REAL AND SYMMETRIC MATRICES

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Abstract

We construct a family of involutions on the space $\mathfrak{gl}'_n(\mathbb{C})$ of $n \times n$ matrices with real eigenvalues interpolating the complex conjugation and the transpose. We deduce from it a stratified homeomorphism between the space $\mathfrak{gl}'_n(\mathbb{R})$ of $n \times n$ real matrices with real eigenvalues and the space $\mathfrak{p}'_n(\mathbb{C})$ of $n \times n$ symmetric matrices with real eigenvalues, which restricts to a real analytic isomorphism between individual $\mathrm{GL}_n(\mathbb{R})$-adjoint orbits and $\mathrm{O}_n(\mathbb{C})$-adjoint orbits. We also establish similar results in more general settings of Lie algebras of classical types and quiver varieties. To this end, we prove a general result about involutions on hyper-Kähler quotients of linear spaces. We provide applications to the (generalized) Kostant–Sekiguchi correspondence, singularities of real and symmetric adjoint orbit closures, and Springer theory for real groups and symmetric spaces.

Contents

1. Introduction

1.1. Main results

A key structural result in Lie theory is Cartan's classification of real forms of a complex reductive Lie algebra $\mathfrak{g}$ in terms of holomorphic involutions. It amounts to a bijection

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$$\{\text{complex conjugations } \eta \text{ of } \mathfrak{g}\}/\text{isom} \longleftrightarrow \{\text{holomorphic involutions } \theta \text{ of } \mathfrak{g}\}/\text{isom} \tag{1.1}$$

between isomorphism classes of complex conjugations and holomorphic involutions of $\mathfrak{g}$. For example, in the case $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$, the complex conjugation $\eta(M) = \overline{M}$ with real form consisting of real matrices $\mathfrak{g}_{\mathbb{R}} = \mathfrak{gl}_n(\mathbb{R})$ corresponds to the involution $\theta(M) = -M^t$ with $(-\theta)$-fixed points consisting of symmetric matrices $\mathfrak{p} = \mathfrak{p}_n(\mathbb{C})$. The interplay between the real $\mathfrak{g}_{\mathbb{R}}$ and symmetric $\mathfrak{p}$ pictures plays a fundamental role in the structure and representation theory of real groups, going back at least to Harish-Chandra's formulation of the representation theory of real groups in terms of $(\mathfrak{g}, K)$-modules.

One of the goals of this article is to get a better understanding of Cartan's bijection and also the real and symmetric pictures for real groups from the geometric point of view. To this end, let η be a conjugation on $\mathfrak{g}$, and let θ be the corresponding involution under (1.1). For simplicity, we assume that η is the split conjugation. Then the subspace $\mathfrak{g}'$ of $\mathfrak{g}$ consisting of elements with real eigenvalues[1] is preserved by both η and $-\theta$ and our first main result here, Theorem 1.4, is a construction of a real analytic family of involutions on $\mathfrak{g}'$

$$\alpha_s : \mathfrak{g}' \longrightarrow \mathfrak{g}', \quad s \in [0, 1], \tag{1.2}$$

interpolating the conjugation η and the holomorphic involution $-\theta$, that is, we have $\alpha_0 = \eta$ and $\alpha_1 = -\theta$, in the case when $\mathfrak{g}$ is of classical type. Using the family of involutions above, we prove the second main result of the article, Theorem 1.3, which says that there exists a stratified homeomorphism

$$\mathfrak{g}'_{\mathbb{R}} \xrightarrow{\sim} \mathfrak{p}' \tag{1.3}$$

between the η and $(-\theta)$-fixed points on $\mathfrak{g}'$ compatible with various structures.[2] The family of involutions in (1.2) and the homeomorphism (1.3) can be thought of as geometric refinements of Cartan's bijection (1.1).

We deduce several applications from the main results. Assume that $\mathfrak{g}$ is of classical type. In Corollary 1.9, we show that there exists a stratified homeomorphism

$$\mathcal{N}_{\mathbb{R}} \xrightarrow{\sim} \mathcal{N}_{\mathfrak{p}}$$

[1] Elements $x \in \mathfrak{g}$ such that the adjoint action $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ has only real eigenvalues.

[2] It is necessary to consider the subspace $\mathfrak{g}' \subset \mathfrak{g}$ but not the whole Lie algebra $\mathfrak{g}$ in the main results because, in general, the fixed points $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}^{\eta}$ and $\mathfrak{p} = \mathfrak{g}^{-\theta}$ have different dimensions and hence cannot be homeomorphic to each other.

between the real nilpotent cone $\mathcal{N}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ and the symmetric nilpotent cone $\mathcal{N}_{\mathfrak{p}} \subset \mathfrak{p}$ providing a lift of the celebrated Kostant–Sekiguchi correspondence between real and symmetric nilpotent orbits. In particular, it implies that $\mathcal{N}_{\mathbb{R}}$ and $\mathcal{N}_{\mathfrak{p}}$ have the same singularities, answering an open question (see, e.g., [11, p. 354]). In Corollary 1.16, we show that Grinberg's nearby cycles sheaf on $\mathcal{N}_{\mathfrak{p}}$ is isomorphic to the real Springer sheaf given by the pushforward of the constant sheaf along the real Springer map, establishing a conjecture of Vilonen, Xue, and the first author.

The key ingredients in the proof are the hyper-Kähler SU(2)-actions on the space of matrices arising from the quiver variety description in [19], [15], [22], [24], and [25], and a general result about involutions on hyper-Kähler quotients of linear spaces (see Theorem 1.6). The techniques used in the proof are not specific to matrices and are applicable to a more general setting. For example, we also establish a quiver variety version of the main results.

We now describe the paper in more detail.

1.1.1. Real-symmetric homeomorphisms for matrices
Let us first illustrate our main results with a notable case accessible to a general audience.

Let $\mathfrak{gl}_n(\mathbb{C}) \simeq \mathbb{C}^{n^2}$ denote the space of $n \times n$ complex matrices. Let $\mathfrak{gl}_n(\mathbb{R}) \subset \mathfrak{gl}_n(\mathbb{C})$ denote the real matrices, that is, those with real entries, and let $\mathfrak{p}_n(\mathbb{C}) \subset \mathfrak{gl}_n(\mathbb{C})$ denote the symmetric matrices, that is, those equal to their transpose. Introduce the following subspaces:

$$\mathfrak{gl}'_n(\mathbb{R}) = \{x \in \mathfrak{gl}_n(\mathbb{R}) \mid \text{eigenvalues of } x \text{ are real}\},$$
$$\mathfrak{p}'_n(\mathbb{C}) = \{x \in \mathfrak{p}_n(\mathbb{C}) \mid \text{eigenvalues of } x \text{ are real}\}.$$

The real general linear group $\mathrm{GL}_n(\mathbb{R})$ and complex orthogonal group $\mathrm{O}_n(\mathbb{C})$ naturally act by conjugation on $\mathfrak{gl}'_n(\mathbb{R})$ and $\mathfrak{p}'_n(\mathbb{C})$, respectively. The real orthogonal group $\mathrm{O}_n(\mathbb{R}) = \mathrm{GL}_n(\mathbb{R}) \cap \mathrm{O}_n(\mathbb{C})$ acts on both $\mathfrak{gl}'_n(\mathbb{R})$ and $\mathfrak{p}'_n(\mathbb{C})$. We also have the natural linear $\mathbb{R}^\times$-actions on both $\mathfrak{gl}'_n(\mathbb{R})$ and $\mathfrak{p}'_n(\mathbb{C})$. Consider the adjoint quotient map $\chi : \mathfrak{gl}_n(\mathbb{C}) \to \mathbb{C}^n$ which associates to each matrix $x \in \mathfrak{gl}_n(\mathbb{C})$ the coefficients of its characteristic polynomial. Equivalently, one can think of it as giving the eigenvalues of the matrix (with multiplicities).

Here is a notable case of our general results.

THEOREM 1.1
There is an $(\mathrm{O}_n(\mathbb{R}) \times \mathbb{R}^\times)$*-equivariant homeomorphism*

$$\mathfrak{gl}'_n(\mathbb{R}) \xrightarrow{\sim} \mathfrak{p}'_n(\mathbb{C}) \tag{1.4}$$

which is compatible with the adjoint quotient map. Furthermore, the homeomorphism restricts to a real analytic isomorphism between individual $\mathrm{GL}_n(\mathbb{R})$*-orbits and* $\mathrm{O}_n(\mathbb{C})$*-orbits.*

We deduce Theorem 1.1 from the following more fundamental structure of linear algebra. Consider the subspace

$$\mathfrak{gl}'_n(\mathbb{C}) = \{x \in \mathfrak{gl}_n(\mathbb{C}) \mid \text{eigenvalues of } x \text{ are real}\}.$$

Let $\chi' : \mathfrak{gl}'_n(\mathbb{C}) \to \mathbb{C}^n$ be the restriction of the adjoint quotient map to $\mathfrak{gl}'_n(\mathbb{C})$.

THEOREM 1.2
There is a continuous one-parameter family of $(\mathrm{O}_n(\mathbb{R}) \times \mathbb{R}^\times)$*-equivariant maps*

$$\alpha_s : \mathfrak{gl}'_n(\mathbb{C}) \longrightarrow \mathfrak{gl}'_n(\mathbb{C}), \qquad s \in [0,1], \tag{1.5}$$

satisfying the following properties.
(1) α_s^2 *is the identity, for all* $s \in [0,1]$.
(2) *We have* $\chi' \circ \alpha_s = \chi' : \mathfrak{gl}'_n(\mathbb{C}) \to \mathbb{C}^n$.
(3) α_s *takes each* $\mathrm{GL}_n(\mathbb{C})$*-orbit real analytically to a* $\mathrm{GL}_n(\mathbb{C})$*-orbit, for all* $s \in [0,1]$.
(4) *At* $s = 0$, *we recover conjugation:* $\alpha_0(A) = \bar{A}$.
(5) *At* $s = 1$, *we recover transpose:* $\alpha_1(A) = A^t$.

1.1.2. Real-symmetric homeomorphisms for Lie algebras
To state a general version of our main results, we next recall some standard constructions in Lie theory, in particular those related to the study of real reductive groups.

Let G be a complex reductive Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{c} = \mathfrak{g}//G$ be the categorical quotient with respect to the adjoint action of G on $\mathfrak{g}$. The adjoint quotient map $\chi : \mathfrak{g} \to \mathfrak{c}$ is the *Chevalley map*.

Let $G_\mathbb{R} \subset G$ be a real form, defined by a conjugation $\eta : G \to G$, with Lie algebra $\mathfrak{g}_\mathbb{R} \subset \mathfrak{g}$. Choose a Cartan conjugation $\delta : G \to G$ that commutes with η, and let $G_c \subset G$ be the corresponding maximal compact subgroup.

Introduce the Cartan involution $\theta = \delta \circ \eta : G \to G$, and let $K \subset G$ be the fixed subgroup of θ with Lie algebra $\mathfrak{k} \subset \mathfrak{g}$. The subgroup K is called the *symmetric subgroup*. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p} \subset \mathfrak{g}$ is the -1-eigenspace of θ. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace contained in $\mathfrak{p}$, and let $\mathfrak{t} \subset \mathfrak{g}$ be a θ-stable Cartan subalgebra containing $\mathfrak{a}$. Let $\mathrm{W}_G = N_G(\mathfrak{t})/Z_G(\mathfrak{t})$ be the Weyl group of G, and let $\mathrm{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ be the little Weyl group of the symmetric pair (G, K). We denote $\mathfrak{p}_\mathbb{R} = \mathfrak{p} \cap \mathfrak{g}_\mathbb{R}$, $\mathfrak{k}_\mathbb{R} = \mathfrak{k} \cap \mathfrak{g}_\mathbb{R}$, $\mathfrak{a}_\mathbb{R} = \mathfrak{a} \cap \mathfrak{g}_\mathbb{R}$, and so on.

One can organize the above groups into the diagram:

$$\begin{array}{ccccc} & & G & & \\ & \nearrow & \uparrow & \nwarrow & \\ K & & G_c & & G_{\mathbb{R}} \\ & \nwarrow & \uparrow & \nearrow & \\ & & K_{\mathbb{R}} & & \end{array} \tag{1.6}$$

Here $K_{\mathbb{R}}$ is the fixed subgroup of θ, δ, and η together (or any two of the three) and the maximal compact subgroup of $G_{\mathbb{R}}$ with complexification K.

