Journal of Engineering Mathematics (2024) 147:1
https://doi.org/10.1007/s10665-024-10372-0

Stabilizing an adverse density difference in the presence of phase change

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Received: 12 January 2024 / Accepted: 4 May 2024

Abstract
Given two phases in equilibrium in a porous solid, the heavy phase lying above the light phase in a gravitational field, we stabilize this adverse density arrangement by heating from below and derive a formula for how steep the temperature gradient must be to do this. The input temperature gradient has two effects on the stability of our system. Its effect on the heat convection is destabilizing, its effect on the heat conduction at the surface is stabilizing. By directing our attention to the case of zero growth rate, we obtain the critical value of the input temperature gradient as it depends on the permeability of the porous solid, the density difference across the surface, the distance between the planes bounding our system, and the physical properties. Our problem makes connections to the Bénard problem where it has two, one, or no critical points, and to the Rayleigh–Taylor problem where it has no critical points.

Keywords Phase-change · Porous media · Rayleigh–Taylor stabilization

1 The problem

In Fig. 1, we have two phases lying in a porous solid in a gravitational field, say, water in equilibrium with water vapor. The phases are separated by the plane $z = 0$. The system is bounded by the planes $z = d$ and $z = -d$. At first, these are no-flow, constant-temperature surfaces. The heavy phase, e.g., water, is the $\star$ phase. It lies between $z = 0$ and $z = d$, above the light phase, water vapor, which lies between $z = -d$ and $z = 0$, whereupon we have a heavy fluid lying above a light fluid. This adverse density difference is unstable. Our view is that heating the light fluid, cooling the heavy fluid, can stabilize this unstable density arrangement. Our aim is to establish that this is so.

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Published online: 18 June 2024

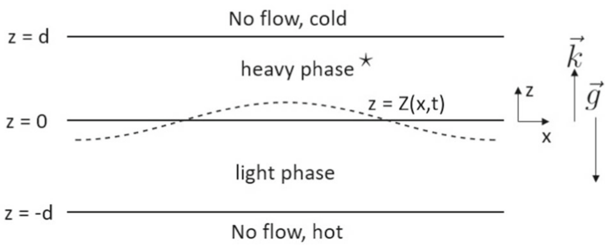


Fig. 1 A heavy phase lying above a light phase in a porous solid

We cite two of the many studies of the vaporization of water in a porous rock. These two, viz., Schubert and Straus [1] and Tsypkin and Il'ichev [2], lead us to try to understand the physics of the stabilization of an adverse density difference. Their models differ in the way the conditions holding at the top and bottom planes bounding the reservoir are set. Their stability predictions result upon solving their dispersion equation for the growth constant, denoted σ, of a small displacement of the surface having a wave number, denoted k. The solution is obtained by numerical methods.

We do not solve our equations numerically. We either drop σ from our domain equations or we direct our attention to the case where σ is zero. This case tells us everything we wish to know about our problem. What draws our attention to this problem is a figure published by Schubert and Straus [1]. It presents a surprising result. In the figure, the growth rate of a small displacement is plotted vs the wave number of the displacement. The surprise is this: the curve may have two neutral points. Yet the Rayleigh–Taylor problem from which this problem is derived has no neutral points. Our aim is to try to understand the existence of two neutral points once two immiscible phases are replaced by two phases in equilibrium and then heated from the high temperature side.

We may also ask: If two phases are in equilibrium in a gravitational field, the heavy phase above the light phase, why would heating from the high temperature side be stabilizing? There are several answers to this question. First, we can see this in nature and that is Schubert and Straus' starting point [1]. Second, heating two gravity-free phases in equilibrium from the high temperature side is known to be stabilizing, cf. Appendix A, and third, we can turn gravity off in the Schubert and Straus problem [1] and obtain stability.

The problem of stabilizing the Rayleigh–Taylor instability by a phase change also arises in the context of film boiling, where a thin layer of vapor lying on a hot solid wall is overlain by its liquid. We direct the readers attention to the work of Tanaka [3] due to the clear physical arguments presented in the construction of a simple model. Likewise, we cite Konalov and Lyubumova [4] who give a more detailed model with clear physical arguments.

Our job is to present a model, i.e., a physical description that does not pretend to be complete but that captures what we think to be the central physical processes at work. Our model will lead to a formula that will explain the effect of these physical processes. We will turn to other conditions holding at the bounding planes, replacing no-flow by constant pressure at the top and bottom (Tsypkin and Il'ichev [2]) and replacing no-flow at the top by constant pressure, retaining no-flow at the bottom (Schubert and Straus [1]).

1.1 Our model

We assume all physical properties are constants. The densities are denoted $\rho^\star$ and ρ. We denote the viscosities by $\mu^\star$ and μ and the thermal diffusivities by $\alpha^\star$ and α. We assume the thermal conductivities are the same and equal to the thermal conductivity of the solid, denoted by λ. This ought to be a fair approximation due to the fact that the porosity of geological rock is of the order of 10^{-2} [1, 5]. The permeability of the porous rock is denoted K.

Surface tension is stabilizing but it is important only at very small wavelengths. Our aim is to stabilize an adverse density difference at large wavelengths. Thus, we do not take surface tension into account.

The pressure and the velocity satisfy Darcy's law in each phase whereupon we have

$$\overrightarrow{v} = -\frac{K}{\mu}(\nabla p - \rho \overrightarrow{g}), \overrightarrow{g} = -g\overrightarrow{k} \tag{1}$$

and

$$\nabla \cdot \overrightarrow{v} = 0 \tag{2}$$

and therefore,

$$\nabla^2 p = 0. \tag{3}$$

The heat equation is

$$\overrightarrow{v} \cdot \nabla T = \alpha \nabla^2 T. \tag{4}$$

Replacing $\overrightarrow{v}$ by $\overrightarrow{v}^\star$, μ by $\mu^\star$, p by $p^\star$, etc. we have the equations holding in the $\star$, or heavy phase. We assume that time derivatives in our domain heat equations need not be taken into account, i.e., we assume the important time dependence is due to the motion of the surface where we denote the surface separating our heavy and light phases by

$$z = Z(x, t).$$

Across this surface, we assume that the temperature and the pressure are continuous and that the mass and the heat balance are given by

$$\rho \overrightarrow{n} \cdot (\overrightarrow{v} - \overrightarrow{u}) = \rho^\star \overrightarrow{n} \cdot (\overrightarrow{v}^\star - \overrightarrow{u}) \tag{5}$$

and

$$\lambda \vec{n} \cdot \nabla T - \lambda \vec{n} \cdot \nabla T^{\star} = \mathcal{L} \vec{n} \cdot (\vec{v} - \vec{u}) \tag{6}$$

where

$$\begin{aligned} \vec{u} &= u \vec{n}, \\ \vec{n} &= \frac{\vec{k} - Z_x \vec{i}}{(1 + Z_x^2)^{\frac{1}{2}}} \end{aligned}$$

and

$$u = \frac{Z_t}{(1 + Z_x^2)^{\frac{1}{2}}}$$

and where $\mathcal{L}$ denotes the latent heat multiplied by ρ. Our problem is closed by assuming that the surface is always at equilibrium, i.e., that the pressure is the vapor pressure at the prevailing temperature.

Our plan is to find out whether or not a small displacement of the equilibrium surface grows or dies out. We write the displacement of the surface

$$Z(x, t) = Z_0 + \epsilon Z_1(x, t),$$

where $Z_1 = \hat{Z}_1 \cos kx \, \mathrm{e}^{\sigma t}$ and our aim is to obtain σ as a function of k.

The base solution, denoted by the subscript zero, is

$$\frac{\mathrm{d}p_0}{\mathrm{d}z} = -\rho g, \quad \frac{\mathrm{d}p_0^{\star}}{\mathrm{d}z} = -\rho^{\star} g$$

and

$$\begin{aligned} \vec{v_0} = \vec{0} &= \vec{v_0}^{\star}, \\ Z_0 &= 0 \end{aligned}$$

and

$$u_0 = 0,$$

where $\frac{\mathrm{d}T_0}{\mathrm{d}z}$, the base temperature gradient, is our input.

