq MPB_i$. It is well-known that every allocation of a Fisher market equilibrium is fPO [51]. A Fisher market equilibrium (A, p) is *price envy-free up to one item* (pEF1) if for every $i, j \in \mathcal{N}$, either $A_i = \emptyset$ or there exists $c \in A_i$ such that $p(A_i \setminus \{c\}) \leq p(A_j)$. It is known that if (A, p) is pEF1, then A is EF1 [35, 39].

4 Fairness with Dubious Chores

We begin by analyzing fairness with dubious chores without any efficiency requirement. We show the computational hardness of finding the smallest k for which DEF-k exists, discuss the relation between DEF-k and EF1, and prove that an existing algorithm can achieve DEF-$(n-1)$ in polynomial time. Later, in Section 5, we study the existence and computation of DEF-k along with fPO.

We start with the decision problem of finding the optimal k for which a DEF-k allocation exists. Bhaskar et al. [17] demonstrated that determining whether for a given instance there exists an EF allocation is NP-complete. We show that for an arbitrary *fixed* constant $k \in \mathbb{Z}_{\geq 0}$, determining whether an instance has a DEF-k allocation is NP-complete, even in the cases of identical or binary valuations.

Theorem 1. *Given an instance $\mathcal{I} = \langle \mathcal{N}, \mathcal{M}, \mathcal{V} \rangle$, for every fixed constant $k \in \mathbb{Z}_{\geq 0}$, deciding if the instance admits a* DEF-k *allocation A is NP-complete, even when valuations are identical or binary.*

The proof uses the techniques developed by Bhaskar et al. [17] and Hosseini et al. [45] and can be found in the full version of the paper [47]. Observe that Theorem 1 implies that finding the minimal $k \in \mathbb{Z}_{\geq 0}$ such that there exists a DEF-k allocation is NP-hard as well. Moreover, since hardness holds also for $k = 0$, there is no polynomial-time constant approximation scheme for this problem, unless P = NP.

Corollary 1. *Given instance $\mathcal{I} = \langle \mathcal{N}, \mathcal{M}, \mathcal{V} \rangle$, unless P=NP, there is no polynomial-time algorithm that gives a constant approximation for the problem of finding* DEF-k *allocation with minimal k.*

Theorem 1 holds when k is a fixed constant. However, if we do not fix k, even the verification problem becomes hard. This means that the natural question of whether a given allocation might be augmented with k dubious chores to an envy-free allocation is computationally intractable.

Theorem 2. *Given an instance $\mathcal{I} = \langle \mathcal{N}, \mathcal{M}, \mathcal{V} \rangle$ with binary valuations and an allocation A, deciding if allocation A is* DEF-k *is NP-complete.*

Proof. Given a dubious allocation A^D, we can verify if the augmented allocation $A^* = A \cup A^D$ is EF in polynomial time, so our problem is in NP. We therefore focus on showing the hardness.

To this end, we follow the reduction from RESTRICTED EXACT COVER BY 3-SETS (RX3C). In RX3C instance $\mathcal{I} = \langle \mathcal{U}, \mathcal{S} \rangle$, we are given a universe of $n = 3k$ elements $\mathcal{U} = (u_1, u_2, \ldots, u_n)$ and a family $\mathcal{S} = (S_1, S_2, \ldots, S_m)$ of three-element subsets of $\mathcal{U}$ such that every element u_i appears in exactly 3 subsets, i.e. $|S_i \in \mathcal{S} : u_j \in S_i| = 3$ for every $j \in [n]$. Hence, necessarily $m = 3k = n$. The question is whether we can find a cover of size k, i.e., a subfamily of subsets $\mathcal{K} \subset \mathcal{S}$ such that $|\mathcal{K}| = k$ and $\bigcup_{S_i \in \mathcal{K}} S_i = \mathcal{U}$. Observe that in such a case every element of $\mathcal{U}$ will appear in subsets in $\mathcal{K}$ exactly once, with means that $\mathcal{K}$ will be an exact cover as well. The problem is known to be NP-hard [42].

