

Lie-algebra centers via de-categorification

Alexandru Chirvasitu

Abstract. Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field $\mathbb{k}$ of characteristic zero. Define the universal grading group $\mathcal{C}(\mathfrak{g})$ as having one generator g_ρ for each irreducible $\mathfrak{g}$ -representation ρ , one relation $g_\pi = g_\rho^{-1}$ whenever π is weakly contained in the dual representation ρ^* (i.e. the kernel of π in the enveloping algebra $U(\mathfrak{g})$ contains that of ρ^*), and one relation $g_\rho = g_{\rho'} g_{\rho''}$ whenever ρ is weakly contained in $\rho' \otimes \rho''$.

The main result is that attaching to an irreducible representation its central character gives an isomorphism between $\mathcal{C}(\mathfrak{g})$ and the dual $\mathfrak{z}^*$ of the center $\mathfrak{z} \leq \mathfrak{g}$ when $\mathfrak{g}$ is (a) finite-dimensional solvable; (b) finite-dimensional semisimple. The group $\mathcal{C}(\mathfrak{g})$ is also trivial when the enveloping algebra $U(\mathfrak{g})$ has a faithful irreducible representation (which happens for instance for various infinite-dimensional algebras of interest, such as $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$). These are analogues of a result of Mürger’s for compact groups and a number of results by the author on locally compact groups, and provide further evidence for the pervasiveness of such center-reconstruction phenomena.

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Introduction

The initial motivation for the material below is a phenomenon noted in [14], which it will be instructive to summarize. Consider a compact group $\mathbb{G}$, and define its *chain group* $\mathcal{C}(\mathbb{G})$ by generators and relations, as follows:

- there is a generator g_V for every irreducible unitary $\mathbb{G}$ -representation V ;
- and a relation $g_U = g_V g_W$ whenever U is contained as a summand in $V \otimes W$.

The main result of [14] (namely [14, Theorem 3.1]) says that assigning to an irreducible representation its central character implements an isomorphism of $\mathcal{C}(\mathbb{G})$ onto the discrete abelian group $\widehat{Z(\mathbb{G})}$, i.e. the Pontryagin dual of the compact (abelian) center $Z(\mathbb{G}) \leq \mathbb{G}$.

In other words, the dual center $\widehat{Z(\mathbb{G})}$ can be recovered from the category of $\mathbb{G}$ -representations by a process of *de-categorification* (hence this paper’s title):

- objects (i.e. $\mathbb{G}$ -representations) are demoted to elements (of $\mathcal{C}(\mathbb{G})$);

- the tensor product bifunctor becomes multiplication;
- and for good measure, dualization in the category of representations corresponds to taking inverses in $\mathcal{C}(\mathbb{G})$: it turns out that the element g_{V^*} corresponding to the dual (or contragredient) representation V^* automatically equals g_V^{-1} .

Note also the general “Tannakian” [19, 20, 6, 23] flavor about the discussion: recovering structure from monoidal categorical data.

The theme is taken up in [3] in the context of *locally* compact groups, where a chain group can be defined analogously (generators again given by irreducible unitary representations), with only the obvious sensible modifications: one imposes a relation $g_U = g_V g_W$ whenever U is *weakly* contained in $V \otimes W$ in the sense of [2, Definition F.1.1] (actual containment would be too much to ask for).

The phenomenon turns out to be remarkably robust: one again has isomorphisms $\mathcal{C}(\mathbb{G}) \cong \bar{Z}(\mathbb{G})$ of (this time topological) groups for broad interesting classes of locally compact groups $\mathbb{G}$: discrete countable with infinite conjugacy classes, connected nilpotent Lie groups, connected semisimple Lie groups, etc.

The present iteration of the project investigates the natural purely algebraic analogues of the above-mentioned objects, constructions and results. Much of the discussion makes sense for Hopf algebras (in place of groups), and we do give definitions in that generality (Definition 2.11), but the substance of the paper mostly concerns Lie algebras. Hence:

Definition 0.1. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{k}$. The *chain group* $\mathcal{C}(\mathfrak{g})$ is defined by

- generators g_ρ for irreducible $\mathfrak{g}$ -representations ρ ;
- and relations $g_\rho = g_{\rho'} g_{\rho''}$ whenever ρ is *weakly contained* in $\rho' \otimes \rho''$ (Definition 2.1): the kernel of ρ , regarded as a morphism from the enveloping algebra $U(\mathfrak{g})$, contains that of $\rho' \otimes \rho''$;
- together with relations $g_\pi = g_\rho^{-1}$ whenever π is weakly contained in the contragredient representation ρ^* .

One might hope, by analogy to everything recalled above, that

- irreducible $\mathfrak{g}$ -representations ρ admit central characters, and in particular give functionals $\mathfrak{z}(\mathfrak{g}) \rightarrow \mathbb{k}$, where $\mathfrak{z}(\mathfrak{g}) \leq \mathfrak{g}$ is the center of the Lie algebra (this is frequently the case, e.g. for finite-dimensional Lie algebras over algebraically closed fields of characteristic zero [5, Proposition 2.6.8]);
- and that this then gives an isomorphism $\mathcal{C}(\mathfrak{g}) \cong \mathfrak{z}(\mathfrak{g})^*$, so that again, the dual center $\mathfrak{z}(\mathfrak{g})^*$ is a de-categorification of the category of $\mathfrak{g}$ -representations.

The main results of the paper confirm that this holds in all cases I have been able to check (i.e. I do not know of any Lie algebras for which it is *not* true). Summarizing Proposition 2.14, Corollary 2.15 and Theorems 2.26 and 2.32:

Theorem 0.2. *Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field $\mathbb{k}$ of characteristic zero and $\mathfrak{z} \leq \mathfrak{g}$ its center.*

- (a) *Associating to an irreducible representation its central character provides an isomorphism $\mathcal{C}(\mathfrak{g}) \cong \mathfrak{z}^*$ if $\mathfrak{g}$ is finite-dimensional and either solvable or semisimple.*
- (b) *Furthermore, $\mathcal{C}(\mathfrak{g})$ is trivial if the enveloping algebra $U(\mathfrak{g})$ has a faithful simple module.*
- (c) *So as a particular case of the previous item, we again have an isomorphism $\mathcal{C}(\mathfrak{g}) \cong \mathfrak{z}^*$ (both sides being trivial) when $\mathfrak{g}$ is one of the infinite-rank complex Lie algebras $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ or $\mathfrak{sp}(\infty)$.*

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1. Preliminaries

We make occasional reference to general ring-theoretic material for which [11, 10], say, are good reference. Coalgebras, bialgebras and Hopf algebras (over fields) will also feature sporadically; [22, 13, 18] all provide good background, which we reference with more specificity where appropriate. All rings are assumed unital.

The phrase ‘representation’ will be employed as a synonym for ‘left module’, appropriately linear when working over a field. Additionally, modules are left unless specified otherwise.

Recall that a proper two-sided ideal $I \trianglelefteq A$ of a ring is

- *primitive* (e.g. [11, Definitions 11.2 and 11.3], [5, §3.1.4]) if it is the kernel of an irreducible A -representation.
- *prime* ([11, Definition 10.1], [5, §3.1.1]) if for ideals J_i , $i = 1, 2$ with $J_1 J_2 \leq I$, I must contain one of the J_i .
- and *semi-prime* ([11, Definition 10.8], [5, §3.1.3]) if for any ideal J with $J^2 \leq I$ we have $J \leq I$.

We apply the term ‘primitive’ universally, whether in the purely algebraic or analytic setting (where algebras are C^* , ideals are closed, representations are on Hilbert spaces with the appropriate notion of irreducibility, etc.).

The following construction features prominently in the discussion below.

Definition 1.1. Consider a set $(S, \triangleleft)$ equipped with a ternary relation, written

$$s \triangleleft (s', s'').$$

- The *chain semigroup* $\mathcal{C} = \mathcal{C}(S, \triangleleft)$ associated to ' $\triangleleft$ ' is defined as having one generator g_s for each $s \in S$ and relations

$$g_s = g_{s'} g_{s''} \quad \text{whenever} \quad s \triangleleft (s', s'').$$

It will occasionally be convenient to enrich the structure of $\mathcal{C}$ as follows.