1.2 The perturbation problem

We denote the perturbation variables by the subscript 1. The perturbation problem is solved on the base domain, the effect of the displacement of the base surface then appears in the boundary conditions. We write our perturbation variables in the form

$$\begin{aligned} p_1 &= \hat{p}_1(z)\cos kx\, \mathrm{e}^{\sigma t}, \\ v_{z_1} &= \hat{v}_{z_1}(z)\cos kx\, \mathrm{e}^{\sigma t}, \\ T_1 &= \hat{T}_1(z)\cos kx\, \mathrm{e}^{\sigma t}, \\ &\text{etc.} \end{aligned}$$

whereupon our perturbation problem on the light and heavy fluid domains is

$$\frac{d^2\hat{p}_1}{\mathrm{d}z^2} - k^2\hat{p}_1 = 0, \tag{7}$$

$$\hat{v}_{z_1} = -\frac{K}{\mu}\frac{d\hat{p}_1}{\mathrm{d}z}, \tag{8}$$

$$\frac{d^2\hat{T}_1}{\mathrm{d}z^2} - k^2\hat{T}_1 = \hat{v}_{z_1}\frac{1}{\alpha}\frac{\mathrm{d}T_0}{\mathrm{d}z}, \tag{9}$$

$$\frac{d^2\hat{p}_1^\star}{\mathrm{d}z^2} - k^2\hat{p}_1^\star = 0, \tag{10}$$

$$\hat{v}_{z_1}^\star = -\frac{K}{\mu^\star}\frac{d\hat{p}_1^\star}{\mathrm{d}z} \tag{11}$$

and

$$\frac{d^2\hat{T}_1^{\ \star}}{\mathrm{d}z^2} - k^2\hat{T}_1^{\ \star} = \hat{v}_{z_1}^\star\frac{1}{\alpha^\star}\frac{\mathrm{d}T_0}{\mathrm{d}z}. \tag{12}$$

At the top and bottom, we have no-flow, constant-temperature planes bounding our system. Thus, we have

$$at\ z = d\colon \quad \hat{T}_1^{\ \star} = 0 = \hat{v}_{z1}^\star \tag{13}$$

and

$$at\ z = -d\colon \quad \hat{T}_1 = 0 = \hat{v}_{z1}. \tag{14}$$

Across the reference interface, we have

$$at\ z = 0\colon$$

$$\hat{p}_1 + \hat{Z}_1\frac{\mathrm{d}p_0}{\mathrm{d}z} - \left(\hat{p}_1^\star + \hat{Z}_1\frac{\mathrm{d}p_0^\star}{\mathrm{d}z}\right) = 0, \tag{15}$$

$$\hat{T}_1 - \hat{T}_1^{\ \star} = 0, \quad \text{due to } \frac{\mathrm{d}T_0}{\mathrm{d}z} = \frac{\mathrm{d}T_0^\star}{\mathrm{d}z}, \tag{16}$$

$$\rho(\hat{v}_{z1} - \sigma \hat{Z}_1) = \rho^\star(\hat{v}^\star_{z1} - \sigma \hat{Z}_1), \quad \text{due to } v_{z_0} = 0 = v^\star_{z0} \tag{17}$$

and

$$\frac{\mathrm{d}\hat{T}_1}{\mathrm{d}z} - \frac{\mathrm{d}\hat{T}_1^{\ \star}}{\mathrm{d}z} = \frac{\mathcal{L}}{\lambda}(\hat{v}_{z_1} - \sigma \hat{Z}_1), \ \text{due to } \frac{d^2 T_0}{\mathrm{d}z^2} = 0 = \frac{d^2 T^\star_0}{\mathrm{d}z^2}. \tag{18}$$

The problem is closed by writing the equilibrium condition upon displacement of the surface. In terms of $\hat{p}_1$ and $\hat{T}_1$ it is, at $z = 0$,

$$\hat{p}_1 + \hat{Z}_1 \frac{\mathrm{d}p_0}{\mathrm{d}z} = \frac{\mathrm{d}P}{\mathrm{d}T}\left(\hat{T}_1 + \hat{Z}_1 \frac{\mathrm{d}T_0}{\mathrm{d}z}\right).$$

Thus, if the temperature changes by $\mathrm{d}T$, the pressure changes by $\mathrm{d}P$ where

$$\mathrm{d}P = \left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)\mathrm{d}T$$

and where $\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)$ along the equilibrium P vs T curve is given by the Clapeyron equation. Hence, we have, at $z = 0$,

$$\hat{p}_1 = \left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)\hat{T}_1 + \left\{\frac{\mathrm{d}P}{\mathrm{d}T}\frac{\mathrm{d}T_0}{\mathrm{d}z} - \frac{\mathrm{d}p_0}{\mathrm{d}z}\right\}\hat{Z}_1 \tag{19}$$

where $\frac{\mathrm{d}p_0}{\mathrm{d}z} < 0$ is set by gravity, where $\frac{\mathrm{d}T_0}{\mathrm{d}z}$ is our input and where we have

$$\left\{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)\frac{\mathrm{d}T_0}{\mathrm{d}z} - \frac{\mathrm{d}p_0}{\mathrm{d}z}\right\} < 0$$

cf., Fig. 2, i.e., to have vapor, the pressure must be less than the equilibrium pressure, to have liquid, it must be greater.

1.3 Solving the perturbation problem

Taking into account that $\hat{v}_{z_1} = 0$ at $z = -d$ and $\hat{v}^\star_{z1} = 0$ at $z = d$, we solve Eqs. (7), (8), (10), and (11) obtaining

$$\begin{aligned}
\hat{p}_1 &= C\left\{\cosh kz + \tanh kd \ \sinh kz\right\}, \\
\hat{v}_{z1} &= -\frac{K}{\mu}kC\left\{\sinh kz + \tanh kd \ \cosh kz\right\}, \\
\hat{p}^\star_1 &= C^\star\left\{\cosh kz - \tanh kd \ \sinh kz\right\}
\end{aligned}$$

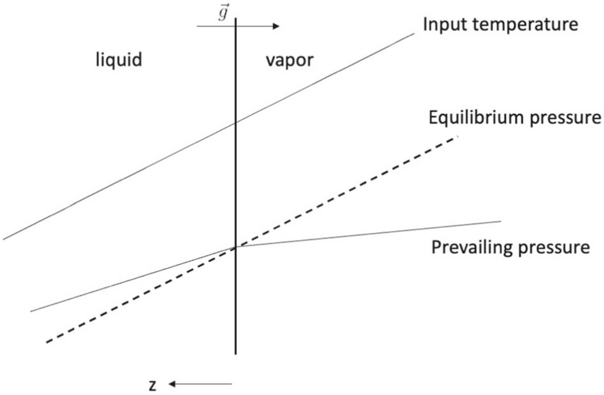


Fig. 2 Gravity acting on liquid and vapor phases in contact in a temperature gradient

and

$$\hat{v}^{\star}_{z1} = -\frac{K}{\mu} \, k \, \frac{\mu}{\mu^{\star}} \, C^{\star} \left\{ \sinh kz - \tanh kd \; \cosh kz \right\}.$$

Then, we solve Eqs. (9) and (12) obtaining

$$\hat{T}_1 = A \cosh kz + B \sinh kz + \frac{K}{\mu} \, kC \left(-\frac{1}{\alpha} \frac{\mathrm{d}T_0}{\mathrm{d}z} \right) \frac{1}{2k} \left\{ z \cosh kz + \tanh kd \; z \sinh kz \right\}$$

and

$$\begin{aligned} \hat{T}_1{}^{\star} &= A^{\star} \cosh kz + B^{\star} \sinh kz \\ &\quad + \frac{K}{\mu} \, k \, \frac{\alpha \mu}{\alpha^{\star} \mu^{\star}} C^{\star} \left(-\frac{1}{\alpha} \frac{\mathrm{d}T_0}{\mathrm{d}z} \right) \frac{1}{2k} \left\{ z \cosh kz - \tanh kd \; z \sinh kz \right\}. \end{aligned}$$

Thus, at $z = 0$, we have

$$\begin{aligned} \hat{p}^{\star}_1 &= C^{\star}, \\ \hat{v}^{\star}_{z1} &= \frac{K}{\mu} k \tanh kd \; \frac{\mu}{\mu^{\star}} C^{\star}, \\ \hat{T}^{\star}_1 &= A^{\star}, \\ \frac{\mathrm{d}\hat{T}^{\star}_1}{\mathrm{d}z} &= kB^{\star} + \frac{K}{\mu} k \left(-\frac{1}{\alpha} \frac{\mathrm{d}T_0}{\mathrm{d}z} \right) \frac{1}{2k} \, \frac{\alpha \mu}{\alpha^{\star} \mu^{\star}} \, C^{\star}, \\ \hat{p}_1 &= C, \\ \hat{v}_{z1} &= -\frac{K}{\mu} k \tanh kd \; C, \\ \hat{T}_1 &= A \end{aligned}$$

and

$$\frac{\mathrm{d}\hat{T}_1}{\mathrm{d}z} = kB + \frac{K}{\mu}k\left(-\frac{1}{\alpha}\frac{\mathrm{d}T_0}{\mathrm{d}z}\right)\frac{1}{2k}\,C.$$

