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Wreath Macdonald operators

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Abstract. We construct a novel family of difference-permutation operators and prove that they are diagonalized by the wreath Macdonald P-polynomials. Our operators arise from the action of the horizontal Heisenberg subalgebra in the vertex representation of the quantum toroidal algebra.

Keywords: Macdonald polynomials, quantum toroidal algebras, shuffle algebras

1 Introduction

Let $X_N = \{x_1, \ldots, x_N\}$ be a set of variables and let $\mathbb{Y}_N$ be the set of partitions with at most N parts. The *Macdonald polynomials* $\{P_\lambda[X_N; q, t] \mid \lambda \in \mathbb{Y}_N\}$ are a basis of the ring of symmetric polynomials $\mathbb{Q}(q,t)[X_N]^{\mathfrak{S}_N}$ that can be characterized as eigenfunctions of a commuting family of difference operators, the *Macdonald operators*: for $1 \leq n \leq N$,

$$D_n(X_N; q, t) := t^{\frac{n(n-1)}{2}} \sum_{\substack{J \subset \{1,\ldots,N\} \\ |J|=n}} \left(\prod_{\substack{i \in J \\ j \notin J}} \frac{tx_i - x_j}{x_i - x_j} \right) \prod_{i \in J} T_{q,x_i} \tag{1.1}$$

$$D_n(X_N; q, t) P_\lambda[X_N; q, t] = e_n(q^{\lambda_1} t^{N-1}, q^{\lambda_2} t^{N-2}, \ldots, q^{\lambda_N}) P_\lambda[X_N; q, t] \tag{1.2}$$

Here T_{q,x_i} is the q-shift operator $T_{q,x_i} x_j = q^{\delta_{i,j}} x_j$ and e_n is the nth elementary symmetric polynomial.

This paper is concerned with *wreath* Macdonald polynomials, a generalization of the Macdonald polynomials defined by Haiman [1]. Let r be a positive integer and let $I = \mathbb{Z}/r\mathbb{Z}$. Fix an r-core partition γ and a vector $N_\bullet = (N_0, N_1, \ldots, N_{r-1}) \in \mathbb{Z}_{\geq 0}^I$ that

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is compatible with γ in the sense given by (3.5) below. Letting $N = |N_\bullet| := \sum_{i\in I} N_i$, we partition the variables $x_1, \ldots, x_N$ into r subsets:

$$X_{N_\bullet} := \bigsqcup_{i=0}^{r-1} \left\{ x_l^{(i)} \mid 1 \le l \le N_i \right\} = \{x_1, \ldots, x_N\}. \tag{1.3}$$

We call the index i the *color* of $x_l^{(i)}$. The product of symmetric groups $\mathfrak{S}_{N_\bullet} := \prod_{i\in I} \mathfrak{S}_{N_i}$ acts on the polynomial ring $\mathbb{Q}(q,t)[X_{N_\bullet}]$ whereby $\mathfrak{S}_{N_i}$ only permutes the variables of color i. Let $\mathbb{Y}_{N,\gamma}$ be the set of partitions with r-core γ and at most N parts. The wreath Macdonald polynomials $\{P_\lambda[X_{N_\bullet}; q, t] \mid \lambda \in \mathbb{Y}_{N,\gamma}\}$ are a basis of the space $\mathbb{Q}(q,t)[X_{N_\bullet}]^{\mathfrak{S}_{N_\bullet}}$ of color-symmetric polynomials. The original Macdonald polynomials $P_\lambda(X_N; q, t)$ [2, p. VI.4] are the case $r = 1$.

Haiman's definition characterizes $P_\lambda[X_{N_\bullet}; q, t]$ using a pair of triangularity conditions. In contrast with the usual Macdonald theory, we *a priori* do not have an analogous characterization as the joint eigenfunction of an explicit family of difference operators. The present work remedies this situation. For each $p \in I$ and positive integer n, define the operators $D_{p,n}$ on $\mathbb{Q}(q,t)[X_{N_\bullet}]$ by

$$\begin{aligned} D_{p,n}(X_{N_\bullet}; q, t) &:= \frac{(-1)^{\frac{n(n-1)}{2}}}{\prod_{k=1}^{n}(1-q^k t^{-k})} \\ &\times \sum_{\underline{\mathbf{J}} \in Sh_p^{[n]}(X_{N_\bullet})} \prod_{a=1}^{\curvearrowright n} \left\{ (1-qt^{-1})^{|\underline{J}_a|} \left(\frac{x_{\underline{J}_a^\nabla}^{(r-1)}}{x_{\underline{J}_a}^{(p)}} \right) \frac{\prod_{\substack{l=1 \\ x_l^{(p)} \notin |\underline{\mathbf{J}}|_{\ge a}}}^{N_p} \left(t x_{\underline{J}_a^\nabla}^{(p-1)} - x_l^{(p)} \right)}{\prod_{\substack{l=1 \\ x_l^{(p)} \notin |\underline{\mathbf{J}}|_{\le a}}}^{N_p} \left(x_{\underline{J}_a}^{(p)} - x_l^{(p)} \right)} \right. \\ &\times \left. \left(\prod_{i\in I\setminus\{p\}} \prod_{\substack{l=1 \\ x_l^{(i)} \ne x_{\underline{J}_a^\nabla}^{(i)}}}^{N_i} \frac{\left(t x_{\underline{J}_a^\nabla}^{(i-1)} - x_l^{(i)} \right)}{\left(x_{\underline{J}_a^\nabla}^{(i)} - x_l^{(i)} \right)} \right) \left(\prod_{i\in J_a\setminus\{p\}} \frac{q^{-1} t T_{\underline{J}_a} x_{\underline{J}_a}^{(i)}}{\left(x_{\underline{J}_a}^{(i)} - T_{\underline{J}_a} x_{\underline{J}_a}^{(i)} \right)} \right) T_{\underline{J}_a} \right\}. \end{aligned} \tag{1.4}$$