- If in addition S is equipped with a distinguished element s_0 , the *chain monoid* $\mathcal{C} = \mathcal{C}(S, \triangleleft, s_0)$ is defined as above, with the additional constraint that g_{s_0} be the trivial element of the monoid.
- Finally, if S is also equipped with a binary relation ' $\sim$ ', the *chain group* $\mathcal{C} = \mathcal{C}(S, \triangleleft, s_0, \sim)$ is the chain monoid, with the additional constraint that

$$g_{s'} = g_s^{-1} \quad \text{whenever} \quad s \sim s'.$$

2. Chain groups and center reconstruction

We will soon specialize the discussion to Lie algebras, but some of it goes through more generally. $\mathbb{k}$ always denotes a field that algebras, Hopf algebras, Lie algebras and so on are understood to be linear over. Additional assumptions will be in force throughout most of the paper, but the reader will be warned when they come into effect.

2.1. Generalities on weak containment for Hopf and Lie algebras

For an arbitrary ring A , $\text{Prim}(A)$ is its space of primitive ideals. It can be equipped with the familiar *Jacobson topology* ([5, §3.2.2] or [15, §7.1.3]): for an ideal $I \trianglelefteq A$, set

$$V(I) := \{P \in \text{Prim}(A) \mid I \leq P\}.$$

The $V(I)$ are precisely the closed sets of the topology. For a Lie algebra $\mathfrak{g}$ with enveloping algebra $U = U(\mathfrak{g})$ the notation $\text{Prim}(\mathfrak{g}) := \text{Prim}(U)$ is an alternative.

We also borrow the usual language of *weak containment* from functional analysis [4, §3.4.5];

Definition 2.1. Let π and ρ two representations of a ring A .

- We say that π is *weakly contained* in ρ (written $\pi \preceq \rho$) if $\ker \pi \supseteq \ker \rho$.
- π and ρ are *weakly equivalent* (written $\pi \approx \rho$) if each of them weakly contains the other.

The terms apply to Lie-algebra representations, where A is taken to be the enveloping algebra.

When convenient, we might also commit mild notational abuse in writing $M \preceq N$ if M and N are the left A -modules carrying the two respective representations.

Remark 2.2. When the kernel of a representation $\rho : A \rightarrow \text{End}(V)$ is an intersection of primitive ideals, we have a weak equivalence

$$\rho \approx \bigoplus \pi, \text{ irreducible } \pi \preceq \rho.$$

When $A = U(\mathfrak{g})$ is the envelope of a finite-dimensional Lie algebra, this is the case precisely if $\ker \rho$ is semi-prime [5, Proposition 3.1.15].

There are alternative characterizations of weak containment, parallel to their analytic counterparts ([4, Theorem 3.4.4], [2, Theorem F.4.4]). First, we need

Definition 2.3. Let V be a $\mathbb{k}$ -vector space. The *weak* topology* on V^* is the weakest topology making all maps

$$V^* \ni f \mapsto f(v) \in \mathbb{k}, \quad v \in V$$

continuous (with $\mathbb{k}$ topologized discretely).

There is also the following notion (following [5, §2.7.8], for instance).

Definition 2.4. Let $\rho : A \rightarrow \text{End}(V)$ be a representation of a $\mathbb{k}$ -algebra on a vector space. The space $MC(\rho)$ of *matrix coefficients* (or just plain ‘coefficients’) of ρ is

$$MC(\rho) := \text{span}\{f(\rho(\cdot)v) \mid v \in V, f \in V^*\} \leq A^*.$$

We follow standard practice (e.g. [5, §1.2.20]) in denoting, with a ‘ $\perp$ ’ superscript, annihilators of vector spaces with respect to a pairing/bilinear form. Specifically, if

$$W \otimes V \xrightarrow{b} \mathbb{k}$$

is such a pairing and $V_0 \leq V$,

$$V_0^\perp := \{w \in W \mid b(w, V_0) = \{0\}\}.$$

The pairing will always be understood, and in fact it will typically be the standard one between a vector space V and its full dual V^* .

We now have the following simple analogue of [4, Theorem 3.4.4].

Lemma 2.5. *For representations $\rho_i : A \rightarrow \text{End}(V_i)$, $i = 1, 2$ of an algebra, the following conditions are equivalent.*

(a) $\rho_1 \preceq \rho_2$ in the sense of Definition 2.1.

(b) We have the inclusion

$$\overline{MC(\rho_1)} \leq \overline{MC(\rho_2)}$$

in A^* , with the bar denoting the weak* closure and MC as in Definition 2.4.

(c) Similarly, we have the inclusion

$$MC(\rho_2)^\perp \leq MC(\rho_1)^\perp$$

in A .

Proof. The central observations are:

(I) For any vector space V , if

$$\text{id} : \text{End}(V) \rightarrow \text{End}(V)$$

is the standard representation of its endomorphism algebra, $MC(\text{id})$ is weak*-dense in $\text{End}(V)^*$.

(II) For any $W \leq V^*$, the weak* closure of W is nothing but the annihilator of $W^\perp \leq V$, i.e. $W^{\perp\perp}$.

This latter remark immediately implies the equivalence of (b) and (c), while (I) shows that (c) reads

$$\ker \rho_2 \leq \ker \rho_1$$

and hence is equivalent to (a). ■

We use the language of *induced representations* of [5, Chapter 5].

Notation 2.6. Let $\mathfrak{h} \leq \mathfrak{g}$ be an inclusion of Lie algebras, and denote the enveloping-algebra construction by $U(\cdot)$.

(a) For a representation $\rho : U(\mathfrak{h}) \rightarrow \text{End}(W)$ the corresponding *induced representation*

$$\text{Ind}(\rho) \quad \text{or} \quad \text{Ind}^{\mathfrak{g}}(\rho) \quad \text{or} \quad \text{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\rho)$$

is $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} W$, with its obvious left $U(\mathfrak{g})$ -module structure.

(b) Assume now that $\mathfrak{g}$ is finite-dimensional.

- In general, for an inclusion $F \leq E$ of finite-dimensional vector spaces left invariant by an operator $T \in \text{End}(E)$, write $\text{tr}_{E/F}(T)$ for the trace of the operator induced by T on the quotient space E/F .

- For $x \in \mathfrak{h}$ set

$$\theta_{\mathfrak{g},\mathfrak{h}}(x) := \frac{1}{2} \text{tr}_{\mathfrak{g}/\mathfrak{h}} ad(x),$$

where $ad : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$ is the adjoint representation.

- For a representation $\rho : \mathfrak{h} \rightarrow \text{End}(V)$, its *twist* $\tilde{\rho}$ is defined by

$$\mathfrak{h} \ni x \xrightarrow{\tilde{\rho}} \rho(x) + \theta_{\mathfrak{g},\mathfrak{h}}(x) \text{id} \in \text{End}(V).$$

Regarding the functional $\theta_{\mathfrak{g},\mathfrak{h}} \in \mathfrak{h}^*$ as a 1-dimensional $\mathfrak{h}$ -representation (which it is, since it vanishes on $[\mathfrak{h}, \mathfrak{h}]$), we also have

$$\tilde{\rho} \cong \rho \otimes \theta_{\mathfrak{g},\mathfrak{h}}.$$

- Finally, *twisted induction* $\widetilde{\text{Ind}}$ is defined by

$$\widetilde{\text{Ind}}_{\mathfrak{h}}^{\mathfrak{g}}(\rho) := \text{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\tilde{\rho}).$$

Remark 2.7. When $\mathfrak{g}$ is nilpotent all traces making a difference between induction and twisted induction in Notation 2.6 vanish, so in that case there is no distinction between Ind and $\widetilde{\text{Ind}}$.

As in the analytic case, the familiar operations of tensoring, induction, etc. respect weak containment (see e.g. [2, §F.3] for the versions pertaining to locally compact groups).