Due to $\hat{T}_1 = 0$ at $z = -d$ and $\hat{T}_1^\star = 0$ at $z = d$, we have

$$A\cosh kd - B\sinh kd - \frac{K}{\mu}\left(-\frac{1}{\alpha}\frac{\mathrm{d}T_0}{\mathrm{d}z}\right)\frac{1}{2k}\frac{kd}{\cosh kd}\,C = 0$$

and

$$A^\star\cosh kd + B^\star\sinh kd + \frac{K}{\mu}\left(-\frac{1}{\alpha}\frac{\mathrm{d}T_0}{\mathrm{d}z}\right)\frac{1}{2k}\frac{kd}{\cosh kd}\frac{\alpha\mu}{\alpha^\star\mu^\star}\,C^\star = 0,$$

whereupon, adding these equations, we have

$$\begin{aligned}(B - B^\star)\tanh kd = A + A^\star - \frac{\mathcal{L}}{\lambda}\frac{K}{\mu}\left(-\frac{1}{\alpha}\frac{\lambda}{\mathcal{L}}\frac{\mathrm{d}T_0}{\mathrm{d}z}\right)\\ \times\frac{1}{2k}\frac{kd}{\cosh^2 kd}\left(C - \frac{\alpha\mu}{\alpha^\star\mu^\star}C^\star\right).\end{aligned} \tag{20}$$

We turn to Eqs. (15), (16), (17), (18), and (19), all at $z = 0$. We write these equations in terms of our unknown constants C, $C^\star$, etc. First, Eq. (16), temperature continuity, tells us $A = A^\star$ and thus $A + A^\star$ in Eq. (20) is $2A$. Then, Eq. (19), the phase equilibrium condition, is

$$C = \frac{\mathrm{d}P}{\mathrm{d}T}A + \left\{\frac{\mathrm{d}P}{\mathrm{d}T}\frac{\mathrm{d}T_0}{\mathrm{d}z} - \frac{\mathrm{d}p_0}{\mathrm{d}z}\right\}\hat{Z}_1 \tag{21}$$

and we turn to Eq. (18), the heat balance across the surface. It is

$$\begin{aligned}k\left(B - B^\star\right) + \frac{\mathcal{L}}{\lambda}\frac{K}{\mu}k\left(-\frac{1}{\alpha}\frac{\lambda}{\mathcal{L}}\frac{\mathrm{d}T_0}{\mathrm{d}z}\right)\frac{1}{2k}\left(C - \frac{\alpha\mu}{\alpha^\star\mu^\star}C^\star\right)\\ = \frac{\mathcal{L}}{\lambda}\left\{-\frac{K}{\mu}k\tanh kd\,C - \sigma\hat{Z}_1\right\}\end{aligned}$$

whereupon dividing by k and multiplying by $\tanh kd$, we have

$$\begin{aligned}\left(B - B^\star\right)\tanh kd = -\frac{\mathcal{L}}{\lambda}\frac{K}{\mu}\tanh kd\left(-\frac{1}{\alpha}\frac{\lambda}{\mathcal{L}}\frac{\mathrm{d}T_0}{\mathrm{d}z}\right)\frac{1}{2k}\left(C - \frac{\alpha\mu}{\alpha^\star\mu^\star}C^\star\right)\\ - \frac{\mathcal{L}}{\lambda}\frac{K}{\mu}\tanh^2 kd\left(C - \frac{\sigma}{\frac{K}{\mu}k\tanh kd}\hat{Z}_1\right).\end{aligned} \tag{22}$$

Then, using Eq. (20) to eliminate $B - B^\star$ and Eq. (21) to eliminate $2A$, we have

$$-\frac{\mathrm{d}P}{\mathrm{d}T}\frac{\mathcal{L}}{\lambda}\frac{K}{\mu}\tanh^2 kd\left\{C + \frac{\sigma}{\frac{K}{\mu}k\tanh kd}\hat{Z}_1\right\} = 2C - 2\left\{\frac{\mathrm{d}P}{\mathrm{d}T}\frac{\mathrm{d}T_0}{\mathrm{d}z} - \frac{\mathrm{d}p_0}{\mathrm{d}z}\right\}\hat{Z}_1$$
$$+\frac{\mathrm{d}P}{\mathrm{d}T}\frac{\mathcal{L}}{\lambda}\frac{K}{\mu}\left(-\frac{1}{\alpha}\frac{\lambda}{\mathcal{L}}\frac{\mathrm{d}T_0}{\mathrm{d}z}\right)\frac{1}{2}d\frac{1}{kd}\left(\tanh kd - \frac{kd}{\cosh^2 kd}\right)\left(C - \frac{\alpha\mu}{\alpha^\star\mu^\star}C^\star\right) \quad (23)$$

and this is an equation in C, $C - \frac{\alpha\mu}{\alpha^\star\mu^\star}C^\star$ and $\hat{Z}_1$.

We turn to Eqs. (15) and (17), pressure continuity, and the mass balance across the surface, to close our problem. Equations (15) and (17) tell us

$$C - C^\star + (\rho^\star - \rho)g\hat{Z}_1 = 0$$

and

$$-\frac{K}{\mu}k\tanh kd\ \rho C - \rho\sigma\hat{Z}_1 = \frac{K}{\mu}k\tanh kd\ \rho^\star\frac{\mu}{\mu^\star}C^\star - \rho^\star\sigma\hat{Z}_1$$

whereupon we have

$$C - \frac{\alpha\mu}{\alpha^\star\mu^\star}C^\star = \left(1 + \frac{\alpha\rho}{\alpha^\star\rho^\star}\right)C - \frac{\alpha}{\alpha^\star}\frac{\rho^\star - \rho}{\rho^\star}\frac{\sigma}{\frac{K}{\mu}k\tanh kd}\hat{Z}_1 \quad (24)$$

and

$$\left(1 + \frac{\mu^\star\rho}{\mu\rho^\star}\right)C = \frac{\mu^\star}{\mu}\frac{(\rho^\star - \rho)}{\rho^\star}\frac{\sigma}{\frac{K}{\mu}k\tanh kd}\hat{Z}_1 - (\rho^\star - \rho)g\hat{Z}_1. \quad (25)$$

We notice that at σ equal to zero

$$\operatorname{sgn}\left(C - \frac{\alpha\mu}{\alpha^\star\mu^\star}C^\star\right) = \operatorname{sgn}(C)$$

and

$$\operatorname{sgn}(C) = -\operatorname{sgn}(\hat{Z}_1).$$

Thus, we have three homogeneous, linear equations in C, $C - \frac{\alpha\mu}{\alpha^\star\mu^\star}C^\star$, and $\hat{Z}_1$, viz., Eqs. (23), (24), and (25). These equations tell us how σ must depend on k in order that we have a solution other than C, $C^\star$, and $\hat{Z}_1$ all zero.

We derive our equation for the growth rate, σ, of a small displacement in Appendix A. It gives the growth rate in terms of the wave number, k, of the displacement and in terms of the input variables, e.g., $-\frac{\mathrm{d}T_0}{\mathrm{d}z}$, $\rho^\star - \rho$, K, etc. It is explicit in σ due to our guess that the only important time derivatives are those having to do with the motion of the surface.

Before we do this, we set σ to zero and obtain a formula which gives us the wave numbers where the growth rates are zero. This will be our most important formula. It will tell us the critical values of the input variables. And, at σ equal to zero, dropping time derivatives in the domain equations is no longer a guess.

To do this, we set σ to zero in Eq. (23), we replace $C - \frac{\alpha\mu}{\alpha^\star\mu^\star}C^\star$ by $\left(1+\frac{\alpha\rho}{\alpha^\star\rho^\star}\right)C$, we multiply by $\left(1+\frac{\mu^\star\rho}{\mu\rho^\star}\right)$ and we set $\left(1+\frac{\mu^\star\rho}{\mu\rho^\star}\right)C$ to $-(\rho^\star-\rho)g\hat{Z}_1$, whereupon we obtain

$$
\begin{aligned}
&-\frac{\mathrm{d}P}{\mathrm{d}T}\frac{\mathcal{L}}{\lambda}\frac{K}{\mu}\tanh^2 kd\,(-(\rho^\star-\rho)g)\\
&= 2(-(\rho^\star-\rho)g) - 2\left(1+\frac{\mu^\star\rho}{\mu\rho^\star}\right)\left\{\frac{\mathrm{d}P}{\mathrm{d}T}\frac{\mathrm{d}T_0}{\mathrm{d}z}-\frac{\mathrm{d}P_0}{\mathrm{d}z}\right\}\\
&\quad+\frac{\mathrm{d}P}{\mathrm{d}T}\frac{\mathcal{L}}{\lambda}\frac{K}{\mu}\left(-\frac{1}{\alpha}\frac{\lambda}{\mathcal{L}}\frac{\mathrm{d}T_0}{\mathrm{d}z}\right)\left(1+\frac{\alpha\rho}{\alpha^\star\rho^\star}\right)\frac{1}{2}d\frac{1}{kd}\\
&\quad\times\left(\tanh kd-\frac{kd}{\cosh^2 kd}\right)\left(-(\rho^\star-\rho)g\right).
\end{aligned}
$$