Let $\mathfrak{g}'_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ (resp., $\mathfrak{p}' \subset \mathfrak{p}$) be the subspace consisting of elements $x \in \mathfrak{g}_{\mathbb{R}}$ (resp., $x \in \mathfrak{p}$) such that the eigenvalues of the adjoint map $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ are real. The real form $G_{\mathbb{R}}$ and the symmetric subgroup K act naturally on $\mathfrak{g}'_{\mathbb{R}}$ and $\mathfrak{p}'$ by the adjoint action. The compact subgroup $K_{\mathbb{R}} = G_{\mathbb{R}} \cap K$ and $\mathbb{R}^\times$ both act on $\mathfrak{g}'_{\mathbb{R}}$ and $\mathfrak{p}'$.

THEOREM 1.3 (Theorem 4.1)
Suppose that $\mathfrak{g}$ is of classical type. There is a $(K_{\mathbb{R}} \times \mathbb{R}^\times)$-equivariant homeomorphism

$$\mathfrak{g}'_{\mathbb{R}} \xrightarrow{\ \sim\ } \mathfrak{p}' \tag{1.7}$$

which is compatible with the adjoint quotient map. Furthermore, it restricts to a real analytic isomorphism between individual $G_{\mathbb{R}}$-orbits and K-orbits.

We deduce Theorem 1.3 from the following. Let $\mathfrak{c}_{\mathfrak{p},\mathbb{R}} \subset \mathfrak{c}$ be the image of the natural map $\mathfrak{a}_{\mathbb{R}} \to \mathfrak{c} = \mathfrak{t}//W_G$. Introduce $\mathfrak{g}' = \mathfrak{g} \times_{\mathfrak{c}} \mathfrak{c}_{\mathfrak{p},\mathbb{R}}$, and let $\chi' : \mathfrak{g}' \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}}$ be the projection map.

THEOREM 1.4 (Theorem 4.2)
Under the same assumption as Theorem 1.3, there is a continuous one-parameter family of $(K_{\mathbb{R}} \times \mathbb{R}^\times)$-equivariant maps

$$\alpha_s : \mathfrak{g}' \longrightarrow \mathfrak{g}', \qquad s \in [0,1], \tag{1.8}$$

satisfying the following properties.

(1) *α_s^2 is the identity, for all $s \in [0,1]$.*
(2) *We have $\chi' \circ \alpha_s = \chi' : \mathfrak{g}' \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}}$.*
(3) *α_s takes each G-orbit real analytically to a G-orbit, for all $s \in [0,1]$.*
(4) *At $s = 0$, we recover the conjugation: $\alpha_0 = \eta$.*
(5) *At $s = 1$, we recover the anti-symmetry: $\alpha_1 = -\theta$.*

Remark 1.5
The special case of Theorems 1.3 and 1.4 stated in Theorems 1.1 and 1.2 is when $G = \mathrm{GL}_n(\mathbb{C})$, $\mathfrak{g} \simeq \mathfrak{gl}_n(\mathbb{C})$, $G_{\mathbb{R}} = \mathrm{GL}_n(\mathbb{R})$, $K = \mathrm{O}_n(\mathbb{C})$, and $K_{\mathbb{R}} = \mathrm{O}_n(\mathbb{R})$.

1.1.3. Involutions on hyper-Kähler quotients
We deduce Theorems 1.3 and 1.4 from a general result about involutions on hyper-Kähler quotients of linear spaces.

Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the quaternions, and let $\mathrm{Sp}(1) \subset \mathbb{H}$ be the group consisting of elements of norm one. Let $\mathbf{M}$ be a finite-dimensional quaternionic representation of a compact Lie group H_u. We assume that the quaternionic representation is unitary; that is, there is a H_u-inner product $(\,,\,)$ on $\mathbf{M}$ which is Hermitian with respect to the complex structures I, J, K on $\mathbf{M}$ given by multiplication by i, j, k, respectively. We have the hyper-Kähler moment map

$$\mu : \mathbf{M} \to \mathrm{Im}\,\mathbb{H} \otimes \mathfrak{h}_u^*$$

vanishing at the origin. Using the isomorphism $\mathrm{Im}\,\mathbb{H} = \mathbb{R} \oplus \mathbb{C}$ sending $x_1 i + x_2 j + x_3 k$ to $(x_1, x_2 + x_3 i)$, we can identify $\mathrm{Im}\,\mathbb{H} \otimes \mathfrak{h}_u^* = \mathfrak{h}_u^* \oplus \mathfrak{h}^*$ and hence obtain a decomposition of the moment map

$$\mu = \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} : \mathbf{M} \to \mathfrak{h}_u^* \oplus \mathfrak{h}^*$$

of μ into real and complex components. We consider the hyper-Kähler quotient

$$\mathfrak{M}_0 = \mu^{-1}(0)/H_u \simeq \mu_{\mathbb{C}}^{-1}(0)//H,$$

where the right-hand side is the categorial quotient of $\mu_{\mathbb{C}}^{-1}(0)$ by the complexification H of H_u, and the second isomorphism follows from a result of Kempf and Ness [14].

The hyper-Kähler quotient $\mathfrak{M}_0$ has the following structures: (1) for a subgroup L of H_u denote by $\mathfrak{M}_{0,(L)}$ the set consisting of orbits through points x whose stabilizer in H_u is conjugate to L. We have an orbit-type stratification

$$\mathfrak{M}_0 = \bigsqcup_{(L)} \mathfrak{M}_{0,(L)}$$

where the summation runs over the set of all conjugacy classes of subgroups of H_u; (2) there is a hyper-Kähler $(\mathrm{SU}(2) = \mathrm{Sp}(1))$-action on $\mathfrak{M}_0$, denoted by $\phi(q) : \mathfrak{M}_0 \to \mathfrak{M}_0$, $q \in \mathrm{Sp}(1)$, coming from the $\mathbb{H}$-module structure on $\mathbf{M}$.

In Section 2, we prove the following general results about involutions on hyper-Kähler quotients.

THEOREM 1.6 (Proposition 2.8, Example 2.13)

(1) *Let η_H and $\eta_{\mathbf{M}}$ be complex conjugations on H and $\mathbf{M}$ which are compatible with the unitary quaternionic representation of H_u on $\mathbf{M}$ (see Definition 2.4 for the precise definition). Then η_H and $\eta_{\mathbf{M}}$ induce an antiholomorphic involution*

$$\eta : \mathfrak{M}_0 \longrightarrow \mathfrak{M}_0 \tag{1.9}$$

such that the composition of η with the hyper-Kähler $\mathrm{SU}(2)$-action of $q_s = \cos(\frac{s\pi}{2})i + \sin(\frac{s\pi}{2})k \in \mathrm{Sp}(1)$ on $\mathfrak{M}_0$, $s \in \mathbb{R}$, gives rise to a continuous family of involutions

$$\alpha_s : \mathfrak{M}_0 \longrightarrow \mathfrak{M}_0, \quad s \in \mathbb{R}, \tag{1.10}$$

interpolating the antiholomorphic involution $\alpha_0 = \phi(i) \circ \eta$ and the holomorphic involution $\alpha_1 = \phi(k) \circ \eta$.

(2) *Let $\mathfrak{M}_0(\mathbb{R})$ and $\mathfrak{M}_0^{\mathrm{sym}}(\mathbb{C})$ be the fixed points of α_0 and α_1 on $\mathfrak{M}_0$, respectively. Then the intersection of the strata $\mathfrak{M}_{0,(L)}$ with $\mathfrak{M}_0(\mathbb{R})$ (resp., $\mathfrak{M}_0^{\mathrm{sym}}(\mathbb{C})$) defines a stratification of $\mathfrak{M}_0(\mathbb{R})$ (resp., $\mathfrak{M}_0^{\mathrm{sym}}(\mathbb{C})$) and there exists a stratified homeomorphism*

$$\mathfrak{M}_0(\mathbb{R}) \xrightarrow{\sim} \mathfrak{M}_0^{\mathrm{sym}}(\mathbb{C}) \tag{1.11}$$

which is real analytic on each stratum.

Remark 1.7

Let G, $G_{\mathbb{R}}$, G_u, $K_{\mathbb{R}}$ be as in Section 1.1.2. Suppose that $\mathbf{M}$ is a unitary quaternionic representation of the larger group $H_u \times G_u$, and suppose that the conjugations $\eta_H \times \eta_G$ and $\eta_{\mathbf{M}}$ on $H \times G$ and $\mathbf{M}$ are compatible with the unitary quaternionic representation. Then the hyper-Kähler quotient $\mathfrak{M}$ carries an action of $K_{\mathbb{R}}$ such that the involutions (1.9) and (1.10) and homeomorphism (1.11) are $K_{\mathbb{R}}$-equivariant.

It is well known that the complex nilpotent cone $\mathcal{N}_n(\mathbb{C}) \subset \mathfrak{gl}_n(\mathbb{C})$ is an example of hyper-Kähler quotients known as *Nakajima's quiver varieties* (see [15], [19], [22], [25]). Applying Theorem 1.6 to this particular example, we obtain a family of $\mathrm{O}_n(\mathbb{C})$-equivariant involutions

$$\alpha_s : \mathcal{N}_n(\mathbb{C}) \longrightarrow \mathcal{N}_n(\mathbb{C}), \quad s \in [0, 1], \tag{1.12}$$

interpolating the complex conjugation $\alpha_0(M) = \overline{M}$ and the transpose $\alpha_1(M) = M^t$, and an $\mathrm{O}_n(\mathbb{C})$-equivariant homeomorphism

$$\mathcal{N}_n(\mathbb{R}) \xrightarrow{\sim} \mathcal{N}_n^{\mathrm{sym}}(\mathbb{C}) \tag{1.13}$$

between real and symmetric nilpotent cones which restricts to a real analytic isomorphism between individual $\mathrm{GL}_n(\mathbb{R})$-orbits and $\mathrm{O}_n(\mathbb{C})$-orbits. This establishes a special case of Theorems 1.3 and 1.4 for the fiber of the adjoint quotient map $\chi' : \mathfrak{gl}_n'(\mathbb{C}) \to \mathbb{C}^n$ over $0 \in \mathbb{C}^n$, that is, matrices with zero eigenvalues. To extend the results to matrices with real eigenvalues, we prove a version of Theorem 1.6 for the family of hyper-Kähler quotients

$$\mathfrak{M}_{Z_{\mathbb{C}}} = \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(Z_{\mathbb{C}})/H_u \to Z_{\mathbb{C}},$$

where $Z_{\mathbb{C}} \subset \mathfrak{h}^*$ is the dual of the center of $\mathfrak{h}$, and then deduce the results using the description of general adjoint orbit closures as quiver varieties in [24]. Finally, we check that the constructions are compatible with inner automorphisms and Cartan involutions and then deduce the case of Lie algebras of classical types from the case of $\mathfrak{gl}_n(\mathbb{C})$.

We would like to emphasize that the keys in the proof of Theorems 1.3 and 1.4 are the symmetries on adjoint orbit closures (or rather, the symmetries on the whole family $\mathfrak{gl}_n'(\mathbb{C}) \to \mathbb{C}^n$) coming from the hyper-Kähler SU(2)-action. Those symmetries are not immediately visible in their original definitions as algebraic varieties.

Remark 1.8
The use of hyper-Kähler SU(2)-actions in the study of geometry of nilpotent orbits goes back to the celebrated work of Kronheimer [20] where he used those symmetries to give a differential-geometric interpretation of Brieskorn's theorem on subregular singularities.

1.2. Applications

We discuss here applications to the Kostant–Sekiguchi correspondence, singularities of real and symmetric adjoint orbit closures, and Springer theory for symmetric spaces.

In the rest of the section, we assume that $\mathfrak{g}$ is of classical type.

1.2.1. Generalized Kostant–Sekiguchi homeomorphisms

The celebrated Kostant–Sekiguchi correspondence is an isomorphism between real and symmetric nilpotent orbit posets

$$|G_{\mathbb{R}}\backslash\mathcal{N}_{\mathbb{R}}| \longleftrightarrow |K\backslash\mathcal{N}_{\mathfrak{p}}|. \tag{1.14}$$

The bijection was proved by Kostant (unpublished) and Sekiguchi [30]. Vergne [31], using Kronheimer's instanton flow in [20], showed that the corresponding orbits are diffeomorphic. Schmid and Vilonen [29] gave an alternative proof and further refinements using Ness's moment map. Barbasch and Sepanski [1] deduced that the bijection is a poset isomorphism from Vergne's results.

We shall state a lift/generalization of the Kostant–Sekiguchi correspondence to stratified homeomorphisms between adjoint orbit closures in the real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ and symmetric subspace $\mathfrak{p}$ whose eigenvalues are real but not necessarily zero.

