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ON THE SINGULAR ABELIAN RANK OF ULTRAPRODUCT II_1 FACTORS

PATRICK HIATT and SORIN POPA

Dedicated to Jacques Dixmier on his 100th birthday

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ABSTRACT. We prove that, under the continuum hypothesis $\mathfrak{c} = \aleph_1$, any ultraproduct II_1 factor $M = \prod_\omega M_n$ of separable finite factors M_n contains more than $\mathfrak{c}$ many mutually disjoint singular MASAs, in other words the *singular abelian rank of* M, $\mathrm{r}(M)$, is larger than $\mathfrak{c}$. Moreover, if the strong continuum hypothesis $2^{\mathfrak{c}} = \aleph_2$ is assumed, then $\mathrm{r}(M) = 2^{\mathfrak{c}}$. More generally, these results hold true for any II_1 factor M with unitary group of cardinality $\mathfrak{c}$ that satisfies the bicommutant condition $(A_0' \cap M)' \cap M = M$, for all $A_0 \subset M$ separable abelian.

KEYWORDS: II_1 *factor, ultraproduct factors, singular* MASA, *singular abelian rank.*

MSC (2020): 46L10, 46L36.

INTRODUCTION

Following Dixmier [3], a maximal abelian $*$-subalgebra (MASA) A in a von Neumann algebra M is called *singular* if the only unitary elements $u \in \mathcal{U}(M)$ that normalize A (i.e., $uAu^* = A$) are the unitaries in A. The existence of such MASAs in the hyperfinite II_1 factor R in [3] was a discovery that led to many interesting developments and subsequent research (see e.g., [8], [10], [11], [14], [18], [19]).

Most recently in this direction, the *singular abelian core* of a II_1 factor M was defined in [2] as the (unique up to unitary conjugacy) maximal abelian $*$-subalgebra $A \subset \mathcal{M} = M \overline{\otimes} \mathcal{B}(\ell^2 K)$, with $|K| \geqslant 2^{|\mathcal{U}(M)|}$, that is generated by finite projections of $\mathcal{M}$, is singular in $1_A \mathcal{M} 1_A$ and is maximal in $\mathcal{M}$ with respect to inclusion. Also, the *singular abelian rank of* M was defined as $\mathrm{r}(M) := \mathrm{Tr}_{\mathcal{M}}(1_A)$, viewed as a cardinality when infinite. Alternatively, $\mathrm{r}(M)$ can be viewed as the "maximal number" of disjoint singular MASAs (or pieces of it) in M. The *sans-core* and respectively, *sans-rank* $\mathrm{r}_{\mathrm{ns}}(M)$ were defined in [2] in a similar way, by considering the maximal singular abelian purely non-separable core $A \subset \mathcal{M} = M\overline{\otimes}\mathcal{B}(\ell^2 K)$ and respectively the semi-finite trace of its support in $\mathcal{M}$.

It was pointed out in [2] that by results in [12], [15], for any separable II_1 factor M one has $\mathrm{r}(M) = \mathfrak{c}$ and that if M is an ultraproduct II_1 factor, $M = \prod_\omega M_n$, associated to a sequence M_n of separable II_1 factors and a free ultrafilter ω on $\mathbb{N}$, then by simply considering ultraproducts of singular MASAs of M_n one obtains $\mathrm{r}(M) = \mathrm{r}_{\mathrm{ns}}(M) \geqslant \mathfrak{c}$. But a more exact calculation of the singular abelian rank of such M was left open.

We prove in this paper that if we assume the continuum hypothesis (CH), $\mathfrak{c} = 2^{\aleph_0} = \aleph_1$, then for any II_1 factor of the form $M = \prod_\omega M_n$, with M_n separable tracial factors with $\dim(M_n) \to \infty$, one has $\mathrm{r}(M) = \mathrm{r}_{\mathrm{ns}}(M) \geqslant 2^{\mathfrak{c}}$, and that if we further assume the strong continuum hypothesis (SCH), $2^{\mathfrak{c}} = \aleph_2$, then we actually have equalities, $\mathrm{r}(M) = \mathrm{r}_{\mathrm{ns}}(M) = 2^{\mathfrak{c}}$ (see Theorem 2.1). Note that in particular this shows that, under CH, an ultraproduct II_1 factor has many more singular MASAs than the ones arising as ultraproducts of MASAs.

To do this calculation, we in fact only use the property of an ultraproduct II_1 factor $M = \prod_\omega M_n$ that any copy $A_0 \subset M$ of the separable diffuse abelian von Neumann algebra $L^\infty[0,1]$ satisfies the bicommutant condition $(A_0' \cap M)' \cap M = A_0$. When viewed as an abstract property of a II_1 factor M, we call this *property* U_0.

We prove that, somewhat surprisingly, a II_1 factor M has property U_0 if and only if it has *property* U_1, requiring that any isomorphism between two copies of $L^\infty[0,1]$ inside M is implemented by a unitary in M (see Theorem 1.2), and call a II_1 factor satisfying any of these equivalent properties a U-*factor*.

We also relate properties U_0, U_1 with the weaker property that any two copies of $L^\infty[0,1]$ inside M are unitary conjugate, already considered in [12], [16], and which we label here U_2. This property for M implies for instance that M is prime and has no Cartan subalgebras and that any MASA in M is purely non-separable (see Proposition 1.4). Thus, for such factors one always has $\mathrm{r}_{\mathrm{ns}}(M) = \mathrm{r}(M)$.

So with this terminology, our main result (Theorem 2.1) shows that if M is a U-factor with unitary group $\mathcal{U}(M)$ having cardinality $|\mathcal{U}(M)| = \mathfrak{c}$, then with the CH assumption we have $\mathrm{r}(M) \geqslant 2^{\mathfrak{c}}$, with equality when SCH is assumed.