Now, for every instance $\mathcal{I}$ of RX3C, we construct a corresponding instance $\mathcal{I}' = \langle \mathcal{N}, \mathcal{M}, \mathcal{V} \rangle$ and an allocation A (for an illustration see Table 2). First, let us take one agent for each element of $\mathcal{U}$ and one *choosing* agent 0. Formally, $\mathcal{N} = \{0, 1, \ldots, n\}$. Next, for every ordered pair of agents $(i, j) \in \{1, \ldots, n\}^2$ let us take 4 *dummy chores*, $d^1_{i,j}, d^2_{i,j}, d^3_{i,j}$, and $d^4_{i,j}$. Also, let us take one chore for every subset in $\mathcal{S}$, i.e., chores $c_1, \ldots, c_m$. This will give us a total of $4n^2 + m$ chores in $\mathcal{M}$. Further, let us describe the valuations of the agents. For the choosing agent 0, we set the value of every chore to 0. In turn, for every agent $i \in \{1, \ldots, n\}$, we set its valuation of dummy chores to -1, if i is the first agent in the subscript, and 0, otherwise. Formally, for every $r \in \{1, 2, 3, 4\}$ and $(j, k) \in \{1, \ldots, n\}^2$ let $v_i(d^r_{j,k}) = -1$, if $j = i$, and $v_i(d^r_{j,k}) = 0$, otherwise. As for chores $c_1, \ldots, c_m$ the valuation depends on whether element u_i belongs to the subset corresponding to the particular chore. Formally, we set $v_i(c_j) = -1$, if $u_i \in S_j$, and $v_i(c_j) = 0$, otherwise. Finally, let us specify the allocation A. First, we give all of the chores $c_1, \ldots, c_m$ to the choosing agent 0, i.e., $A_0 = \{c_1, \ldots, c_m\}$. Then, every agent $i \in \{1, \ldots, n\}$, receives all of the dummy chores in which it is in the second position of the subscript, i.e., $A_i = \bigcup_{r=1}^4 \{d^r_{1,i}, d^r_{2,i}, \ldots, d^r_{n,i}\}$.

In the remainder of the proof, let us show that A is DEF-k, if and only if, there exists an exact cover $\mathcal{K}$ in the original instance $\mathcal{I}$. To this end, observe that for every two agents $i, j \in \{1, 2, \ldots, n\}$ we have $v_i(A_j) = v_i(A_i) = -4$. Hence, there is no envy among these agents. Furthermore, for the choosing agent 0, we have $v_0(S) = 0$ for every $S \subseteq \mathcal{M}$, thus it does not envy any other agents. However, for every agent $i \in \{1, 2, \ldots, n\}$ we have that $v_i(A_0)$ is equal to the number of subsets in $\mathcal{S}$ that include element u_i times -1, which is -3. Hence, the choosing agent is envied by every other agent. Now,

agent	c_1	c_2	$\cdots$	c_n
1	(-1)	0		0
2	0	(-1)		$\cdots$
$\vdots$		$\vdots$	$\ddots$	$\vdots$
n	0	0	$\cdots$	(-1)

Table 3. EF1 allocation that is not DEF-k for any $k < n(n-1)$.

in order to eliminate such envy, we have to give the choosing agent 0 at least one dubious copy of a chore that has value -1 for each of agents $1, \ldots, n$. Since we have $3k$ such agents, each chore has value -1 for maximally 3 agents, and we can add up to k dubious chores, we have to choose a subset of k chores from $\{c_1, c_2, \ldots, c_m\}$ in such a way that every agent from $\{1, \ldots, n\}$ has value -1 for at least one of them. But that is possible if and only if there exists a set cover in the original instance $\mathcal{I}$. □

In lieu of this computational hardness, we establish upper-bounds on the required number of dubious chores to make a real allocation envy-free. While EF1 allocations of chores do exist and can be computed in polynomial time [9, 17], such allocations may require many (i.e., $n(n-1)$) dubious chores to become envy-free. This is because for each pairwise envy relation between agents, the envied agent can dubiously receive the "worst" chore of the envious agent to remove envy. Example 2 demonstrates that this bound is in fact tight.

Proposition 1. *Given an instance $\mathcal{I} = \langle \mathcal{N}, \mathcal{M}, \mathcal{V} \rangle$, every* EF1 *allocation A is also* DEF-$n(n-1)$.

Example 2. *The instance presented in Table 3 demonstrates an* EF1 *allocation A that is* DEF-$n(n-1)$ *but not* DEF-k *for any $k < n(n-1)$. Note that A is extremely ineffective—any other allocation would yield a Pareto improvement. Furthermore, we note that this allocation could be an output of the* `Envy Graph` *algorithm.*

We next demonstrate stronger results for allocations produced by `Round Robin`, which require at most $n-1$ dubious chores to become envy-free. The improvement arises since each agent chooses the best of the remaining chores for each round. Hence, in each round, there is a single agreed-upon "worst" chore, the latest chore allocated, that can eliminate all envy-relations when dubiously allocated to all envious agents.

Theorem 3. *Given an instance $\mathcal{I} = \langle \mathcal{N}, \mathcal{M}, \mathcal{V} \rangle$,* `Round Robin` *returns a* DEF-$(n-1)$ *allocation.*