Denote by $\mathcal{N}_\xi = \chi^{-1}(\xi)$ the fiber of the Chevalley map $\chi : \mathfrak{g} \to \mathfrak{c}$ over $\xi \in \mathfrak{c}$. In [16], Kostant proved that there are finitely many G-orbits in $\mathcal{N}_\xi$ and there is a unique closed orbit $\mathcal{O}_\xi^{\mathfrak{s}}$ consisting of semisimple elements and a unique open orbit $\mathcal{O}_\xi^{\mathfrak{r}}$ consisting of regular elements. Moreover, we have $\mathcal{N}_\xi = \overline{\mathcal{O}_\xi^{\mathfrak{r}}}$.

Assume that $\xi \in \mathfrak{c}_{\mathfrak{p},\mathbb{R}} \subset \mathfrak{c}$. Then ξ is fixed by the involutions on $\mathfrak{c}$ induced by η and $-\theta$ and hence the fiber $\mathcal{N}_\xi$ is stable under η and $-\theta$. We write

$$\mathcal{N}_{\xi,\mathbb{R}} = \mathcal{N}_\xi \cap \mathfrak{g}_{\mathbb{R}}, \qquad \mathcal{N}_{\xi,\mathfrak{p}} = \mathcal{N}_\xi \cap \mathfrak{p}$$

for the fixed points. There are finitely many $G_{\mathbb{R}}$-orbits and K-orbits on $\mathcal{N}_{\xi,\mathbb{R}}$ and $\mathcal{N}_{\xi,\mathfrak{p}}$,

$$\mathcal{N}_{\xi,\mathbb{R}} = \bigsqcup_l \mathcal{O}_{\mathbb{R},l}, \qquad \mathcal{N}_{\xi,\mathfrak{p}} = \bigsqcup_l \mathcal{O}_{\mathfrak{p},l}.$$

COROLLARY 1.9
There is a $K_{\mathbb{R}}$-equivariant stratified homeomorphism

$$\mathcal{N}_{\xi,\mathbb{R}} \xrightarrow{\sim} \mathcal{N}_{\xi,\mathfrak{p}} \tag{1.15}$$

which restricts to real analytic isomorphisms between individual $G_{\mathbb{R}}$-orbits and K-orbits. The homeomorphism induces an isomorphism between $G_{\mathbb{R}}$-orbit and K-orbit posets

$$|G_{\mathbb{R}} \backslash \mathcal{N}_{\xi,\mathbb{R}}| \longleftrightarrow |K \backslash \mathcal{N}_{\xi,\mathfrak{p}}|. \tag{1.16}$$

Proof
This follows immediately from Theorem 1.3. □

Remark 1.10
Thanks to the work of Vergne [31], it is known that under the Kostant–Sekiguchi bijection the corresponding orbits are diffeomorphic. It is an open question whether the corresponding orbit closures have the same singularities (see, e.g., [11, Introduction]). Corollary 1.9 gives a positive answer in the case of classical Lie algebras.

Remark 1.11
In [3] and [4], the authors proved an extended Kostant–Sekiguchi correspondence for

certain adjoint orbits. We expect that their correspondence is compatible with the one in (1.16).

Remark 1.12
In Theorem 3.2, we also establish a Kostant–Sekiguchi correspondence between real and symmetric leaves for quiver varieties.

1.2.2. Derived categories
Let $D_{G_{\mathbb{R}}}(\mathcal{N}_{\xi,\mathbb{R}})$, $D_K(\mathcal{N}_{\xi,\mathfrak{p}})$ denote the respective equivariant derived categories of sheaves (over any commutative ring). Since $K_{\mathbb{R}} \to G_{\mathbb{R}}$, $K_{\mathbb{R}} \to K$ are homotopy equivalences, the forgetful functors $D_{G_{\mathbb{R}}}(\mathcal{N}_{\mathbb{R}}) \to D_{K_{\mathbb{R}}}(\mathcal{N}_{\mathbb{R}})$, $D_K(\mathcal{N}_{\mathfrak{p}}) \to D_{K_{\mathbb{R}}}(\mathcal{N}_{\mathfrak{p}})$ to $K_{\mathbb{R}}$-equivariant complexes are fully faithful with essential image those complexes constructible along the respective orbits of $G_{\mathbb{R}}$ and K.

Transport along the homeomorphism of Theorem 1.9 immediately provides the following.

COROLLARY 1.13
Pushforward along the homeomorphism (1.15) provides an equivalence of equivariant derived categories

$$D_{G_{\mathbb{R}}}(\mathcal{N}_{\xi,\mathbb{R}}) \simeq D_K(\mathcal{N}_{\xi,\mathfrak{p}}). \tag{1.17}$$

1.2.3. Vanishing of odd-dimensional intersection cohomology
Theorem 1.9 implies that the singularities of symmetric nilpotent orbit closures $\bar{\mathcal{O}}_{\mathfrak{p}} \subset \mathcal{N}_{\mathfrak{p}}$ are homeomorphic to the singularities of the corresponding real nilpotent orbit closures $\bar{\mathcal{O}}_{\mathbb{R}} \subset \mathcal{N}_{\mathbb{R}}$. Thus we can deduce results about one from the other.

Here is a notable example. Let $\mathrm{IC}(\mathcal{O}_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}})$ be the intersection cohomology sheaf of a real nilpotent orbit $\mathcal{O}_{\mathbb{R}} \subset \mathcal{N}_{\mathbb{R}}$ with coefficients in a $G_{\mathbb{R}}$-equivariant local system $\mathcal{L}_{\mathbb{R}}$. (Recall that all nilpotent orbits $\mathcal{O} \subset \mathcal{N}$ have even complex dimension, so all real nilpotent orbits $\mathcal{O}_{\mathbb{R}} \subset \mathcal{N}_{\mathbb{R}}$ have even real dimension, hence middle perversity makes sense.)

COROLLARY 1.14
The cohomology sheaves $\mathcal{H}^i(\mathrm{IC}(\mathcal{O}_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}))$ vanish for $i - \dim_{\mathbb{R}} \mathcal{O}_{\mathbb{R}}/2$ odd.

Proof
Using the equivalence (1.17), it suffices to prove the asserted vanishing for the intersection cohomology sheaf $\mathrm{IC}(\mathcal{O}_{\mathfrak{p}}, \mathcal{L}_{\mathfrak{p}})$ of a symmetric nilpotent orbit $\mathcal{O}_{\mathfrak{p}} \subset \mathcal{N}_{\mathfrak{p}}$ with

coefficients in a K-equivariant local system $\mathcal{L}_{\mathfrak{p}}$, and $i - \dim_{\mathbb{C}} \mathcal{O}_{\mathfrak{p}}$ odd. This is proved in [21, Theorem 14.10].[3] □

Remark 1.15
The proof of [21, Theorem 14.10] makes use of Deligne's theory of weights and the theory of canonical bases, and hence does not have an evident generalization to a real algebraic setting.

1.2.4. Formula for the sheaf of symmetric nearby cycles
Consider the quotient map $\chi_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{c}_{\mathfrak{p}} = \mathfrak{p}//K$. According to [18], the generic fiber of $\chi_{\mathfrak{p}}$ is a single K-orbit through a semisimple element in $\mathfrak{p}$, and the special fiber over the basepoint $\chi_{\mathfrak{p}}(0) \in \mathfrak{c}_{\mathfrak{p}}$ is the symmetric nilpotent cone $\mathcal{N}_{\mathfrak{p}}$. Following Grinberg [8] (see also [9], [10]), we consider the sheaf $\mathcal{F}_{\mathfrak{p}} \in D_K(\mathcal{N}_{\mathfrak{p}})$ of nearby cycles along the special fiber $\mathcal{N}_{\mathfrak{p}}$ in the family $\chi_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{c}_{\mathfrak{p}}$ (see Section 5.3 for the precise definition). We will call $\mathcal{F}_{\mathfrak{p}}$ the *sheaf of symmetric nearby cycles*.

Let $B_{\mathbb{R}} \subset G_{\mathbb{R}}$ be a minimal parabolic subgroup with Lie algebra $\mathfrak{b}_{\mathbb{R}} = \mathfrak{m}_{\mathbb{R}} + \mathfrak{a}_{\mathbb{R}} + \mathfrak{n}_{\mathbb{R}}$, where $\mathfrak{m}_{\mathbb{R}} = Z_{\mathfrak{k}_{\mathbb{R}}}(\mathfrak{a}_{\mathbb{R}})$ and $\mathfrak{n}_{\mathbb{R}}$ is the nilpotent radical. Consider the real Springer map

$$\pi_{\mathbb{R}} : \widetilde{\mathcal{N}}_{\mathbb{R}} \to \mathcal{N}_{\mathbb{R}},$$

where $\widetilde{\mathcal{N}}_{\mathbb{R}} = G_{\mathbb{R}} \times^{B_{\mathbb{R}}} \mathfrak{n}_{\mathbb{R}}$ and $\pi_{\mathbb{R}}(g, v) = \mathrm{Ad}_g\, v$.

We have the following formula for the sheaf of symmetric nearby cycles.

COROLLARY 1.16 (Theorem 5.3)
Under the equivalence $D_K(\mathcal{N}_{\mathfrak{p}}) \simeq D_{G_{\mathbb{R}}}(\mathcal{N}_{\mathbb{R}})$ (1.17), the sheaf of symmetric nearby cycles $\mathcal{F}_{\mathfrak{p}}$ becomes the real Springer sheaf $\mathcal{S}_{\mathbb{R}} := (\pi_{\mathbb{R}})_! \mathbb{C}[\dim_{\mathbb{R}} \mathcal{N}_{\mathbb{R}}/2]$. In particular, the real Springer sheaf $\mathcal{S}_{\mathbb{R}}$ is a perverse sheaf.

In fact, Theorem 5.3 is slightly stronger than the one stated here. We also prove a formula for the sheaf of symmetric nearby cycles with coefficients in K-equivariant local systems and we show that, for any $\mathfrak{g}_{\mathbb{R}}$ (not just for classical types), the real Springer sheaf is isomorphic to the *sheaf of real nearby cycles* $\mathcal{F}_{\mathbb{R}}$ introduced in Section 5.2.

Remark 1.17
The formula above for symmetric nearby cycles was originally conjectured by Vilonen, Xue, and the first author. It can be viewed as a symmetric space version of the

[3]In fact, [21] establishes the odd vanishing in the more general setting of graded Lie algebras.

well-known result that the sheaf of nearby cycles along the special fiber $\mathcal{N}$ in the family $\chi : \mathfrak{g} \to \mathfrak{c}$ is isomorphic to the Springer sheaf.

Remark 1.18
In [7], the authors used the sheaves of symmetric nearby cycles (with coefficients) to produce all cuspidal complexes on $\mathcal{N}_{\mathfrak{p}}$ and use them to establish a Springer correspondence for the split symmetric pair of type A (see [32] for the cases of classical symmetric pairs). The formula established in Corollary 1.16 provides new insights and methods into the study of Springer theory for general symmetric pairs and real groups. We will give one example below. The details will be discussed in a sequel [6].

1.2.5. Real Springer theory and Hecke algebras at roots of unity
In [8], Grinberg gave a generalization of Springer theory using nearby cycles. One of the main results there is a description of the endomorphism algebra $\mathrm{End}(\mathcal{F}_{\mathfrak{p}})$ of the sheaf of symmetric nearby cycles as a certain Hecke algebra at roots of unity.[4] To explain his result, let $(\Phi, \mathfrak{a}_{\mathbb{R}}^*)$ be the root system (possibly nonreduced) of $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{a}_{\mathbb{R}})$. For each $\alpha \in \Phi$, we denote by $\mathfrak{g}_{\mathbb{R},\alpha} \subset \mathfrak{g}_{\mathbb{R}}$ the corresponding α-eigenspace. Choose a system of simple roots $\Delta \subset \Phi$, and let $\mathrm{S} \subset \mathrm{W}$ be the set of simple reflections of the little Weyl group associated to Δ. Consider the algebra

$$\mathcal{H}_{G_{\mathbb{R}}} := \mathbb{C}[B_{\mathrm{W}}]/(T_s - 1)\big(T_s + (-1)^{d_s}\big)_{s\in \mathrm{S}},$$

where $\mathbb{C}[B_{\mathrm{W}}]$ is the group algebra of the braid group B_{W} of W with generators T_s, $s \in \mathrm{S}$, and d_s is the integer given by

$$d_s = \sum_{\alpha\in\Delta, s_\alpha = s} \dim_{\mathbb{R}}(\mathfrak{g}_{\mathbb{R},\alpha}),$$

where s_α denotes the reflection corresponding to the simple root $\alpha \in \Delta$.[5] For examples, if $G_{\mathbb{R}}$ is a split real form, then we have $d_s = 1$ for all $s \in \mathrm{S}$ and $\mathcal{H}_{G_{\mathbb{R}}}$ is isomorphic to the Hecke algebra associated to W at $q = -1$. On the other hand, if $G_{\mathbb{R}}$ is a complex group, then we have $d_s = 2$ and $\mathcal{H}_{G_{\mathbb{R}}}$ is isomorphic to the group algebra $\mathbb{C}[\mathrm{W}]$.

