THE MATHEMATICAL THEORY BEHIND THE ACCESSIBLE CALCULUS PROJECT

William Crombie
The Algebra Project, Inc.

This paper presents a theoretical reconstruction and critique of the elementary differential calculus that provides the mathematical basis for the Accessible Calculus Project (*ACP*)*. The ACP is a three-year project to field test materials that cover the fundamentals of the differential and integral calculus in the context of Algebra II. The ACP is developed at the level of Algebra II to ensure that all students passing through the US education system have a foundation in calculus and, consequently, the mathematical sciences that depend upon it.*

INTRODUCTION

The concepts of an order relation and an ordered set are fundamental to mathematics and mathematics education. In mathematics, we are all familiar with the conventional ordering of the real numbers on a number line. In mathematics education, the notion of a learning progression applies an order of complexity and sophistication to a set of concepts in a specified knowledge domain. But both disciplines also recognize that a given set or domain can support many different order relations. In the Algebra Project, we have investigated different developments of the Algebra to Calculus transition that support different orderings of the elementary to advanced learning continuum. We have found that whether Algebra is a subset of Calculus or Calculus of Algebra depends upon the architecture of the knowledge domain that is constructed.

This distinction between Algebra and Calculus is made by not conflating Calculus with Analysis. We are defining Calculus by its constitutive or defining problems: the problem of rate of change (the domain of the differential calculus) and the problem of net change (the domain of the integral calculus). We do not define the elementary calculus by the procedures, namely approximation and limits, conventionally used in the definition and exposition of its principal concepts. In fact, this study of advanced topics in an elementary setting is made possible by separating advanced topics into their basic concepts and advanced procedures. These basic concepts are then developed using the tools available at the more elementary level to yield the same results as the more advanced procedures.

The purpose of this paper is to outline the mathematical theory behind the ACP, not its pedagogical development for students in the classroom. In that sense, its audience is researchers and teachers interested in mathematics qua mathematics, in particular, the development of alternative pathways through some well-trodden mathematics. In this approach to the differential calculus, we do not attempt to calculate the derivative through successive approximations on a non-linear graph. Rather, we find the straight line whose slope is exactly equal to the rate of change of the graph at the given point. But how do we know how fast the non-linear graph changes at a point if we don't measure its rate of change? We don't. But we know whether the rate of change of a straight line is faster or slower than the rate of change of the graph by whether the line passes or is passed by the non-linear graph. Passing is the order relation at the heart of this approach.

THE ROLE OF THE TRANSITION LINE

This approach to the derivative is based upon an order relation, the idea of passing, as opposed to the notion of successive approximations. The Swiss cognitive scientist Piaget outlined the basic idea behind this approach in Psychology and Epistemology (Piaget, 1972). Piaget explained,

> Observation shows, in fact, that there exists a basic intuition of speed, independent of any idea of duration and resulting from the primal concept of order … namely the intuition of kinematic *overtaking*. If a moving object A is behind B at instant T1 and passes in front of moving object B at instant T2, it is judged to be faster, and this holds for all ages: nothing intervenes here except temporal order (T1 before T2) and spatial order (behind and in front of), and there is no consideration of duration or space traversed. Speed is therefore initially independent of durations.

In applying this description of an order relation in the context of Algebra, consider a smooth non-linear polynomial graph as pictured in Figure I. We will consider an arbitrary point on the graph and all the non-vertical straight lines passing through the specified point. The lines that pass from below the graph to above the graph at the specified point have a slope that is greater than the rate of change of the graph at that point. The straight lines that pass from above the graph to below the graph at the specified point have a slope that is less than the rate of change of the graph at the point. We are interested in the line that separates the lines that are faster than the graph from the lines that are slower than the graph at the specified point. We will call that line the Transition Line. The Transition Line cuts the plane into two half-planes: one where the straight lines pass by the graph and the other where the straight lines are passed by the graph. Trichotomy implies that the rate of change of the non-linear polynomial graph at the point of interest is exactly equal to the slope of the Transition Line. If we can determine the Transition Line, we have the rate of change of the graph at the point of interest.

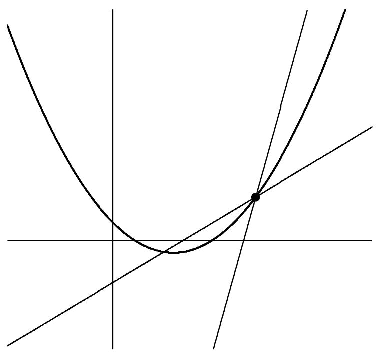

Figure I

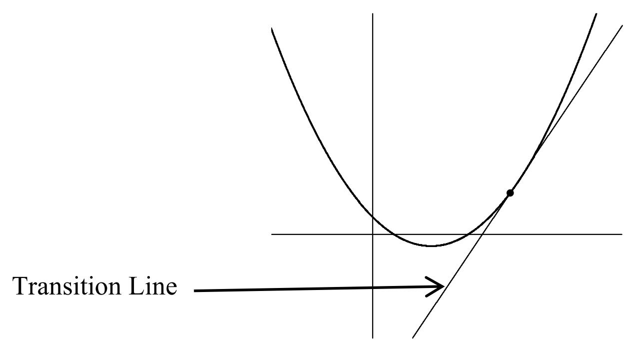

Figure II

The Transition Line (Figure I) has three basic properties outlined below.

I. The Transition Line does exist.
   Consider all the non-vertical lines at the point of intersection. The values of the slopes of this collection of straight lines are complete. They fill the real line. Thus, there is a slope value corresponding to the transition line.