Proposition 2.8. *The weak containment relation $\preceq$ of Definition 2.1 is compatible with the following operations.*

(a) *Direct sums, in the sense that if $\pi_i \preceq \rho_i$ for a family $i \in I$ of representations of a ring A , we also have*

$$\bigoplus_I \pi_i \preceq \bigoplus_I \rho_i.$$

(b) *Tensor products, for bialgebras over fields:*

$$\pi_i \preceq \rho_i, \ i = 1, 2 \Rightarrow \pi_1 \otimes \pi_2 \preceq \rho_1 \otimes \rho_2$$

for representations π_i and ρ_i of a bialgebra H over any field.

(c) *Scalar extension (or induction), for free ring extensions: suppose $A \rightarrow B$ is a ring embedding with B/A free as a right A -module. If M and N are two left A -modules,*

$$M \preceq N \implies B \otimes_A M \preceq B \otimes_A N.$$

(d) *Induction, for Lie algebras: if $\mathfrak{h} \leq \mathfrak{g}$ is an inclusion of Lie algebras and $\pi \preceq \rho$ are $\mathfrak{h}$ -representations, then*

$$\text{Ind}^{\mathfrak{g}}(\pi) \preceq \text{Ind}^{\mathfrak{g}}(\rho).$$

(e) *Twisted induction: same as in (d), but with $\widetilde{\text{Ind}}$ in place of Ind (assuming $\mathfrak{g}$ is finite-dimensional).*

Proof. Considering the claims in the stated order:

(a) follows from the fact that

$$\ker \left(\bigoplus_I \pi_i \right) = \bigcap_I \ker \pi_i$$

and similarly for the ρ_i , so if $\ker \pi_i \geq \ker \rho_i$ the same goes for I -fold intersections.

(b) Consider a bialgebra H as in the statement. We then have

$$\ker(\pi_1 \otimes \pi_2) = \ker \pi_1 \wedge \ker \pi_2,$$

where

$$V \wedge W := \ker \left(H \xrightarrow{\Delta} H \otimes H \rightarrow (H/V) \otimes (H/W) \right)$$

is the *wedge product* of two subspaces $V, W \leq H$ ([22, p.179] or [13, proof of Theorem 5.2.2]). That “wedging” respects inclusion is elementary, hence the conclusion.

- (c) Let J_M and J_N be the respective kernels of the module-structure maps

$$A \rightarrow \text{End}(M) \quad \text{and} \quad A \rightarrow \text{End}(N),$$

the freeness assumption ensures that the proof of [5, Proposition 5.1.7 (i)] replicates to show that

$$\text{Ann}_B(M) = BJ_M$$

and similarly for N , where

$$M \leq B \otimes_A M, \quad N \leq B \otimes_A N$$

via the canonical maps. But as in [5, Proposition 5.1.7 (ii)], the annihilators of $M \leq B \otimes_A M$ in B are the largest two-sided ideals contained in

$$\text{Ann}_B(M) = BJ_M \quad \text{and} \quad \text{Ann}_B(N) = BJ_N :$$

this is an instance of the general remark that for any set S of generators for a left R -module V , the annihilator of V is the largest bilateral ideal contained in the annihilator of S .

The assumption

$$J_M \geq J_N \implies BJ_M \geq BJ_N$$

then implies the same ordering between the largest 2-sided ideals respectively contained in the two, and we are done.

- (d) is a consequence of (c), applied to the ring inclusion $U(\mathfrak{h}) \leq U(\mathfrak{g})$, the freeness assumption being a consequence of the *Poincaré-Birkhoff-Witt theorem* ([5, Theorem 2.1.11] or [15, Theorem 6.1.1]).
- (e) follows from points (b) and (d), given the definition

$$\widetilde{\text{Ind}}^{\mathfrak{g}}(\pi) = \text{Ind}^{\mathfrak{g}}(\pi \otimes \theta_{\mathfrak{g}, \mathfrak{h}})$$

of Notation 2.6 and its ρ analogue.

This concludes the proof. ■

Remark 2.9. (1) An alternative proof of Proposition 2.8 (b) could have used Lemma 2.5: we are assuming

$$\overline{MC(\pi_i)} \leq \overline{MC(\rho_i)},$$

whence also

$$\overline{MC(\pi_1) \otimes MC(\pi_2)} \leq \overline{MC(\rho_1) \otimes MC(\rho_2)}.$$

The conclusion then follows from the fact that in general, $V^* \otimes W^*$ is weak*-dense in $(V \otimes W)^*$.

- (2) In reference to Proposition 2.8 (c), the issue of weak-containment permanence under scalar extension is a bit delicate. On the one hand that statement made a fairly strong freeness assumption. On the other hand though, even *faithful flatness* [10, §4I] of B as a right A -module would not quite have been sufficient, as Example 2.10 shows.

Example 2.10. Let $p \neq q$ be two prime numbers, and denote by A the *localization* [1, Chapter 3] of $\mathbb{Z}$ away from the prime ideals (p) and (q) : the ring obtained from $\mathbb{Z}$ by inverting all primes distinct from p and q .

We consider two A modules:

$${}_pM := \bigoplus_n A/p^n A$$

and similarly for ${}_qM$ (with q in place of p). Both are faithful, in the sense that their annihilators are trivial. In particular, ${}_pM \approx {}_qM$ (Definition 2.1).

The ring extension $A \rightarrow B$ will now be the (pq) -adic completion of [1, discussion following Proposition 10.5]:

$$B := \varprojlim_n A/(pq)^n A.$$

That $A \rightarrow B$ is faithfully flat follows, say, from [1, Chapter 10, Exercise 7]. But B is easily computed to be the product

$$B \cong \mathbb{Z}_p \times \mathbb{Z}_q$$

of the rings of p -adic and q -adic integers ([1, p.105, Example 2]) respectively. Because p is invertible in $\mathbb{Z}_q$, the ideal

$$\mathbb{Z}_q \trianglelefteq B \cong \mathbb{Z}_p \times \mathbb{Z}_q$$

annihilates ${}_pM$ and hence $B \otimes_A {}_pM$. With just a trace amount of additional effort one can show that in fact

$$\text{Ann}_N(B \otimes_A {}_pM) = \mathbb{Z}_q \quad \text{and} \quad \text{Ann}_N(B \otimes_A {}_qM) = \mathbb{Z}_p.$$

In particular, the two modules have annihilators that are incomparable under containment, so the weak containment relation has not survived the faithfully flat scalar extension along $A \rightarrow B$.

Definition 2.11. (1) Let H be a Hopf algebra over a field $\mathbb{k}$. The *chain group* $\mathcal{C}(H)$ is that of Definition 1.1, for

- the set S of isomorphism classes of simple H -modules;
- the ternary relation

$$\rho \triangleleft (\rho', \rho'') \iff \rho \preceq \rho' \otimes \rho''; \tag{1}$$

- the distinguished element of S corresponding to the trivial H -module $\mathbb{k}$ induced by the counit $\varepsilon : H \rightarrow \mathbb{k}$;
- and the binary relation ‘ $\sim$ ’ defined by

$$\rho \sim \text{any irreducible representation weakly contained in the dual } \rho^*.$$

- (2) Similarly, the chain group $\mathcal{C}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is that of its universal enveloping algebra with its usual Hopf-algebra structure [13, Example 1.5.4]: $\mathcal{C}(\mathfrak{g}) := \mathcal{C}(U(\mathfrak{g}))$.

Remark 2.12. (1) References in Definition 2.11 to primitive ideals containing arbitrary ideals are unproblematic, so that $\mathcal{C}(H)$ is indeed a well-defined group: every proper ideal is contained in a maximal one (for any unital ring, by Zorn's lemma), and in turn maximal ideals are primitive [5, §3.1.6].

- (2) Suppose the Lie algebra $\mathfrak{g}$ of Definition 2.11 is finite-dimensional, and let ρ be an irreducible $\mathfrak{g}$ -representation.