The notation used in this formula is defined in Section 4. Our main result is the following,

Theorem 1.1. *Let $N_\bullet$ be compatible with γ and $\lambda \in \mathbb{Y}_{N,\gamma}$. Then the number of $1 \le i \le N$ with $i - \lambda_i - 1 \equiv p \bmod r$ is equal to N_p and $P_\lambda[X_{N_\bullet}; q, t]$ satisfies the eigenfunction equation*

$$D_{p,n}(X_{N_\bullet}; q, t) P_\lambda[X_{N_\bullet}; q, t] = e_n \left[\sum_{\substack{i=1 \\ i-\lambda_i-1 \equiv p \bmod r}}^{N} q^{\lambda_i} t^{-(N-i)} \right] P_\lambda[X_{N_\bullet}; q, t]. \tag{1.5}$$

The eigenvalue uses plethystic notation: the N_p variables of the elementary symmetric polynomial are replaced by the summands of the form $q^a t^b$. Our operators (1.4) are much more complicated than the original Macdonald operators (1.1). In the case $r = 1$, we do indeed obtain (1.1) after some simplification. When $r > 1$, the q-shift operator T_{q,x_i} is replaced with what we call a *cyclic-shift operator* $T_{\underline{I}_a}$, which cyclically permutes variables of different colors in addition to multiplying by a power of q.

1.1 Integral formulas

Our strategy for deriving the eigenoperators (1.4) and establishing the eigenfunction equation (1.5) uses work of the third author [5]. We study the wreath Macdonald polynomials using the *quantum toroidal algebra* $\ddot{U} = U_{\mathfrak{q},\mathfrak{d}}(\ddot{\mathfrak{sl}}_r)$ and its *vertex representation* W. The aforementioned work proves that infinite-variable wreath Macdonald polynomials can be naturally embedded inside W such that they diagonalize a large commutative subalgebra of $\ddot{U}$ known as the *horizontal Heisenberg subalgebra*. This alone is insufficient for obtaining explicit formulas. We also need work of Neguț [3] realizing $\ddot{U}$ as a *shuffle algebra*. The shuffle algebra is a space of rational functions endowed with an exotic product structure, and it is isomorphic to a part of $\ddot{U}$ via a map that is morally (but not precisely) an integration map. Writing its action on W and then specializing from infinite to finite variables, we obtain integral formulas. Finally, to pin down the eigenvalues, we use the twisted isomorphism established by Tsymbaliuk [4] between the vertex representation and the *Fock representation*. We apply this process to the shuffle realizations of well-chosen elements of the horizontal Heisenberg subalgebra which were found in [5].

1.2 Towards duality

In the case $r = 1$, the eigenfunction equation (1.2) is particularly interesting when juxtaposed with the *Pieri rules* [2]. To make this apparent, introduce a continuous extension of the discrete parameters $\lambda = (\lambda_1, \dots, \lambda_N)$: $s_i := q^{\lambda_i} t^{N-i}$, $S_N := \{s_1, \dots, s_N\}$. We call the variables X_N the *position variables* and S_N the *spectral variables*. It is natural to interpret the q-shift $T_{q,s_i} P_\lambda[X_N; q, t]$ by adding a box to row i of the partition λ. For a certain renormalization $\tilde{P}_\lambda[X_N; q, t]$ of $P_\lambda[X_N; , q, t]$, we can write the Pieri rules as

$$e_n(x_1, \dots, x_N) \tilde{P}_\lambda[X_N; q, t] = t^{\frac{n(n-1)}{2}} \sum_{\substack{I \subset \{1,\dots,N\} \\ |I|=n}} \left(\prod_{\substack{i \in I \\ j \notin I}} \frac{t s_i - s_j}{s_i - s_j} \right) \prod_{i \in I} T_{q,s_i} \tilde{P}_\lambda[X_N; q, t]. \tag{1.6}$$

The fact that no shift operator T_{q,s_i} appears more than once enforces the well known support condition of the Pieri rules: the $\tilde{P}_\mu[X_N; q, t]$ that appear on the right hand side of (1.6) are such that $\mu \backslash \lambda$ contains no horizontally adjacent boxes. On the other hand, we

can view the eigenfunction equation (1.2) as describing multiplication by $e_n(s_1, \ldots, s_N)$. The similarity between (1.2) and (1.6) is reflective of a symmetry $X_N \leftrightarrow S_N$. One manifestation of this symmetry is the well-known *evaluation duality* [2]:

$$\tilde{P}_\lambda(q^{\mu_1}t^{N-1}, q^{\mu_2}t^{N-2}, \ldots, q^{\mu_N}) = \tilde{P}_\mu(q^{\lambda_1}t^{N-1}, q^{\lambda_2}t^{N-2}, \ldots, q^{\lambda_N}).$$