We then rewrite this

$$
\begin{aligned}
\frac{\frac{-2\left(1+\frac{\mu^\star\rho}{\mu\rho^\star}\right)\left\{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)\frac{\mathrm{d}T_0}{\mathrm{d}z}-\frac{\mathrm{d}p_0}{\mathrm{d}z}\right\}}{(\rho^\star-\rho)g}-2}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)\frac{\mathcal{L}}{\lambda}\frac{K}{\mu}} &= \left(-\frac{1}{\alpha}\frac{\lambda}{\mathcal{L}}\frac{\mathrm{d}T_0}{\mathrm{d}z}\right)\left(1+\frac{\alpha\rho}{\alpha^\star\rho^\star}\right)\frac{1}{2}d\frac{1}{kd}\\
&\quad\times\left(\tanh kd-\frac{kd}{\cosh^2 kd}\right)+\tanh^2 kd
\end{aligned}
$$

and defining $\mathcal{A}$ and $\mathcal{B}$ via

$$
\mathcal{A}=\frac{\frac{-2\left(1+\frac{\mu^\star\rho}{\mu\rho^\star}\right)\left\{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)\frac{\mathrm{d}T_0}{\mathrm{d}z}-\frac{\mathrm{d}p_0}{\mathrm{d}z}\right\}}{(\rho^\star-\rho)g}-2}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)\frac{\mathcal{L}}{\lambda}\frac{K}{\mu}}
$$

and

$$
\mathcal{B}=\left(-\frac{1}{\alpha}\frac{\lambda\frac{\mathrm{d}T_0}{\mathrm{d}z}}{\mathcal{L}}\right)\frac{1}{2}d\left(1+\frac{\alpha\rho}{\alpha^\star\rho^\star}\right)>0,
$$

we have our formula for the wave numbers where a small displacement is neither growing nor dying out, i.e., where the growth rate is zero. It is

$$
\mathcal{A}=\mathcal{B}\frac{1}{kd}\left\{\tanh kd-\frac{kd}{\cosh^2 kd}\right\}+\tanh^2 kd\equiv\text{RHS}, \tag{26}
$$

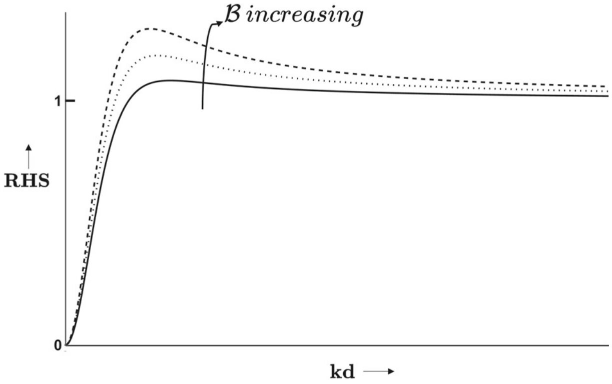


Fig. 3 The curve RHS vs kd at increasing values of $\mathcal{B}$

where $\mathcal{A}$ and $\mathcal{B}$ do not depend upon k and where $\left\{\frac{\mathrm{d}P}{\mathrm{d}T}\frac{\mathrm{d}T_0}{\mathrm{d}z} - \frac{\mathrm{d}p_0}{\mathrm{d}z}\right\}$ must be negative. This formula is written in terms of dimensionless variables, i.e.,

$$kd, \frac{\left\{\frac{\mathrm{d}P}{\mathrm{d}T}\frac{\mathrm{d}T_0}{\mathrm{d}z} - \frac{\mathrm{d}p_0}{\mathrm{d}z}\right\}}{(\rho^\star - \rho)g}, \frac{\mathrm{d}P}{\mathrm{d}T}\frac{\mathcal{L}}{\lambda}\frac{K}{\mu} \text{ and } \left(-\frac{1}{\alpha}\frac{\lambda}{\mathcal{L}}\frac{\mathrm{d}T_0}{\mathrm{d}z}\right)\frac{1}{2}d$$

are all dimensionless.

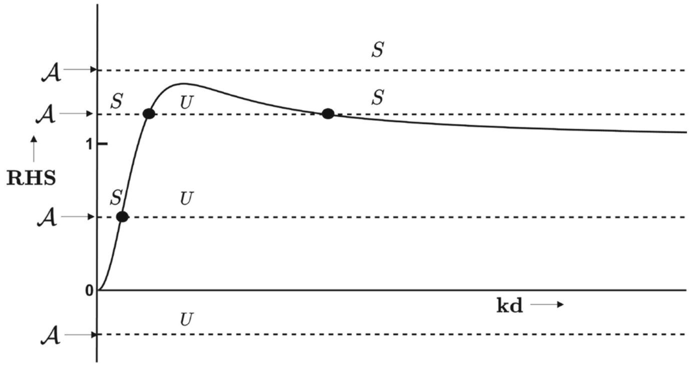
The right-hand side, viz.,

$$\text{RHS} = \mathcal{B}\frac{1}{kd}\left\{\tanh kd - \frac{kd}{\cosh^2 kd}\right\} + \tanh^2 kd$$

is not negative. It is bounded and, unless $\mathcal{B}$ is near zero, as kd increases from zero it rises from zero to its peak value before falling and leveling off at 1.

To obtain the wave numbers where growth rates are zero, we first plot the right-hand side of Eq. (26) vs kd, cf., Fig. 3. Along each curve growth rates are zero, below each curve growth rates are positive, above each curve growth rates are negative. The larger the value of $\mathcal{B}$ the larger the region of positive growth rates, i.e., increasing $\mathcal{B}$ is destabilizing. This is one effect of $\frac{\mathrm{d}T_0}{\mathrm{d}z}$ and it tells us that heat convection is destabilizing.

Then, having set the value of $\mathcal{B}$ and having drawn the RHS vs kd curve, we obtain the kd's at zero growth rate by setting $\mathcal{A}$ and noticing where the horizontal line along which $\mathcal{A}$ is constant intersects the RHS vs kd curve. Thus, we have different numbers of neutral points depending on the value of $\mathcal{A}$. If $\mathcal{A}$ is negative, all displacements are unstable. If $\mathcal{A}$ lies on (0,1), we have one neutral point. If $\mathcal{A}$ is greater than 1 but less than the peak of the RHS vs kd curve, we have two neutral points and if $\mathcal{A}$ lies above the peak all displacements are stable, cf., Fig. 4. Above $\mathcal{A}$ equal to 1, we have 2 or 1 or 0 neutral points. Thus, our problem has a connection to the Bénard problem, cf., Appendix D. Below $\mathcal{A}$ equal to 1 and especially below $\mathcal{A}$ equal to zero, our problem makes its connection to the Rayleigh–Taylor problem. Figure 4 shows a



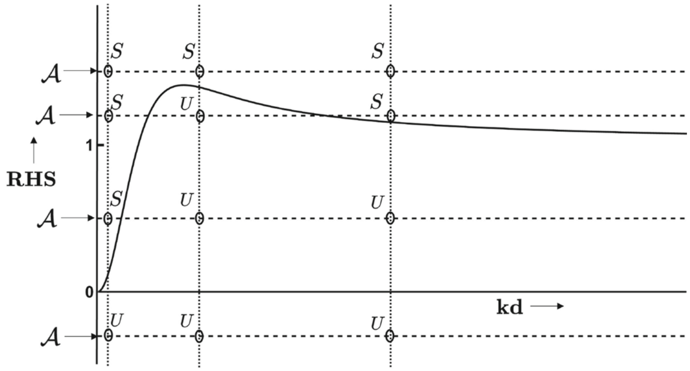


Fig. 4 Given $\mathcal{B}$ and the curve RHS vs. kd, **a** neutral points, denoted by •'s, at increasing values of $\mathcal{A}$ and **b** stability, denoted by S or U, of small displacements of wave numbers kd at increasing values of $\mathcal{A}$

sketch revealing the information contained in Eq. (26). Neutral points are apparent. But there is more. Given $\mathcal{A}$ and $\mathcal{B}$, we can say whether a displacement of a given wave number is or is not stable. The larger the value of $\mathcal{A}$, the greater the stability, and all displacements are stable whenever $\mathcal{A}$ lies above the peak of the RHS vs kd curve at $\mathcal{B}$. This is the second effect of $\frac{\mathrm{d}T_0}{\mathrm{d}z}$ and it is stabilizing.