We mention that Gao, Kunnawalkam Elayavalli, Patchell and Tan have recently been able to construct (under CH) examples of II_1 U-factors M with $|\mathcal{U}(M)|$ equal to $\mathfrak{c}$ but which cannot be decomposed as an ultraproduct of separable finite factors [7].

Throughout this paper we will systematically use notations, terminology and basic results from [13] (for all things concerning ultraproduct II_1 factors) and [14] (for intertwining of subalgebras and disjointness in II_1 factors, in particular for MASAs, especially singular ones). Our work here has been especially motivated by remarks and considerations in [2], notably Sections 2.3, 2.4 and the remarks therein. We comment at length about this in Section 3 of this paper.

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1. SOME ABSTRACT PROPERTIES OF ULTRAPRODUCT II_1 FACTORS

While any separable approximately finite dimensional (AFD) tracial von Neumann algebra (B_0, τ) can be embedded into any II_1 factor M [9], when M is an ultraproduct II_1 factor, $M = \prod_{\omega} M_n$, such an embedding $(B_0, \tau) \hookrightarrow M$ follows even unique up to unitary conjugacy in M. Also, any separable AFD subalgebra $B_0 \subset M$ satisfies the bicommutant condition $(B_0' \cap M)' \cap M = B_0$ (see e.g. Theorem 2.1 in [13]).

In particular, the uniqueness of the embedding and the bicommutant property hold true when (B_0, τ) is the separable diffuse abelian von Neumann algebra $(L^\infty[0,1], \int \cdot \, \mathrm{d}\lambda)$. In this section we will consider these two properties as abstract properties of a II_1 factor M and prove that they are in fact equivalent. We also discuss the apriori weaker condition that any two copies of $L^\infty[0,1]$ inside M are unitary conjugate.

DEFINITION 1.1. Given a II_1 factor M, we consider the following three properties:

(U_0) any separable abelian von Neumann subalgebra $A_0 \subset M$ satisfies the bicommutant property $(A_0' \cap M)' \cap M = A_0$;

(U_1) any trace preserving isomorphism between two separable diffuse abelian von Neumann subalgebras of M is implemented by a unitary element in M;

(U_2) any two separable diffuse abelian von Neumann subalgebras of M are unitary conjugate;

For each $i = 0, 1, 2$, we say that M has *stable property* U_i, if M^t satisfies U_i for any $t > 0$.

THEOREM 1.2. *Conditions* $\mathrm{U}_0, \mathrm{U}_1$ *for a* II_1 *factor* M *are equivalent and they are both stable properties, i.e, if* M *satisfies property* U_i, *for some* $i = 0, 1$, *then* M^t *satisfies it for any* $t > 0$.

Proof. Let us first show that U_1 is stable. So assume M satisfies U_1. We first show that $N = \mathbb{M}_n(M)$ satisfies U_1 as well. Let $A_1, A_2 \subset N$ be separable diffuse abelian von Neumann algebras and $\theta : A_1 \simeq A_2$ an isomorphism preserving the trace on N. Then A_1 contains a partition of 1 with projections $\{p_j^1\}_{j=1}^n$ of trace equal $1/n$. Let $p_j^2 = \theta(p_j^1)$. By conjugating with appropriate unitaries $u_1, u_2 \in N$ we may assume $p_j^i = e_{jj}$, $1 \leqslant j \leqslant n$, $i = 1, 2$, where $\{e_{ij} : 1 \leqslant i, j \leqslant n\} \subset \mathbb{M}_n(\mathbb{C})$, are the matrix units. Denoting by θ_j the restriction of θ to $A_1 e_{jj} \simeq A_2 e_{jj}$ and viewing them both as subalgebras in $M \simeq e_{jj}Ne_{jj}$, by the U_1 property for M it follows that θ_j is implemented by $u_j \in e_{jj}Ne_{jj}$. But then $u = \sum_j u_j \in \mathcal{U}(N)$ implements $\theta : A_1 \simeq A_2$.

We now show that if $p \in \mathcal{P}(M)$ then pMp satisfies U_1. If $A_1, A_2 \subset pMp$ are separable diffuse abelian von Neumann algebras and $\theta : A_1 \simeq A_2$ an isomorphism preserving the trace on pMp, then there exist separable diffuse abelian von Neumann subalgebras $\widetilde{A}_i \subset M$ such that $p \in \widetilde{A}_i$, and $\widetilde{A}_i p = A_i$, $i = 1, 2$, as well as a trace preserving isomorphism $\widetilde{\theta} : \widetilde{A}_1 \simeq \widetilde{A}_2$ whose restriction to A_1 is equal to θ. If $u \in \mathcal{U}(M)$ implements $\widetilde{\theta}$, then $up \in \mathcal{U}(pMp)$ implements θ. Thus, U_1 is stable.

Let us now prove that conditions U_0, U_1 are equivalent. Let $A_0 \subset M$ be a separable diffuse abelian von Neumann algebra. Denote $B = A_0' \cap M$ and $Z = B' \cap M$. Note that $Z = \mathcal{Z}(B)$. Indeed, because any element in M that commutes with all elements in $B = A_0' \cap M$ must in particular commute with A_0, so $B' \cap M \subset B$, which is equivalent to $B' \cap M = \mathcal{Z}(B)$.