In [8, Theorem 6.1], Grinberg showed that there is a canonical isomorphism of algebras

$$\mathrm{End}(\mathcal{F}_{\mathfrak{p}}) \simeq \mathcal{H}_{G_{\mathbb{R}}}. \tag{1.18}$$

[4] In fact, he works in a more general setting of polar representations.
[5] Since the root system might not be reduced, there might be more than one simple root α such that $s_\alpha = s$.

Since the algebra $\mathcal{H}_{G_{\mathbb{R}}}$ is in general not semisimple, as an interesting corollary of (1.18), we see that the sheaf of symmetric nearby cycles $\mathcal{F}_{\mathfrak{p}}$ is not semisimple in general.

Now combining Corollary 1.16 with Grinberg's theorem, we obtain the following result in real Springer theory.

COROLLARY 1.19
We have a canonical isomorphism of algebras

$$\mathrm{End}(\mathcal{S}_{\mathbb{R}}) \simeq \mathcal{H}_{G_{\mathbb{R}}}.$$

In particular, the real Springer sheaf $\mathcal{S}_{\mathbb{R}}$ is in general not semisimple and, for any $x \in \mathcal{N}_{\mathbb{R}}$, the cohomologies $\mathrm{H}^*(\mathcal{B}_x, \mathbb{C})$ of the real Springer fiber $\mathcal{B}_x = \pi_{\mathbb{R}}^{-1}(x)$ carry a natural action of the algebra $\mathcal{H}_{G_{\mathbb{R}}}$.

Remark 1.20
In [6], we will give an alternative proof of Corollary 1.19 (for all types) following the classical arguments in Springer theory. In particular, combining with Corollary 1.16, we obtain a new proof of Grinberg's theorem on the endomorphism algebra of $\mathcal{F}_{\mathfrak{p}}$.

1.3. Previous work
In our previous work [5], we establish Corollary 1.9 for the nilpotent cone of $\mathfrak{gl}_n(\mathbb{C})$ using the geometry of moduli spaces of quasimaps associated to a symmetric pair (G, K). In more detail, we use the factorization properties of the moduli space of quasimaps to establish a real-symmetric homeomorphism in the setting of Beilinson–Drinfeld Grassmannians (for any reductive group G) and then deduce Corollary 1.9 using the Lusztig embedding of the nilpotent cone for $\mathfrak{gl}_n(\mathbb{C})$ into the affine Grassmannian for $\mathrm{GL}_n(\mathbb{C})$. The result in the present paper suggests that there should be a hyper-Kähler geometry interpretation of the results in [5]. This will be discussed in detail in a sequel.

We conclude the introduction with the following conjecture.

CONJECTURE 1.21
Theorems 1.3 and 1.4 remain true when $\mathfrak{g}$ is of exceptional type.

1.4. Organization
We briefly summarize here the main goals of each section. In Section 2 immediately to follow, we study involutions on hyper-Kähler quotients of linear spaces. In Section 3, we apply the results established in the previous section to the case of quiver varieties. In Section 4, we establish our main results, Theorems 4.1 and 4.2. In Section 5, we discuss applications to Springer theory for real groups and symmetric spaces.

2. A family of involutions on hyper-Kähler quotients

In this section, we introduce a family of involutions on hyper-Kähler quotients of linear spaces with remarkable properties. The main references for hyper-Kähler quotients are [12] and [13].

2.1. *Quaternions*

Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the quaternions. For any $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$, we denote by $\bar{x} = x_0 - x_1 i - x_2 j - x_3 k$. Then the pairing $(x, x') = \mathrm{Re}(x\bar{x}')$ defines a real-valued inner product on $\mathbb{H}$. We denote by $\mathrm{Im}(\mathbb{H}) = \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ the pure imaginary quaternions, and by $\mathrm{Sp}(1) = \{x \in \mathbb{H} \mid (x, x) = 1\}$ the group of quaternions of norm one.

2.2. *Hyper-Kähler quotient of linear spaces*

Let H be a complex reductive group with compact real form H_u. Let $\mathbf{M}$ be a quaternionic representation of H_u, that is, $\mathbf{M}$ is a finite-dimensional left quaternionic vector space together with an $\mathbb{H}$-linear action of H_u. We assume that the quaternionic representation is unitary; that is, there is an H_u-invariant inner product $(\,,\,)$ on $\mathbf{M}$ (as a real vector space) which is Hermitian with respect to the complex structures I, J, K on $\mathbf{M}$ given by multiplication by i, j, k, respectively. We have a natural complex representation of H on $\mathbf{M}$ preserving the complex symplectic form $\omega_{\mathbb{C}}(v, v') = (Jv, v') + i(Kv, v')$ on $\mathbf{M}$.

We have the hyper-Kähler moment map

$$\mu : \mathbf{M} \to \mathrm{Im}\,\mathbb{H} \otimes_{\mathbb{R}} \mathfrak{h}_u^*$$

satisfying

$$\langle \xi, \mu(\phi) \rangle = (I\xi\phi, \phi)i + (J\xi\phi, \phi)j + (K\xi\phi, \phi)k \in \mathrm{Im}\,\mathbb{H},$$

where $\xi \in \mathfrak{h}_u$, $\phi \in \mathbf{M}$, and $\langle\,,\,\rangle$ is the paring between $\mathfrak{h}_u^*$ and $\mathfrak{h}_u$. The map μ has the following equivariant properties: (1) it intertwines the $(\mathrm{Sp}(1) \times H_u)$-action on $\mathbf{M}$ and the one on $\mathrm{Im}(\mathbb{H}) \otimes_{\mathbb{R}} \mathfrak{h}_u^*$ given by $(q, h)(w, u) = (\mathrm{Ad}_q\, w, \mathrm{Ad}_h\, u)$; (2) we have $\mu(tv) = t^2 \mu(v)$ for $t \in \mathbb{R}^\times$, $v \in \mathbf{M}$.

Using the isomorphism $\mathrm{Im}\,\mathbb{H} = \mathbb{R} \oplus \mathbb{C}$ sending $x_1 i + x_2 j + x_3 k$ to $(x_1, x_2 + x_3 i)$, we can identify $\mathrm{Im}\,\mathbb{H} \otimes \mathfrak{h}_u^* = \mathfrak{h}_u^* \oplus \mathfrak{h}^*$ and hence obtain a decomposition of the moment map

$$\mu = \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} : \mathbf{M} \to \mathfrak{h}_u^* \oplus \mathfrak{h}^*$$

of μ into real and complex components. The map $\mu_{\mathbb{C}} : \mathbf{M} \to \mathfrak{h}^*$ is holomorphic with respect to the complex structure I on $\mathbf{M}$ and satisfies

$$\langle \xi, \mu_{\mathbb{C}}(\phi) \rangle = \omega_{\mathbb{C}}(\xi\phi, \phi),$$

where $\xi \in \mathfrak{h}$ and $\phi \in \mathbf{M}$. Moreover, it is H-equivariant with respect to the complex representation of H on $\mathbf{M}$ and the adjoint representation on $\mathfrak{h}^*$.

Let $Z = \{v \in \mathfrak{h}_u^* \mid \mathrm{Ad}_h(v) = v \text{ for all } h \in H_u\}$ and $Z_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} Z$. Then we have $\mathrm{Im}\,\mathbb{H} \otimes_{\mathbb{R}} Z = Z \oplus Z_{\mathbb{C}}$. For any $\zeta_{\mathbb{C}} \in Z_{\mathbb{C}}$, we can consider the hyper-Kähler quotient

$$\mathfrak{M}_{\zeta_{\mathbb{C}}} = \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}})/H_u. \tag{2.1}$$

We have the holomorphic description

$$\mathfrak{M}_{\zeta_{\mathbb{C}}} \simeq \mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}})//H,$$

where the right-hand side is the categorial quotient of $\mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}})$ by H. One can form a perturbed hyper-Kähler quotient

$$\mathfrak{M}_{(\zeta_{\mathbb{R}},\zeta_{\mathbb{C}})} = \mu_{\mathbb{R}}^{-1}(-\zeta_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}})/H_u$$

with not necessarily zero real component $\zeta_{\mathbb{R}}$. The composition $\mu_{\mathbb{R}}^{-1}(-\zeta_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}}) \to \mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}}) \to \mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}})//H$ gives rise to a map

$$\pi : \mathfrak{M}_{(\zeta_{\mathbb{R}},\zeta_{\mathbb{C}})} \to \mathfrak{M}_{\zeta_{\mathbb{C}}} \tag{2.2}$$

which is holomorphic with respect to the complex structure I.

From now on, we will fix a real parameter $\zeta_{\mathbb{R}}$. For any subset $S \subset Z_{\mathbb{C}}$, we can consider the following family of hyper-Kähler quotients:

$$\chi_S : \mathfrak{M}_S = \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(-S)/H_u \to S,$$
$$\tilde{\chi}_S : \mathfrak{M}_{(\zeta_{\mathbb{R}},S)} = \mu_{\mathbb{R}}^{-1}(-\zeta_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(-S)/H_u \to S.$$

Then the map (2.2) gives rise to a map

$$\pi_S : \mathfrak{M}_{(\zeta_{\mathbb{R}},S)} \to \mathfrak{M}_S \tag{2.3}$$

compatible with the projection maps to S. If S is semialgebraic, then $\mathfrak{M}_S$ is also semialgebraic, and if S is a complex algebraic variety, then we have the holomorphic description $\mathfrak{M}_S \simeq \mu_{\mathbb{C}}^{-1}(-S)//H$.

2.3. *A stratification*

Let $\zeta_{\mathbb{C}} \in Z_{\mathbb{C}}$. Let L be a subgroup of H_u. We denote by $\mathbf{M}_{(L)}$ the set of all points in $\mathbf{M}$ whose stabilizer is conjugate to L. A point in $\mathfrak{M}_{\zeta_{\mathbb{C}}}$ is said to be of stabilizer type

(L) if it has a representative in $\mathbf{M}_{(L)}$. The set of all points of stabilizer type (L) is denoted by $\mathfrak{M}_{\zeta_{\mathbb{C}},(L)}$. We have an *orbit-type* stratification

$$\mathfrak{M}_{\zeta_{\mathbb{C}}} = \bigsqcup_{(L)} \mathfrak{M}_{\zeta_{\mathbb{C}},(L)}, \tag{2.4}$$

where the union runs over the set of all conjugacy classes of subgroups of H_u. Each stratum $\mathfrak{M}_{\zeta_{\mathbb{C}},(L)}$ is a smooth hyper-Kähler manifold; moreover, it is a symplectic variety with respect to the complex structure I.

2.4. Symmetries of hyper-Kähler quotients

Let G be another complex reductive group with a compact real form G_u. Consider a unitary representation of G_u on $\mathbf{M}$ commuting with the H_u-action on $\mathbf{M}$. Then for any semialgebraic subset $S \subset Z_{\mathbb{C}}$, the action of the complexification G on $\mathbf{M}$ descends to an action on the hyper-Kähler quotient $\mathfrak{M}_S$ which is compatible with the projection map to S, and the induced action of the fibers $\mathfrak{M}_{\zeta_{\mathbb{C}}}$ are holomorphic with respect to the complex structure I.

Assume that $S \subset Z_{\mathbb{C}}$ is an $\mathbb{R}$-linear subspace. Then the action of $\mathbb{R}^{\times}$ on $\mathbf{M}$ descends to a G-equivariant $\mathbb{R}^{\times}$-action on $\mathfrak{M}_S$:

$$\phi(t) : \mathfrak{M}_S \to \mathfrak{M}_S, \quad t \in \mathbb{R}^{\times}. \tag{2.5}$$

Moreover, we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{M}_S & \xrightarrow{\phi(t)} & \mathfrak{M}_S \\ \downarrow & & \downarrow \\ S & \xrightarrow{t^2(-)} & S \end{array}$$

where the bottom arrow is the multiplication by t^2.