For the wreath case $r > 1$, the spectral variables should also have color. We assign $s_l^{(i)}$ to some b such that $b - \lambda_b \equiv i + 1 \bmod r$: $s_l^{(i)} := q^{\lambda_b} t^{|N_\bullet| - b}$. Here, we point out a natural motivation for imposing our compatibility condition between the r-core of λ and $N_\bullet$—it forces there to also be N_i spectral variables of color i. The eigenfunction equation (1.5) then describes multiplication by $e_n(s_1^{(p)}, \ldots, s_{N_p}^{(p)})$. Note that adding a box to a row will not only contribute a q-shift but also change the color, and that is precisely what the cyclic-shift operators $T_{\underline{J}_a}$ do. Work of the third author [5] provides one constraint on the support of the wreath Pieri rules. Namely, for a box (a, b), if we call the class of $b - a$ mod r its color, then $P_\mu[X_{N_\bullet}; q, t]$ appears as a summand of $e_n(x_{p,1}, \ldots, x_{p,N_p}) P_\lambda[X_{N_\bullet}; q, t]$ only if $\mu \backslash \lambda$ consists of n boxes of each color such that no boxes of color p and $p + 1$ are horizontally adjacent. One can check that the combinations of $T_{\underline{J}_a}$ appearing in (1.4) enforce this condition after swapping $x_l^{(i)} \leftrightarrow s_l^{(i)}$. Computer calculations done by the second author also confirm a wreath analogue of evaluation duality. While we are still a long way from establishing a wreath analogue of the $X_N \leftrightarrow S_N$ symmetry, our strange operators seem to go out of their way to say it must be true.

1.3 Outline

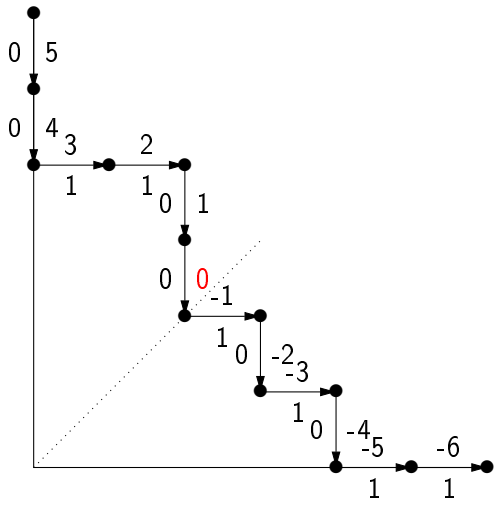
Section 2 discusses the core-quotient combinatorics in detail. Section 3 defines wreath Macdonald symmetric functions and their finitizations. Section 4 defines the ingredients of the formula (1.4) of our wreath Macdonald eigenoperators.

2 Abaci, cores, quotients, and root lattice combinatorics

Fix a positive integer r and let $I = \mathbb{Z}/r\mathbb{Z}$.

2.1 Partitions and cores

Let $\mathbb{Y}$ be the set of all integer partitions. The diagram of a partition $\mu = (\mu_1, \mu_2, \ldots) \in \mathbb{Y}$ is by definition the French-style diagram $D(\mu) = \{(a, b) \in (\mathbb{Z}_{\geq 0})^2 : 0 \leq a < \mu_{b+1}\}$ which uses 0-based first quadrant Cartesian (x, y)-coordinates for the elements of $D(\mu)$. The *residue* or *color* of $(a, b) \in \mathbb{Z}^2$ is the element $b - a \in \mathbb{Z}/r\mathbb{Z}$. This is the residue mod r of the *negative* of Macdonald's content of a cell [2, Ex. I.1.3].



i	$\cdots$	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6	$\cdots$
b_i	0	0	0	1	1	0	0	1	0	1	0	1	1	1

$\text{charge}(b) = 0$ $\quad$ $\text{shape}(b) =$ 

Figure 1: The shape and charge of an edge sequence

For $(a,b) \in D(\mu)$ the hook length $h_\mu(a,b)$ is the number of elements $(a',b') \in D(\mu)$ that are in the same row as (a,b) and weakly to the right (that is, $(a',b') = (a',b)$ with $a' \geq a$) or in the same column as (a,b) and weakly above (that is, $(a',b') = (a,b')$ with $b' \geq b$). A partition $\mu \in \mathbb{Y}$ is an r-core if $h_\mu(a,b) \neq r$ for all $(a,b) \in D(\mu)$. We denote by $\mathcal{C} = \mathcal{C}_r$ the set of r-cores.

2.2 Edge sequences and partitions

A function $b : \mathbb{Z} \to \{0,1\}$ can be viewed as an infinite indexed binary word

$$\cdots b(1)b(0)b(-1)\cdots;$$

notice that in writing such a word we index the positions in reverse order. An *inversion* of b is a pair of integers $i > j$ such that $b(i) > b(j)$, a 1 to the left of a 0. An *edge sequence* (or Maya diagram) is a function $b : \mathbb{Z} \to \{0,1\}$ such that $b(i) = 0$ for $i \gg 0$ and $b(i) = 1$ for $i \ll 0$, that is, b has finitely many inversions. Let ES denote the set of edge sequences. The *shape* of $b \in$ ES is the partition whose French partition diagram has boundary traced out by the values of b from top left to bottom right, where 0 (resp. 1) indicates a vertical downward (resp. horizontal rightward) unit vector. See Figure 1, in which the 0s and 1s

of the edge sequence are written to the left and below their corresponding unit vectors, and the index i of the bit $b(i)$ is written to the right or above the unit vector.