Assume M satisfies U_1. If $Z \neq A_0$, then there exists a projection $p \in Z$ with $b = E_{A_0}(p) \neq p$. There exists a projection $q \in A_0$ majorized by the support $s = s(b)$ of b such that $cq \leqslant qb \leqslant (1-c)q$ for some $c > 0$. Thus, by replacing p by qp we may assume p itself satisfies $cs \leqslant b = E_{A_0}(p) \leqslant (1-c)s$. Denote $B_0 = A_0 s \vee \{p\} \subset Zs$. Note that the inclusion $L^\infty X \simeq A_0 s \subset B_0 \simeq L^\infty Y$ is given by a surjective measure preserving map $\alpha : Y \to X$ with two-points fiber $\forall t \in X$. Consider then the trace preserving embedding of (B_0, τ_{B_0}) into a tracial von Neumann algebra $Q \simeq A_0 s \overline{\otimes} R$, endowed with the trace $\tau_{A_0 s} \otimes \tau_R$, such that $A_0 s$ identifies with the center $\mathcal{Z}(Q) = A_0 s \otimes 1 \simeq L^\infty X$ and such that when we view p as a measurable field $p_t, t \in X$, with $p_t \in \mathcal{P}(R)$, we have $\tau_R(p_t) = b_t$, where $(b_t)_t = b$.

Since Q with its trace can be embedded into any II_1 factor, we can view it as a von Neumann subalgebra of sMs and then by using U_1 for $A_0 s \subset sMs$ we may assume the center of Q coincides with $A_0 s$ and B_0 with $A_0 s \vee \{p\}$. So $1 \otimes R$ is in the commutant of $A_0 s$, and hence of A_0. Since $p \in Z$, we should thus have $1 \otimes R$ commute with p. But by averaging p over the unitaries in $1 \otimes R$ we get b, which is not equal to p, a contradiction.

Thus, we must have $(A_0' \cap M)' \cap M = A_0$, showing that U_0 is satisfied.

Conversely, assume M satisfies the bicommutant condition U_0. Let $A_1, A_2 \subset M^{1/2}$ be separable diffuse abelian and $\theta : A_1 \simeq A_2$ be an isomorphism preserving the restrictions of the trace on $M^{1/2}$ to A_1, A_2. Let $A = \{ae_{11} + \theta(a)e_{22} : a \in A_1\}$ which we view as a (separable abelian diffuse) von Neumann subalgebra of $M = \mathbb{M}_2(M^{1/2})$. Then $(A' \cap M)' \cap M = A$ implies in particular that the projections $e_{11}, e_{22} \in A' \cap M$ are equivalent in $A' \cap M$, via some partial isometry $v = ue_{12}$ where u is a unitary in $e_{11}Me_{11} = M^{1/2}$. But this means $\theta(a) = uau^*$ for any $a \in A_1$.

We have thus proved that if M satisfies U_0 then $M^{1/2}$ satisfies U_1. Since we already showed that U_1 is a stable property, this implies M satisfies U_1. Thus, U_0, U_1 are equivalent, and since U_1 was shown to be stable, U_0 follows stable as well. ∎

DEFINITION 1.3. We say that a II_1 factor M is a U-*factor* if it satisfies the equivalent conditions U_0, U_1.

We already mentioned that ultraproduct II_1 factors $M = \prod_\omega M_n$ satisfy the bicommutant property U_0 and the unique (up to unitary conjugacy) embedding property U_1. They are the typical examples of U-factors.

Since property U_1 for a II_1 factor M trivially implies the unitary conjugacy of any two copies of $L^\infty[0,1]$ inside M, i.e., condition U_2, any U-factor satisfies U_2 as well. Condition U_2 was already considered as an abstract property of II_1 factors in Proposition 2.3 of [12], where it was noticed that the arguments in Section 7 of [10], showing that an ultraproduct II_1 factor M has no Cartan subalgebras and all its MASAs are purely non-separable, only use the fact that M satisfies condition U_2. It was further noticed in [16] that U_2 factors are prime and have the property that the commutant of any separable abelian $*$-subalgebra is of type II_1.

We restate all these results here, including their proofs from [10], [12], [16], for the reader's convenience.

PROPOSITION 1.4 ([10], [12], [16]). *Assume a* II_1 *factor M satisfies property* U_2 (*for instance, if M is a* U*-factor*). *Then M automatically satisfies the following properties:*

(i) *for any* MASA A *in* M, *there exists a diffuse abelian von Neumann subalgebra* $B_0 \subset M$ *orthogonal to* A;

(ii) *any separable abelian von Neumann subalgebra* $A_0 \subset M$ *has type* II_1 *relative commutant* $A_0' \cap M$;

(iii) *any* MASA *in* M *is purely non-separable;*

(iv) M *has no Cartan* MASA;

(v) M *is prime.*

Proof. (i) Let $A \subset M$ be a MASA. Let $D \subset A$ be a separable diffuse von Neumann subalgebra. Since any two separable diffuse abelian subalgebras in M are unitary conjugate and since M contains copies of the hyperfinite II_1 factor (by [9]), we may assume D is the Cartan subalgebra of such a subfactor $R \subset M$, represented as $D = D_2^{\otimes\infty} \subset M_{2\times 2}(\mathbb{C})^{\otimes\infty} = R$. Let $D_2^0 \subset M_{2\times 2}(\mathbb{C})$ be a maximal abelian subalgebra of $M_{2\times 2}(\mathbb{C})$ that is perpendicular to D_2 and denote $D^0 = {D_2^0}^{\otimes\infty} \subset R$. Then $D \perp D^0$ and since both D, D^0 are MASAs in R, we have $E_{D'\cap M}(D^0) = E_{D'\cap R}(D^0) = E_D(D^0) = \mathbb{C}$, i.e. $D^0 \perp D' \cap M \supset A$, proving (i).