Let $q \in \mathrm{Sp}(1)$ be such that $\mathrm{Ad}_q(S) \subset S$. Here $\mathrm{Ad}_q : \mathrm{Im}\,\mathbb{H} \otimes Z \to \mathrm{Im}\,\mathbb{H} \otimes Z$ is the map $\mathrm{Ad}_q(w,u) = (\mathrm{Ad}_q\, w, u)$ and we identify $Z_{\mathbb{C}} = (\mathbb{R}j \oplus \mathbb{R}k) \otimes Z$, and hence S, as a subspace of $\mathrm{Im}\,\mathbb{H} \otimes Z$ with zero i-component. The action of $q \in \mathrm{Sp}(1)$ on $\mathbf{M}$ gives rise to a G_u-equivariant map

$$\phi(q) : \mathfrak{M}_S \to \mathfrak{M}_S \tag{2.6}$$

commuting with the $\mathbb{R}^{\times}$-actions. In addition, we have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{M}_S & \xrightarrow{\phi(q)} & \mathfrak{M}_S \\ \downarrow & & \downarrow \\ S & \longrightarrow & S \end{array}$$

where the bottom arrow is $\mathrm{Ad}_q : S \to S$.

It is straightforward to check that the stratum $\mathfrak{M}_{\zeta_{\mathbb{C}},(L)}$ in (2.4) is stable under the G- and $\mathbb{R}^{\times}$-actions. Moreover, for any $q \in \mathrm{Sp}(1)$ (resp., $t \in \mathbb{R}^{\times}$) and S as above, the map $\phi(q)$ (resp., $\phi(t)$) is compatible with the stratifications in the sense that it maps the stratum $\mathfrak{M}_{\zeta_{\mathbb{C}},(L)}$ in the fiber $\chi_S^{-1}(\zeta_{\mathbb{C}}) = \mathfrak{M}_{\zeta_{\mathbb{C}}}$ to the corresponding stratum $\mathfrak{M}_{\zeta'_{\mathbb{C}},(L)}$ in the fiber $\chi_S^{-1}(\zeta'_{\mathbb{C}}) = \mathfrak{M}_{\zeta'_{\mathbb{C}}}$, where $\zeta'_{\mathbb{C}} = \mathrm{Ad}_q\, \zeta_{\mathbb{C}}$ (resp., $\zeta'_{\mathbb{C}} = t^2\zeta_{\mathbb{C}}$).

Example 2.1
Let $S = 0$. Then we have $\mathrm{Ad}_q(0) = 0$ for all $q \in \mathrm{Sp}(1)$ and the family of maps $\phi(q)$ in (2.6) gives rise to a $(G_u \times \mathbb{R}^{\times})$-equivariant $\mathrm{Sp}(1)$-action on $\mathfrak{M}_0$, called the *hyper-Kähler* $\mathrm{Sp}(1)$*-action*. Moreover, the stratum $\mathfrak{M}_{0,(L)}$ is stable under the $\mathrm{Sp}(1)$-action.

2.5. *Conjugations on* $\mathfrak{M}_{Z_{\mathbb{C}}}$

Definition 2.2
Let η_H and $\eta_{\mathbf{M}}$ be conjugations on H and $\mathbf{M}$, respectively. We say that η_H and $\eta_{\mathbf{M}}$ are compatible with the symplectic representation of H on $\mathbf{M}$ if the following hold.
(1) We have $\eta_{\mathbf{M}}(hv) = \eta_H(h)\eta_{\mathbf{M}}(v)$ for all $h \in H$ and $v \in \mathbf{M}$.
(2) We have $\omega_{\mathbb{C}}(\eta_{\mathbf{M}}(v), \eta_{\mathbf{M}}(v')) = \overline{\omega_{\mathbb{C}}(v, v')}$ for all $v, v' \in \mathbf{M}$.

LEMMA 2.3
Let η_H and $\eta_{\mathbf{M}}$ be conjugations on H and $\mathbf{M}$ compatible with the symplectic representation of H on $\mathbf{M}$. Then the complex moment map $\mu_{\mathbb{C}} : \mathbf{M} \to \mathfrak{h}^*$ intertwines $\eta_{\mathbf{M}}$ and η_H.

Proof
For any $\xi \in \mathfrak{h}$, $v \in \mathbf{M}$, we have

$$\begin{aligned}\big\langle \xi, \mu_{\mathbb{C}}\big(\eta_{\mathbf{M}}(v)\big)\big\rangle &= \omega_{\mathbb{C}}\big(\xi\eta_{\mathbf{M}}(v), \eta_{\mathbf{M}}(v)\big) = \omega_{\mathbb{C}}\big(\eta_{\mathbf{M}}\big(\eta_H(\xi)v\big), \eta_{\mathbf{M}}(v)\big) \\ &= \overline{\omega_{\mathbb{C}}\big(\eta_H(\xi)v, v\big)} = \overline{\big\langle \eta_H(\xi), \mu_{\mathbb{C}}(v)\big\rangle} = \big\langle \xi, \eta_H\big(\mu_{\mathbb{C}}(v)\big)\big\rangle.\end{aligned}$$

This implies that $\mu_{\mathbb{C}}(\eta_{\mathbf{M}}(v)) = \eta_H(\mu_{\mathbb{C}}(v))$ for all $v \in \mathbf{M}$. The lemma follows. □

Let η_H and $\eta_{\mathbf{M}}$ be as in Lemma 2.3. Then the center of $\mathfrak{h}$, and hence $Z_{\mathbb{C}}$, is stable under η_H. It follows that, for any $\zeta_{\mathbb{C}} \in Z_{\mathbb{C}}$, the conjugation $\eta_{\mathbf{M}}$ on $\mathbf{M}$ descends to a map

$$\mathfrak{M}_{\zeta_{\mathbb{C}}} = \mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}})//H \to \mathfrak{M}_{\eta_H(\zeta_{\mathbb{C}})} = \mu_{\mathbb{C}}^{-1}\big(-\eta_H(\zeta_{\mathbb{C}})\big)//H \tag{2.7}$$

which is antiholomorphic with respect to the complex structure I. Moreover, it maps the stratum $\mathfrak{M}_{\zeta_{\mathbb{C}},(L)}$ to the corresponding stratum $\mathfrak{M}_{\eta_H(\zeta_{\mathbb{C}}),(L)}$. As $\zeta_{\mathbb{C}}$ varies over $Z_{\mathbb{C}}$, the maps (2.7) organize into a map

$$\eta_{Z_{\mathbb{C}}} : \mathfrak{M}_{Z_{\mathbb{C}}} \to \mathfrak{M}_{Z_{\mathbb{C}}} \tag{2.8}$$

making the following diagram commute:

$$\begin{array}{ccc} \mathfrak{M}_{Z_{\mathbb{C}}} & \xrightarrow{\eta_{Z_{\mathbb{C}}}} & \mathfrak{M}_{Z_{\mathbb{C}}} \\ \downarrow & & \downarrow \\ Z_{\mathbb{C}} & \xrightarrow{\eta_H} & Z_{\mathbb{C}} \end{array} \tag{2.9}$$

We will call $\eta_{Z_{\mathbb{C}}}$ the *conjugation* on $\mathfrak{M}_{Z_{\mathbb{C}}}$ associated to the conjugations η_H and $\eta_{\mathbf{M}}$.

2.6. *Compatibility with symmetries*

Recall the $\mathbb{R}$-subspace $Z \subset Z_{\mathbb{C}}$. For any $s \in \mathbb{R}$, let

$$q_s = \cos\Big(\frac{s\pi}{2}\Big) i + \sin\Big(\frac{s\pi}{2}\Big) k \in \mathrm{Sp}(1). \tag{2.10}$$

A direct computation shows that Ad_{q_s} preserves the subspace $Z = \mathbb{R}j \otimes_{\mathbb{R}} Z \subset \mathrm{Im}\,\mathbb{H} \otimes_{\mathbb{R}} Z$ and its restriction to Z is given by $-\mathrm{id}_Z$.[6] Consider the family of hyper-Kähler quotients

$$\mathfrak{M}_Z = \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(-Z)/H_u \tag{2.11}$$

over Z. Then the discussion in the previous section shows that there is a family of maps

$$\phi_s = \phi(q_s) : \mathfrak{M}_Z \to \mathfrak{M}_Z, \quad s \in \mathbb{R}, \tag{2.12}$$

making the following diagram commute:

$$\begin{array}{ccc} \mathfrak{M}_Z & \xrightarrow{\phi_s} & \mathfrak{M}_Z \\ \downarrow & & \downarrow \\ Z & \xrightarrow{-\mathrm{id}_Z} & Z \end{array} \tag{2.13}$$

[6]Note that $Z_{\mathbb{C}} = (\mathbb{R}j + \mathbb{R}k) \otimes_{\mathbb{R}} Z$ is not stable under the family of maps Ad_{q_s}.

Consider $j \in \mathrm{Sp}(1)$. Since $\mathrm{Ad}_j = \mathrm{id}_Z$ on Z, we have a map

$$\phi(j) : \mathfrak{M}_Z \to \mathfrak{M}_Z \tag{2.14}$$

making the following diagram commute:

$$\begin{array}{ccc} \mathfrak{M}_Z & \xrightarrow{\phi(j)} & \mathfrak{M}_Z \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\mathrm{id}_Z} & Z \end{array} \tag{2.15}$$

Note that

$$\phi_s^2 = \phi(j)^2 = \phi(-1). \tag{2.16}$$

In particular, if $\phi(-1)$ is equal to the identity map, then ϕ_s and $\phi(j)$ are involutions on $\mathfrak{M}_Z$.

Our next goal is to study the compatibility between the maps ϕ_s, $\phi(j)$, and the conjugation $\eta_{Z_\mathbb{C}}$ introduced in Section 2.5.

Definition 2.4

Let η_H and $\eta_\mathbf{M}$ be conjugations on H and $\mathbf{M}$, respectively. We say that η_H and $\eta_\mathbf{M}$ are compatible with the unitary quaternionic representation of H_u on $\mathbf{M}$ if the following hold.

(1) The pair $(\eta_H, \eta_\mathbf{M})$ is compatible with the symplectic representation of H on $\mathbf{M}$ (see Definition 2.2).

(2) $\eta_\mathbf{M}$ preserves the inner product $(\, , \,)$; that is, we have $(\eta_\mathbf{M}(v), \eta_\mathbf{M}(v')) = (v, v')$ for $v, v' \in \mathbf{M}$.

(3) η_H commutes with the Cartan conjugation δ_H.

PROPOSITION 2.5

Let η_H and $\eta_\mathbf{M}$ be conjugations on H and $\mathbf{M}$ compatible with the unitary quaternionic representation of H_u on $\mathbf{M}$. Let $\eta_{Z_\mathbb{C}} : \mathfrak{M}_{Z_\mathbb{C}} \to \mathfrak{M}_{Z_\mathbb{C}}$ be the conjugation in (2.8).[7] *Then the subspace $\mathfrak{M}_Z \subset \mathfrak{M}_{Z_\mathbb{C}}$ is stable under $\eta_{Z_\mathbb{C}}$. Denote by $\eta_Z : \mathfrak{M}_Z \to \mathfrak{M}_Z$ the resulting map. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{M}_Z & \xrightarrow{\eta_Z} & \mathfrak{M}_Z \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\mathrm{id}_Z} & Z \end{array} \tag{2.17}$$

[7] $\eta_{Z_\mathbb{C}}$ is well defined since η_H and $\eta_\mathbf{M}$ are compatible with the symplectic representation of H on $\mathbf{M}$.

Moreover, we have the following equality of maps on $\mathfrak{M}_Z$*:*

$$\phi_s \circ \eta_Z = \phi(-1) \circ \eta_Z \circ \phi_s, \qquad \phi_s \circ \phi(j) = \phi(-j) \circ \phi_s, \qquad \phi(j) \circ \eta_Z = \eta_Z \circ \phi(j). \tag{2.18}$$

Proof
The commutativity of (2.17) is clear. We shall prove the equality of maps in (2.18). Since η_H commutes with the Cartan conjugation δ_H, the center of $\mathfrak{h}_u$, and hence its real dual $Z \subset \mathfrak{h}_u^*$, is stable under η_H and (2.9) implies that $\mathfrak{M}_Z$ is preserved by the conjugation $\eta_{Z_\mathbb{C}}$.

We claim that conditions (1) and (2) in Definition 2.4 imply that $\eta_\mathbf{M}$ commutes with J and preserves $\mu_\mathbb{R}^{-1}(0)$. Assume the claim for the moment. Then using the equality $I \circ \eta_\mathbf{M} = -\eta_\mathbf{M} \circ I$ and $K = IJ$, a direct computation shows that we have the following equality of maps on $\mu_\mathbb{R}^{-1}(0) \cap \mu_\mathbb{C}^{-1}(-Z)$:

$$\begin{aligned} \big(\cos(a)I + \sin(a)K\big) \circ \eta_\mathbf{M} &= -\eta_\mathbf{M} \circ \big(\cos(a)I + \sin(a)K\big), \\ \big(\cos(a)I + \sin(a)K\big) \circ J &= -J \circ \big(\cos(a)I + \sin(a)K\big), \\ J \circ \eta_\mathbf{M} &= \eta_\mathbf{M} \circ J \end{aligned}$$

compatible with the H_u-action. The desired equality (2.18) follows.