The kernel of ρ^* is easily seen to be $S(\ker \rho)$, where $S : U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ is the antipode. Since S is an anti-automorphism (the *principal anti-automorphism* $x \mapsto x^T$ of [5, §2.2.18]), $\ker \rho^*$ is semi-prime and thus an intersection of primitive ideals (Remark 2.2). But this means that

$$\rho^* \approx \bigoplus \pi$$

for irreducible $\pi \preceq \rho^*$ as in Remark 2.2. There are, in particular, “enough” irreducible representations that play the role of the inverse to ρ in Definition 2.11.

- (3) And in fact, if furthermore $\mathfrak{g}$ is solvable (as well as finite-dimensional), the primitive ideals of its enveloping algebra $U := U(\mathfrak{g})$ are precisely those prime ideals $I \trianglelefteq U$ for which the intersection of the primes $I' \supsetneq I$ contains I strictly [5, Theorem 4.5.7].

It follows from this characterization that any anti-automorphism of U sends primitive ideals to primitive ideals, and hence $\ker \rho^*$ is *primitive* (as opposed to just (semi-)prime). The same remark is made in [7, Chapitre I, §8], along with related comments.

- (4) The generator $g_\rho \in \mathcal{C}(H)$ of the chain group does not actually depend on the isomorphism class of ρ , but rather only on the primitive ideal $\ker \rho$. This is immediate from Proposition 2.8 (b), which implies that

$$\rho \preceq \rho' \otimes \rho''$$

entails the same relation upon substituting for ρ'' (say) any irreducible representation weakly equivalent to it (i.e. having the same kernel).

We will often take this observation for granted in the sequel, and extend the notation g_ρ (for irreducible representations ρ) to g_J (for primitive ideals $J = \ker \rho$).

In reference to Remark 2.12 (4), we observe that passing to the chain group obliterates the distinction between primitive ideals ordered by inclusion:

Lemma 2.13. *Let $J \leq J' \trianglelefteq H$ be two primitive ideals of a Hopf algebra. In the notation of Remark 2.12 (4), we have*

$$g_J = g_{J'} \text{ in } \mathcal{C}(H).$$

Proof. Let ρ and ρ' be irreducible representations with kernels J and J' respectively. We have $\rho \preceq \rho'$ by assumption, so that

$$\rho \otimes \rho' \preceq \rho' \otimes \rho'$$

by Proposition 2.8 (b). But then irreducible representations weakly contained in the left-hand side are also weakly contained in the right-hand side, whence

$$g_J g_{J'} = g_\rho g_{\rho'} = g_{\rho'} g_{\rho'} = g_{J'} g_{J'}$$

in $\mathcal{C}(\mathfrak{g})$. Since the latter is a group, this indeed implies $g_J = g_{J'}$. $\blacksquare$

As a simple consequence, chain groups are easy to understand for Hopf algebras with a “large” simple representation.

Proposition 2.14. *The chain group $\mathcal{C}(H)$ of a Hopf algebra with a faithful simple module is trivial.*

Proof. Since $\{0\}$ is primitive and contained in any other primitive ideal $J \trianglelefteq H$, Lemma 2.13 says that $g_J = g_{\{0\}}$ for all J . In short, the chain group is a singleton (and hence trivial). $\blacksquare$

In particular, Proposition 2.14 applies to infinite-dimensional Lie algebras that have gained some recent attention: the infinite-rank $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$ obtained by embedding $\mathfrak{sl}(n) \subset \mathfrak{sl}(n+1)$ in the obvious fashion, as upper left-hand corner matrices (and similarly for orthogonal and symplectic matrices). The primitive ideals of their enveloping algebras are classified in [16, 17], and as observed in [16, Introduction], these enveloping algebras have (many) faithful simple modules; consequently:

Corollary 2.15. *The infinite-rank classical complex Lie algebras $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$ of [16, §1] have trivial chain groups.*

The significance of this remark in the present context will become apparent later, when we seek to identify (typically for *finite*-dimensional Lie algebras) the chain group $\mathcal{C}(\mathfrak{g})$ with the dual $\mathfrak{z}^*$ of the center $\mathfrak{z} \leq \mathfrak{g}$. Because $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$ have trivial centers, Corollary 2.15 fits into the same pattern as Theorems 2.26 and 2.32 below.

Remark 2.16. Although Corollary 2.15 focuses on a few specific infinite-dimensional Lie algebras, *finite*-dimensional examples fitting into the mold of Proposition 2.14 exist: according to [5, Theorem 6.1.1 and Lemma 6.1.2 (i)], for instance, the non-abelian 2-dimensional Lie algebra has the requisite property.

Convention 2.17. We henceforth focus on

- finite-dimensional Lie algebras;
- over algebraically closed fields $\mathbb{k}$ of characteristic zero.

Unless specified otherwise (e.g. in reverting to the general case by explicitly mentioning ‘arbitrary fields’ or some such phrase), these assumptions are in place throughout the sequel.

The next few subsections are titled for the various classes of Lie algebras under consideration therein.

2.2. Nilpotent

Assume until further notice that $\mathfrak{g}$ is finite-dimensional. Because furthermore the ground field is algebraically closed, every irreducible representation has a *central character* [5, §2.6.7 and Proposition 2.6.8]: the center $Z(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$ acts via an algebra morphism $\chi : Z(\mathfrak{g}) \rightarrow \mathbb{k}$. In particular, the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ acts via a linear functional

$$\chi|_{\mathfrak{z}(\mathfrak{g})} \in \mathfrak{z}(\mathfrak{g})^*;$$

this provides a canonical group morphism

$$\text{CAN} = \text{CAN}_{\mathfrak{g}} : \mathcal{C}(\mathfrak{g}) \rightarrow \mathfrak{z}(\mathfrak{g})^*, \quad (2)$$

and it will be of interest to determine when/whether this map is a group isomorphism. Its additivity is clear, and moreover surjectivity is unproblematic:

Lemma 2.18. *For any finite-dimensional Lie algebra $\mathfrak{g}$ over the algebraically closed field $\mathbb{k}$ the morphism (2) is onto.*

Proof. Consider an arbitrary functional f on the center $\mathfrak{z} := \mathfrak{z}(\mathfrak{g})$, regarded as a 1-dimensional representation of $\mathfrak{z}$. Any irreducible $\mathfrak{g}$ -representation

$$\rho \preceq \text{Ind}_{\mathfrak{z}}^{\mathfrak{g}}(f)$$

(such representations do exist by Remark 2.12 (1)) will then be acted upon by $\mathfrak{z}$ via f , so f is the image of g_{ρ} through (2). ■

The main result to be discussed here is

Theorem 2.19. *Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra over the algebraically closed field $\mathbb{k}$ of characteristic zero.*

The canonical map (2) is an isomorphism.

We need some preparation. First, the following simple observation is an analogue of sorts (albeit a much less precise and powerful one) for the usual induction-restriction result on unitary representations of locally compact groups [12, Theorem 12.1].

Lemma 2.20. *Let $\mathfrak{h}, \mathfrak{g}' \leq \mathfrak{g}$ be inclusions of Lie algebras and $\rho : \mathfrak{h} \rightarrow \text{End}(W)$ an $\mathfrak{h}$ -representation. We then have the inclusion*

$$\ker \text{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\rho)|_{\mathfrak{g}'} \leq \ker \text{Ind}_{\mathfrak{h} \cap \mathfrak{g}'}^{\mathfrak{g}'}(\rho|_{\mathfrak{h} \cap \mathfrak{g}'}) \quad (3)$$

Proof. We write $W_{\mathfrak{g}}$ and $W_{\mathfrak{g}'}$ for the carrier spaces of the two representations

$$\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\rho)|_{\mathfrak{g}'} \quad \text{and} \quad \mathrm{Ind}_{\mathfrak{h} \cap \mathfrak{g}'}^{\mathfrak{g}'}(\rho|_{\mathfrak{h} \cap \mathfrak{g}'})$$

respectively, and denote by $J \leq U(\mathfrak{h})$ the kernel of ρ .