The *charge* of $b \in \mathrm{ES}$ is the index of the unit vector that touches the main diagonal from above and left, or equivalently the index of the rightmost 0 in the edge sequence of the form $\cdots 0011 \cdots$ obtained from b by repeatedly swapping adjacent pairs 10 to 01 until none remain. There is a bijection

$$\mathrm{ES} \to \mathbb{Z} \times \mathbb{Y}, \qquad b \mapsto (\mathrm{charge}(b), \mathrm{shape}(b)). \tag{2.1}$$

2.3 Cores and quotients

Our goal is to define the bijection

$$\mathbb{Y} \cong \mathcal{C}_r \times \mathbb{Y}^r, \qquad \lambda \mapsto (\mathrm{core}_r(\lambda), \mathrm{quot}_r(\lambda)) \tag{2.2}$$

where core_r is the *r-core* and quot_r is the *r-quotient* map.

In the following diagram all horizontal maps are bijections and vertical maps are inclusions.

$$\begin{array}{ccccccccc}
\mathbb{Z} \times \mathbb{Y} & \xleftarrow{\mathrm{charge}\times\mathrm{shape}} & \mathrm{ES} & \longrightarrow & \mathrm{ES}^r & \xrightarrow{c^\bullet \times \mathrm{quot}_r} & \mathbb{Z}^r \times \mathbb{Y}^r & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\{0\} \times \mathbb{Y} & \longleftarrow & \mathrm{ES}_0 & \longrightarrow & (\mathrm{ES}^r)_0 & \longrightarrow & Q \times \mathbb{Y}^r & \xrightarrow{\kappa^{-1}\times \mathrm{id}} & \mathcal{C}_r \times \mathbb{Y}^r \\
\uparrow & & & & & & \uparrow & & \\
\{0\} \times \mathcal{C}_r & & & \xrightarrow{\kappa} & & & Q \times \varnothing^r & &
\end{array}$$

Elements $b^\bullet = (b^0, b^1, \ldots, b^{r-1}) \in \mathrm{ES}^r$ are called *abaci*. We may write them as $\{0, 1, \ldots, r-1\} \times \mathbb{Z}$ matrices with entries in $\{0, 1\}$ where a 0 is a bead and a 1 is a hole (position with no bead) and the i-th row represents the edge sequence b^i and is the i-th runner in the abacus.

There is a bijection $\mathrm{ES} \to \mathrm{ES}^r$ sending b to $(b^0, b^1, \ldots, b^{r-1})$ by letting b^i select the bits in b indexed by integers congruent to i mod r: $b^i(j) = b(rj + i)$ for $0 \leq i < r$ and $j \in \mathbb{Z}$. The inverse map is given by interleaving the sequences $b^0, b^1, \ldots, b^{r-1}$. This bijection is charge-additive: $\mathrm{charge}(b) = \sum_{j=0}^{r-1} \mathrm{charge}(b^j)$. The r-fold product of the bijection (2.1) yields the bijection $\mathrm{ES}^r \cong \mathbb{Z}^r \times \mathbb{Y}^r$. Denote this by $b^\bullet = (b^0, \ldots, b^{r-1}) \mapsto ((c_0, \ldots, c_{r-1}), \lambda^\bullet)$. We write $\lambda^\bullet = \mathrm{quot}_r(b^\bullet)$; this is the r-quotient. Call $(c_0, \ldots, c_{r-1}) = c^\bullet(b^\bullet)$ the charge vector. This indicates the position on each runner where the beads end after pushing all beads to the left within each runner. This defines the bijections going across the top row of the diagram.

We now restrict all these bijections. Let $\mathrm{ES}_0 = \{b \in \mathrm{ES} \mid \mathrm{charge}(b) = 0\}$ and $(\mathrm{ES}^r)_0 = \{b^\bullet \in \mathrm{ES}^r \mid \sum_{i=0}^{r-1} c_i(b^\bullet) = 0\}$. Then $c^\bullet(b^\bullet)$ can be viewed as an element of the $\mathfrak{sl}_r$ root

lattice Q (and belongs to the zero lattice $Q = 0$ when $r = 1$). The second row of the diagram (save the last map) is given by suitable restrictions of the top row of bijections.

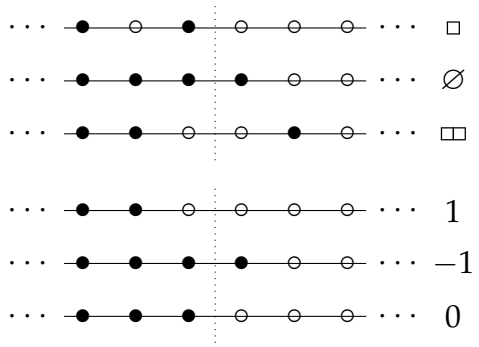
Let $\gamma \in \mathbb{Y}$ and $b \in \mathrm{ES}$ with $\gamma = \mathrm{shape}(b)$. We see that the following are equivalent: γ is an r-core; $h_\gamma(i,j) \neq r$ for all $(i,j) \in \gamma$; there is no index k such that $b(k) = 1$ and $b(k+r) = 0$; $\mathrm{quot}_r(\gamma)$ is the empty multipartition $(\varnothing^I)$. Therefore the bijection $\{0\} \times \mathbb{Y} \cong Q \times \mathbb{Y}^r$ restricts to the bijection $\{0\} \times \mathcal{C}_r \cong Q \times (\varnothing^r)$, that is, $\mathcal{C}_r \cong Q$. We call this bijection κ.