Proof of the claim. For any $\xi \in \mathfrak{h}_u$ and $v \in \mathbf{M}$, we have

$$\begin{aligned} \big\langle \xi, \mu_\mathbb{R}\big(\eta_\mathbf{M}(v)\big)\big\rangle &= \big(I\xi\eta_\mathbf{M}(v), \eta_\mathbf{M}(v)\big) = -\big(\eta_\mathbf{M}\big(I\eta_H(\xi)v\big), \eta_\mathbf{M}(v)\big) \\ &= -\big(I\eta_H(\xi)v, v\big) = \big\langle -\eta_H(\xi), \mu_\mathbb{R}(v)\big\rangle \\ &= \big\langle \xi, -\eta_H\big(\mu_\mathbb{R}(v)\big)\big\rangle. \end{aligned}$$

Thus we have $\mu_\mathbb{R}(\eta_\mathbf{M}(v)) = -\eta_H(\mu_\mathbb{R}(v))$ and it follows that $\mu_\mathbb{R}^{-1}(0)$ is stable under the conjugation $\eta_\mathbf{M}$. Recall that $\omega_\mathbb{C}(v, v') = (Jv, v') + i(Kv, v')$. Thus the equality $\omega_\mathbb{C}(\eta_\mathbf{M}(v), \eta_\mathbf{M}(v')) = \overline{\omega_\mathbb{C}(v, v')}$ is equivalent to

$$\big(J\eta_\mathbf{M}(v), \eta_\mathbf{M}(v')\big) + i\big(K\eta_\mathbf{M}(v), \eta_\mathbf{M}(v')\big) = (Jv, v') - i(Kv, v'),$$

which implies that

$$\big(J\eta_\mathbf{M}(v), \eta_\mathbf{M}(v')\big) = (Jv, v').$$

Since $\eta_\mathbf{M}$ preserves $(\, , \,)$, the above equality implies that

$$\big(J\eta_\mathbf{M}(v), \eta_\mathbf{M}(v')\big) = \big(\eta_\mathbf{M}J(v), \eta_\mathbf{M}(v')\big)$$

and it follows that $J \circ \eta_\mathbf{M} = \eta_\mathbf{M} \circ J$. This finishes the proof of the claim. □

Remark 2.6
The proof above shows that condition (2) in Definition 2.4 is equivalent to the condition that $\eta_{\mathbf{M}}$ commutes with J.

2.7. A family of involutions
Let η_H and $\eta_{\mathbf{M}}$ be conjugations on H and $\mathbf{M}$ compatible with the unitary quaternionic representation of H_u on $\mathbf{M}$. Let G be another complex reductive group with a compact real form G_u, and let η_G be a conjugation on G with real form $G_{\mathbb{R}}$. Suppose that $\mathbf{M}$ is a unitary quaternionic representation of the larger group $H_u \times G_u$ and that the conjugations $\eta_H \times \eta_G$ and $\eta_{\mathbf{M}}$ are compatible with the unitary quaternionic representation. Then the maps η_Z, ϕ_s, $\phi(j)$ in Proposition 2.5 are $K_{\mathbb{R}}$-equivariant, where $K_{\mathbb{R}} = G_{\mathbb{R}} \cap G_u$ is a maximal compact subgroup of $G_{\mathbb{R}}$.

Introduce the maps

(1) $\alpha_s = \phi_s \circ \eta_Z : \mathfrak{M}_Z \to \mathfrak{M}_Z$,

(2) $\beta = \phi(j) \circ \eta_Z : \mathfrak{M}_Z \to \mathfrak{M}_Z$.

PROPOSITION 2.7
We have $\alpha_s \circ \beta = \beta \circ \alpha_s$ for all $s \in \mathbb{R}$.

Proof
By Proposition 2.5, we have

$$\alpha_s \circ \beta = \phi_s \circ \eta_Z \circ \phi(j) \circ \eta_Z = \phi_s \circ \phi(j)$$

and

$$\alpha_s \circ \beta = \phi(j) \circ \eta_Z \circ \phi_s \circ \eta_Z = \phi(j) \circ \phi(-1) \circ \phi_s = \phi_s \circ \phi(j).$$

The result follows. □

PROPOSITION 2.8
The continuous family of maps

$$\alpha_s : \mathfrak{M}_Z \longrightarrow \mathfrak{M}_Z, \quad s \in \mathbb{R},$$

satisfies the following.

(1) *α_s^2 is equal to identity, for all $s \in \mathbb{R}$.*

(2) *α_s is $K_{\mathbb{R}}$-equivariant and commutes with the $\mathbb{R}^\times$-action.*

(3) *We have $\chi_Z \circ \alpha_s = \chi_Z : \mathfrak{M}_Z \to Z$, where χ_Z is the natural projection map, and the induced involutions on the fibers $\alpha_s : \mathfrak{M}_{\zeta_{\mathbb{C}}} \to \mathfrak{M}_{\zeta_{\mathbb{C}}}$, $\zeta_{\mathbb{C}} \in Z$, preserve the stratification $\mathfrak{M}_{\zeta_{\mathbb{C}}} = \bigsqcup_{(L)} \mathfrak{M}_{\zeta_{\mathbb{C}},(L)}$.*

(4) *At* $s = 0$, *we have* $\alpha_0 = \phi(i) \circ \eta_Z$ *which is an antiholomorphic involution.*
(5) *At* $s = 1$, *we have* $\alpha_1 = \phi(k) \circ \eta_Z$ *which is a holomorphic involution.*

Proof
According to (2.10), we have $q_s^2 = -1$. Thus $\phi_s^2 = \phi(q_s^2) = \phi(-1)$ and Proposition 2.5 implies that

$$\alpha_s^2 = (\phi_s \circ \eta_Z)^2 = \phi_s \circ \eta_Z \circ \phi_s \circ \eta_Z = \phi_s^2 \circ \phi(-1) \circ \eta_Z^2 = \mathrm{id}.$$

Part (1) follows. Parts (2), (3), (4), and (5) follow from the construction. □

PROPOSITION 2.9
The map

$$\beta : \mathfrak{M}_Z \longrightarrow \mathfrak{M}_Z$$

satisfies the following.
(1) *We have* $\beta^2 = \phi(-1)$.
(2) β *is* $K_{\mathbb{R}}$*-equivariant and commutes with the* $\mathbb{R}^\times$*-action.*
(3) β *induces a holomorphic map between fibers* $\beta : \mathfrak{M}_{\zeta_{\mathbb{C}}} \to \mathfrak{M}_{-\zeta_{\mathbb{C}}}$ *which takes the stratum* $\mathfrak{M}_{\zeta_{\mathbb{C}},(L)}$ *to the stratum* $\mathfrak{M}_{-\zeta_{\mathbb{C}},(L)}$.

Proof
Since $\beta^2 = \phi(j) \circ \eta_Z \circ \phi(j) \circ \eta_Z = \phi(j)^2 = \phi(-1)$, part (1) follows. Parts (2) and (3) follow from the construction. □

Remark 2.10
Unlike the family of involutions α_s, the map β is well defined on the whole family $\mathfrak{M}_{Z_{\mathbb{C}}}$ (see footnote 6).

2.8. *A stratified homeomorphism*
Our aim is to trivialize the family of fixed points of the involutions α_s. To that end, we will invoke the following lemma.

Recall that a subset S of a real analytic manifold M (resp., real algebraic variety M) is called *semianalytic* (resp., *semialgebraic*) if any point $s \in S$ has an open neighborhood U (resp., a Zariski affine open neighborhood U) such that the intersection $S \cap U$ is a finite union of sets of the form

$$\{x \in U \mid f_1(x) = \cdots = f_r(x) = 0, g_1(x) > 0, \ldots, g_l(x) > 0\},$$

where the f_i and g_j are real analytic functions on U (resp., polynomial functions on U). Let S^1 be the unit circle with the following.

LEMMA 2.11
Let M and N be two semianalytic sets, and let $f: M \to N$ be a continuous map. Let

$$\alpha_s : M \to M, \quad s \in \mathbb{R},$$

be a continuous family of involutions over N.

(1) *Assume that α_s preserves a semianalytic stratification[8] of M and restricts to a real analytic map on each stratum. Then the fixed points of the strata are real analytic manifolds and the α_s-fixed points M^{α_s} are stratified by the fixed points of the strata.*

(2) *Assume further that there is a continuous $\mathbb{R}_{>0}$-action on M (resp., N) real analytic on strata and a proper continuous map $\|-\| : M \to \mathbb{R}_{\geq 0}$ such that (i) $f: M \to N$ is $\mathbb{R}_{>0}$-equivariant, (ii) the $\mathbb{R}_{>0}$-action on M has a unique fixed point $o_M \in M$, which is also a stratum, and (iii) $\|tm\| = t\|m\|$ and $\|\alpha_s(m)\| = \|m\|$ for $t \in \mathbb{R}_{>0}$, $s \in \mathbb{R}$, $m \in M$. Then for any $s, s' \in \mathbb{R}$ there is an $\mathbb{R}_{>0}$-equivariant stratified homeomorphism*

$$M^{\alpha_s} \simeq M^{\alpha_{s'}} \tag{2.19}$$

that is real analytic on each stratum and compatible with the natural maps to N.

(3) *Assume further that there is a continuous action of a compact group L on M satisfying that (i) the action commutes with the map $f: M \to N$, the involutions α_s, and the $\mathbb{R}_{>0}$-action, and is real analytic on each stratum, and (ii) the map $\|-\| : M \to \mathbb{R}_{\geq 0}$ is L-invariant. Then the homeomorphism in (2.19) is L-equivariant.*

Proof
(1) Only the first claim requires a proof and it follows from the general fact that the fixed points M^α of a real analytic involution α on a real analytic manifold M is again a real analytic manifold.

(2) *Step 1.* Let $M_0 = M \setminus \{o_M\}$ and $C = \{m \in M_0 \mid \|m\| = 1\}$. Since $\|-\| : M \to \mathbb{R}_{\geq 0}$ is α_s-invariant and proper, C is compact and stable under the α_s-action. Since $\mathbb{R}_{>0}$ acts freely on M_0 and $\|-\|$ is $\mathbb{R}_{>0}$-equivariant, the restriction $\|-\||_{M_0} : M_0 \to \mathbb{R}_{>0}$ is a stratified submersion (where $\mathbb{R}_{>0}$ is equipped with the trivial stratification). It follows that $C = \|-\|^{-1}(1) \subset M_0$ is stratified by the intersection of the strata with C.

[8] A stratification of a semianalytic set is called *semianalytic* if each stratum is a real analytic manifold.

Step 2. We shall show that there exists a stratified homeomorphism

$$\nu : C^{\alpha_s} \simeq C^{\alpha_{s'}} \tag{2.20}$$

which is real analytic on each stratum and is compatible with natural maps to N. Consider the involution $\alpha : \mathbb{R} \times C \to \mathbb{R} \times C$, $\alpha(s,m) = (s, \alpha_s(m))$. Let $w = (\partial_s \times 0) + \alpha_*(\partial_s \times 0)$ be the average of the vector field $\partial_s \times 0$ on $\mathbb{R} \times C$ with respect to the $\mathbb{Z}/2\mathbb{Z}$-action given by the involution α. Since C is compact and the $\mathbb{Z}/2\mathbb{Z}$-action is real analytic on each stratum, the α-invariant vector field w is complete and the integral curves of w define the desired stratified homeomorphism $\nu : C^{\alpha_s} \simeq C^{\alpha_{s'}}$, $s, s' \in \mathbb{R}$ between the fibers of the α-fixed point $(\mathbb{R} \times C)^\alpha$ along the projection map to $\mathbb{R}$.