Proof. Without loss of generality, let A be a real allocation generated by `Round Robin` by the order $\sigma = \{1, 2, \ldots, n\}$. Let T be the last agent allocated a chore and let $T' = (T+1) \ mod \ n$ be the subsequent agent in σ. We know that A is EF1 and, specifically $\forall i \neq T'$ and $k \neq i$, $v_i(A_i \backslash \{\tilde{c}_i\}) \geq v_i(A_k)$, where $\tilde{c}_i$ is the last chore allocated to each agent i by `Round Robin` [9, 27]. The subsequent agent T' is not envious of any other agent i, since for every chore c of T' there is one chore of i that was chosen after c. We also know that in the last n rounds of `Round Robin`, each agent $i \neq T$ chose their chore $\tilde{c}_i$ over the last allocated chore $\tilde{c}_T$, meaning that $v_i(A_i) \geq v_i(A_i \cup \{\tilde{c}_T\} \backslash \{\tilde{c}_i\})$. Therefore for every agent $i \neq T'$ and $k \neq i$, we have $v_i(A_i) \geq v_i(A_k \cup \{\tilde{c}_i\}) \geq v_i(A_k \cup \{\tilde{c}_T\})$. As a result, if a dubious copy of $\tilde{c}_T$ were given to every agent except T, no agent would envy any other. This is $n-1$ dubious chores. □

agent	c_1	c_2	c_3	c_4	c'_1	c'_2
1	(-3)	(-2)	-1	-5	-3	-2
2	-1	-1	(-1)	-1	(0)	(0)
3	-1	-1	-1	(-1)	-1	-1

Table 4. An example allocation that is DEF-2 but not EF1.

Allotting $n-1$ dubious chores for allocations produced by `Round Robin` is similar to the case of goods allocations satisfying *strong envy-freeness up to one good* (sEF1) [34], in which all envy to each agent i can be eliminated by removing a single good in i's bundle A_i. Hosseini et al. [45] observe that hiding $n-1$ of these goods can make sEF1 allocations envy-free and that this bound is tight for the existence of any hidden envy-free allocation. Likewise, with chores, Example 1 above demonstrates that the existence of *any* DEF-k allocation is tight at $k = n-1$. This is because some agent i must receive c_n in the real allocation and they will envy every other agent by a value of at least 1. Therefore, at least $n-1$ chores must be dubiously copied to make i not envious, so any real allocation cannot be DEF-$(n-2)$.

Conitzer et al. [34] discuss that both `Round Robin` and `Envy Graph` yield sEF1 allocations for goods; however, only `Round Robin` of these yields DEF-$(n-1)$ allocations for chores, which are notably also EF1. Example 2 previously demonstrated that EF1 allocations furnished by `Envy Graph` may be not DEF-$(n-1)$. Finally, DEF-$(n-1)$ allocations are not necessarily EF1, as in Example 3.

Example 3. *Fix $n = 3$ and $m = 4$ in the instance depicted by Table 4. The real allocation indicated by circled valuations is* DEF-2 *with respect to the dubious set $D = \{c'_1, c'_2\}$ and allocation $A^D = \{\emptyset, \{c'_1, c'_2\}, \emptyset\}$, but it is not* EF1.

5 Fairness and Efficiency

In this section, we proceed to demonstrate the existence and computation of DEF-k along with fractional Pareto optimality. Specifically, we show that we can efficiently compute DEF-$(n-1)$ and fPO allocations in four special cases: if agents have identical, binary, or bivalued valuations, or when there are two types of chores. However, we start by a general result that a DEF-$(2n-2)$ and fPO allocation always exist. Our approach is based on the algorithm for finding fair and efficient allocations for goods developed by Barman and Krishnamurthy [11]. We note the same algorithm was utilized by Akrami et al. [3] to attain an EF1 and fPO allocation by introducing at most $n-1$ actual copies of chores.

Theorem 4. *Given an instance $\mathcal{I} = \langle \mathcal{N}, \mathcal{M}, \mathcal{V} \rangle$, there always exists an* DEF-$(2n-2)$ *and* fPO *allocation.*

Proof. Barman and Krishnamurthy [11] showed[1] that there always exists a Fisher market equilibrium (A, p) such that for every agent $i \in \mathcal{N}$,

$$p(A_i) \leq 1 \text{ and } \exists c \in \mathcal{M} \text{ such that } p(A_i \cup \{c\}) \geq 1, \text{ or}$$
$$p(A_i) > 1 \text{ and } \exists c \in A_i \text{ such that } p(A_i \setminus \{c\}) < 1. \quad (1)$$

[1] Barman and Krishnamurthy [11] only considered goods but the same algorithm works for chores [20]. However, the algorithm is based on finding *competitive equilibrium with equal incomes* (CEEI), which can be computed in polynomial time for goods. Only an XP algorithm for chores parameterized by the number of agents or chores is known; the existence of nonparameterized polynomial-time algorithm is unknown.

Since (A, p) is an equilibrium, we know that A is fPO. In what follows, we show that A is DEF-$(2n-2)$ as well.

To this end, let us denote by $i_{\max} \in \arg\max_{i \in \mathcal{N}} p(A_i)$ an agent with maximally priced bundle and by $c_{\max} \in \arg\max_{c \in \mathcal{M}} p(c)$ a chore with the highest price. Let D consist of $2n-2$ dubious copies of chore $c_{\max}$, which we denote $c'_{\max}$, and let A^D be a dubious allocation in which each agent in $\mathcal{N} \setminus \{i_{\max}\}$ receives two dubious copies of this chore. Next we show the augmented allocation $A^* = A \cup A^D$ is envy-free.