Example 2.1. Let $b \in \mathrm{ES}_0$ be as in the previous example. We have $\lambda = \mathrm{shape}(b) = (4,3,2,2)$. Set $r = 3$. We map $b \mapsto (b^0, b^1, b^2)$ which are pictured in the matrix below. Reading up the columns of the $\{0,1,2\} \times \mathbb{Z}$ matrix we recover b. Each runner of the abacus is an edge sequence; the corresponding shapes give the 3-quotient of $(4,3,2,2)$, which is $((1), \varnothing, (2))$.

To get the 3-core of λ we move all beads to the left in each runner. This produces the second abacus. Reading up columns we obtain the edge sequence $\cdots 0001|1011 \cdots$. Therefore $\mathrm{core}_3(4,3,2,2) = (2)$. The charge sequence is $(1,-1,0) \in Q$.

i	$\cdots$	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6	$\cdots$
b_i	$\cdots$	0	0	1	1	0	0	1	0	1	0	1	1	$\cdots$

	2	1	0	-1	-2	-3
b^0	0	1	0	1	1	1
b^1	0	0	0	0	1	1
b^2	0	0	1	1	0	1



core

i	$\cdots$	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6	$\cdots$
a_i	0	0	0	0	0	0	1	1	0	1	1	1	1	1

□

When considering a fixed r, we simply write $\mathrm{core} = \mathrm{core}_r$ and $\mathrm{quot} = \mathrm{quot}_r$.

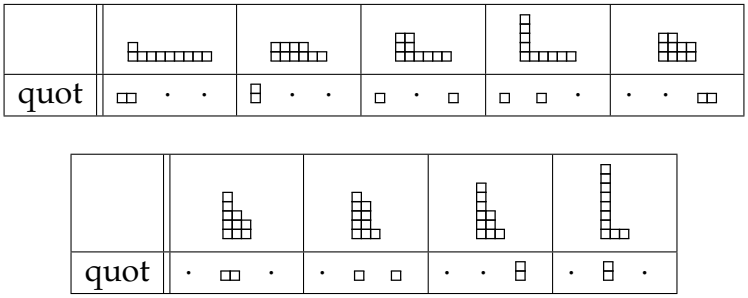
2.4 Cores to root lattice

Recall that $Q := \left\{(c_0, \ldots, c_{r-1}) \in \mathbb{Z}^I \,\middle|\, \sum_{i\in I} c_i = 0\right\}$ denotes the $\mathfrak{sl}_r$ root lattice (or $Q = 0$ in the case $r = 1$), realized as the zero sum elements in the lattice $\mathbb{Z}^I$. Let $\epsilon_i \in \mathbb{Z}^I$ be the i-th coordinate vector. Then Q is the spanned by the elements $\{\alpha_i := \epsilon_{i-1} - \epsilon_i \mid i \in I\}$. We realize the simple roots of $\mathfrak{sl}_r$ as the α_i for $i \neq 0$.

Define the map $\kappa : \mathbb{Y} \to Q$ by $\kappa(\mu) = -\sum_{(p,q)\in\mu} \alpha_{q-p}$. Then the restriction of κ to $\mathcal{C}$ is the same as the bijection $\mathcal{C} \cong Q$ constructed above.

Example 2.2. Let $r = 3$ and consider the 3-core $\gamma = \ydiagram{2,2}$. We have $\kappa(\ydiagram{2,2}) = -(\alpha_0 + \alpha_2) = \alpha_1$, which agrees with the charge sequence $(1, -1, 0) \in Q$ computed above.

We list (in reverse lex order, which refines dominance order) the partitions with core $\gamma = \ydiagram{2,2}$ and 3-quotient of size 2.



3 Wreath Macdonald symmetric functions and finitization

3.1 Tensor symmetric functions

Let Λ be the algebra of symmetric functions over $\mathbb{K} = \mathbb{Q}(q,t)$ [2, §I.2]. Denote by $\Lambda^I = \Lambda^{\otimes I}$ the I-fold tensor power of Λ over $\mathbb{K}$. For $f \in \Lambda$, we write $f[X^{(i)}]$ to indicate the element of Λ^I with 1 in tensor factors $j \neq i$ and f in factor i. Λ^I is a polynomial ring over $\mathbb{K}$ with polynomial generators given by the power sums $p_k[X^{(i)}]$ for $i \in I$ and $k > 0$. We write $X^\bullet$ for the I-tuple of alphabets $(X^{(0)}, \ldots, X^{(r-1)})$ and often denote by $f[X^\bullet]$ a generic element of Λ^I. For an I-tuple of partitions $\lambda^\bullet = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r-1)}) \in \mathbb{Y}^I$, define the tensor Schur function $s_{\lambda^\bullet} = \bigotimes_{i\in I} s_{\lambda^{(i)}} = \prod_{i\in I} s_{\lambda^{(i)}}[X^{(i)}]$. The tensor Hall pairing on Λ^I is defined by the orthonormality of the tensor Schur basis: $\langle s_{\lambda^\bullet}, s_{\mu^\bullet}\rangle = \delta_{\lambda^\bullet,\mu^\bullet}$ for $\lambda^\bullet, \mu^\bullet \in \mathbb{Y}^I$.

For any $a = a(q,t) \in \mathbb{K}$, define the $\mathbb{K}$-algebra automorphism $\mathcal{P}_{\mathrm{id}-a\chi^{-1}}$ of Λ^I by $\mathcal{P}_{\mathrm{id}-a\chi^{-1}}(p_k[X^{(i)}]) = p_k[X^{(i)}] - a(q^k, t^k)p_k[X^{(i-1)}]$ for all $i \in I$ and $k > 0$.

