THE TEACHING AND LEARNING OF ALGEBRA: SIX ELEMENTS THAT BUILD ALGEBRAIC FLUENCY FROM CONCEPTUAL UNDERSTANDINGS

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This work presents six research-based elements that align with building algebraic fluency from conceptual understandings in the teaching and learning of algebra. The six elements are: symbol sense, processes/relationships of algebra, process as an object, anticipating solution strategies, anticipating solution formats, and relationships among representations.

Keywords: Algebra and algebraic thinking, algebraic fluency, conceptual understanding

The purpose of this work is to describe how six research-based elements contribute to students building algebraic fluency from conceptual understandings. First, we use Kaput's (2008) two core aspects of algebra to illustrate the difference between algebra readiness and conceptual understandings of algebra. Second, we provide a characterization of algebraic fluency from conceptual understandings of algebra. Third, we describe six research-based elements of algebraic fluency and illustrate connections to the teaching and learning of algebra.

Core Aspects of Algebra, Algebra Readiness, and Conceptual Understandings

Kaput (2008) defines algebra using two core aspects. First, "[a]lgebra as systematically symbolizing generalizations of regularities and constraints" (p. 11). One example of a regularity of our number system is the commutative property of addition. We note that $7 + 3 = 3 + 7$ and can generalize this regularity symbolically as $a + b = b + a$. Another example of symbolizing generalizations of regularities is finding a rule for the n^{th} term in a visual pattern like in Figure 1.

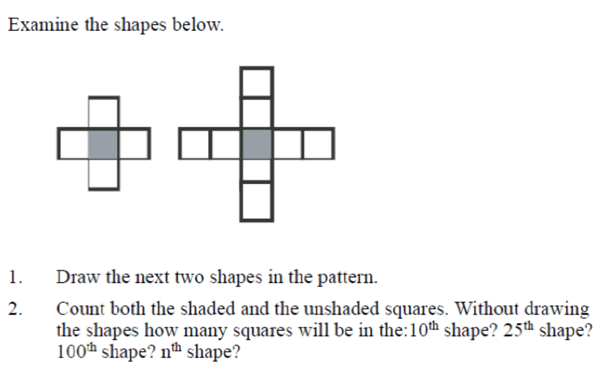

Figure 1: Visual Pattern, Growing Squares

The first core aspect aligns with an algebra readiness perspective (Feikes et al., 2022). Algebra readiness involves helping students see regularities, describe generalizations from the regularities, and represent these generalizations symbolically. The growing squares pattern

(Figure 1) illustrates this as students can generalize and represent a regularity symbolically by expressing the n^{th} shape as "$4n + 1$".

However, algebra entails more than just generalization and symbolic representation. Kaput's (2008) second core aspect is: "algebra as syntactically guided reasoning and actions on generalizations expressed in conventional symbols systems" (p.11). We look at this second core aspect as the processes, properties, procedures, and symbolic generalizations which allow for the abstract manipulation of algebraic objects. This characteristic of algebra allows for the modeling of real-life situations, the creation of abstractions, the manipulation of algebraic objects, and the application of abstractions to real-life situations. An example consistent with this core aspect of algebra is the derivation of the quadratic formula by completing the square (See Figure 2).

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Figure 2: Quadratic Formula

The derivation of the quadratic formula by performing actions on variables *a*, *b*, and *c* through completing the square is a process which becomes a mathematical object. This object can then be used in other algebraic work or manipulations. The core aspects of algebra are what allow for the manipulation of algebraic generalizations.

Kaput's second core aspect of algebra aligns with our perspective of developing conceptual understandings of algebra. We understand conceptual understandings in a way that is consistent with the National Research Council (NRC) (2001) and National Council of Teachers of Mathematics (NCTM) (2014) descriptions of students having an integrated and functional grasp of mathematical ideas. Opportunities to develop conceptual understandings of algebra occur when students are provided problems where they can develop and manipulate symbolic generalizations of regularities in meaningful ways (Feikes et al., 2021; Feikes et al., 2022).

Algebraic Fluency from Conceptual Understandings of Algebra

Our perspective of algebraic fluency is based on the NCTM (2023, p. 1) position statement on procedural fluency. Our work defines algebraic fluency as the ability to apply algebraic processes, properties, and procedures with efficiency, flexibility, and accuracy; to transfer algebraic processes, properties, and procedures to different problems and contexts; to build or modify algebraic processes, properties, and procedures from other processes, properties, and procedures; and to recognize when a particular algebraic process, property, or procedure is more appropriate than another. Algebraic fluency entails understanding how to carry out procedures, why procedures can be performed, and which is more appropriate. Research on procedural fluency is relevant to our perspective on algebraic fluency. Efficiency, flexibility, and accuracy involve the knowledge of multiple strategies and the ability to apply them in different contexts (Star, 2005), including procedures and processes in algebra.

Algebraic fluency builds from conceptual understandings so that students become skillful in using procedures appropriately, flexibly, and efficiently, considering different representations, using reasoning to apply these representations to different purposes, and producing accurate answers (NCTM, 2014). Students who build conceptual understandings are more likely to remember and use topics, concepts, and procedures without error due to reasoning and understandings of mathematical relationships (e.g., Fuson et al., 2005; Hiebert & Carpenter, 1992; Hiebert et al., 1997). Alternatively, mindlessly manipulating symbols or learning tricks like memorizing formulas or mnemonics are typically applied to specific problems, likely to be misused in different mathematical problems, and often quickly forgotten (NRC, 2001).