According to [5, Proposition 5.1.7 (i)], the annihilator of $W \leq W_{\mathfrak{g}}$ in $U(\mathfrak{g})$ is the left ideal $U(\mathfrak{g})J$. Choose an ordered basis for $\mathfrak{g}$ consisting, in this order, of

- a basis for a subspace of $\mathfrak{g}$ supplementing $\mathfrak{h} + \mathfrak{g}'$;
- a basis for a subspace of $\mathfrak{g}'$ supplementing $\mathfrak{h} \cap \mathfrak{g}'$;
- one for $\mathfrak{h} \cap \mathfrak{g}'$;
- and finally, one for a subspace of $\mathfrak{h}$ supplementing $\mathfrak{h} \cap \mathfrak{g}'$.

Plugging that basis into the PBW theorem ([5, Theorem 2.1.11] or [15, Theorem 6.1.1]), it will follow easily that an element of $U(\mathfrak{g}')$ annihilates $W \leq W_{\mathfrak{g}}$ if and only if it belongs to the left ideal

$$U(\mathfrak{g}') \cdot (\text{kernel of } \rho|_{\mathfrak{h} \cap \mathfrak{g}'}). \quad (4)$$

But that means that the kernel of the left-hand side of (3) is a bilateral ideal of $U(\mathfrak{g}')$ contained in the right-hand side of (4), whereas by [5, Proposition 5.1.7 (ii)] the right-hand side of (3) is the *largest* such ideal. The inclusion (3) follows. ■

Recall now, briefly, how the classification of primitive ideals for solvable Lie algebras proceeds (the process is summarized in [5, §6.1.5]); this will also serve to fix some notation.

Construction 2.21. Throughout, $\mathfrak{g}$ is assumed solvable (as always, over an algebraically-closed characteristic-0 field).

- Consider an element $f \in \mathfrak{g}^*$, i.e. a linear functional on $\mathfrak{g}$.
- To it, associate any *polarization* $\mathfrak{h} \leq \mathfrak{g}$; recall [5, 1.12.8] that this means
 - $\mathfrak{h}$ is *subordinate* to f in the sense that $f|_{[\mathfrak{h}, \mathfrak{h}]} \equiv 0$;
 - and its dimension achieves the theoretical maximum:

$$\dim \mathfrak{h} = \frac{1}{2} (\dim \mathfrak{g} + \dim \mathfrak{g}^f),$$

where

$$\mathfrak{g}^f := \{x \in \mathfrak{g} \mid f([x, y]) = 0, \forall y \in \mathfrak{g}\}$$

is the Lie subalgebra of $\mathfrak{g}$ leaving f invariant under the coadjoint action.

Polarizations always exist for solvable Lie algebras over algebraically closed fields [5, Proposition 1.12.10].

- Being f -subordinate, $\mathfrak{h}$ carries a one-dimensional representation induced by f ; we denote it by the same symbol (i.e. ' f ').

- Now form the twisted induced representation $\widetilde{\text{Ind}}_{\mathfrak{h}}^{\mathfrak{g}}(f)$ as in Notation 2.6.
- Set

$$I(f) := \ker \widetilde{\text{Ind}}_{\mathfrak{h}}^{\mathfrak{g}}(f).$$

This turns out to be a primitive ideal of $U(\mathfrak{g})$.

- That primitive ideal does not actually depend on the polarization $\mathfrak{h}$ or indeed even on f itself, but only on the orbit of f under the action of the algebraic adjoint group $\mathcal{A}$ attached to $\mathfrak{g}$.
- And the resulting map

$$\bar{I} : \mathfrak{g}^*/\mathcal{A} \rightarrow \text{Prim}(U)$$

is bijective.

Notation 2.22. As a matter of convenience, we occasionally write ρ_f for an irreducible representation with kernel $I(f)$ as in Construction 2.21. Such a representation is in general not unique subject to this condition, but this will not matter whenever the notation is in use.

In reference to all of this, we now have

Lemma 2.23. *Let $\mathfrak{g}' \leq \mathfrak{g}$ be an inclusion of finite-dimensional nilpotent Lie algebras and $f \in \mathfrak{g}^*$. In the notation of Construction 2.21, we have the inclusion*

$$I(f) \cap U(\mathfrak{g}') \leq I(f|_{\mathfrak{g}'}). \quad (5)$$

Proof. Because we are working with *nilpotent* Lie algebras, twisted induction is just plain induction (Remark 2.7). Choose a polarization $\mathfrak{h}$ for f , so that

$$I(f) = \ker \text{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(f).$$

By Lemma 2.20, its intersection with $U(\mathfrak{g}')$ (i.e. the left-hand side of (5)) is contained in the kernel of the induced representation $\text{Ind}_{\mathfrak{h} \cap \mathfrak{g}'}^{\mathfrak{g}'}(f|_{\mathfrak{h} \cap \mathfrak{g}'}).$ But because

$$\mathfrak{h} \cap \mathfrak{g}' \leq \mathfrak{g}'$$

is subordinate to $f|_{\mathfrak{g}'}$, that kernel is in turn contained in the right-hand side of (5) by [5, Lemma 6.4.3]. ■

Lemma 2.24. *Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra and $f, f' \in \mathfrak{g}^*$ two functionals with respective polarizations $\mathfrak{h}, \mathfrak{h}' \leq \mathfrak{g}$.*

We then have the inclusion

$$\ker (\text{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(f) \otimes \text{Ind}_{\mathfrak{h}'}^{\mathfrak{g}}(f')) \leq I(f + f'). \quad (6)$$

Proof. This is a fairly straightforward application of Lemma 2.23 to the diagonal inclusion $\mathfrak{g} \leq \mathfrak{g} \oplus \mathfrak{g}$:

- The enveloping algebra $U(\mathfrak{g} \oplus \mathfrak{g})$ is the tensor square $U(\mathfrak{g})^{\otimes 2}$ [5, Proposition 2.2.10].
- The subalgebra

$$\mathfrak{h} \oplus \mathfrak{h}' \leq \mathfrak{g} \oplus \mathfrak{g}$$

is a polarization for $f + f' \in \mathfrak{g}^* \oplus \mathfrak{g}^*$.

- Induction plays well with external tensor products:

$$\mathrm{Ind}_{\mathfrak{h} \oplus \mathfrak{h}'}^{\mathfrak{g} \oplus \mathfrak{g}}(f + f') \cong \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(f) \otimes \mathrm{Ind}_{\mathfrak{h}'}^{\mathfrak{g}}(f')$$

as modules over

$$U(\mathfrak{g} \oplus \mathfrak{g}) \cong U(\mathfrak{g}) \otimes U(\mathfrak{g}).$$

- And finally, the *internal* tensor product appearing on the left-hand side of (6) is the restriction of that same tensor product regarded as a $(U(\mathfrak{g}) \otimes U(\mathfrak{g}))$ -module along the comultiplication [5, §2.7.1]

$$U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$$

that lifts the diagonal embedding $\mathfrak{g} \rightarrow \mathfrak{g} \oplus \mathfrak{g}$.

This proves the claim. ■

Remark 2.25. Lemma 2.24 is an additivity result of sorts for the map

$$\mathfrak{g}^* \ni f \mapsto I(f) \in \mathrm{Prim}(U(\mathfrak{g}))$$

for nilpotent $\mathfrak{g}$ (where “addition” on the right-hand side corresponds to tensoring representations). The same map (under the same hypotheses) is also compatible with respect to “taking inverses”: according to [7, Lemme 8.1], for nilpotent $\mathfrak{g}$ we have

$$I(-f) = S(I(f)),$$

where S is the antipode of the enveloping algebra $U := U(\mathfrak{g})$. If I is the kernel of a U -representation ρ then $S(I)$ is that of the dual ρ^* : the representation-theoretic analogue of an “inverse”.