(ii) By [9], one has $R\overline{\otimes}R \simeq R$ and so $R\overline{\otimes}R$ embedds into M. If one takes any MASA $B_0 \subset R \otimes 1 \subset R\overline{\otimes}R \simeq R$, then $B_0' \cap M \supset 1 \otimes R$, implying that $B_0' \cap M$ is type II_1. Since A_0, B_0 are unitary conjugate in M, $A_0' \cap M$ is II_1 as well.

(iii) Let A be a MASA in M. If Ap is separable for some projection $p \in M$, then by taking a smaller p if necessary we may assume $\tau(p) = 1/n$ for some integer $n \geqslant 1$. Let $v_1 = p, v_2, \dots, v_n \in M$ be partial isometries with $v_i^* v_i = p$, $\forall 1 \leqslant i \leqslant n$, and $\sum_i v_i v_i^* = 1$ and define $B = \sum_i v_i (Ap) v_i^*$. Then B is a separable MASA in M. But then taking $B_0 \subset B$ to be any diffuse proper von Neumann subalgebra of B, it cannot be unitary conjugate to B because B_0 is not a MASA while B is, contradiction.

(iv) Let $A \subset M$ be a MASA. By part (i), there exist separable diffuse abelian subalgebras D, D^0 in M such that $D \subset A$ and $D^0 \perp A$. Let $u \in \mathcal{U}(M)$ be so that $uDu^* = D^0$. Then u is perpendicular to the normalizer of A in M. Indeed, for any $v \in \mathcal{N}_M(A)$ and any partition $p_i \in D$ of mesh $\leqslant \varepsilon$, we have

$$|\tau(uv)|^2 = \Big|\tau\Big(\sum_i p_i uv p_i\Big)\Big|^2 \leqslant \Big\|\sum_i p_i uv p_i\Big\|_2^2 = \sum_i \tau(u^* p_i uv p_i v^*) = \sum_i \tau(p_i)^2 \leqslant \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\tau(uv) = 0$. Thus $u \perp \mathcal{N}_M(A)''$.

(v) If $M = M_1 \overline{\otimes} M_2$ with M_1, M_2 of type II_1 then there exist separable diffuse abelian von Neumann subalgebras $A_i \subset M_i$. By hypothesis, there exists a unitary $u \in M$ such that $uA_1u^* = A_2 \perp A_1$. From the argument in (iv), it follows that for any unitaries $v_1 \in M_1$, $v_2 \in M_2$ one has $\tau(uv_1v_2) = \tau(v_2uv_1) = 0$. Taking span of v_i and using that the $\|\cdot\|_2$ closure of the span of $1 \otimes M_2 \cdot M_1 \otimes 1$ is M, it follows that $\tau(uu^*) = 0$, contradiction. ∎

COROLLARY 1.5. *If a* II_1 *factor* M *satisfies property* U_2 *(e.g., if* M *is a* U*-factor), then* $\mathrm{r}_{\mathrm{ns}}(M) = \mathrm{r}(M)$.

Proof. By part (iii) of Proposition 1.4, any MASA in a U_2-factor is purely non-separable. ∎

Let us also mention that it was shown in [12, 2.3.1° (c)] that the Kadison–Singer paving problem over a MASA in a factor satisfying the stable U_2 property reduces to paving of projections having scalar expectation on the MASA. (Note that by Theorem 3.3 in [17], in order for a MASA A in a II_1 factor M to have the paving property, it is necessary that A be purely non-separable.) Whether U_2 is a stable property was however left open in [12], but upon reading a preliminary draft of our paper Adrian Ioana pointed out to us that an argument in the same vein as the proof of Theorem 1.2 easily implies U_2 stability as well. We thank him for sharing this with us.

PROPOSITION 1.6. *Condition* U_2 *is a stable property.*

Proof. Assume the II_1 factor M satisfies U_2. Since this trivially implies that $\mathbb{M}_n(M)$ satisfies U_2, $\forall n$, to prove the stability it is sufficient to show that pMp satisfies U_2 for any projection $p \in M$. Let $A_1, A_2 \subset pMp$ be separable diffuse abelian von Neumann algebras. Let $R \subset M$ be a copy of the hyperfinite II_1 factor with $D \subset R$ its Cartan subalgebra and so that $p \in D$. Let also $\widetilde{A}_i \subset M, i =$

1, 2, be separable diffuse abelian von Neumann algebras containing p and such that $\widetilde{A}_i p = A_i$. By the U_2 property of M, there exist unitaries $u_i \in M$ such that $u_i \widetilde{A}_i u_i^* = D$. Since $D \subset R$ is Cartan, there exist $v_i \in \mathcal{N}_R(D)$ such that $v_i(u_i p u_i^*)v_i^* = p$, $i = 1, 2$. But this means $w_i = v_i u_i p$ are unitaries in pMp that conjugate A_i onto Dp, $i = 1, 2$. Thus, A_1, A_2 are unitary conjugate as well. ∎

COROLLARY 1.7. *If a II_1 factor M satisfies property U_2 (e.g., if M is a U-factor), then a* MASA *$A \subset M$ has the paving property if and only if any projection $q \in M$ with $E_A(q) \in \mathbb{C}1$ can be paved.*

Proof. By Proposition 1.6 above, property U_2 is stable, so the statement follows from Proposition 2.3.1° (c) in [12]. ∎

REMARK 1.8. (i) While U_1 trivially implies U_2, we have no examples of a II_1 factor satisfying U_2 but not U_1. Note in this respect that if M satisfies property U_2 and $A_0, A_1 \simeq L^\infty[0,1]$ are von Neumann subalgebras of M then by conjugating by a unitary in M we may assume $A_0 = A_1$ and then property U_1 amounts to whether any automorphism of (A_0, τ) is implemented by a unitary in M. Thus, the following two additional properties of a II_1 factor M are relevant:

(U_3) given any separable diffuse abelian von Neumann subalgebra $A_0 \subset M$, any automorphism of (A_0, τ) is implemented by a unitary in M;

(U_3') there exists a separable diffuse abelian von Neumann subalgebra $A_0 \subset M$ such that any automorphism of (A_0, τ) is implemented by a unitary in M.