To develop fluency, students need to practice strategies and procedures to solidify their knowledge (NCTM, 2014, p. 45). For example, multiplying two binomials or two larger polynomials (like a binomial and a trinomial) can be taught as an application of the distributive property. This would be a conceptual way to teach and learn this skill because it presents this skill as part of the coherent whole of algebra. As students practice the skill to develop proficiency, they can develop fluency by being asked about patterns and strategies when multiplying the polynomial expressions.

The following example illustrates algebraic fluency with conceptual understandings of algebra when a student recognizes properties of a given equation and related procedures. To solve for x in $\frac{1}{2}(x+4)-1=5$, students could consider a variety of mathematical concepts. They may recognize that the equation represents $6-1=5$, such that $\frac{1}{2}(x+4)$ should equal 6. The distributive property could be used to create an equivalent expression for $\frac{1}{2}(x+4)$ or each term could be multiplied by 2 so that $\frac{1}{2}$ is no longer part of the equation. Students who have developed algebraic fluency from conceptual understandings could consider the benefits and drawbacks of different ways of finding a solution.

Six Elements that Algin with Building Algebraic Fluency from Conceptual Understandings

We have identified six research-based elements which build algebraic fluency from conceptual understandings of algebra and allow students to comprehend algebraic notation or symbols and operate within the processes, properties, and procedures of algebra.

1. Developing *symbol sense* by learning the constructs that algebraic symbols convey.
2. Understanding *processes/relationships of algebra* and how to express these with symbols, e.g., $2n$ is “$n + n$” or “$2 \ x \ n$”; $y = 3x$.
3. Conceptualizing a *process as an object*, often called process object duality or procept.
4. Understanding, anticipating, and being proficient with *solution strategies*.
5. Anticipating *solution formats*, like solutions as a single number, a graph, or a function.
6. Noticing and expressing *relationships among representations*, like relationships between algebraic expressions, tables, and graphs.

Discussion of the Six Elements

The six elements that align with building algebraic fluency with conceptual understandings have important implications for the teaching and learning of algebra. For example, Arcavi et al. (2017) describe *symbol sense* as giving meaning to symbols and expressions and connecting the symbols to underlying concepts (p. 94). Students need to have an understanding of the constructs conveyed with symbols. Becoming fluent with the abstract ideas represented by the symbols and examples that help these representations become transparent is a key to helping students learn to think with algebra (Kieran, 2007).

Understanding *processes/relationships of algebra* is significant because it helps students express the generalizable from the particular. Mason and Sutherland (2002) emphasize that algebra is about processes and understanding them through the symbols used to represent the processes. Students need to learn to “see through” the symbols by being aware of the processes and applying them (Wheeler, 1989).

Algebra includes processes that are represented with symbols. A conceptual leap occurs when students begin to see and act on *processes as objects* (c.f. Arcavi et al., 2017; Kieran, 1992). Sfard (1991) describes recognizing processes as objects as reification. Warren et al. (2016) have suggested that one of the benefits for students seeing a process as an object is when algebraic objects become accessible which leads to identifying algebraic structures. Further, students should be able to unpack an objectified process into objects related by processes (Tall & Gray, 1994). Instructors of algebra need to encourage students to see processes as objects by providing examples, discussing how processes act as objects, and working on examples that both compress processes into objects and decompress objects to processes (Tall & Gray, 1994).

Understanding, anticipating, and demonstrating proficiency with *solution strategie*s and anticipating *solution formats* require more attention during instruction and work on mathematical problems. Students need to anticipate aspects of solution formats so that possible strategies can be considered (Booth, 1988; Boero, 2001). To help students develop fluency with algebraic solutions formats and strategies, instructors should discuss possible solutions, different ways the solution could be conveyed, different solution strategies, and examples where when a solution does not meet the anticipated expectation.

The final feature of *relationships among representations* is central for creating meaning in algebra (Kieran, 2007, p. 712). Encouraging multiple representations provides opportunities for students to make sense of algebra as represented in different types of thinking and allows students more ways to express their algebraic understandings (Kieran, 2006). We need to help students analyze multiple representations, encouraging them to notice what is similar and what is different about each (Jacobs et al., 2010).

The six elements are interrelated and build upon each other. For example, Kieran (1992) notes, "the development of algebraic symbolism … allowed the symbolic forms to be used structurally as objects" (p. 391). This statement relates to the *process as an object* feature and shows the importance of *symbol sense* in having a structural perspective of algebra.

Additional Considerations and Conclusion

We have identified six elements that align with building algebraic fluency from conceptual understandings of algebra to address theoretical and practitioner needs. Each one of the six elements are research-based and together they form a unique framework for examining the teaching and learning of algebra. Building algebraic fluency should position students as capable, using reasoning and decision-making to improve skill and understanding (NCTM, 2023). We suggest that student learning of algebra can be improved by developing skills and understandings around the six elements. This can occur in classrooms by discussing conceptual aspects of algebra and providing opportunities for students to build understandings about algebraic symbols, processes, properties, and procedures. Research is needed to understand how the six elements of algebraic fluency impact student achievement, equitable classroom practices, and the development of assessments for algebra.

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