Step 3. We have a natural map $M_0^{\alpha_s} \to C^{\alpha_s}$ sending m to $\frac{m}{\|m\|}$. Consider the following map:

$$M_0^{\alpha_s} \to M_0^{\alpha_{s'}}, \qquad m \to \|m\| \nu\Big(\frac{m}{\|m\|}\Big). \tag{2.21}$$

Note that M^{α_s} is homeomorphic to the cone $C(M_0^{\alpha_s}) = M_0^{\alpha_s} \cup \{o_M\}$ of $M_0^{\alpha_s}$. Thus by the functoriality of cone, the map (2.21) extends to a homeomorphism

$$M^{\alpha_s} \to M^{\alpha_{s'}} \tag{2.22}$$

sending o_M to o_M. It is straightforward to check that (2.22) is an $\mathbb{R}_{>0}$-equivariant stratified homeomorphism which is real analytic on each stratum and compatible with the natural maps to N. This finishes the proof of part (2). Part (3) is clear from the construction of (2.22). □

Remark 2.12

In fact, one can also use the Thom–Mather theory to obtain the homeomorphism (2.20) in Step 2. Indeed, under the assumption that the stratification is Whitney, the Thom–Mather theory shows that the vector field ∂_s on $\mathbb{R}$ admits a lift to a controlled vector field w on the α-fixed points $(\mathbb{R} \times C)^\alpha$ along the projection map $(\mathbb{R} \times C)^\alpha \to \mathbb{R}$ and, as the projection map is proper, the integral curves of w give rise to a trivialization of $(\mathbb{R} \times C)^\alpha \to \mathbb{R}$, and hence the homeomorphism (2.20) between fibers (see, e.g., [23, Corollary 10.2]). In our case, due to the fact that $(\mathbb{R} \times C)^\alpha \to \mathbb{C}$ is the α-fixed points subset of the trivial family $\mathbb{R} \times C \to \mathbb{R}$, we have a direct construction of the controlled vector field as the average vector field w of the *canonical* controlled vector field $\partial_s \times 0$ on $\mathbb{R} \times C$, and thus do not need to invoke the Thom–Mather theory.

Example 2.13

We preserve the setup in Section 2.7. The map $\|-\| : \mathfrak{M}_Z = \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(Z)/$

$H_u \to \mathbb{R}_{\geq 0}$ given by $\|m\| = (\tilde{m}, \tilde{m})^{\frac{1}{2}}$, where $\tilde{m} \in \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(Z)$ is a lift of m, is a $(K_{\mathbb{R}} \times \alpha_s)$-invariant proper real analytic map satisfying $\|\phi(t)m\| = t\|m\|$, $t \in \mathbb{R}_{>0}$. Let $\mathfrak{M}_0 = \mu^{-1}(0)/H_u$, let $\alpha_s : \mathfrak{M}_0 \to \mathfrak{M}_0$ be the family of involutions in Proposition 2.8, and let $\phi(t) : \mathfrak{M}_0 \to \mathfrak{M}_0$ be the $\mathbb{R}_{>0}$-action in (2.5). Denote by

$$\mathfrak{M}_0(\mathbb{R}) = \mathfrak{M}_0^{\alpha_0}, \qquad \mathfrak{M}_0^{\mathrm{sym}}(\mathbb{C}) = \mathfrak{M}_0^{\alpha_1}$$

the fixed points of α_0 and α_1 on $\mathfrak{M}_0$, respectively. Applying Lemma 2.11 to the case $M = \mathfrak{M}_0$ with the stratification $\mathfrak{M}_0 = \bigsqcup_{(L)} \mathfrak{M}_{0,(L)}$, $N = 0$, $L = K_{\mathbb{R}}$, and the restriction $\|-\||_M : M = \mathfrak{M}_0 \to \mathbb{R}_{\geq 0}$ of the function $\|-\|$ above to $\mathfrak{M}_0 \subset \mathfrak{M}_Z$, we see that there is a $(K_{\mathbb{R}} \times \mathbb{R}_{>0})$-equivariant stratified homeomorphism

$$\mathfrak{M}_0(\mathbb{R}) \xrightarrow{\sim} \mathfrak{M}_0^{\mathrm{sym}}(\mathbb{C}) \tag{2.23}$$

which is real analytic on each stratum. Note that whereas $\mathfrak{M}_0^{\mathrm{sym}}(\mathbb{C})$ is complex analytic, $\mathfrak{M}_0(\mathbb{R})$ is not; it is a real form of $\mathfrak{M}_0$.

3. Quiver varieties

In this section, we consider the examples when the hyper-Kähler quotients are Nakajima's quiver varieties. We show that any quiver variety has a canonical conjugation called the *split conjugation* and hence has a canonical family of involutions α_s introduced in Section 2.7. The main reference for quiver varieties is [25].

3.1. Split conjugations

Let $Q = (Q_0, Q_1)$ be a quiver, where Q_0 is the set of vertices and Q_1 is the set of arrows. For any Q_0-graded Hermitian vector space $V = \bigoplus_{k \in Q_0} V_k$, we write $\mathrm{GL}(V) = \prod_{k \in Q_0} \mathrm{GL}(V_i)$ and $\mathrm{U}(V) = \prod_{k \in Q_0} \mathrm{U}(V_k)$, where $\mathrm{U}(V_k)$ is the unitary group associated to the Hermitian vector space V_k. We denote by $\mathfrak{gl}(V)$ and $\mathfrak{u}(V)$ the Lie algebras of $\mathrm{GL}(V)$ and $\mathrm{U}(V)$, respectively.

Let $V = \bigoplus_{k \in Q_0} V_k$ and $W = \bigoplus_{k \in Q_0} W_k$ be two Q_0-graded Hermitian vector spaces. Define

$$\begin{aligned} \mathbf{M} &= \mathbf{M}(V, W) \\ &= \Big(\bigoplus_{h \in Q_1} \mathrm{Hom}(V_{o(h)}, V_{i(h)}) \oplus \mathrm{Hom}(V_{i(h)}, V_{o(h)}) \Big) \\ &\quad \oplus \Big(\bigoplus_{k \in Q_0} \mathrm{Hom}(W_k, V_k) \oplus \mathrm{Hom}(V_k, W_k) \Big). \end{aligned} \tag{3.1}$$

Here $o(h)$ and $i(h)$ are the outgoing and incoming vertices of the oriented arrow $h \in Q_1$, respectively.

We consider the $\mathbb{H}$-vector space structure on $\mathbf{M}$ given by the original complex structure I together with the new complex structure J given by

$$J(X,Y,x,y) = (-Y^\dagger, X^\dagger, -y^\dagger, x^\dagger), \tag{3.2}$$

where $(X,Y,x,y) \in \mathrm{Hom}(V_{o(h)}, V_{i(h)}) \oplus \mathrm{Hom}(V_{i(h)}, V_{o(h)}) \oplus \mathrm{Hom}(W_k, V_k) \oplus \mathrm{Hom}(V_k, W_k)$ and $(-)^\dagger$ is the Hermitian adjoint.

The Hermitian inner products on V_k and W_k induce a Hermitian inner product on $\mathrm{Hom}(V_k, W_k)$ (resp., $\mathrm{Hom}(V_k, V_{k'})$) given by $(f,g) = \mathrm{tr}(fg^\dagger)$. We consider the Hermitian inner product on $\mathbf{M}$ induced from the ones on V_k and W_k.

Let $H = \mathrm{GL}(V)$ and $G = \mathrm{GL}(W)$ with compact real forms $H_u = \mathrm{U}(V)$ and $G_u = \mathrm{U}(W)$. Then action of $H \times G = \mathrm{GL}(V) \times \mathrm{GL}(W)$ on $\mathbf{M}$ given by the formula

$$(g,g')(X,Y,x,y) = \big(gXg^{-1}, gYg^{-1}, gx(g')^{-1}, g'yg^{-1}\big)$$

defines a unitary quaternionic representation of $\mathrm{U}(V) \times \mathrm{U}(W)$ on $\mathbf{M}$. The holomorphic symplectic form $\omega_{\mathbb{C}}$ is given by

$$\omega_{\mathbb{C}}\big((X,Y,x,y),(X',Y',x',y')\big) = \mathrm{tr}(XY' - YX') + \mathrm{tr}(xy' - x'y). \tag{3.3}$$

We denote by

$$\mu : \mathbf{M} \to \mathrm{Im}(\mathbb{H}) \otimes \mathfrak{u}(V)^* = \mathrm{Im}(\mathbb{H}) \otimes \mathfrak{u}(V) \tag{3.4}$$

the hyper-Kähler moment map with respect to the $\mathrm{U}(V)$-action. Here we identify $\mathfrak{u}(V)$ with its dual space $\mathfrak{u}(V)^*$ via the above Hermitian inner product. We have the following formulas for the real and complex moment maps:

$$\mu_{\mathbb{R}}(X,Y,x,y) = \frac{i}{2}(XX^\dagger - Y^\dagger Y + xx^\dagger - y^\dagger y) \in \mathfrak{u}(V),$$
$$\mu_{\mathbb{C}}(X,Y,x,y) = [X,Y] + xy \in \mathfrak{gl}(V) = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{u}(V).$$

The hyper-Kähler quotient $\mathfrak{M}_\xi$ is called the *quiver variety*.

LEMMA 3.1
Let η_V and η_W be conjugations on V and W compatible with the Q_0-grading,[9] *and let η_H, η_G, and $\eta_{\mathbf{M}}$ be the induced conjugations on $H = \mathrm{GL}(V)$, $G = \mathrm{GL}(W)$, and $\mathbf{M}$, respectively. Assume that η_H and η_G commute with the Cartan conjugations on H and G given by the Hermitian adjoint. Then the conjugations $\eta_H \times \eta_G$ and $\eta_{\mathbf{M}}$ are compatible with the unitary quaternionic representation of $H_u \times G_u$ on $\mathbf{M}$.*

[9] That is, we have $\eta_V(V_k) = V_k$, $\eta_W(W_k) = W_k$ for all $k \in Q_0$.

Proof
The conjugation $\eta_H \times \eta_G$ commutes with the Cartan involution on $H \times G$ by assumption. Using (3.2) and (3.3), it is straightforward to check that $\eta_{\mathbf{M}}$ commutes with J and $\eta_H \times \eta_G$ and that $\eta_{\mathbf{M}}$ are compatible with the symplectic representation of $H \times G$ on $\mathbf{M}$. In view of Remark 2.6, we see that $\eta_H \times \eta_G$ and $\eta_{\mathbf{M}}$ satisfy (1), (2), and (3) in Definition 2.4. The lemma follows. ☐

Choose $\mathbf{v} = (\mathbf{v}_k)_{k \in Q_0}$, $\mathbf{w} = (\mathbf{w}_k)_{k \in Q_0} \in \mathbb{Z}_{\geq 0}^{Q_0}$, and let $\mathbf{M}(\mathbf{v}, \mathbf{w}) = \mathbf{M}(V, W)$, where $V = \bigoplus_{k \in Q_0} \mathbb{C}^{\mathbf{v}_k}$ and $W = \bigoplus_{k \in Q_0} \mathbb{C}^{\mathbf{w}_k}$ equipped with the standard Hermitian inner products. The standard complex conjugations on V and W induce the split conjugations on $H = \mathrm{GL}(V)$ and $G = \mathrm{GL}(W)$ commuting with the Cartan conjugations, and hence give rise to involutions η_H, η_G and $\eta_{\mathbf{M}}$ compatible with the unitary quaternionic representation. We will call the conjugation

$$\eta_{Z_{\mathbb{C}}} : \mathfrak{M}_{Z_{\mathbb{C}}} \to \mathfrak{M}_{Z_{\mathbb{C}}}$$

on the family of quiver varieties $\mathfrak{M}_{Z_{\mathbb{C}}}$ associated to $\eta_H \times \eta_G$ and $\eta_{\mathbf{M}}$ the *split conjugation*.

3.2. *Real-symmetric homeomorphisms for quiver varieties*

Let $\mathrm{O}(W_{\mathbb{R}}) = \mathrm{U}(W) \cap \mathrm{GL}(W_{\mathbb{R}})$ be the real orthogonal group. By Propositions 2.5 and 2.8, the split conjugation $\eta_{Z_{\mathbb{C}}}$ on $\mathfrak{M}_{Z_{\mathbb{C}}}$ preserves the subspace $\mathfrak{M}_Z \subset \mathfrak{M}_{Z_{\mathbb{C}}}$ and gives rise to a family of $\mathrm{O}(W_{\mathbb{R}})$-equivariant involutions

$$\alpha_s : \mathfrak{M}_Z \to \mathfrak{M}_Z, \quad s \in \mathbb{R}, \tag{3.5}$$

interpolating the antiholomorphic involution $\alpha_0 = \phi(i) \circ \eta_Z$ and the holomorphic involution $\alpha_1 = \phi(k) \circ \eta_Z$, and preserving the strata $\mathfrak{M}_{\zeta_{\mathbb{C}},(L)}$ of the fiber $\mathfrak{M}_{\zeta_{\mathbb{C}}}$ for $\zeta_{\mathbb{C}} \in Z$.