Thus, we see that $U_1 \Rightarrow U_3 \Rightarrow U_3'$, $U_1 \Leftrightarrow (U_2 + U_3') \Leftrightarrow (U_2 + U_3)$, and that both U_3, U_3' are stable properties (proof being similar to the proof of the stability of U_1, U_2). Thus, an example of a II_1 factor M satisfying U_2 but not U_1 (so M not a U-factor) should contain a copy of the non-atomic probability space $([0,1], \lambda)$ whose normalizer in M does not implement all of its automorphism group.

(ii) The equivalence between the bicommutant property U_0 and the conjugacy of embeddings U_1 for $B = L^\infty[0,1]$ in Theorem 1.2 raises the possibility that a correlation between these two properties may occur for other tracial von Neumann algebras (B, τ). If one takes B to be the hyperfinite II_1 factor, $B = R$, then it is easy to see that both U_0 and U_1 are stable and that the proof of $U_1 \Rightarrow U_0$ goes exactly the same way as in the case $B = L^\infty[0,1]$ in Theorem 1.2. It would be interesting to see if one has $U_0 \Rightarrow U_1$ as well.

2. CONSTRUCTING DISJOINT SINGULAR MASAS IN U-FACTORS

We show in this section that, under the continuum hypothesis, the size of the singular abelian core of any U-factor is quite "large" and can be estimated.

We briefly recall (see e.g., [11]) that if M is a II_1 factor and $A \subset M$ is a MASA, then A is singular in M if and only if any partial isometry $v \in M$ satisfying $v^*v, vv^* \in A$, $vAv^* \subset A$ must be contained in A. Also, using notations from intertwining theory (see e.g., 1.5 in [13], for 1.3 in [14]) given two MASAs $A_1, A_2 \subset M$ one has $A_1 \prec_M A_2$ if and only if there exists a non-zero partial isometry $v \in M$ such that $v^*v \in A_1, vv^* \in A_2$ and $vA_1v^* \subset A_2$ (note this is symmetric, i.e. $A_1 \prec_M A_2$ if and only if $A_2 \prec_M A_1$). If there exists no such v we write $A_1 \not\prec_M A_2$ (equivalently $A_2 \not\prec_M A_1$) and say that A_1, A_2 are *disjoint*.

THEOREM 2.1. *Let M be a* II_1 U*-factor M with the property that the cardinality of its unitary group $\mathcal{U}(M)$ is equal to $\mathfrak{c}$. If the continuum hypothesis, $\mathfrak{c} = \aleph_1$, is assumed, then M contains more than $\mathfrak{c}$ many mutually disjoint singular* MASA*s, i.e.,* $\mathrm{r}(M) > \mathfrak{c}$. *Moreover, if the strong continuum hypothesis $2^{\mathfrak{c}} = \aleph_2$ is assumed, then* $\mathrm{r}(M) = 2^{\mathfrak{c}}$.

Proof. Denote by $(I, <)$ the set of ordinals $< \aleph_1 = \mathfrak{c}$ endowed with its well ordered relation. Since $|\mathcal{U}(M)| = \mathfrak{c}$, it follows that $|\mathcal{P}(M)| = \mathfrak{c}$, and thus the cardinality of the set $\mathcal{V} = \mathcal{V}(M) = \{up : u \in \mathcal{U}(M), p \in \mathcal{P}(M)\}$ of partial isometries of M is equal to $\mathfrak{c}$ as well. Let $\{v_i\}_{i\in I}$ be an enumeration with repetition of $\mathcal{V}$, where each $v \in \mathcal{V}$ appears $\mathfrak{c}$-many times.

Let $\mathcal{A}$ be a maximal family of disjoint singular abelian wo-closed subalgebras $A \subset 1_A M 1_A$ (which apriori may be an empty set). Assume $|\mathcal{A}| \leqslant \mathfrak{c} = \aleph_1$. Let $\{A_i\}_{i\in I}$ be a family of MASAs in M indexed by our set I, such that each $A \in \mathcal{A}$ appears as a direct summand of some A_i.

Note that if we can show that under these assumptions there exists a singular MASA $B \subset M$ such that $B \not\prec_M A_i, \forall i \in I$, then this would contradict the fact that $\{A_i\}_{i\in I}$ contains all of $\mathcal{A}$, which was chosen to be the maximal singular core for M. This contradiction would show that one necessarily have $|\mathcal{A}| > \mathfrak{c}$, thus finishing the proof of the first part. If in addition we have $2^{\mathfrak{c}} = \aleph_2$, since the total number of distinct MASAs in a II_1 factor M with $|\mathcal{U}(M)| = \mathfrak{c}$ is obviously majorized by $2^{\mathfrak{c}}$, it would then also follow that $\mathrm{r}(M) = |\mathcal{A}| = 2^{\mathfrak{c}}$.

We construct B as the wo-closure of the union of an increasing family $\{B_i\}_{i\in I}$ of separable diffuse abelian von Neumann subalgebras of M, which we construct by transfinite induction over $i \in I$, in the following way.

Assume that B_j have been constructed for all $j < i$. We want to construct B_i so that v_i is not intertwining B_i into $B_i' \cap M$, nor B_i into A_j for $j \leqslant i$. To this end, we proceed as follows:

(a) Denote $B_i^0 = \overline{\bigcup_{j<i} B_j}$. Note that B_i^0 is separable abelian diffuse.