Observe that every agent $i \in \mathcal{N}$ does not envy agent $i_{\max}$, even in the real allocation A. Indeed, since i receives only its minimum pain per buck chores,

$$v_i(A_i) = -p(A_i) \cdot \min_{c \in \mathcal{M}} \tfrac{|v_i(c)|}{p(c)} \geq p(A_{i_{\max}}) \cdot \max_{c \in \mathcal{M}} \tfrac{v_i(c)}{p(c)}$$
$$\geq \textstyle\sum_{c \in A_{i_{\max}}} p(c) \cdot \frac{v_i(c)}{p(c)} = v_i(A_{i_{\max}}).$$

Hence, in A^* no agent envies $i_{\max}$ as well. Now, let us take arbitrary $i, j \in \mathcal{N}$ such that $i \neq i_{\max}$ and show that j does not envy i. Observe that

$$\begin{aligned} v_j(A_i^*) &= v_j(A_i) + 2v_j(c'_{\max}) \\ &= \textstyle\sum_{c \in A_i} \left(p(c)\frac{v_j(c)}{p(c)}\right) + 2p(c_{\max})\frac{v_j(c'_{\max})}{p(c_{\max})} \\ &\leq -(p(A_i) + 2p(c_{\max})) \cdot \min_{c \in \mathcal{M}} \tfrac{|v_j(c)|}{p(c)}. \end{aligned}$$

From Equation (1) we know that $p(A_i) + 2p(c_{\max}) \geq p(A_j)$. Hence, $v_j(A_i^*) \leq -p(A_j) \cdot \min_{c \in \mathcal{M}} \frac{|v_j(c)|}{p(c)} = v_j(A_j) = v_j(A_j^*)$. Thus, A^* is envy-free, so A is DEF-$(2n-2)$. ☐

Next, we focus on four restricted domains of preferences. We utilize their structure to strengthen our result and prove the existence of algorithms that find DEF-$(n-1)$ and fPO allocations in polynomial time. We begin by considering identical and then binary valuations.

Theorem 5. *For every instance $\mathcal{I} = \langle \mathcal{N}, \mathcal{M}, \mathcal{V} \rangle$ with identical valuations, there exists a* DEF-$(n-1)$ *and* fPO *allocation and it can be computed in polynomial time.*

Proof. Observe that with identical valuations every allocation is fPO. Hence, by Theorem 3, `Round Robin` algorithm returns a DEF-$(n-1)$ and fPO allocation in such case. ☐

Theorem 6. *For every instance $\mathcal{I} = \langle \mathcal{N}, \mathcal{M}, \mathcal{V} \rangle$ with binary valuations, there exists a* DEF-$(n-1)$ *and* fPO *allocation and it can be computed in polynomial time.*

Proof. The following algorithm returns a DEF-$(n-1)$ and fPO allocation. First, while possible, assign every chore to any agent that values it at 0. This ensures that the final allocation will have the minimal possible total cost for the agents, also among fractional allocations. Thus, it will be fPO. Let us assign the remaining *common chores*, i.e., those valued -1 by all agents, by `Round Robin` algorithm. Then, either every agent receives exactly the same number of common chores or there are two groups of agents with k and $k-1$ such chores for some $k \in \mathbb{N}$. In the former case, the allocation is envy-free, hence DEF-$(n-1)$ as well. In the latter case, if we give a dubious copy of some common chore to every agent that receives $k-1$ common chores (and there are at most $n-1$ such), we obtain an envy-free augmented allocation as well. ☐

We note that the algorithm described above also guarantees that the output allocation satisfies EFX.

For bivalued valuations, we will build on the algorithms introduced independently by Ebadian et al. [35] and Garg et al. [39] that for every such instance find an EF1 and PO allocation. The algorithm relies on Fisher market equilibrium and in order to use it, we first show the following lemma.

Lemma 1. *Given an instance $\mathcal{I} = \langle \mathcal{N}, \mathcal{M}, \mathcal{V} \rangle$, and a* pEF1 *Fisher market equilibrium (A, p), it holds that A is a* DEF-$(n-1)$ *and* fPO *allocation.*

Proof. Since an allocation in Fisher market equilibrium is always fPO [51], it suffices to show that if (A, p) is pEF1, then allocation A is DEF-$(n-1)$.

To this end, let us denote by $i_{\max}$ the agent with maximally priced bundle, i.e., $i_{\max} \in \arg\max_{i \in \mathcal{N}} p(A_i)$, and by $c_{\max}$ a chore with the highest price, i.e., $c_{\max} \in \arg\max_{c \in \mathcal{M}} p(c)$. Now, we construct a dubious allocation A^D where D contains $n-1$ copies of $c_{\max}$, denoted by $c'_{\max}$, which are allocated one each to agents $\mathcal{N} \setminus \{i_{\max}\}$. We show that the obtained augmented allocation $A^* = A \cup A^D$ is envy-free as follows.