In our view, $\frac{\mathrm{d}T_0}{\mathrm{d}z}$ is the input variable, all else remaining fixed. Both $\mathcal{A}$ and $\mathcal{B}$ depend on $\frac{\mathrm{d}T_0}{\mathrm{d}z}$. In Fig. 5, we plot two cases, one where $|\frac{\mathrm{d}T_0}{\mathrm{d}z}|$ is large and $\mathcal{A}$ lies above the peak of the RHS vs kd curve, the other where $|\frac{\mathrm{d}T_0}{\mathrm{d}z}|$ is small and $\mathcal{A}$ lies below the peak. Thus, if we start a series of experiments by setting $|\frac{\mathrm{d}T_0}{\mathrm{d}z}|$ to a large value, at first $\mathcal{A}$ lies above the peak of the RHS vs kd curve and all displacements are stable. Then upon decreasing $|\frac{\mathrm{d}T_0}{\mathrm{d}z}|$, both will fall and $\mathcal{A}$ will reach the peak of the RHS vs kd curve, also falling, whereupon $|\frac{\mathrm{d}T_0}{\mathrm{d}z}|$ will have reached its critical value, i.e., a further decrease will open an interval of unstable wave numbers, i.e., displacements having positive growth rates. Figure 6 illustrates the result of this critical point construction.

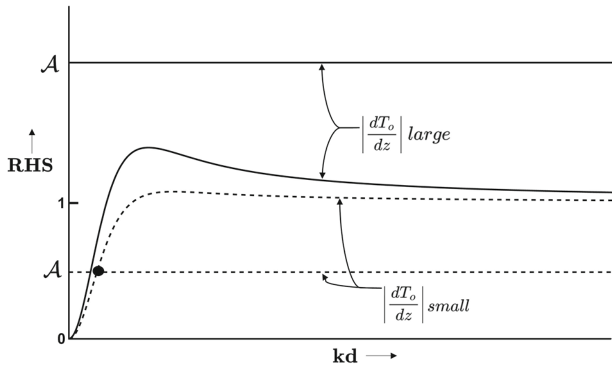


Fig. 5 The line $\mathcal{A}$ constant and the curve RHS vs kd at two values of $|\frac{\mathrm{d}T_0}{\mathrm{d}z}|$, one large, the other small

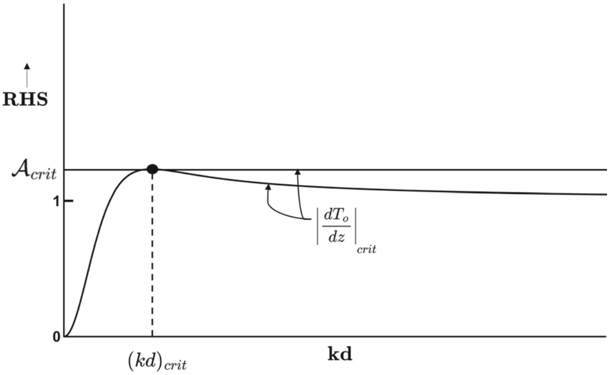


Fig. 6 Obtaining the critical value of the input temperature gradient

We can ask: Can the permeability be too large, i.e., is it always true, upon increasing $|\frac{\mathrm{d}T_0}{\mathrm{d}z}|$, that $\mathcal{A}$ rises faster than the peak of the curve RHS vs kd? Surely, if the permeability is large enough $\mathcal{A}$ cannot rise fast enough. We can find an upper bound on K at large values of the temperature gradient. At larger K's , $\mathcal{A}$ does not rise faster than the peak of the curve. Thus, to have $\mathcal{A}$ rise fast enough, assuming we have large values of $-\frac{\mathrm{d}T_0}{\mathrm{d}z}$, we must have

$$\frac{1}{K} > \frac{1}{4}(\rho^\star - \rho)g\,d\,\frac{\frac{1}{\alpha}\left(1+\frac{\alpha\rho}{\alpha^\star\rho^\star}\right)}{\mu\left(1+\frac{\mu^\star\rho}{\mu\rho^\star}\right)} \times \text{greatest value of}\frac{1}{kd}\left\{\tanh kd - \frac{kd}{\cosh^2 kd}\right\},$$

where the greatest value of $\frac{1}{kd}\left\{\tanh kd - \frac{kd}{\cosh^2 kd}\right\}$ is about $\frac{1}{2}$ and where the smaller d, the larger the range of stable K's.

1.4 The far-field conditions of Tsypkin and Il'ichev [2] and those of Schubert and Straus [1]—the case of zero growth rate

Tsypkin and Il'ichev [2] assume the far-field boundaries are constant pressure planes. Thus, assuming $\hat{p}_1^\star = 0$ at $z = d$ and $\hat{p}_1 = 0$ at $z = -d$, we obtain, at $\sigma = 0$, the formula

$$\mathcal{A} = \mathcal{B}\frac{\tanh kd}{kd} + 1 \equiv \text{RHS}. \tag{27}$$

For large values of d, Eq. (27) agrees with the result of setting d to infinity obtained in Appendix C. Unlike Tsypkin and Il'ichev, we have retained heat convection, though heat convection is always unimportant at large values of kd.

We plot RHS vs kd in Fig. 7. Along the curve σ is zero, above it σ is negative, and below it σ is positive. For $\mathcal{A}$ less than 1, all displacements are unstable. For $\mathcal{A}$ greater than $\mathcal{B}+1$, all displacements are stable, and for $1 < \mathcal{A} < \mathcal{B}+1$, there is one neutral point. Strangely, the formula is $\mathcal{A} = 1$ if heat convection is dropped.

If we assume that only the top boundary is a constant pressure plane, we have the model of Schubert and Straus, and we obtain, at $\sigma = 0$, the formula

$$\begin{aligned}
\text{LHS} &= \frac{\dfrac{-2\left(1+\frac{\mu^\star\rho}{\mu\rho^\star}\tanh^2 kd\right)\left\{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)\frac{\mathrm{d}T_0}{\mathrm{d}z}-\frac{\mathrm{d}p_0}{\mathrm{d}z}\right\}}{(\rho^\star-\rho)g} - 2}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)\frac{\mathcal{L}}{\lambda}\frac{K}{\mu}} \\
&= \left(-\frac{1}{\alpha}\frac{\lambda}{\mathcal{L}}\frac{\mathrm{d}T_0}{\mathrm{d}z}\right)\frac{1}{2}d\,\frac{1}{kd}\left(\tanh kd - \frac{kd}{\cosh^2 kd} + \frac{\alpha\rho}{\alpha^\star\rho^\star}\tanh^3 kd\right) + \tanh^2 kd \\
&= \text{RHS},
\end{aligned}$$

where both LHS and RHS depend on kd and where LHS goes to $\mathcal{A}$ and RHS goes to 1 as kd grows large.

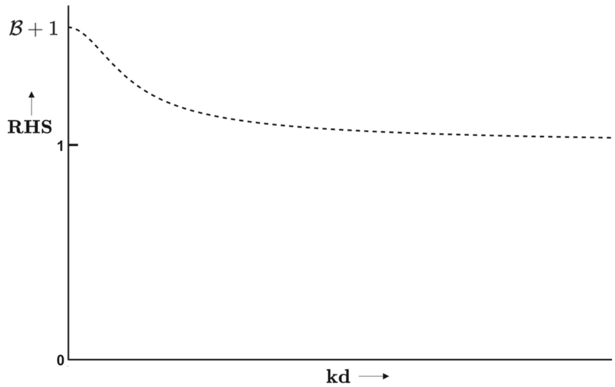
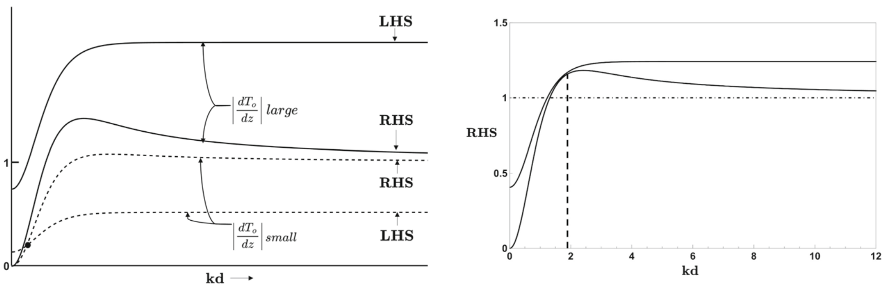


Fig. 7 The curve RHS vs kd for the model of Tsypkin and Il'ichev [2]



(a) The curves LHS and RHS vs kd given a large value of $|\frac{dT_0}{dz}|$ where all displacements are stable and a small value of $|\frac{dT_0}{dz}|$ where there is one neutral point due to $\mathcal{A} < 1$ and where all displacements to the right of the neutral point are unstable.