(b) If $v_i^* v_i \notin B_i^0$ then by U_0 there exists a self-adjoint element $a \in (B_i^0)' \cap M$ such that $[v_i^* v_i, a] \neq 0$ and we let $B_i = B_i^0 \vee \{a\}$. Note that B_i is then still separable abelian and $[v_i^* v_i, B_i] \neq 0$.

(c) If $v_i^* v_i \in B_i^0$ then we let $K_i = \{j \in I, j \leqslant i : v_i B_i^0 v_i^* \not\subset A_j\}$ and $L_i = \{j \in I, j \leqslant i : v_i B_i^0 v_i^* \subset A_j\}$. Note that K_i, L_i are disjoint, countable sets, with $K_i \cup L_i = \{j \in I : j \leqslant i\}$. Denote $p_i = v_i^* v_i \in B_i^0$ and notice that for each $j \in L_i$ we have $v_i^* A_j v_i \subset Q_i^0 \overset{\text{def}}{=} (B_i^0 p_i)' \cap p_i M p_i$, with $v_i^* A_j v_i$ a MASA in Q_i^0. Thus, if we denote $S_i := \bigcup_{j \in L_i} v_i^* A_j v_i$ then the set $S_i \subset Q_i^0$ is a countable union of abelian von Neumann algebras (even MASAs) in the II_1 von Neumann algebra Q_i^0, so $Q_i^0 \setminus S_i$ is a G_δ dense subset of Q_i^0.

Note already that if $a_0 \in Q_i^0 \setminus S_i$ is a self-adjoint element then any separable abelian von Neumann algebra that contains the abelian algebra $B_i^1 = B_i^0 \vee \{a_0\}$ cannot be intertwined by the partial isometry v_i into A_j for any $j \leqslant i$.

In order to choose $B_i \supset B_i^1$ so that to exclude v_i from properly normalizing any MASA B containing B_i, let us note that there are several possibilities:

(i) $v_i \in B_i^1$, in which case we just put $B_i = B_i^1$.

(ii) $v_i B_i^1 v_i^* \not\subset (B_i^1)' \cap M$, in which case we again let $B_i = B_i^1$.

(iii) $v_i B_i^1 v_i^* \subset (B_i^1)' \cap M$ but $v_i B_i^1 v_i^* \not\subset B_i^1$. This means there exists $a \in B_i^1 p_i$ such that $v_i a v_i^* \in ((B_i^1)' \cap M) \setminus B_i^1$, and by applying U_0 there exists $a_1 = a_1^* \in (B_i^1)' \cap M$ such that $[a_1, v_i a v_i^*] \neq 0$. We then let $B_i = B_i^1 \vee \{a_1\}$.

(iv) $v_i B_i^1 v_i^* \subset B_i^1$ but $v_i B_i^1 v_i^* \neq B_i^1 v_i v_i^*$. In this case we have that $v_i^* B_i^1 v_i$ strictly contains $B_i^1 v_i^* v_i$. Like in (iii) above, by U_0 there exist $a' \in B_i^1$ and a self-adjoint $a_1' \in (B_i^1)' \cap M$ such that $[v_i^* a' v_i, a_1'] \neq 0$. We then define $B_i = B_i^1 \vee \{a_1'\}$.

(v) $v_i B_i^1 v_i^* = B_i^1 v_i v_i^*$ but $v_i \notin B_i^1$. This implies the partial isometry v_i normalizes the II_1 von Neumann algebra $Q_i = (B_i^1)' \cap M$, acting non-trivially on it, having left and right supports in $\mathcal{Z}(Q_i) = B_i^1$. There are two possibilities:

(v1) $v_i \in Q_i$. In this case $v_i^* v_i = v_i v_i^* = p_i \in \mathcal{Z}(Q_i)$ and so v_i is a non-central unitary in the II_1 von Neumann algebra $Q_i p_i$.

We claim that if this is the case, then there exists a unitary $u \in Q_i p_i$ such that $v_i u v_i^*$ does not commute with u.

To see this, first note that by Proposition 1.4(ii), $Q_i p_i$ is of type II_1, so $Q_i p_i \not\prec_N \mathcal{Z}(Q_i p_i)$ in any ambient II_1 factor N that we would embed $Q_i p_i$. Taking N to be a free product of $Q_i p_i$ with a diffuse tracial algebra, we can assume $Q_i p_i$ is embedded in a II_1 factor N so that its relative commutant in N is equal to $\mathcal{Z}(Q_i p_i)$. But then we can apply Theorem 0.1 (a) in [13] to get a Haar unitary $u \in Q_i p_i$ that is approximately free to $x = v_i - E^N_{\mathcal{Z}(Q_i p_i)}(v_i) \neq 0$. In particular, one can take u to be ε 4-independent to x, which for $\varepsilon > 0$ sufficiently small insures that $[v_i u v_i^*, u] \neq 0$.

Taking now $u \in Q_i p_i$ to be any unitary satisfying this property, we define $B_i = B_i^1 \vee \{u\}$.

(v2) $v_i \notin Q_i$. In this case v_i acts non-trivially on the center of Q_i, so there exists mutually orthogonal projections $z_1, z_2 \in \mathcal{Z}(Q_i)$ such that $z_1 \leqslant v_i^* v_i$, $z_2 \leqslant v_i v_i^*$ and $v_i z_1 v_i^* = z_2$. Since $Q_i z_1$ is II_1, there exists a copy of $\mathbb{M}_2(\mathbb{C})$ inside it. So there exist self-adjoint unitaries $u, w \in Q_i z_1$ such that $uw = -wu$. Let $c = u + v_i w v_i^*$ and define $B_i = B_i^1 \vee \{c\}$. Note that c, z_1, z_2 are elements in B_i such that $[v_i(cz_1)v_i^*, cz_2] \neq 0$.