First, observe that no agent envies $i_{\max}$, even in the real allocation A. Indeed, every agent $i \in \mathcal{N}$ receives only its minimum pain per buck chores, therefore

$$v_i(A_i) = -p(A_i) \cdot \min_{c \in \mathcal{M}} \tfrac{|v_i(c)|}{p(c)} \geq p(A_{i_{\max}}) \cdot \max_{c \in \mathcal{M}} \tfrac{v_i(c)}{p(c)}$$
$$\geq \textstyle\sum_{c \in A_{i_{\max}}} p(c) \cdot \frac{v_i(c)}{p(c)} = v_i(A_{i_{\max}}).$$

Hence, in A^* also no agent envies $i_{\max}$. Thus, let us take arbitrary $i, j \in \mathcal{N}$ such that $i \neq i_{\max}$ and prove that j does not envy i. Observe that

$$\begin{aligned} v_j(A_i^*) &= v_j(A_i) + v_j(c'_{\max}) \\ &= \textstyle\sum_{c \in A_i} \left(p(c)\frac{v_j(c)}{p(c)}\right) + p(c_{\max})\frac{v_j(c'_{\max})}{p(c_{\max})} \\ &\leq -(p(A_i) + p(c_{\max})) \cdot \min_{c \in \mathcal{M}} \tfrac{|v_j(c)|}{p(c)}. \end{aligned}$$

Since (A, p) is pEF1, we get that $p(A_j) - p(c) \leq p(A_i)$ for some $c \in \mathcal{M}$. Hence, $p(A_j) \leq p(A_i) + p(c_{\max})$ and $v_j(A_i^*) \leq -p(A_j) \cdot \min_{c \in \mathcal{M}} \frac{|v_j(c)|}{p(c)} = v_j(A_j) = v_j(A_j^*)$. In conclusion, A^* is envy-free, so A is DEF-$(n-1)$. ☐

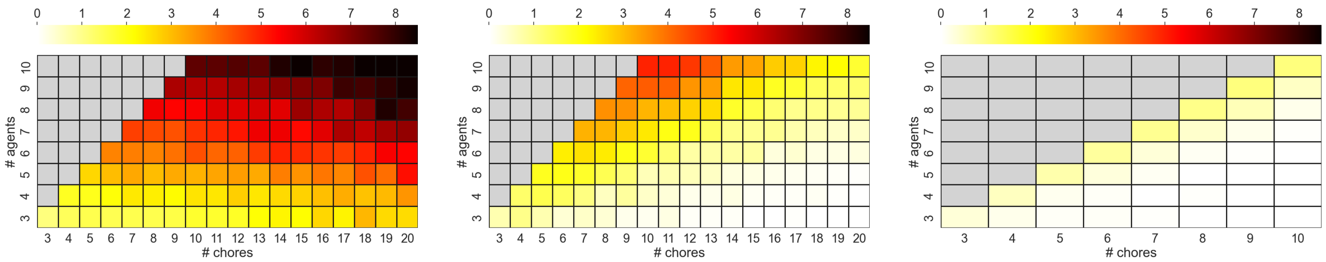
Theorem 7. *Given an instance $\mathcal{I} = \langle \mathcal{N}, \mathcal{M}, \mathcal{V} \rangle$ with bivalued valuations, there exists a* DEF-$(n-1)$ *and* fPO *allocation and it can be computed in polynomial time.*

Proof. Garg et al. [39] and Ebadian et al. [35] provided a polynomial -time algorithm that finds a pEF1 Fisher market equilibrium for every instance with bivalued valuations. This, combined with Lemma 1, yields the thesis. ☐

Finally, let us consider the case of two types of chores.

Theorem 8. *Given an instance $\mathcal{I} = \langle \mathcal{N}, \mathcal{M}, \mathcal{V} \rangle$ with two types of chores, there exists a* DEF-$(n-1)$ *and* fPO *allocation and it can be computed in polynomial time.*

Proof. By definition, since the instance has two types of chores, we can split $\mathcal{M}$ into two sets X and Y such that for every agent $i \in \mathcal{N}$ and chores c, c' in one of the sets we have $v_i(c) = v_i(c')$. Thus, for every agent $i \in \mathcal{N}$, by v_i^X and v_i^Y let us denote the agent's valuations of chores from sets X and Y respectively. Aziz et al. [10] provided a polynomial-time algorithm that for every instance with two types of chores finds an fPO allocation A and agent $i^* \in \mathcal{N}$ such that:



(a) `Envy Graph` (b) `Round Robin` (c) `PO`

Figure 1. Results for synthetic data for (a) `Envy Graph`, (b) `Round Robin`, and (c) `PO` algorithms. Grey tiles indicate no samples.