(b) The curves LHS and RHS vs kd at the critical value of $|\frac{dT_0}{dz}|$

Fig. 8 The model of Schubert and Straus [1]

In Fig. 8a, we plot LHS and RHS vs kd for two cases, one where $|\frac{\mathrm{d}T_0}{\mathrm{d}z}|$ is large, and the other where it is small. In the first case, all displacements are stable, in the second all kd's lying to the right of the neutral point are unstable. Then, reducing $|\frac{\mathrm{d}T_0}{\mathrm{d}z}|$ from its original large value, we reach the critical value of $|\frac{\mathrm{d}T_0}{\mathrm{d}z}|$, cf., Fig. 8b.

1.5 The advantage of having a formula

Equation (26) tells us the wave numbers of displacements whose growth rates are zero. This information and more is displayed in Fig. 4. Thus, having $\mathcal{A}$ and $\mathcal{B}$, we can say whether a displacement of wave number kd is or is not stable, though we cannot say what the growth or decay rate is. To do this, the formula for σ in Appendix A is needed.

We rewrite $-\{\frac{\mathrm{d}P}{\mathrm{d}T}(\frac{\mathrm{d}T_0}{\mathrm{d}z})-\frac{\mathrm{d}p_0}{\mathrm{d}z}\}$ as $\{\frac{\mathrm{d}P}{\mathrm{d}T}(-\frac{\mathrm{d}T_0}{\mathrm{d}z})-\rho g\}$. Then, we observe that g appears twice in $\mathcal{A}$ but not in $\mathcal{B}$. Its effect is always destabilizing. The larger the g, the smaller the $\mathcal{A}$ and the larger the range of unstable wave numbers.

Likewise $\frac{\mathrm{d}P}{\mathrm{d}T}$ appears twice in $\mathcal{A}$ but not in $\mathcal{B}$. Its two effects oppose one another and for large enough $\frac{\mathrm{d}P}{\mathrm{d}T}$, i.e., large enough $\mathcal{L}$, its effect on $\mathcal{A}$ is small.

The inputs $\rho^\star - \rho$ and K appear in $\mathcal{A}$ but not in $\mathcal{B}$, the input d appears in $\mathcal{B}$ but not in $\mathcal{A}$ and the input $-\frac{\mathrm{d}T_0}{\mathrm{d}z}$ appears in both $\mathcal{A}$ and $\mathcal{B}$.

Upon setting $\mathcal{B}$, and thus the RHS vs kd curve, at small enough values of either $\rho^\star - \rho$ or K, $\mathcal{A}$ is large enough that all the displacements are stable. Then, we can obtain a critical value of $\mathcal{A}$ by increasing either $\rho^\star - \rho$ or K.

From $\mathcal{A}_{crit}$ there obtains $(\rho^\star - \rho)_{crit}$ or K_{crit}, all else being held fixed. Increasing either $\rho^\star - \rho$ or K increases the flow and thus the convection of heat.

Upon setting $\mathcal{A}$ greater than 1, at small enough values of d, $\mathcal{B}$ is small, and the peak of the RHS vs kd curve is near 1. All displacements are then stable. This is a low flow limit where the far-field planes are blocking the upward and downward flows. Increasing d, i.e., increasing $\mathcal{B}$, the flows are increasing and the peak of the RHS vs kd

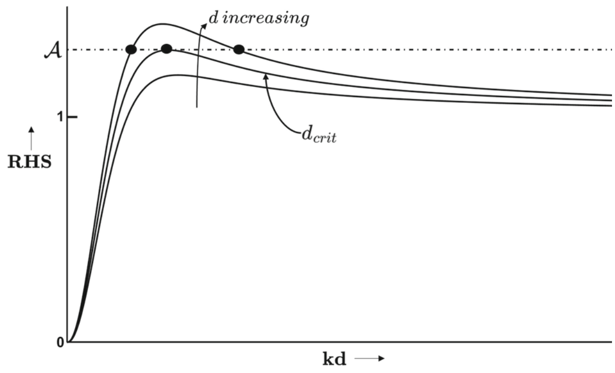


Fig. 9 Given $\mathcal{A}$ greater than 1, the curve RHS vs kd at increasing values of d passing through d_{crit}

curve rises to the value of $\mathcal{A}$, cf., Fig. 9. Thus, d reaches its critical value. Increasing d a little more opens an interval of unstable wave numbers.

Turning to the input temperature gradient we see that $\{\frac{\mathrm{d}P}{\mathrm{d}T}(-\frac{\mathrm{d}T_0}{\mathrm{d}z}) - \rho g\}$ must be positive and therefore $-\frac{\mathrm{d}T_0}{\mathrm{d}z}$ must be positive. This maintains liquid over vapor in the face of the pressure decreasing downward. Then, we see that unless

$$\frac{2\left(1+\frac{\mu^\star \rho}{\mu \rho^\star}\right)\left\{\frac{\mathrm{d}P}{\mathrm{d}T}\left(-\frac{\mathrm{d}T_0}{\mathrm{d}z}\right)-\rho g\right\}}{(\rho^\star-\rho)g}$$

exceeds 2, we have $\mathcal{A}$ negative and all displacements are unstable. Upon increasing $-\frac{\mathrm{d}T_0}{\mathrm{d}z}$, we have $\mathcal{A}$, $\mathcal{B}$, and the peak of the RHS vs kd curve all rising. Rising $\mathcal{A}$ is stabilizing, rising $\mathcal{B}$ is destabilizing. These are the two effects of $-\frac{\mathrm{d}T_0}{\mathrm{d}z}$. Their physics is explained in the next section.

1.6 What is going on?

We can attach a physical interpretation to what our equations are telling us. To do this, we direct our attention to a neutral state, i.e., a steady state where the growth rate is zero and where the displacement is then a standing wave whose wavelength is the prediction of our model. Upon setting σ to zero, we have

$$\mathrm{sgn}(C) = -\mathrm{sgn}(\hat{Z}_1),$$

whereupon water is evaporating at a trough and water vapor is condensing at a crest. The flows are sketched in Fig. 10. Heat must be supplied by our input temperature gradient to account for the evaporation at a trough and downward heat convection must be overcome to do this.

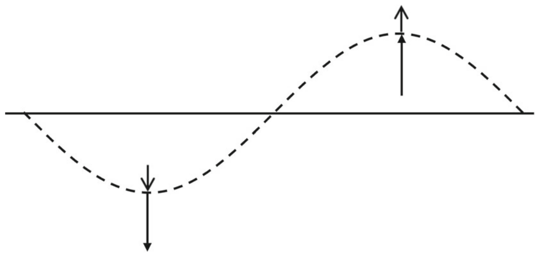


Fig. 10 A sketch of the flows at a neutral state, downward at a trough, upward at a crest

Setting σ to zero in Eq. (22) and then eliminating $(\mathcal{B}-\mathcal{B}^\star)$ via Eq. (20), we have

$$-\frac{\mathcal{L}}{\lambda}\frac{k}{\mu}\tanh^2 kd\, C = 2A + \frac{\mathcal{L}}{\lambda}\frac{K}{\mu}\left(-\frac{1}{\alpha}\frac{\lambda}{\mathcal{L}}\frac{\mathrm{d}T_0}{\mathrm{d}z}\right)\frac{1}{2}d\frac{1}{kd}\left(\tanh kd - \frac{kd}{\cosh^2 kd}\right)\left(C - \frac{\alpha\mu}{\alpha^\star\mu^\star}C^\star\right).$$

Then, setting σ to zero in Eq. (24), we have

$$\left(C - \frac{\alpha\mu}{\alpha^\star\mu^\star}C^\star\right) = \left(1 + \frac{\alpha\rho}{\alpha^\star\rho^\star}\right)C,$$

whereupon we have

$$\mathrm{sgn}(A) = -\mathrm{sgn}(C),$$

where $A = \hat{T}_1(z=0)$.

Thus, at a trough $\hat{T}_1$ must be negative and therefore upon displacement the temperature gradient is sharpened. It then remains to find the wave number at which the sharpened temperature gradient overcomes the adverse heat convection and balances the heat of vaporization at a trough.

In fact, the stronger the base temperature gradient, the more it is sharpened on displacement. This is explained in the Appendix B. But we can turn to

$$\hat{p}_1 + \hat{Z}_1\frac{\mathrm{d}p_0}{\mathrm{d}z} = \frac{\mathrm{d}P}{\mathrm{d}T}\left(\hat{T}_1 + \hat{Z}_1\frac{\mathrm{d}T_0}{\mathrm{d}z}\right),$$

which holds at the base surface assuming the displaced surface is in equilibrium. Then, if $\frac{\mathrm{d}P}{\mathrm{d}T}$ is large enough, we see that $\hat{T}_1$ is nearly $-\hat{Z}_1\frac{\mathrm{d}T_0}{\mathrm{d}z}$. Thus at a trough $\hat{T}_1$ is more negative the more negative $\frac{\mathrm{d}T_0}{\mathrm{d}z}$, i.e., the sharper the base temperature the more it is sharpened upon displacement of the surface.