Finally, we define $B = \overline{\bigcup_i B_i}^{\mathrm{wo}}$. Let us first show that B is a MASA in M, i.e., $B = B' \cap M$. To see this, it is sufficient to prove that any selfadjoint unitary

$v \in B' \cap M$ lies in B. Since $v \in \mathcal{V}$, it is of the form v_i for some $i \in I$. This means v_i is being considered in step i of the induction and we see that we are necessarily in the situation (v1), where we have chosen B_i (which is a subalgebra of B) so that to contain some b such that $v_i b v_i^* b \neq b v_i b v_i^*$, contradicting $[B, v_i] = 0$.

Assume now that B is not singular. This implies there exists a non-zero partial isometry $w \in M$ with w^*w, ww^* mutually orthogonal projections in B. Thus $w \in \mathcal{V}$ so $w = v_i$ for some $i \in I$ and so we have considered w at step i of the induction, and we are necessarily in one of the situations (iii), (iv), (v1), (v2), which all lead to contradictions.

Finally, assume $B \prec_M A_j$ for some countable ordinal $j \in I$. This means there exists a partial isometry $v \in M$ such that $v^*v \in B$, $vv^* \in A_j$ and $vBv^* = A_j vv^*$. Because of our choice of repeating v $\mathfrak{c}$-many times in $\{v_i\}_{i \in I}$, there exists $i \in I$ such that $i > j$ and $v = v_i$. But then the choices we made in (ii), (iii) for the algebra $B_i \subset B$, easily imply that we cannot have $v_i B v_i^* \subset A_i$. ∎

COROLLARY 2.2. *Let $\{M_n\}_{n \geqslant 1}$ be a sequence of separable tracial factors with* $\dim(M_n) \to \infty$ *and ω a free ultrafilter on $\mathbb{N}$. Denote $M = \prod_\omega M_n$ the associated ultraproduct* II_1 *factor. If we assume the continuum hypothesis then* $\mathrm{r}(M) > \mathfrak{c}$. *If we further assume the strong continuum hypothesis, then* $\mathrm{r}(M) = 2^{\mathfrak{c}}$.

Proof. Since any ultraproduct II_1 factor $M = \prod_\omega M_n$ satisfies the bicommutant axiom U_0, it is a U-factor. If in addition M_n are all separable, then $|\mathcal{U}(M_n)| = \mathfrak{c}$, so $|\mathcal{U}(M)| = \mathfrak{c}^{\aleph_0} = \mathfrak{c}$. Thus, we can apply Theorem 2.1 to conclude that under the CH condition we have $\mathrm{r}(M) > \mathfrak{c}$. Since the total number of distinct MASAs in M is majorised by the number of subsets of $\mathcal{U}(M)$, it is bounded by $2^{\mathfrak{c}}$. Thus, $\mathrm{r}(M) \leqslant 2^{\mathfrak{c}}$. So, if SCH is assumed then $\mathrm{r}(M) = 2^{\mathfrak{c}}$. ∎

3. FURTHER CONSIDERATIONS

The motivation behind our calculations of singular abelian rank of ultraproduct II_1 factors was the hope that this invariant might be able to differentiate among some of these factors (for instance, between $\prod_\omega \mathbb{M}_{k_n}(\mathbb{C})$, with $k_n \nearrow \infty$, and M^ω, for a separable non-Gamma II_1 factor M). But our calculations, which anyway depend on CH/SCH, show that, like in the separable case where one has $\mathrm{r}(M) = \mathfrak{c}$ for any separable II_1 factor M (cf. [12], [15]; see Remark 2.7 in [2]), the singular abelian rank is the same, equal to $2^{\mathfrak{c}}$, for all ultraproducts II_1 factors.

One can try to "diminish" the number of disjoint singular MASAs by restricting our attention to MASAs that satisfy various stronger versions of singularity, thus attempting to bring them to a "small cardinality", even finite if possible.

Thus, in the spirit of the terminology in Definitions 2.5, 2.9 in [2], let us denote by $\mathcal{A}^*_M$ a maximal family of disjoint "special" singular MASAs in the II_1 factor M satisfying a "generic" stronger singularity property $*$. As in [2], we will in fact view $\mathcal{A}^*_M$ in "unfolded" form, as one single singular abelian wo-closed $*$-subalgebra generated by finite projections in the II_∞ factor $\mathcal{M} = M\overline{\otimes}\mathcal{B}(\ell^2K)$, where K is a set of sufficiently large cardinality ($K \geqslant 2^{|\mathcal{U}(M)|}$ will do), which is so that any of its finite corners has the property $*$, and which is maximal (with respect to inclusion) with these properties. Note that these requirements force the definition of disjointness to be taken possibly stronger as well.

One then takes the corresponding rank $\mathrm{r}_*(M)$ to be the trace $\mathrm{Tr}_\mathcal{M}$ of the support of $\mathcal{A}^*_M \subset \mathcal{M}$. Like in [2] one clearly has the amplification formula $\mathrm{r}_*(M^t) = \mathrm{r}_*(M)/t, \forall t > 0$, making such considerations particularly interesting if the rank of the "special" singular core could be shown finite.

We illustrate below with four examples of such a possible strengthening.

3.1. THE SUPERSINGULAR ABELIAN CORE. Following [12], we will say that a wo-closed abelian $*$-subalgebra A in a II_1 factor M is *supersingular* if there is no automorphism $\theta \in \mathrm{Aut}(M)$ such that $\theta(Ap) \subset A$ for some non-zero $p \in \mathcal{P}(A)$ other than the inner automorphisms of M that act trivially on pMp. Two such supersingular abelian subalgebras $A_1, A_2 \subset M$ are *disjoint* if there exists no automorphism θ of M satisfying $\theta(A_1p_1) \subset A_2$ for some non-zero projection $p_1 \in A_1$. Note that this is the same as requiring that $A_1 \oplus A_2$ be supersingular in $M^2 = \mathbb{M}_2(M)$.