Proof of Theorem 2.19. The classification of the primitive ideals of $U := U(\mathfrak{g})$ via coadjoint orbits outlined in [5, §6.1.5] and recalled in Construction 2.21 goes through. By Lemma 2.24 the composite map

$$\mathfrak{g}^* \ni f \mapsto I(f) \mapsto g_{I(f)} \in \mathcal{C}(\mathfrak{g})$$

is additive, and since it sends $0 \in \mathfrak{g}^*$ to the trivial element it must in fact be a group morphism. Since on the other hand it also factors through the orbit space $\mathfrak{g}^*/\mathcal{A}$ for the action of the algebraic adjoint group $\mathcal{A}$ of $\mathfrak{g}$ (Construction 2.21), it must descend to a group morphism

$$\mathfrak{g}_{\mathcal{A}}^* \rightarrow \mathcal{C}(\mathfrak{g}) \tag{7}$$

from the group (also vector space) of coinvariants in $\mathfrak{g}^*$ under the coadjoint action. That morphism, composed with (2), is nothing but the usual identification

$$\mathfrak{g}_{\mathcal{A}}^* \cong (\mathfrak{g}^{\mathcal{A}})^* \cong \mathfrak{z}^*, \quad \mathfrak{z} := \text{the center of } \mathfrak{g}, \quad (8)$$

with the dual to the vector space $\mathfrak{g}^{\mathcal{A}} \cong \mathfrak{z}$ of $\mathcal{A}$ -invariants in $\mathfrak{g}$ (i.e. the latter's center). Since (7) is surjective and its composition with (2) is the isomorphism (8), (2) itself must be an isomorphism. ■

2.3. Solvable

The following result supersedes Theorem 2.19, but that earlier argument is useful to have as a reference.

Theorem 2.26. *Let $\mathfrak{g}$ be a finite-dimensional solvable Lie algebra over the algebraically closed field $\mathbb{k}$ of characteristic zero.*

The canonical map (2) is an isomorphism.

Once more, some preparatory remarks are necessary.

Lemma 2.27. *Let*

- $\mathfrak{g}$ be a finite-dimensional solvable Lie algebra over the algebraically closed field $\mathbb{k}$;
- $f, \lambda \in \mathfrak{g}^*$ with λ annihilating $[\mathfrak{g}, \mathfrak{g}]$;
- and ρ_f and $\rho_{f+\lambda}$ irreducible representations as in Notation 2.22.

We then have

$$\rho_{f+\lambda} \preceq \rho_f \otimes \lambda, \quad (9)$$

where $\lambda : \mathfrak{g} \rightarrow \mathbb{k}$ is regarded as a 1-dimensional representation.

Proof. The claim is that the kernel $I(f + \lambda)$ of $\rho_{f+\lambda}$ contains that of the right-hand side of (9).

Let $\mathfrak{h} \leq \mathfrak{g}$ be a polarization for f , so that

$$I(f) = \ker \widetilde{\text{Ind}}_{\mathfrak{h}}^{\mathfrak{g}}(f) = \ker \text{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(f \otimes \theta_{\mathfrak{g}, \mathfrak{h}})$$

(see Construction 2.21). The same Lie algebra $\mathfrak{h}$ is then also subordinate to $f + \lambda$ (because λ by assumption vanishes on $[\mathfrak{g}, \mathfrak{g}] \supseteq [\mathfrak{h}, \mathfrak{h}]$), so by [5, Lemma 6.4.2] we have

$$\ker \text{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(f \otimes \theta_{\mathfrak{g}, \mathfrak{h}} \otimes \lambda) = \ker \widetilde{\text{Ind}}_{\mathfrak{h}}^{\mathfrak{g}}(f + \lambda) \leq I(f + \lambda) \quad (10)$$

Now, because the $\mathfrak{h}$ -representation λ is restricted from $\mathfrak{g}$, the push-pull formula for induction/restriction of Hopf-algebra modules shows that the leftmost induced representation in (10) is

$$\text{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(f \otimes \theta_{\mathfrak{g}, \mathfrak{h}}) \otimes \lambda :$$

Apply Lemma 2.28 with

- $H = U(\mathfrak{g})$ and $H' = U(\mathfrak{h})$;
- with modules/representations $W = f \otimes \theta_{\mathfrak{g}, \mathfrak{h}}$ and $V = \lambda$.

We thus have

$$\rho_{f+\lambda} \preceq \text{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(f \otimes \theta_{\mathfrak{g}, \mathfrak{h}}) \otimes \lambda,$$

hence (9), via Proposition 2.8 (b), since the left-hand tensorands have the same kernel. $\blacksquare$

Lemma 2.28. *Let $H' \leq H$ be an inclusion of Hopf algebras over an arbitrary field, W a left H' -module and V a left H -module. We then have an isomorphism*

$$H \otimes_{H'} (W \otimes V) \cong (H \otimes_{H'} W) \otimes V$$

of left H -modules.

Proof. Using Sweedler notation $x \mapsto x_1 \otimes x_2$ for Hopf algebra comultiplications ([13, Notation 1.4.2]) and ‘ S ’ for antipodes, we leave it to the reader to check that

$$\begin{array}{ccc} H \otimes_{H'} (W \otimes V) & \begin{array}{c} \xrightarrow{h \otimes w \otimes v \mapsto h_1 \otimes w \otimes h_2 v} \\ \xleftarrow{h_1 \otimes w \otimes S(h_2) v \longleftarrow h \otimes w \otimes v} \end{array} & (H \otimes_{H'} W) \otimes V \end{array}$$

are mutually inverse module morphisms. $\blacksquare$

Note, incidentally, the following consequence of Lemma 2.28 on the relation between Ind and $\widetilde{\text{Ind}}$. The statement refers to the *nilradical* $\mathfrak{n} \leq \mathfrak{g}$, i.e. the largest nilpotent ideal of $\mathfrak{g}$ (discussed e.g. in [5, Proposition 1.4.9]). When $\mathfrak{g}$ is solvable the nilradical contains the derived ideal $[\mathfrak{g}, \mathfrak{g}]$ [21, Corollary V.5.3].

Lemma 2.29. *Let $\mathfrak{h} \leq \mathfrak{g}$ be an inclusion of finite-dimensional solvable Lie algebras over a field $\mathbb{k}$ of characteristic zero and $\rho : \mathfrak{h} \rightarrow \text{End}(W)$ an $\mathfrak{h}$ -representation.*

(1) *If $\sigma \in \mathfrak{h}^*$ is a functional vanishing on $\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$ then*

$$\widetilde{\text{Ind}}_{\mathfrak{h}}^{\mathfrak{g}}(\rho \otimes \sigma) \cong \text{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\rho) \otimes \lambda$$

for some functional $\lambda \in \mathfrak{g}^$ annihilating $[\mathfrak{g}, \mathfrak{g}] + (\mathfrak{n} \cap \ker \sigma)$, where $\mathfrak{n} := \mathfrak{n}(\mathfrak{g})$ is the nilradical.*

(2) *In particular,*

$$\widetilde{\text{Ind}}_{\mathfrak{h}}^{\mathfrak{g}}(\rho) \cong \text{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\rho) \otimes \lambda$$

for some $\lambda \in \mathfrak{g}^$ annihilating the nilradical $\mathfrak{n} \trianglelefteq \mathfrak{g}$.*

Proof. Part (2) is indeed an instance of (1): simply take $\sigma = 0$. We thus focus on (1).