(1) only agent i^* may receive chores of two different types,
(2) every agent $i \in \mathcal{N}$ for which $v_i^X/v_{i^*}^X < v_i^Y/v_{i^*}^Y$ receives only items of type X,
(3) every agent $i \in \mathcal{N}$ for which $v_i^X/v_{i^*}^X > v_i^Y/v_{i^*}^Y$ receives only items of type Y, and
(4) in the instance with identical valuations obtained from $\mathcal{I}$ by changing valuations of every agent to that of agent i^*, i.e., $\mathcal{I}' = \langle \mathcal{N}, \mathcal{M}, v_{i^*} \rangle$, allocation A is EF1.

Thus, it suffices to show that the allocation satisfying these conditions is also DEF-$(n-1)$. To this end, for every chore c of type X or Y let us set $p(c) = v_{i^*}^X$ or $p(c) = v_{i^*}^Y$, respectively. First, let us show that in such a case (A, p) is a Fisher market equilibrium. To this end, observe that for every agent $i \in \mathcal{N}$ with $v_i^X/v_{i^*}^X < v_i^Y/v_{i^*}^Y$ we have that MPB_i consists of all chores of type X. And by condition (2), these are the only chores that agent i receives. Analogously, from condition (3) we get that every agent $i \in \mathcal{N}$ for which $v_i^X/v_{i^*}^X > v_i^Y/v_{i^*}^Y$ receives only its minimum pain per buck chores as well. Finally, for agent i^*, we get that $MPB_{i^*} = \mathcal{M}$. Hence, every agent receives only its minimal pain per buck chores, which means that (A, p) is indeed a Fisher market equilibrium.

Now, let us show that (A, p) is pEF1. Fix arbitrary agents $i, j \in \mathcal{N}$. If $A_i \neq \emptyset$, then by condition (4) there exists $c \in A_i$ such that $v_{i^*}(A_i \setminus \{c\}) \geq v_{i^*}(A_j)$. Thus, we get $p(A_i \setminus \{c\}) \geq p(A_j)$, which means that (A, p) indeed is pEF1. Therefore, the thesis follows from Lemma 1. □

6 Experiments

We experimentally investigate the minimal number of dubious chores needed to make an allocation EF. We generated a synthetic data set varying the number of agents n from 3 to 10 and chores m from 3 to 10 or 20. For each pair (n, m), we generated 100 instances with independent binary valuations such that for each $i \in \mathcal{N}$ and $c \in \mathcal{M}$ the valuation $v_i(c)$ is -1 with probability 0.7 and 0 with probability 0.3. To avoid the trivial EF case, we assert $m \geq n$ and set the last chore in each instance to be valued at -1 for each agent.

Next, we computed allocations using three different algorithms: (a) `Envy Graph`, (b) `Round Robin` , and (c) `PO`. The first two produce a single allocation per instance that satisfies EF1. The last algorithm searches through all PO allocations and returns one that is DEF-k for minimal k. Figure 1 shows heatmaps of average minimum number of dubious chores needed to make the output allocations EF.

In Figure 1a, we see that this minimum count for `Envy Graph` allocations increase with the number of agents or chores. For `Round Robin`, in Figure 1b, we see that this count also increases with the number of agents but decreases as we have more chores. This may be explained by the fact that with a large number of chores, it is easier for agents to choose chores valued 0 to them but possibly -1 to other agents. With many chores assigned this way, the envy can be reduced using a smaller number of dubious chores. Moreover, observe that the average number of dubious chores needed is generally smaller for allocation `Round Robin` than with `Envy Graph`. The difference between these two algorithms matches our theoretical findings, as we have an $n-1$ bound for `Round Robin` (Theorem 3), but the same bound for `Envy Graph` does not hold (Example 2).

Finally, when looking at the optimal PO solution in Figure 1c, we see that the required number of dubious chores increases with the increase in the number of agents, but drops sharply, when we increase the number of chores. In fact, whenever the number of chores is not almost equal to the number of agents, we can obtain envy-freeness with one or zero dubious chores in most cases. All in all, our experiment shows that in practice the number of dubious chores needed is much smaller than our theoretical guarantees indicate.

7 Conclusion

We have proposed a novel epistemic framework for fair allocations of chores through the introduction of dubious chores and DEF-k, an approximation of envy-freeness which details the trade-off between fairness and transparency of an allocation. Although finding DEF-k allocations with small k is computationally hard, we have provided several guarantees for the existence of such allocations with and without Pareto optimality. Our experimental results suggest that the number of dubious chores required to make an allocation free of envy is lower in practice than our theoretical guarantees indicate.

Some of the problems considered in this paper are still not fully resolved. In particular, the existence of a DEF-$(n-1)$ and PO chore allocation in every instance for agents with additive valuations seems possible. Proving this can be a valuable step to resolve the open existence problem of EF1 and PO allocations for chores.

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References

[1] R. Abebe, J. Kleinberg, and D. Parkes. Fair division via social comparison. In *Proceedings of AAMAS*, 2017.

[2] H. Akrami, R. Rezvan, and M. Seddighin. An EF2X allocation protocol for restricted additive valuations. In *Proceedings of IJCAI*, 2022.