As we mentioned above, like in [2], we in fact view any family $\mathcal{A}$ of disjoint (in this stronger sense) supersingular abelian subalgebras in M in its "unfolded" form, as one single supersingular abelian algebra generated by finite projections in $M\overline{\otimes}\mathcal{B}(\ell^2K)$, for a sufficiently large K. One clearly has a maximal such algebra with respect to inclusion, $\mathcal{A}^{\mathrm{ss}}_M$, which is moreover unique up to unitary conjugacy in $\mathcal{M}$, and which we will call the *superrsingular abelian core*. The corresponding *supersingular rank* $\mathrm{r}_{\mathrm{ss}}(M)$ is then given by the trace $\mathrm{Tr}_\mathcal{M}$ of the support of $\mathcal{A}^{\mathrm{ss}}_M$ in $\mathcal{M}$, viewed as a cardinality when infinite.

3.2. THE COARSE ABELIAN CORE. In the same spirit, this time following [15], one can take in $\mathcal{M} = M\overline{\otimes}\mathcal{B}(\ell^2K)$ the *coarse abelian core* to be a wo-closed abelian $*$-subalgebra $\mathcal{A}^{\mathrm{c}}_M \subset \mathcal{M}$ generated by finite projections with the property that $\mathcal{A}p$ is coarse in $p\mathcal{M}p$ for any finite projection $p \in \mathcal{A}^{\mathrm{c}}_M$, and which is maximal with respect to inclusion. Note that disjointness for coarse abelian $A_1, A_2 \subset M$ amounts to A_1, A_2 being a coarse pair (as defined in [15]).

The coarse core this way defined is clearly unique in $\mathcal{M}$ up to unitary conjugacy. The *coarse abelian rank* is then $\mathrm{r}_{\mathrm{c}}(M) = \mathrm{Tr}_\mathcal{M}(1_{\mathcal{A}^{\mathrm{c}}_M})$.

Note however that by results in [15], for any separable M one has $\mathrm{r}_{\mathrm{c}}(M) > \aleph_0$, so if we assume CH then $\mathrm{r}_{\mathrm{c}}(M) = \mathfrak{c} = \aleph_1$.

3.3. THE MAXIMAL AMENABLE ABELIAN CORE. We define the *maximal amenable abelian core* $\mathcal{A}^{\mathrm{ma}}_M$ of the II_1 factor M as the wo-closed abelian $*$-subalgebra $\mathcal{A} =$

$\mathcal{A}_M^{\rm ma} \subset \mathcal{M} = M\overline{\otimes}\mathcal{B}(\ell^2 K)$ generated by finite projections with the property that $\mathcal{A}$ is maximal amenable in $\mathcal{M}$, and which is maximal with respect to inclusion. Its *maximal amenable abelian rank* is $\mathrm{r}_{\rm ma}(M) = \mathrm{Tr}_{\mathcal{M}}(1_{\mathcal{A}_M^{\rm ma}})$.

While it is not clear how this invariant fares for separable II_1 factors, note that by Theorem 5.3.1 in [12] any ultraproduct $A = \prod_\omega A_n$ of singular MASAs in II_1 factors $A_n \subset M_n$, is maximal amenable in $M = \prod_\omega M_n$. Thus, for such factors one has $\mathrm{r}_{\rm ma}(M) \geqslant \mathfrak{c}$. It would be interesting to know whether any singular MASA in an ultraproduct II_1 factor (and more generally in a U-factor) is automatically maximal amenable.

3.4. THE SINGULAR S-MASA CORE. Following [14], a MASA A in a II_1 factor M is an s-MASA if $A \vee A^{\rm op}$ is a MASA in $\mathcal{B}(L^2M)$. By a well know result of Feldman and Moore [6], any Cartan subalgebra satisfies this property. It has been shown in [14] that if the II_1 factor M is separable and has s-MASAs, then it has singular s-MASAs, and in fact it has $> \aleph_0$ many disjoint s-MASAs.

One defines the *s-MASA core* of a II_1 factor M, as the wo-closed abelian $*$-subalgebra $\mathcal{A} = \mathcal{A}_M^{\rm s} \subset \mathcal{M}$ generated by finite projections with the property that $\mathcal{A}p$ is a singular s-MASA in $p\mathcal{M}p$ for any finite projection $p \in \mathcal{A}$, and which is maximal with respect to inclusion. Again, this is obviously unique in $\mathcal{M}$ up to unitary conjugacy. The *s-MASA rank* of M is then $\mathrm{r}_{\rm s}(M) = \mathrm{Tr}_{\mathcal{M}}(1_{\mathcal{A}_M^{\rm s}})$. So by [14], in this case as well the associated rank is huge, $\mathrm{r}_{\rm s}(M) > \aleph_0$, so equal to $\mathfrak{c}$ when CH is assumed. It is not clear if ultraproduct factors, or even more generally U-factors, can have singular s-MASAs at all. Since existence of an s-MASA in a II_1 factor is a "thinness" property that ultraproducts are unlikely to have, it seems that such factors cannot have s-MASAs, but this remains an open problem.

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PATRICK HIATT, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA LOS ANGELES, LOS ANGELES, CA 90095, U.S.A.
E-mail address: pjhiatt@math.ucla.edu

SORIN POPA, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA LOS ANGELES, LOS ANGELES, CA 90095, U.S.A.
E-mail address: popa@math.ucla.edu

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