By definition (Notation 2.6), we have

$$\widetilde{\text{Ind}}_{\mathfrak{h}}^{\mathfrak{g}}(\rho \otimes \sigma) \cong \text{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\rho \otimes \theta_{\mathfrak{g}, \mathfrak{h}} \otimes \sigma).$$

All eigenvalues of $ad(x)$ for $x \in \mathfrak{n}$ vanish, so the trace defining $\theta_{\mathfrak{g},\mathfrak{h}} \in \mathfrak{h}^*$ vanishes on $\mathfrak{h} \cap \mathfrak{n}$. It follows that $\theta_{\mathfrak{g},\mathfrak{h}}$ and σ both vanish on

$$\mathfrak{h} \cap ([\mathfrak{g}, \mathfrak{g}] + (\mathfrak{n} \cap \ker \sigma)) = (\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]) + (\mathfrak{n} \cap \ker \sigma),$$

so $\theta_{\mathfrak{g},\mathfrak{h}} + \lambda$ can be extended to a functional $\lambda \in \mathfrak{g}^*$ vanishing on the target space $[\mathfrak{g}, \mathfrak{g}] + (\mathfrak{n} \cap \ker \sigma)$. In particular λ can be regarded as a 1-dimensional $\mathfrak{g}$ -representation, and the conclusion then follows from the push-pull formula again: Lemma 2.28 with

$$H = U(\mathfrak{g}), \quad H' = U(\mathfrak{h}), \quad V = \text{the 1-dimensional } \mathfrak{g}\text{-module attached to } \lambda.$$

■

Solvable analogues of Lemmas 2.23 and 2.24, needed below:

Lemma 2.30. *Let $\mathfrak{g}$ be a finite-dimensional solvable Lie algebra over an algebraically closed field of characteristic zero.*

- (1) *Let $\mathfrak{g}' \leq \mathfrak{g}$ be a Lie subalgebra, $f \in \mathfrak{g}^*$ and $f' := f|_{\mathfrak{g}'}$. Recalling Notation 2.22, we have*

$$\ker \rho_f \cap U(\mathfrak{g}') \leq \ker (\rho_{f'} \otimes \lambda)$$

for some 1-dimensional representation $\lambda \in (\mathfrak{g}')^$ annihilating the intersection $\mathfrak{g}' \cap \mathfrak{n}$ with the nilradical $\mathfrak{n} := \mathfrak{n}(\mathfrak{g}) \trianglelefteq \mathfrak{g}$.*

- (2) *For two functionals $f, f' \in \mathfrak{g}^*$ we have*

$$\ker (\rho_f \otimes \rho_{f'}) \leq \ker (\rho_{f+f'} \otimes \lambda)$$

for some 1-dimensional representation $\lambda \in (\mathfrak{g})^$ annihilating the nilradical of $\mathfrak{g}$.*

Proof. The arguments are minor adaptations of those respectively employed in the proofs of Lemmas 2.23 and 2.24.

- (1) Following the same line of reasoning as in Lemma 2.23: choose a polarization $\mathfrak{h} \leq \mathfrak{g}$ for f , giving

$$I(f) = \ker \text{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(f \otimes \theta_{\mathfrak{g},\mathfrak{h}}).$$

Then, by Lemma 2.20 again,

$$I(f) \cap U(\mathfrak{g}') \leq \ker \text{Ind}_{\mathfrak{h} \cap \mathfrak{g}'}^{\mathfrak{g}'}(f \otimes \theta_{\mathfrak{g},\mathfrak{h}}) = \ker \widetilde{\text{Ind}}_{\mathfrak{h} \cap \mathfrak{g}'}^{\mathfrak{g}'}(f \otimes \theta_{\mathfrak{g},\mathfrak{h}} \otimes \theta_{\mathfrak{g}',\mathfrak{h} \cap \mathfrak{g}'}^*), \quad (11)$$

where as usual, the ‘ $*$ ’ superscript denotes the dual representation.

Now note that both $\theta_{\mathfrak{g},\mathfrak{h}}|_{\mathfrak{h} \cap \mathfrak{g}'}$ and $\theta_{\mathfrak{g}',\mathfrak{h} \cap \mathfrak{g}'}$ vanish on $\mathfrak{h} \cap \mathfrak{g}' \cap \mathfrak{n}$, so they extend to functionals on $\mathfrak{g}'$ vanishing on $[\mathfrak{g}', \mathfrak{g}'] + (\mathfrak{g}' \cap \mathfrak{n})$. But now, by Lemma 2.29 (1), the rightmost representation in (11) is

$$\widetilde{\text{Ind}}_{\mathfrak{h} \cap \mathfrak{g}'}^{\mathfrak{g}'}(f \otimes \theta_{\mathfrak{g},\mathfrak{h}} \otimes \theta_{\mathfrak{g}',\mathfrak{h} \cap \mathfrak{g}'}^*) \cong \widetilde{\text{Ind}}_{\mathfrak{h} \cap \mathfrak{g}'}^{\mathfrak{g}'}(f) \otimes \lambda$$

for some $\lambda \in (\mathfrak{g}')^*$ as in the statement.

- (2) This follows from part (1) the same way Lemma 2.24 follows from Lemma 2.23, taking into account the fact that for the diagonal embedding $\mathfrak{g} \leq \mathfrak{g} \oplus \mathfrak{g}$ the intersection

$$\mathfrak{g} \cap \mathfrak{n}(\mathfrak{g} \oplus \mathfrak{g})$$

is nothing but the nilradical $\mathfrak{n}(\mathfrak{g})$.

This concludes the proof of the two claims. ■

We will prove Theorem 2.26 by induction on the dimension of $\mathfrak{g}$, with the following result carrying the brunt of the iterative load.

Proposition 2.31. *Let $\mathfrak{g}$ be a finite-dimensional solvable Lie algebra over the algebraically closed field $\mathbb{k}$, and assume the conclusion of Theorem 2.26 holds for all solvable Lie algebras of smaller dimension.*

The following conditions are equivalent.

- (a) *Theorem 2.26 holds for $\mathfrak{g}$.*
- (b) *The canonical map (2) is injective.*
- (c) *For every 1-dimensional $\mathfrak{g}$ -representation $\lambda \in \mathfrak{g}^*$ annihilating the center $\mathfrak{z} := \mathfrak{z}(\mathfrak{g})$ the generator $g_\lambda \in \mathcal{C}(\mathfrak{g})$ is trivial.*
- (d) *Same as (c), but only for those 1-dimensional representations $\lambda \in \mathfrak{g}^*$ annihilating the nilradical $\mathfrak{n} \trianglelefteq \mathfrak{g}$.*

Proof. That (a) and (b) are equivalent is a general remark (Lemma 2.18). Naturally, (b) implies (c), and the latter is formally stronger than (d) (since the center is contained in the nilradical, so *fewer* functionals will annihilate it). It thus remains to prove (d) $\implies$ (a), which implication we henceforth focus on.

By Lemma 2.30 (2), Lemma 2.24 holds with the caveat that one might have to tensor by 1-dimensional representations $\lambda \in \mathfrak{g}^*$ annihilating the nilradical $\mathfrak{n} \trianglelefteq \mathfrak{g}$. Since we are assuming

$$1 = g_\lambda \in \mathcal{C}(\mathfrak{g}) \text{ for all such } \lambda,$$

we can conclude as in the proof of Theorem 2.19: the map

$$\mathfrak{g}^* \ni f \mapsto I(f) \mapsto g_{I(f)} \in \mathcal{C}(\mathfrak{g})$$

is additive, gives an isomorphism (8) upon further composition with (2), and we are done. ■

Proof of Theorem 2.26. As announced, we proceed by induction on $\dim \mathfrak{g}$, with the base case(s) being simple exercises. This leaves the induction step which, per Proposition 2.31, amounts to showing that

$$1 = g_\lambda \in \mathcal{C}(\mathfrak{g}), \quad \forall \lambda \in \mathfrak{n}^\perp \leq \mathfrak{g}^*$$

(where as before, $\mathfrak{n} \trianglelefteq \mathfrak{g}$ is the nilradical).

Fix such a functional $\lambda \in \mathfrak{n}^\perp$ (non-zero, or there is nothing to prove). Because the chain group is functorial for surjections, the induction hypothesis (together with Proposition 2.31) tells us that λ cannot annihilate the center of any proper quotient of $\mathfrak{g}$. Since $\ker \lambda$ contains some 1-dimensional ideal

$$\mathfrak{k} := \mathbb{k}z \trianglelefteq \mathfrak{g},$$

because we are working over an algebraically closed field (e.g. [5, 1.3.12]). It follows that the quotient $\mathfrak{g}/\mathfrak{k}$ splits as

$$\mathfrak{g}/\mathfrak{k} = (\ker \lambda|_{\mathfrak{g}/\mathfrak{k}}) \oplus (\text{image of } \mathbb{k}x)$$

for some $x \in \mathfrak{g}$ not annihilated by λ .