1.7 Conclusion

Assuming we have two phases in equilibrium in a gravitational field, the heavy phase lying above the light phase, we can stabilize this adverse density difference by heating from below and by setting the temperature gradient strong enough. We derive a formula from which we can obtain the critical value of this input temperature gradient where for all stronger gradients, all small displacements are stable.

The input temperature gradient exerts two effects on the stability of a small displacement. It strengthens the convection of heat. This is destabilizing. It steepens the temperature gradient near a displaced surface. This is stabilizing. We can be sure that these two effects come into balance at a critical input gradient by making sure the permeability of the porous rock is small enough. The dependence of the stability of a small displacement of a given wave number on the temperature gradient, the density difference, the permeability, the vertical height of our two-phase system, etc. are predicted by our zero growth rate formula.

Appendix A: Obtaining a formula for the growth rate of a small displacement

Just before we set σ to zero we have Eqs. (23), (24), and (25). They are

$$\begin{aligned} &-\frac{\mathrm{d}P}{\mathrm{d}T}\frac{\mathcal{L}}{\lambda}\frac{K}{\mu}\tanh^2 kd\left\{C+\frac{\sigma}{\frac{K}{\mu}k\tanh kd}\hat{Z}_1\right\} \\ &= 2C - 2\left\{\frac{\mathrm{d}P}{\mathrm{d}T}\frac{\mathrm{d}T_0}{\mathrm{d}z}-\frac{\mathrm{d}p_0}{\mathrm{d}z}\right\}\hat{Z}_1 \\ &\quad+\frac{\mathrm{d}P}{\mathrm{d}T}\frac{\mathcal{L}}{\lambda}\frac{K}{\mu}\left(-\frac{1}{\alpha}\frac{\lambda}{\mathcal{L}}\frac{\mathrm{d}T_0}{\mathrm{d}z}\right)\frac{1}{2}d\frac{1}{kd}\left(\tanh kd-\frac{kd}{\cosh^2 kd}\right)\left(C-\frac{\alpha\mu}{\alpha^\star\mu^\star}C^\star\right), \end{aligned} \tag{28}$$

$$C-\frac{\alpha\mu}{\alpha^\star\mu^\star}C^\star=\left(1+\frac{\alpha\rho}{\alpha^\star\rho^\star}\right)C-\frac{\alpha}{\alpha^\star}\frac{\rho^\star-\rho}{\rho^\star}\frac{\sigma}{\frac{K}{\mu}k\tanh kd}\hat{Z}_1 \tag{29}$$

and

$$\left(1+\frac{\mu^\star\rho}{\mu\rho^\star}\right)C=\frac{\mu^\star}{\mu}\frac{(\rho^\star-\rho)}{\rho^\star}\frac{\sigma}{\frac{K}{\mu}k\tanh kd}\hat{Z}_1-(\rho^\star-\rho)g\,\hat{Z}_1. \tag{30}$$

These are linear, homogeneous equations for C, $C^\star$, and $\hat{Z}_1$ and we wish to have a non-zero solution. Thus, eliminating $C-\frac{\alpha\mu}{\alpha^\star\mu^\star}C^\star$ and then C and requiring that $\hat{Z}_1$ not be zero, we obtain our equation for the growth rate of a small disturbance. It is

$$\begin{aligned} &\mathcal{A}-\tanh^2(kd)-\mathcal{B}\frac{1}{kd}\left(\tanh(kd)-\frac{kd}{\cosh^2(kd)}\right) \\ &\quad=-\frac{\sigma}{\frac{K}{\mu}k\tanh(kd)}\frac{d}{(\rho^\star-\rho)g}\left\{\left(1+\frac{\mu^\star}{\mu}\right)\tanh^2(kd)+\frac{2\frac{\mu^\star}{\mu}\left(1-\frac{\rho}{\rho^\star}\right)}{\frac{\mathrm{d}P}{\mathrm{d}T}\frac{\mathcal{L}}{\lambda}\frac{K}{\mu}}\right. \\ &\qquad\left.+\frac{1-\frac{\rho}{\rho^\star}}{1+\frac{\alpha\rho}{\alpha^\star\rho^\star}}\left(\frac{\mu^\star}{\mu}-\frac{\alpha}{\alpha^\star}\right)\mathcal{B}\frac{1}{kd}\left(\tanh(kd)-\frac{kd}{\cosh^2(kd)}\right)\right\}, \end{aligned} \tag{31}$$

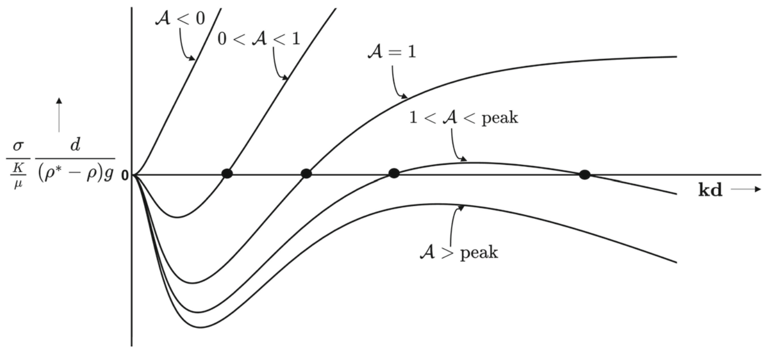


Fig. 11 The growth rate of a small displacement as a function of its wave number. The curve RHS vs kd is held fixed. The values of $\mathcal{A}$ are increasing such that at the least $\mathcal{A}$ all wave numbers are unstable until at the greatest $\mathcal{A}$ all wave numbers are stable

where ordinarily we have $\frac{\mu^\star}{\mu} > 1$ and $\frac{\alpha}{\alpha^\star} < 1$.

We sketch $\frac{\sigma}{\frac{K}{\mu}} \frac{d}{(\rho^\star - \rho)g}$ vs kd in Fig. 11 for $\mathcal{A} < 0$, $0 < \mathcal{A} < 1$, $\mathcal{A} = 1$, $1 < \mathcal{A} <$ the peak of the curve $\mathcal{B}\frac{1}{kd}\left(\tanh kd - \frac{kd}{\cosh^2 kd}\right) + \tanh^2 kd$ vs kd and $\mathcal{A} >$ the peak of this curve, where $\mathcal{B}$ is held fixed.

Limiting formulas

We give a few limiting forms of Eq. (31). First, we assume that the latent heat is very large. Then, we have

$$-\tanh^2 kd = -\frac{\sigma}{\frac{K}{\mu} k\, d\ \tanh kd} \frac{d}{(\rho^\star - \rho)g} \left(1 + \frac{\mu^\star}{\mu}\right) \tanh^2 kd$$

or

$$\left(1 + \frac{\mu^\star}{\mu}\right) \frac{\sigma}{\frac{K}{\mu} k \tanh kd} = (\rho^\star - \rho)g.$$

This is the Rayleigh–Taylor formula. Large latent heat implies small vaporization and in this limit, we recover the immiscible fluids result, e.g., water lying above steam in a porous rock.

Second, we turn gravity off. To do this we multiply our equation by $(\rho^\star - \rho)g$, set g to zero and obtain

$$\frac{-2\left(1+\frac{\mu^\star\rho}{\mu\rho^\star}\right)\left\{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)\frac{\mathrm{d}T_0}{\mathrm{d}z}-\frac{\mathrm{d}p_0}{\mathrm{d}z}\right\}}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)\frac{\mathcal{L}}{\lambda}\frac{K}{\mu}}$$

$$= -\frac{\sigma}{\frac{K}{\mu}k\tanh kd}\left\{\left(1+\frac{\mu^\star}{\mu}\right)\tanh^2 kd + \frac{2\frac{\mu^\star}{\mu}\left(1-\frac{\rho}{\rho^\star}\right)}{\frac{\mathrm{d}P}{\mathrm{d}T}\frac{\mathcal{L}}{\lambda}\frac{K}{\mu}} + \text{etc.}\right\}.$$

Thus, given $\frac{\mathrm{d}T_0}{\mathrm{d}z} < 0$, we have σ negative for all positive wave numbers. It is for this reason that we propose to stabilize an adverse density difference by heating from below.

Now assume that kd is large. Then, we have

$$\mathcal{A}-1 = -\frac{\sigma}{\frac{K}{\mu}}\frac{d}{(\rho^\star-\rho)g}\frac{1}{kd}\left\{\left(1+\frac{\mu^\star}{\mu}\right)+\frac{2\frac{\mu^\star}{\mu}\left(1-\frac{\rho}{\rho^\star}\right)}{\frac{\mathrm{d}P}{\mathrm{d}T}\frac{\mathcal{L}}{\lambda}\frac{K}{\mu}}\right\},$$

whereupon σ is a negative multiple of $kd(\mathcal{A}-1)$. Thus, as kd increases, σ is positive and increasing or negative and decreasing according as $\mathcal{A}$ is greater or less than 1.