II. The Transition Line is unique.
Suppose there are two distinct Transition Lines. If the Transition Lines are distinct and pass through the given point on the graph, there is a non-zero angle between them. Consider the lines that pass through this angle. By one of the Transition Lines, the lines within this angle pass above the curve. By the other Transition Line, they pass below. Since the same lines can not both pass above and below the curve, the two Transition Lines must coincide.

III. The Transition Line is the closest line to the graph at the point of intersection.
Suppose that there is a line passing between the graph and the Transition Line. If there was such a line, the Transition Line would not separate the line passing above the graph from the lines passing below the curve. Therefore, there can be no line passing between the curve and the Transition Line.

The Transition Line, the closest line to the graph, is historically called the Tangent Line. This is essentially Euclid's definition of the tangent line in Proposition 16, Book III of the Elements. As formulated here, the transition line/tangent line is a Dedekind cut in the pencil of all lines through the point of intersection on the polynomial graph. Determining the tangent line as the closest line to a polynomial graph is a straightforward exercise in Algebra.

THE TRANSITION LINE IS THE CLOSEST LINE TO A POLYNOMIAL GRAPH

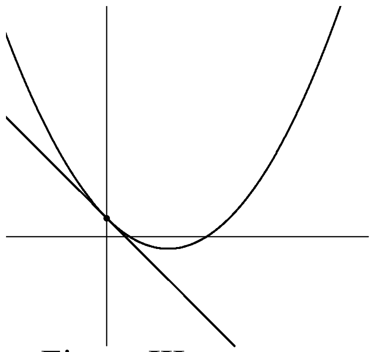
Figure III

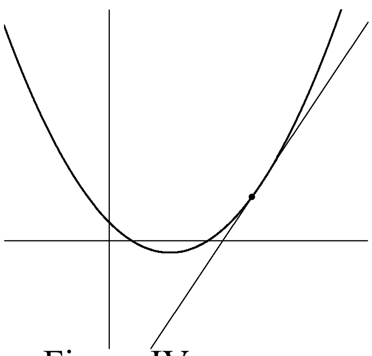
Figure IV

We will present the argument for a quadratic function, but the basic argument applies to a polynomial of any degree. The place to begin is with the polynomial function at the y-intercept (Figure III). Consider the closest line to the polynomial graph, $y = ax^2 + bx + c$, at the y-intercept $(0,c)$. The closest line to the graph requires the closest numerical agreement between the coordinates of the graph and the coordinates of the line. For the coordinates of the graph and the line to have the closest numerical agreement, the polynomials that define the graph and the line must maximally agree. The polynomials of the graph and the line maximally agree when their coefficients are the same to 1st order. Therefore, the tangent line at the y-intercept, $x = 0$, is the line that agrees with the graph to 1st order.

$$y = ax^2 + bx + c$$

$$y_t = bx + c$$

The tangent line at any other point on the graph (Figure IV) can be found by translating the y-axis and applying the condition of 1st-order agreement.

So, the closest line to a polynomial graph, $y = ax^2 + bx + c$, at the point (x_0, y_0) is found by a change of coordinates. The tangent line at $x = x_0$ becomes the tangent line at $x' = 0$, where a coordinate change of the form $x = x_0 + x'$ has transformed $y = ax^2 + bx + c$ into $y = a(x')^2 + b'(x') + c'$ and the tangent line is given by $y_t = b'(x') + c'$ The tangent line at $x = x_0$ is the line that agrees with the polynomial graph, $y = ax^2 + bx + c,$ at $x = x_0$ to 1st order, namely, $y_t = b'(x - x_0) + c'$ where $b' = 2a + b$.

This approach to the tangent line can be characterized by an equivalence relation: agreement to 1st order: the tangent line to a polynomial graph is the line that agrees with the graph at the point of tangency to 1st order. Consequently, tangent lines to any polynomial at any point can be found by expanding the polynomial to 1st order around the point of interest. The equation of the tangent line is simply the linear portion of the expanded polynomial.

This argument can, with a little work, be extended to include rational and algebraic functions by considering polynomials in two variables, $P(x, y)$, rather than simply one variable.

CONCLUSION

The architecture of a knowledge domain refers to how its concepts are structured and developed in a meaningful and logical manner. In the realm of the elementary calculus, initially formulated by Newton and Leibniz in the 17th century, this architecture has undergone several revisions, reflecting the evolution of mathematical thought over some 300 years. This evolution underscores how the organization of a knowledge domain can either facilitate or hinder access to its concepts by learners in addition to informed practitioners.

Our investigations into the architecture of the elementary calculus reveal that its historically derived form places unnecessary restrictions on developing its basic concepts. Specifically, the traditional approach often introduces complex ideas/techniques early on, which are simply unnecessary to reach the targeted concept and prevent learners from approaching the topic earlier in their academic careers.

To address this issue, we propose an alternative construction of the basic concept of the derivative. Our approach focuses on building this foundational concept using only basic Algebra, avoiding introducing more advanced concepts/techniques until they are necessary. By this restructuring of the learning process, we aim to make calculus more accessible and comprehensible to a wider range of students, ultimately enhancing their ability to engage with and master this fundamental branch of mathematics.

Here is a question that in some way summarizes the present issue. What is the common difference between Calculus and Pluto? The answer: the difference between a future versus a past demotion. Both the differential and integral calculus, based on their mathematical content, should be placed in the secondary mathematics curriculum as a topic within the domain of basic Algebra. The present position of Calculus as the capstone course of high school mathematics does little more than maintain a standard of preference and social inequity that serves neither the students with access to Calculus nor those without access. Calculus, as a subset of Algebra, raises the level of math literacy for all students. Calculus is still a filter rather than a pump. Algebra is the place to prime the pump.