[3] H. Akrami, B. R. Chaudhury, J. Garg, K. Mehlhorn, and R. Mehta. Fair and efficient allocation of indivisible chores with surplus. In *Proceedings of IJCAI*, 2023.

[4] M. Aleksandrov. Envy freeness up to one item: Shall we duplicate or remove resources? In *EPIA Conference on Artificial Intelligence*, pages 727–738. Springer, 2022.

[5] M. Aleksandrov, H. Aziz, S. Gaspers, and T. Walsh. Online fair division: Analysing a food bank problem. In *Proceedings of IJCAI*, pages 2540–2546, 2015.

[6] G. Amanatidis, G. Birmpas, A. Filos-Ratsikas, and A. A. Voudouris. Fair Division of Indivisible Goods: A Survey. In *Proceedings of IJCAI*, pages 5385–5393, 2022.

[7] H. Aziz, G. Rauchecker, G. Schryen, and T. Walsh. Algorithms for maxmin share fair allocation of indivisible chores. *Proceedings of AAAI*, pages 335–341, 2017.

[8] H. Aziz, S. Bouveret, I. Caragiannis, I. Giagkousi, and J. Lang. Knowledge, Fairness, and Social Constraints. In *Proceedings of AAAI*, pages 4638–4645, 2018.

[9] H. Aziz, I. Caragiannis, A. Igarashi, and T. Walsh. Fair allocation of indivisible goods and chores. *Autonomous Agents and Multi-Agent Systems*, 36:1–21, 2022.

[10] H. Aziz, J. Lindsay, A. Ritossa, and M. Suzuki. Fair allocation of two types of chores. In *Proceedings of AAMAS*, pages 143–151, 2023.

[11] S. Barman and S. K. Krishnamurthy. On the proximity of markets with integral equilibria. In *Proceedings of AAAI*, pages 1748–1755, 2019.

[12] S. Barman, S. K. Krishnamurthy, and R. Vaish. Finding fair and efficient allocations. In *Proceedings of ACM EC*, pages 557–574, 2018.

[13] X. Bei, Y. Qiao, and S. Zhang. Networked fairness in cake cutting. In *Proceedings of IJCAI*, pages 3632–3638, 2017.

[14] B. A. Berendsohn, S. Boyadzhiyska, and L. Kozma. Fixed-point cycles and approximate EFX allocations. In *MFCS*, volume 17, pages 1–13, 2022.

[15] B. Berger, A. Cohen, M. Feldman, and A. Fiat. Almost full EFX exists for four agents. In *Proceedings of AAAI*, pages 4826–4833, 2022.

[16] A. Beynier, Y. Chevaleyre, L. Gourvès, A. Harutyunyan, J. Lesca, N. Maudet, and A. Wilczynski. Local envy-freeness in house allocation problems. *Autonomous Agents and Multi-Agent Systems*, 33:591–627, 2019.

[17] U. Bhaskar, A. Sricharan, and R. Vaish. On Approximate Envy-Freeness for Indivisible Chores and Mixed Resources. In *APPROX/RANDOM*, pages 1–23, 2021.

[18] I. Bliznets, A. Bukov, and D. Sagunov. Fair division with minimal withheld information in social networks. In *28th International Conference on Computing and Combinatorics*, pages 126–137. Springer, 2022.

[19] A. Bogomolnaia, H. Moulin, F. Sandomirskiy, and E. Yanovskaia. Dividing bads under additive utilities. *Social Choice and Welfare*, 52: 395–417, 2019.

[20] S. Brânzei and F. Sandomirskiy. Algorithms for competitive division of chores. *Mathematics of Operations Research*, 49:398–429, 2023.

[21] R. Bredereck, A. Kaczmarczyk, and R. Niedermeier. Envy-free allocations respecting social networks. *Artificial Intelligence*, 305:103664, 2022.

[22] J. Brustle, J. Dippel, V. V. Narayan, M. Suzuki, and A. Vetta. One dollar each eliminates envy. In *Proceedings of ACM EC*, pages 23–39, 2020.

[23] E. Budish. The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes. *Journal of Political Economy*, 119:1061–1103, 2011.

[24] E. Budish, G. P. Cachon, J. B. Kessler, and A. Othman. Course match: A large-scale implementation of approximate competitive equilibrium from equal incomes for combinatorial allocation. *Operations Research*, 65:314–336, 2017.

[25] I. Caragiannis and S. D. Ioannidis. Computing envy-freeable allocations with limited subsidies. In *Proceedings of WINE*, pages 522–539. Springer, 2021.

[26] I. Caragiannis, N. Gravin, and X. Huang. Envy-freeness up to any item with high nash welfare: The virtue of donating items. In *Proceedings of ACM EC*, pages 527–545, 2019.

[27] I. Caragiannis, D. Kurokawa, H. Moulin, A. D. Procaccia, N. Shah, and J. Wang. The unreasonable fairness of maximum Nash welfare. *ACM TEAC*, 7:1–32, 2019.