If on the other hand, kd is small and approaching zero we observe that σ is a positive multiple of $-(kd)^2\mathcal{A}$ whereupon it approaches zero through positive or negative values as kd approaches zero according as $\mathcal{A}$ is negative or positive.

Appendix B: Why do we set $\frac{\mathrm{d}T_0}{\mathrm{d}z}$ less than zero in order to stabilize two phases in equilibrium?

In Fig. 12, we have a solid phase in equilibrium with its liquid. The pressure is set using up one degree of freedom, otherwise it has no role. Thus, the equilibrium temperature is fixed and upon displacement we have

$$\hat{T}_1 + \hat{Z}_1\frac{\mathrm{d}T_0}{\mathrm{d}z} = 0 \quad \text{at } z = 0.$$

This connection between $\hat{T}_1$ and $\frac{\mathrm{d}T_0}{\mathrm{d}z}$ is not quite so clear in our problem where we have

$$\hat{p}_1 + \hat{Z}_1\frac{\mathrm{d}p_0}{\mathrm{d}z} = \frac{\mathrm{d}P}{\mathrm{d}T}\left(\hat{T}_1 + \hat{Z}_1\frac{\mathrm{d}T_0}{\mathrm{d}z}\right).$$

We assume no gravity and no-flow, we drop time derivatives from our domain equations and set d to infinity. Our perturbation problem is then

Solid: $\frac{d^2\hat{T}_1^\star}{\mathrm{d}z^2} - k^2\hat{T}_1^\star = 0 \implies \hat{T}_1^\star = D^\star e^{-kz}$

Liquid: $\frac{d^2\hat{T}_1}{\mathrm{d}z^2} - k^2\hat{T}_1 = 0 \implies \hat{T}_1 = Ce^{kz}$

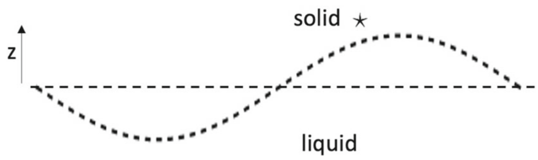


Fig. 12 A sketch of a solid and a liquid phase in equilibrium. The liquid is heated, the solid is cooled

At $z = 0$ we have $\hat{T}_1 = \hat{T}_1^{\star} \implies C = D^{\star}$, $\hat{T}_1 + \hat{Z}_1 \frac{dT_0}{dz} = 0 \implies C = -\hat{Z}_1 \frac{dT_0}{dz}$ and

$$\frac{d\hat{T}_1}{dz} - \frac{d\hat{T}_1^{\star}}{dz} = \frac{\mathcal{L}}{\lambda}(-\sigma \hat{Z}_1) \implies 2kC = -\frac{\mathcal{L}}{\lambda}\sigma \hat{Z}_1.$$

Thus, we obtain

$$2k\frac{dT_0}{dz} = \frac{\mathcal{L}}{\lambda}\sigma,$$

whereupon a displaced interface returns to zero if and only if $\frac{dT_0}{dz}$ is less than zero.

Appendix C: Setting the bounding planes far apart

We might imagine that we could have learned all that we wish to know by setting d to infinity at the outset. This is not so. If we do this, we obtain the k's at zero growth rate via

$$\mathcal{A} = \left(-\frac{1}{\alpha}\frac{\lambda}{\mathcal{L}}\frac{dT_0}{dz}\right)\left(1 + \frac{\alpha\rho}{\alpha^{\star}\rho^{\star}}\right)\frac{1}{2k} + 1 = \text{RHS}$$

where RHS vs k is sketched in Fig. 13.

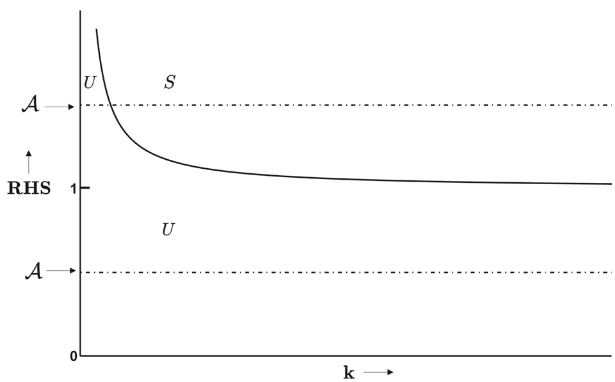


Fig. 13 The curve RHS vs k in the case where the planes bounding the system are very far apart

If $\mathcal{A}$ is less than 1 all displacements are unstable, if $\mathcal{A}$ is greater than 1 there is one value of k where σ is zero.

This limit agrees with the case where at finite values of d the bounding planes are held at constant pressure and are not no-flow planes.

Appendix D: The Bénard problem

Our problem has a Rayleigh–Taylor side. To see it, turn to Fig. 11 in Appendix A and look at the σ vs kd curves at $\mathcal{A} < 1$. We see two critical points at $\mathcal{A} > 1$. This tells us that our problem also has a Bénard side, and to see that this is so we derive the neutral curve for the Bénard problem.

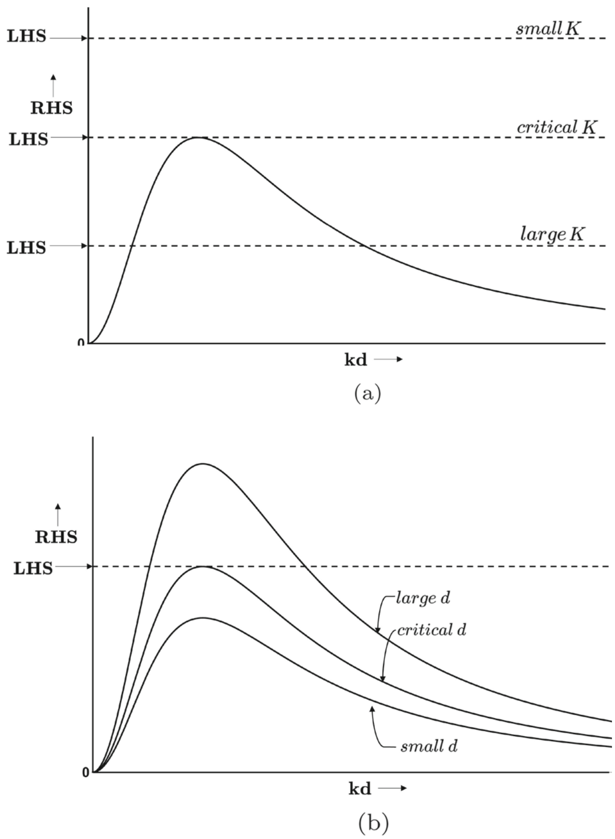


Fig. 14 The Bénard problem. The line LHS constant and the curve RHS vs kd, **a** LHS at small, critical and large values of K, **b** RHS vs kd at small, critical, and large values of d

A fluid lies in a porous solid between a hot, no-flow plane at $z = 0$, and a cold no-flow plane at $z = d$. The density of the fluid depends on the temperature via

$$\rho = \rho_0 - \rho_0 \beta (T - T_0), \ \beta > 0.$$

There is one phase yet heavy cold fluid lies above light hot fluid and this adverse density difference will be unstable if the temperature gradient is steep enough or the permeability of the porous solid is large enough. The fluid is at rest in the base state. Our job is to find out if a perturbation of wave number k grows or dies. It is neutral if, given the Rayleigh number, k satisfies

$$\frac{(\pi^2 + k^2 d^2)^2}{k^2 d^2} = R_a = \frac{K}{\mu} \left(-\frac{1}{\alpha} \frac{\mathrm{d}T_0}{\mathrm{d}z} \right) d^2 \ (\rho_0 \ \beta).$$

Then eliminating $\rho_0 \ \beta$, we have a formula that is like Eq. (26), viz.,

$$LHS \equiv \frac{\left(-\frac{\mathrm{d}T_0}{\mathrm{d}z} \right)}{\frac{K}{\mu} (\rho^\star - \rho) g} = \left(-\frac{1}{\alpha} \frac{\mathrm{d}T_0}{\mathrm{d}z} \right) d \ \frac{k^2 d^2}{(\pi^2 + k^2 d^2)^2} \equiv \text{RHS}, \tag{32}$$

where $\rho^\star - \rho = \rho(z = d) - \rho(z = 0)$ and where the RHS vs kd is not negative, rising to its peak value as kd increases then falling back to zero as kd continues its increase. In Fig. 14, we sketch the line LHS constant and the curve RHS vs kd in Eq. (32). The LHS is changed by changing the permeability of the solid rock. The RHS is changed by changing the distance between the bounding planes.

This figure suggests that the part of Fig. 4 lying above $\mathcal{A} = 1$ is Bénard-like.

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