If x were to commute with z then λ would fail to annihilate the nilpotent ideal $\text{span}\{x, z\}$, contradicting the choice $\lambda \in \mathfrak{n}^\perp$. We can thus assume that $[x, z] = z$; because z commutes with $\ker \lambda$ modulo z , we have a decomposition

$$\ker \lambda = \mathbb{k}z \oplus \mathfrak{h}, \quad \mathfrak{h} := C_{\ker \lambda}(x) = \{x' \in \ker \lambda \mid [x, x'] = 0\}.$$

There are now some possibilities to consider:

- (a) **The Lie algebra $\mathfrak{h}$ centralizes z .** In this case $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, and the quotient $\mathfrak{g}/\mathfrak{h}$ is (isomorphic to) the non-nilpotent 2-dimensional Lie algebra generated by x and z . This is the $\mathfrak{g}_2$ of [5, Lemma 6.1.2 (i)], and has faithful irreducible representations.

It follows from Proposition 2.14 that *all* irreducible $(\mathfrak{g}/\mathfrak{h})$ -representations (in particular λ) are trivial in the chain group, hence also in $\mathcal{C}(\mathfrak{g})$ by the above-mentioned chain-group functoriality under quotients.

- (b) $[\mathfrak{h}, z] \neq \{0\}$. In this case there is some $y \in \mathfrak{h} \leq \ker \lambda$ with $[y, z] = z$, and λ fails to annihilate the $\mathfrak{g}$ -central element $x - y$. This contradicts our assumption that $\lambda \in \mathfrak{n}^\perp$, and finishes the proof. $\blacksquare$

2.4. Semisimple

The main result we address here is

Theorem 2.32. *Let $\mathfrak{g}$ be a finite-dimensional, semisimple Lie algebra over the algebraically closed field $\mathbb{k}$ of characteristic zero.*

The chain group $\mathcal{C}(\mathfrak{g})$ is trivial, and hence (2) is an isomorphism.

We use some of the language familiar in the theory of semisimple Lie algebras as covered, say, in [9], [5, Chapter 1], etc. In particular:

- $\mathfrak{h} \leq \mathfrak{g}$ will be a *Cartan subalgebra* of $\mathfrak{g}$ ([9, §15], [5, §1.9]).
- This then induces a *root-space decomposition* for $\mathfrak{g}$ [9, §8], for which we assume we have chosen a *base* $\Delta \subset \mathfrak{h}^*$ [9, §10.1].

- We denote by $\delta \in \mathfrak{h}^*$ the half-sum of the *positive roots* [9, §10.1] attached to the choice of Δ . This element is discussed in [9, §13.3] as well as [5, §11.1.13], where it is denoted by the same symbol.
- Let $\lambda \in \mathfrak{h}^*$. Following [5, §7.1.4] or [8, §I.1], we denote by $M(\lambda)$ the *Verma module* of highest weight $f - \delta$. For comparison: in [9, §20.3] $M(\lambda)$ would rather be denoted by $Z(\lambda - \delta)$.
- The simple quotient of $M(\lambda)$ is $L(\lambda)$, again following either [5, §7.1.12] or [8, §I.1]. [9, §20.3] would set $L(\lambda) = V(\lambda - \delta)$.
- We write W for the *Weyl group* of $\mathfrak{g}$ ([9, §10.3], [5, §1.10.10]); it is a finite group of linear automorphisms of $\mathfrak{h}^*$, generated by reflections.

Proof of Theorem 2.32. The notation outlined above is in force throughout. We will also (somewhat abusively) identify representations with their carrier spaces (e.g. by referring to a representation of $U := U(\mathfrak{g})$ on V as just plain V , regarded as a U -module). Finally, recall from Remark 2.12 (4) that we may as well attach generators g_J of the chain group to primitive ideals $J \leq U$ (rather than actual representations); we do this below.

According to [5, Theorem 8.4.4 (iv)], the respective kernels J_λ of $M(\lambda)$, $\lambda \in \mathfrak{h}^*$ are precisely the minimal primitive ideals of U . An arbitrary primitive ideal J will thus contain some J_λ , and by Lemma 2.13 we have

$$J \geq J_\lambda \Rightarrow g_J = g_{J_\lambda} \in \mathcal{C}(\mathfrak{g}). \quad (12)$$

[5, Theorem 8.4.4 (ii)] moreover shows that $J_\lambda = J_{\lambda'}$ whenever λ' is in the Weyl-group orbit $W\lambda$; the same must be true of chain-group generators then:

$$g_{J_\lambda} = g_{J_{\lambda'}}, \quad \forall \lambda' \in W\lambda.$$

One last observation: the tensor product $M(\lambda) \otimes M(\lambda')$ (for arbitrary $\lambda, \lambda' \in \mathfrak{h}^*$) contains a highest-weight vector of weight $\lambda + \lambda' - 2\delta$, so it has a submodule surjecting onto the simple module $L(\lambda + \lambda' - \delta)$. Because the latter's kernel contains $J_{\lambda + \lambda' - \delta}$ (and hence is identified to it in $\mathcal{C}(\mathfrak{g})$ by (12)), and writing $g_\lambda := g_{J_\lambda}$ for better readability, we have

$$g_\lambda g_{\lambda'} = g_{\lambda + \lambda' - \delta}, \quad \forall \lambda, \lambda' \in \mathfrak{h}^*. \quad (13)$$

In summary:

- The map

$$\mathfrak{h}^* \ni \lambda \mapsto g_\lambda := g_{J_\lambda} \in \mathcal{C}(\mathfrak{g})$$

is onto;

- and invariant under the Weyl-group action:

$$g_{w\lambda} = g_\lambda, \quad \forall w \in W, \quad \forall \lambda \in \mathfrak{h}^*; \quad (14)$$

- and “ δ -shifted-additive” in the sense of (13).

Applying (13) to $w\lambda$ and $w\lambda'$ instead, using (14) once and absorbing $\lambda + \lambda'$ into a single λ , we obtain

$$g_{\lambda-w^{-1}\delta} = g_{w\lambda-\delta} = g_{\lambda-\delta}, \quad \forall w \in W, \quad \forall \lambda \in \mathfrak{h}^*. \quad (15)$$

Because the longest element $w_0 \in W$ ([5, §7.2.3]; this is the σ of [9, Exercise 10.9]) sends δ to $-\delta$, the leftmost term of (15) can be set to $g_{\lambda+\delta}$. But then the right-hand side of (13) also holds with ‘ $+\delta$ ’ in place of ‘ $-\delta$ ’, whereupon the substitutions $\lambda \mapsto \lambda - \delta$ and $\lambda' \mapsto \lambda' - \delta$ yield

$$g_{\lambda-\delta}g_{\lambda'-\delta} = g_{\lambda+\lambda'-\delta}, \quad \forall \lambda, \lambda' \in \mathfrak{h}^*.$$

Together with (15), this finally tells us that

$$\mathfrak{h}^* \ni \lambda \mapsto g_{\lambda-\delta} \in \mathcal{C}(\mathfrak{g})$$

descends to a surjection from the group

$$\mathfrak{h}^* / \langle w\lambda - \lambda, w \in W, \lambda \in \mathfrak{h}^* \rangle.$$

of coinvariants under the action of the Weyl group. That group of coinvariants is of course trivial, because the representation of W on $\mathfrak{h}^*$ contains no copies of the trivial representation (obviously: by definition, W is generated by non-trivial reflections). ■

Remark 2.33. Note, incidentally, that in none of the proofs thus far have we had to make direct use of dual representations (via the binary relation ‘ $\sim$ ’ of Definition 2.11): additivity in the form of (1) was enough, essentially for the familiar reason that if a semigroup happens to be a group, that group structure is unique.

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Alexandru Chirvasitu
Department of Mathematics
University at Buffalo
Buffalo, NY 14260-2900, USA
achirvas@buffalo.edu