[28] I. Caragiannis, J. Garg, N. Rathi, E. Sharma, and G. Varricchio. New fairness concepts for allocating indivisible items. In *Proceedings of IJCAI*, 2023.

[29] H. Chan, J. Chen, B. Li, and X. Wu. Maximin-aware allocations of indivisible goods. In *Proceedings of IJCAI*, pages 137–143, 2019.

[30] B. R. Chaudhury, J. Garg, P. McGlaughlin, and R. Mehta. Dividing bads is harder than dividing goods: On the complexity of fair and efficient division of chores. *arXiv preprint arXiv:2008.00285*, 2020.

[31] B. R. Chaudhury, T. Kavitha, K. Mehlhorn, and A. Sgouritsa. A little charity guarantees almost envy-freeness. *SIAM Journal on Computing*, 50:1336–1358, 2021.

[32] Y. Chen and N. Shah. Ignorance is often bliss: Envy with incomplete information. Technical report, Working paper, Harvard University, 2017.

[33] Y. Chevaleyre, U. Endriss, and N. Maudet. Distributed fair allocation of indivisible goods. *Artificial Intelligence*, 242:1–22, 2017.

[34] V. Conitzer, R. Freeman, N. Shah, and J. W. Vaughan. Group fairness for the allocation of indivisible goods. In *Proceedings of AAAI*, volume 33, pages 1853–1860, 2019.

[35] S. Ebadian, D. Peters, and N. Shah. How to fairly allocate easy and difficult chores. In *Proceedings of AAMAS*, pages 372–380, 2022.

[36] D. K. Foley. *Resource Allocation and The Public Sector*. Yale University, 1967.

[37] R. Freeman, S. Sikdar, R. Vaish, and L. Xia. Equitable allocations of indivisible chores. In *Proceedings of AAMAS*, pages 384–392, 2020.

[38] J. Garg and A. Murhekar. Computing fair and efficient allocations with few utility values. In *Proceedings of SAGT*, pages 345–359, 2021.

[39] J. Garg, A. Murhekar, and J. Qin. Fair and efficient allocations of chores under bivalued preferences. In *Proceedings of AAAI*, volume 36, pages 5043–5050, 2022.

[40] J. Garg, A. Murhekar, and J. Qin. New algorithms for the fair and efficient allocation of indivisible chores. In *Proceedings of IJCAI*, 2023.

[41] J. Goldman and A. D. Procaccia. Spliddit: Unleashing fair division algorithms. *ACM SIGecom Exchanges*, 13:41–46, 2015.

[42] T. F. Gonzalez. Clustering to minimize the maximum intercluster distance. *Theoretical computer science*, 38:293–306, 1985.

[43] P. Gorantla, K. Marwaha, and S. Velusamy. Fair allocation of a multiset of indivisible items. In *Proceedings of SODA*, pages 304–331, 2023.

[44] D. Halpern and N. Shah. Fair division with subsidy. In *Proceedings of SAGT*, pages 374–389, 2019.

[45] H. Hosseini, S. Sikdar, R. Vaish, J. Wang, and L. Xia. Fair Division through Information Withholding. In *Proceedings of AAAI*, pages 2014–2021, 2020.

[46] H. Hosseini, A. Searns, and E. Segal-Halevi. Ordinal maximin share approximation for chores. In *Proceedings AAMAS*, pages 597–605, 2022.

[47] H. Hosseini, J. Kavner, T. Wąs, and L. Xia. Distribution of chores with information asymmetry. *arXiv preprint arXiv:2305.02986*, 2023.

[48] H. Hosseini, S. Sikdar, R. Vaish, and L. Xia. Fairly dividing mixtures of goods and chores under lexicographic preferences. In *Proceedings of AAMAS*, pages 152–160, 2023.

[49] R. J. Lipton, E. Markakis, E. Mossel, and A. Saberi. On Approximately Fair Allocations of Indivisible Goods. In *Proceedings of ACM EC*, pages 125–131, 2004.

[50] R. Mahara. Extension of additive valuations to general valuations on the existence of EFX. *Mathematics of Operations Research*, 66:1–15, 2023.

[51] A. Mas-Colell, M. Whinston, and J. R. Green. *Microeconomic Theory*. Oxford University Press, 1995.

[52] A. Murhekar and J. Garg. On fair and efficient allocations of indivisible goods. In *Proceedings of AAAI*, volume 35, pages 5595–5602, 2021.

[53] D. C. Parkes, A. D. Procaccia, and N. Shah. Beyond dominant resource fairness: Extensions, limitations, and indivisibilities. *ACM TEAC*, 3: 1–22, 2015.

[54] P. A. Pathak, T. Sönmez, M. U. Ünver, and M. B. Yenmez. Fair allocation of vaccines, ventilators and antiviral treatments: leaving no ethical value behind in health care rationing. In *Proceedings of ACM EC*, pages 785–786, 2021.