3.2 Wreath Macdonald H and P functions

For $\lambda \in \mathbb{Y}$ let $H_\lambda[X^\bullet; q, t]$ be the wreath Macdonald H function [1], as defined in [5]. These are characterized by the conditions

$$\mathcal{P}_{\mathrm{id}-q\chi^{-1}} H_\lambda[X^\bullet; q, t] \in \mathbb{K}^\times s_{\mathrm{quot}(\lambda)} + \bigoplus_{\substack{\nu > \lambda \\ \kappa(\nu)=\kappa(\lambda)}} \mathbb{K} s_{\mathrm{quot}(\nu)} \tag{3.1}$$

$$\mathcal{P}_{\mathrm{id}-t^{-1}\chi^{-1}} H_\lambda[X^\bullet; q, t] \in \mathbb{K}^\times s_{\mathrm{quot}(\lambda)} + \bigoplus_{\substack{\nu < \lambda \\ \kappa(\nu)=\kappa(\lambda)}} \mathbb{K} s_{\mathrm{quot}(\nu)} \tag{3.2}$$

$$\langle s_{(n)}[X^{(0)}], H_\lambda[X^\bullet; q, t] \rangle = 1. \tag{3.3}$$

where $n = |\mathrm{quot}(\lambda)|$ and $<$ is the (strict) dominance order on partitions [2, §I.1].

For any $\lambda \in \mathbb{Y}$ the wreath Macdonald P-function $P_\lambda[X^\bullet; q, t]$ is defined so that $P_\lambda[X^\bullet; q, t^{-1}]$ is the scalar multiple of $\mathcal{P}_{\mathrm{id}-t^{-1}\chi^{-1}}(H_\lambda[X^\bullet; q, t])$ in which the coefficient of $s_{\mathrm{quot}(\lambda)}$ is 1. In particular, $P_\lambda[X^\bullet; q, t]$ satisfies the unitriangularity

$$P_\lambda[X^\bullet; q, t] \in s_{\mathrm{quot}(\lambda)} + \bigoplus_{\substack{\nu < \lambda \\ \kappa(\nu)=\kappa(\lambda)}} \mathbb{K}\, s_{\mathrm{quot}(\nu)}$$

For any fixed $\alpha \in Q$, $\{P_\lambda[X^\bullet; q, t] \mid \lambda \in \mathbb{Y} \text{ such that } \kappa(\lambda) = \alpha\}$ is a homogeneous basis of Λ^I, with $P_\lambda[X^\bullet; q, t]$ having degree $|\mathrm{quot}(\lambda)|$.

Example 3.1. For $r = 1$, $P_\lambda[X^\bullet; q, t]$ is the usual Macdonald P-function.

Example 3.2. If λ is an r-core than $H_\lambda[X^\bullet; q, t] = 1$ and $P_\lambda[X^\bullet; q, t] = 1$.

Example 3.3. Let $r = 2$ and $\lambda = (1, 1)$. We have $H_{11}(X^\bullet; q, t) = s_1[X^{(0)}] + t s_1[X^{(1)}]$. To verify this we consider the partitions $(2) \trianglerighteq (1, 1)$. We need only check

$$\begin{aligned} \mathcal{P}_{\mathrm{id}-t^{-1}\chi^{-1}}(s_1[X^{(0)}] + t s_1[X^{(1)}]) &= s_1[X^{(0)}] - t^{-1} s_1[X^{(1)} + t(s_1[X^{(1)} - t^{-1} s_1[X^{(0)}]) \\ &= (t - t^{-1}) s_1[X^{(1)}]. \end{aligned}$$

and note that $s_{\mathrm{quot}((1,1))} = s_{(\varnothing,(1))} = s_1[X^{(1)}]$. It follows that $P_{(1,1)}[X^\bullet; q, t^{-1}] = s_1[X^{(1)}]$.

3.3 Symmetric polynomials

Recall the sets of variables (1.3). There is a restriction map

$$\pi_{N_\bullet} : \Lambda^I \to \Lambda^I_{N_\bullet} := \bigotimes_{i \in I} \mathbb{K}\left[x_1^{(i)}, \ldots, x_{N_i}^{(i)}\right]^{\mathfrak{S}_{N_i}} \tag{3.4}$$

given by the tensor product $\pi_{N_\bullet} = \otimes_{i \in I} \pi_{N_i}$, where $\pi_N : \Lambda \to \mathbb{K}[x_1, \ldots, x_N]^{\mathfrak{S}_N}$ is the standard projection to symmetric polynomials. We also write $\pi_{N_\bullet}(f) = f[X_{N_\bullet}]$.

Let γ be an r-core and $\kappa(\gamma) = (c_0, c_1, \ldots, c_{r-1}) \in Q \subset \mathbb{Z}^I$. We say that $N_\bullet \in \mathbb{Z}^I_{\geq 0}$ is compatible with γ if

$$N_i - N_{i-1} = (\alpha_i^\vee, \kappa(\gamma)) = c_{i-1} - c_i, \quad \text{for all } i \in I, \tag{3.5}$$

where $(-,-) : Q^\vee \times Q \to \mathbb{Z}$ is the standard pairing between $\mathfrak{sl}_r$ root and coroot lattices. All $N_\bullet$ compatible with a given γ differ from each other by a vector in $\mathbb{Z}(1^r)$.

Example 3.4. In the setting of Example 2.2, the core is $\gamma = (2)$ and $\kappa(\gamma) = (1, -1, 0)$. The smallest $N_\bullet$ compatible with γ is $N_\bullet = (0, 2, 1) = -\kappa(\lambda) + (1, 1, 1)$. To this we can add the vector $(1, 1, 1)$ any number of times to obtain the other $N_\bullet$ compatible with γ.

Lemma 3.5. *Let γ be an r-core and $N_\bullet \in \mathbb{Z}^I_{\geq 0}$ be compatible with γ. Then*

1. *The quantity $N = |N_\bullet| := \sum_{i \in I} N_i$ is divisible by r.*
2. *For all $\lambda \in \mathbb{Y}_{N,\gamma}$ and all $p \in I$,*

$$N_p = |\{1 \leq i \leq N : i - \lambda_i - 1 \equiv p \bmod r\}|.$$

   In particular, $\mathrm{quot}(\lambda) = \lambda^\bullet$ *satisfies $\ell(\lambda^{(i)}) \leq N_i$ for all $i \in I$.*

3. *For any $\lambda^\bullet \in \mathbb{Y}^I$ satisfying $\ell(\lambda^{(i)}) \leq N_i$ for all i, the unique partition λ with core γ and quotient $\lambda^\bullet$ satisfies $\lambda \in \mathbb{Y}_{N,\gamma}$.*

An immediate consequence of parts (2) and (3) of Lemma 3.5 is the following:

Proposition 3.6. *Let γ be an r-core and $N_\bullet$ be compatible with γ. Then $\{P_\lambda[X_{N_\bullet}; q, t] \mid \lambda \in \mathbb{Y}_{N,\gamma}\}$ forms a basis of the space of color symmetric polynomials $\Lambda^I_{N_\bullet}$.*

4 Wreath Macdonald eigenoperators

Fix an r-core γ and compatible $N_\bullet \in \mathbb{Z}^I_{\geq 0}$. As before, let $X_{N_\bullet}$ be the variables as in (1.3) satisfying (3.5). Define a *shift pattern of* $X_{N_\bullet}$ to be a subset of $X_{N_\bullet}$ that contains no more than one variable of each color. A shift pattern *contains the color* $p \in I$ if it contains a variable of color p. Let $Sh_p(X_{N_\bullet})$ denote the set of all shift patterns containing p.

For a shift pattern $\underline{J}$, let $J \subset I$ denote the set of colors of the variables in $\underline{J}$. We denote the variables in $\underline{J}$ by $x^{(i)}_{\underline{J}}$, so $\underline{J} = \{x^{(i)}_{\underline{J}}\}_{i \in J}$. To $\underline{J}$ we associate the following:

1. *Gap labels:* For $i \in I$ let $i^\nabla \in J$ be the first element less than or equal to i in the cyclic order. We stipulate that $0 \leq i - i^\nabla \leq r - 1$. Define

$$x^{(i)}_{\underline{J}^\nabla} = q^{(i - i^\nabla)} x^{(i^\nabla)}_{\underline{J}}.$$

   In particular $x^{(i)}_{\underline{J}^\nabla} = x^{(i)}_{\underline{J}}$ if $i \in J$. Thus, while $\underline{J}$ gives a list of variables colored by $J \subset I$, we 'fill in the gaps' for values $i \in I \backslash J$ with q-shifts of the elements of $\underline{J}$.

2. *A cyclic-shift operator:* For $i \in J$, let $i^{\blacktriangledown} \in J$ be the first element *strictly* less than i in the cyclic order. We set $1 \leq i - i^{\blacktriangledown} \leq r$, where r occurs if and only if $|J| = \{i\}$. We then define the operator $T_{\underline{J}}$ on $\mathbb{K}[X_{N_\bullet}]$ as the $\mathbb{K}$-algebra map induced by

$$T_{\underline{J}}(x_l^{(i)}) = \begin{cases} q^{(i-i^{\blacktriangledown})} x_{\underline{J}}^{(i^{\blacktriangledown})} & \text{if } i \in J \text{ and } x_l = x_{\underline{J}}^{(i)} \\ x_l^{(i)} & \text{otherwise.} \end{cases}$$

For an n-tuple $\underline{\mathbf{J}} = (\underline{J}_1, \ldots, \underline{J}_n)$ of shift patterns and $0 \leq k \leq n$, we denote

$$|\underline{\mathbf{J}}| = \underline{J}_1 \cup \cdots \cup \underline{J}_n \subset X_{N_\bullet}, \quad |\underline{\mathbf{J}}|_{\leq k} = \underline{J}_1 \cup \cdots \cup \underline{J}_k \subset X_{N_\bullet}, \quad |\underline{\mathbf{J}}|_{\geq k} = \underline{J}_k \cup \cdots \cup \underline{J}_n \subset X_{N_\bullet}.$$

If $\underline{\mathbf{J}}$ is an n-tuple of shift patterns all containing color p, we say $\underline{\mathbf{J}}$ is *p-distinct* if the p-colored variables $x_{\underline{J}_k}^{(p)}$ are all distinct. Let $Sh_p^{[n]}(X_{N_\bullet})$ denote the set of all p-distinct n-tuples of shift patterns containing color p.

Finally, if $a_1, a_2, \ldots, a_n$ are operators then define $\overset{\curvearrowright}{\prod_{j=1}^{n}} a_j := a_1 \circ a_2 \circ \cdots \circ a_n$. This completes the definition of the operators $D_{p,n}$ in (1.4).

Example 4.1. Let $r = 1$, $p = 1$, $N_\bullet = (2, 1, 0)$, and $\lambda = (3, 1, 1)$. In this case, λ is a 3-core and by Example 3.2 $P_\lambda[X^\bullet; q, t] = 1$ so that $P_\lambda[X_{N_\bullet}; q, t] = 1$. There are three shift patterns containing 1: $\underline{J}_1 = \{x_1^{(1)}\}$, $\underline{J}_2 = \{x_1^{(0)}, x_1^{(1)}\}$, and $\underline{J}_3 = \{x_2^{(0)}, x_1^{(1)}\}$. We have

$$D_{1,1}(X_{N_\bullet}; q, t) = q \left(\frac{qtx_1^{(1)} - x_1^{(0)}}{q^2 x_1^{(1)} - x_1^{(0)}} \right) \left(\frac{qtx_1^{(1)} - x_2^{(0)}}{q^2 x_1^{(1)} - x_2^{(0)}} \right) T_{\underline{J}_1} \tag{4.1}$$

$$+ (1 - qt^{-1}) q \left(\frac{qtx_1^{(1)} - x_2^{(0)}}{x_1^{(0)} - x_2^{(0)}} \right) \left(\frac{qtx_1^{(1)}}{x_1^{(0)} - q^2 x_1^{(1)}} \right) T_{\underline{J}_2} \tag{4.2}$$

$$+ (1 - qt^{-1}) q \left(\frac{qtx_1^{(1)} - x_1^{(0)}}{x_2^{(0)} - x_1^{(0)}} \right) \left(\frac{qtx_1^{(1)}}{x_2^{(0)} - q^2 x_1^{(1)}} \right) T_{\underline{J}_3}. \tag{4.3}$$

The cyclic-shift operators act trivially on $P_\lambda(X_{N_\bullet}; q, t) = 1$. After a miraculous simplification one obtains $D_{1,1}(X_{N_\bullet}; q, t) P_\lambda[X_{N_\bullet}; q, t] = q P_\lambda[X_{N_\bullet}; q, t]$.

Example 4.2. Let $r = 2$, $p = 0$, $N_\bullet = (1, 1)$, and $\lambda = (1, 1)$. We have $P_{(1,1)}[X^\bullet; q, t] = s_1[X^{(1)}]$ from Example 3.3. Finitizing we obtain $P_\lambda[X_{N_\bullet}; q, t] = x_1^{(1)}$. There are two shift patterns containing 0: $\underline{J}_1 = \{x_1^{(0)}\}$, $\underline{J}_2 = \{x_1^{(0)}, x_1^{(1)}\}$. We then have

$$D_{0,1}(X_{N_\bullet}; q, t) = q \left(\frac{tx_1^{(0)} - x_1^{(1)}}{qx_1^{(0)} - x_1^{(1)}} \right) T_{\underline{J}_1} + (1 - qt^{-1}) \frac{x_1^{(1)}}{x_1^{(0)}} \left(\frac{tx_1^{(0)}}{x_1^{(1)} - qx_1^{(0)}} \right) T_{\underline{J}_2}.$$

Observe that $T_{\underline{J}_1} x_1^{(1)} = x_1^{(1)}$ and $T_{\underline{J}_2} x_1^{(1)} = q x_1^{(0)}$. Altogether then,

$$\begin{aligned} D_{0,1}(X_{N_\bullet}; q, t) P_\lambda[X_{N_\bullet}; q, t] &= q \left(\frac{t x_1^{(0)} - x_1^{(1)}}{q x_1^{(0)} - x_1^{(1)}} \right) x_1^{(1)} + (1 - qt^{-1}) \frac{x_1^{(1)}}{x_1^{(0)}} \left(\frac{t x_1^{(0)}}{x_1^{(1)} - q x_1^{(0)}} \right) q x_1^{(0)} \\ &= q x_1^{(1)} \left(\frac{t x_1^{(0)} - x_1^{(1)} - (t-q) x_1^{(0)}}{q x_1^{(0)} - x_1^{(1)}} \right) = q P_\lambda[X_{N_\bullet}; q, t]. \end{aligned}$$

Example 4.3. Let $r = 2$, $p = 1$, $N_\bullet = (0, 2)$, and $\lambda = (1)$. Because λ is a 2-core, $P_\lambda[X_{N_\bullet}; q, t] = 1$. There are only two shift patterns containing 1: $\underline{J}_1 = \{x_1^{(1)}\}$, $\underline{J}_2 = \{x_2^{(1)}\}$. Note that $T_{\underline{J}_1} x_1^{(1)} = q^2 x_1^{(1)}, T_{\underline{J}_1} x_2^{(1)} = x_1^{(1)}, T_{\underline{J}_2} x_1^{(1)} = x_1^{(1)}, T_{\underline{J}_2} x_2^{(1)} = q^2 x_2^{(1)}$. Therefore,

$$\begin{aligned} D_{1,1}(X_{N_\bullet}; q, t) P_\lambda[X_\bullet; q, t] &= \frac{(-1)(1 - qt^{-1})}{1 - q^2 t^{-2}} \left\{ \frac{q t x_2^{(1)} - q^2 x_1^{(1)}}{x_1^{(1)} - x_2^{(1)}} + \frac{q t x_1^{(1)} - q^2 x_2^{(1)}}{x_2^{(1)} - x_1^{(1)}} \right\} \\ &= \frac{(-1)(1 - qt^{-1})(-qt - q^2)}{1 - q^2 t^{-2}} = q t P_\lambda[X_{N_\bullet}; q, t]. \end{aligned}$$

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