The involutions in (3.5) restrict to a family of involutions $\alpha_a : \mathfrak{M}_0 \to \mathfrak{M}_0$. Write $\mathfrak{M}_0(\mathbb{R}) = \mathfrak{M}_0^{\alpha_0}$ and $\mathfrak{M}_0^{\mathrm{sym}}(\mathbb{C}) = \mathfrak{M}_0^{\alpha_1}$ for the fixed points of α_0 and α_1. The intersections of the stratum $\mathfrak{M}_{0,(L)}$ with $\mathfrak{M}_0(\mathbb{R})$ and $\mathfrak{M}_0^{\mathrm{sym}}(\mathbb{C})$ are unions of components

$$\mathfrak{M}_{0,(L)} \cap \mathfrak{M}_0(\mathbb{R}) = \bigsqcup \mathcal{O}_l(\mathbb{R}), \qquad \mathfrak{M}_{0,(L)} \cap \mathfrak{M}_0^{\mathrm{sym}}(\mathbb{C}) = \bigsqcup \mathcal{O}_l^{\mathrm{sym}}(\mathbb{C}).$$

In [2, Theorem 1.9], Bellamy and Schedler proved that the strata $\mathfrak{M}_{0,(L)}$ are symplectic leaves of $\mathfrak{M}_0$. We will call the components $\mathcal{O}_l(\mathbb{R})$ and $\mathcal{O}_l^{\mathrm{sym}}(\mathbb{C})$ above the *real leaves* and *symmetric leaves*, respectively.

The following proposition follows from Example 2.13.

THEOREM 3.2
There is an $(\mathrm{O}(W_{\mathbb{R}}) \times \mathbb{R}^{\times})$*-equivariant stratified homeomorphism*

$$\mathfrak{M}_0(\mathbb{R}) \xrightarrow{\sim} \mathfrak{M}_0^{\mathrm{sym}}(\mathbb{C}) \tag{3.6}$$

which restricts to real analytic $\mathrm{O}(W_{\mathbb{R}})$*-equivariant isomorphisms between individual real and symmetric leaves. The homeomorphism induces a bijection*

$$\{\mathcal{O}_l(\mathbb{R})\}_l \longleftrightarrow \{\mathcal{O}_l^{\mathrm{sym}}(\mathbb{C})\}_l \tag{3.7}$$

between real and symmetric leaves preserving the closure relation.

In the next section, we shall see that the nilpotent cone $\mathcal{N}_n(\mathbb{C})$ in $\mathfrak{gl}_n(\mathbb{C})$ is an example of a quiver variety and the homeomorphism (3.6) in this case becomes an $(\mathrm{O}_n(\mathbb{R}) \times \mathbb{R}^\times)$-equivariant homeomorphism

$$\mathcal{N}_n(\mathbb{R}) \simeq \mathcal{N}_n^{\mathrm{sym}}(\mathbb{C})$$

between the real nilpotent cone in $\mathfrak{gl}_n(\mathbb{R})$ and the symmetric nilpotent cone in the space of symmetric matrices $\mathfrak{p}_n(\mathbb{C})$, and the bijection (3.7) is the well-known Kostant–Sekiguchi bijection between $\mathrm{GL}_n(\mathbb{R})$-orbits in $\mathcal{N}_n(\mathbb{R})$ and $\mathrm{O}_n(\mathbb{C})$-orbits in $\mathcal{N}_n^{\mathrm{sym}}(\mathbb{C})$. Thus one can view (3.6) as Kostant–Sekiguchi homeomorphisms for quiver varieties.

4. Real-symmetric homeomorphisms for Lie algebras

4.1. Main results

Let us return to the Cartan subgroup $T \subset G$, stable under η and θ, and maximally split with respect to η. Let $\mathfrak{t} \subset \mathfrak{g}$ denote its Lie algebra, let $\mathrm{W}_G = N_G(\mathfrak{t})/Z_G(\mathfrak{t})$ be the Weyl group, and introduce the affine quotient $\mathfrak{c} = \mathfrak{g}//G = \mathrm{Spec}(\mathcal{O}(\mathfrak{g})^G) \simeq \mathfrak{t}//\mathrm{W}_G = \mathrm{Spec}(\mathcal{O}(\mathfrak{t})^{\mathrm{W}_G})$. Let $\chi : \mathfrak{g} \to \mathfrak{c}$ be the natural map.

Next, let $\mathfrak{a} = \mathfrak{t} \cap \mathfrak{p}$ be the -1-eigenspace of θ, and write $\mathfrak{a}_{\mathbb{R}} = \mathfrak{a} \cap \mathfrak{g}_{\mathbb{R}}$ for the real form of $\mathfrak{a}$ with respect to η. Let $\mathrm{W} = N_{K_{\mathbb{R}}}(\mathfrak{a}_{\mathbb{R}})/Z_{K_{\mathbb{R}}}(\mathfrak{a}_{\mathbb{R}}) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ be the "little Weyl group," and introduce the affine quotient $\mathfrak{c}_{\mathfrak{p}} = \mathfrak{p}//K = \mathrm{Spec}(\mathcal{O}(\mathfrak{p})^K) \simeq \mathfrak{a}//\mathrm{W} = \mathrm{Spec}(\mathcal{O}(\mathfrak{a})^{\mathrm{W}})$. Let $\chi_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{c}_{\mathfrak{p}}$ denote the natural map.

Let $\mathfrak{c}_{\mathfrak{p},\mathbb{R}} \subset \mathfrak{c}$ be the image of the natural map $\mathfrak{a}_{\mathbb{R}} \to \mathfrak{c}$. Since the map $\mathfrak{a}_{\mathbb{R}} \to \mathfrak{c}$ is a polynomial map, by the Tarski–Seidenberg theorem, its image $\mathfrak{c}_{\mathfrak{p},\mathbb{R}}$ is semialgebraic. For example, if $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}_2(\mathbb{R})$, then $\mathfrak{c} = \mathbb{C}$ and $\mathfrak{c}_{\mathfrak{p},\mathbb{R}} = \mathbb{R}_{\leq 0}$.

Consider the following semialgebraic subsets of $\mathfrak{g}$, $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{p}$:

$$\mathfrak{g}' = \mathfrak{g} \times_{\mathfrak{c}} \mathfrak{c}_{\mathfrak{p},\mathbb{R}}, \qquad \mathfrak{g}'_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R}} \times_{\mathfrak{c}} \mathfrak{c}_{\mathfrak{p},\mathbb{R}}, \qquad \mathfrak{p}' = \mathfrak{p} \times_{\mathfrak{c}} \mathfrak{c}_{\mathfrak{p},\mathbb{R}}. \tag{4.1}$$

We have

$$\mathfrak{g}'_{\mathbb{R}} = \{x \in \mathfrak{g}_{\mathbb{R}} \mid \text{eigenvalues of } \mathrm{ad}_x \text{ are real}\}, \tag{4.2}$$

$$\mathfrak{p}' = \{x \in \mathfrak{p} \mid \text{eigenvalues of } \mathrm{ad}_x \text{ are real}\}. \tag{4.3}$$

Note that G, $G_{\mathbb{R}}$, and K naturally act on $\mathfrak{g}'$, $\mathfrak{g}'_{\mathbb{R}}$, and $\mathfrak{p}'$, respectively, and the actions are along the fibers of the natural projections

$$\mathfrak{g}' \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}}, \qquad \mathfrak{g}'_{\mathbb{R}} \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}}, \qquad \mathfrak{p}' \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}}. \tag{4.4}$$

THEOREM 4.1
Suppose that all simple factors of the complex reductive Lie algebra $\mathfrak{g}$ are of classical type. There is a $K_{\mathbb{R}}$-equivariant homeomorphism

$$\mathfrak{g}'_{\mathbb{R}} \xrightarrow{\sim} \mathfrak{p}' \tag{4.5}$$

compatible with the natural projections to $\mathfrak{c}_{\mathfrak{p},\mathbb{R}}$. Furthermore, the homeomorphism restricts to a real analytic isomorphism between individual $G_{\mathbb{R}}$-orbits and K-orbits.

We deduce the theorem above from the following.

THEOREM 4.2
Suppose that all simple factors of the complex reductive Lie algebra $\mathfrak{g}$ are of classical type. There is a continuous one-parameter family of maps

$$\alpha_s : \mathfrak{g}' \longrightarrow \mathfrak{g}', \quad s \in \mathbb{R},$$

satisfying the following.
(1) α_s^2 *is the identity, for all* $s \in \mathbb{R}$.
(2) *At* $s = 0$, *we have* $\alpha_0(M) = \eta(M)$.
(3) *At* $s = 1$, *we have* $\alpha_1(M) = -\theta(M)$.
(4) α_s *is* $K_{\mathbb{R}}$*-equivariant and takes a* G*-orbit real analytically to a* G*-orbit.*
(5) *We have* $\chi_{\mathfrak{g}'} \circ \alpha_a = \chi_{\mathfrak{g}'} : \mathfrak{g}' \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}}$, *where* $\chi_{\mathfrak{g}'}$ *is the projection map in (4.4).*

4.2. Quiver varieties of type A and conjugacy classes of matrices

Consider the type A_n quiver:

$$Q : \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet} \longrightarrow \cdots \longrightarrow \overset{n-2}{\bullet} \longrightarrow \overset{n-1}{\bullet} \longrightarrow \overset{n}{\bullet}$$

Let $\mathbf{v} = (n, n-1, \ldots, 2, 1) \in \mathbb{Z}^n_{\geq 0}$ and $\mathbf{w} = (n, 0, \ldots, 0, 0) \in \mathbb{Z}^n_{\geq 0}$. Consider the unitary quaternionic representation $\mathbf{M}(\mathbf{v}, \mathbf{w})$ of $H_u = \prod_{k=1}^n \mathrm{U}(k)$ in Section 3.1. A vector in $\mathbf{M}(\mathbf{v}, \mathbf{w})$ can be represented as a diagram:

$$\begin{array}{l} \mathbb{C}^n \\ y \big\uparrow \big\downarrow x \\ \mathbb{C}^n \underset{Y}{\overset{X}{\rightleftarrows}} \mathbb{C}^{n-1} \underset{Y}{\overset{X}{\rightleftarrows}} \mathbb{C}^{n-2} \underset{Y}{\overset{X}{\rightleftarrows}} \cdots \underset{Y}{\overset{X}{\rightleftarrows}} \mathbb{C}^3 \underset{Y}{\overset{X}{\rightleftarrows}} \mathbb{C}^2 \underset{Y}{\overset{Y}{\rightleftarrows}} \mathbb{C}^1 \end{array} \tag{4.6}$$

Let $\mathfrak{M}_{Z_{\mathbb{C}}} = \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(-Z_{\mathbb{C}})/H_u \to Z_{\mathbb{C}}$ be the family of quiver varieties associated to $\mathbf{M}(\mathbf{v}, \mathbf{w})$.

Denote $\mathfrak{g}_n = \mathfrak{gl}_n(\mathbb{C})$, let $\mathfrak{t}_n \subset \mathfrak{g}_n$ be the subspace of diagonal matrices, let $\mathfrak{c}_n = \mathfrak{g}_n//\mathrm{GL}_n(\mathbb{C})$, and let $\chi_n : \mathfrak{g}_n \to \mathfrak{c}_n$ be the Chevalley map. We will fix an identification $\mathfrak{c}_n = \mathbb{C}^n$ so that the map $\chi_n : \mathfrak{g}_n \to \mathfrak{c}_n = \mathbb{C}^n$ is given by $\chi_n(M) = (c_1, \ldots, c_n)$, where $T^n + c_1 T^{n-1} + \cdots + c_n$ is the characteristic polynomial of M. Consider the maps

$$\tilde{\phi}_{n,\mathbb{C}} : \mathfrak{M}_{Z_{\mathbb{C}}} \to \mathfrak{g}_n \times \mathfrak{t}_n, \qquad [X, Y, x, y] \to (yx, \zeta_{\mathbb{C}}), \tag{4.7}$$

$$\iota_{n,\mathbb{C}} : Z_{\mathbb{C}} \to \mathfrak{t}_n, \qquad \zeta_{\mathbb{C}} \to (c_1, \ldots, c_n), \tag{4.8}$$

where $\zeta_{\mathbb{C}} = (\zeta_1, \ldots, \zeta_n)$ is the image of $[X, Y, x, y] \in \mathfrak{M}_{Z_{\mathbb{C}}}$ under the projection map $\chi_{Z_{\mathbb{C}}} : \mathfrak{M}_{Z_{\mathbb{C}}} \to Z_{\mathbb{C}}$ and $c_i = \zeta_1 + \cdots + \zeta_i$, $1 \le i \le n$. Note that the map $\tilde{\phi}_{n,\mathbb{C}}$ intertwines the $(\mathrm{GL}_n(\mathbb{C}) \times \mathbb{R}^\times)$-action on $\mathfrak{M}_{Z_{\mathbb{C}}}$ with the one on $\mathfrak{g}_n \times \mathfrak{t}_n$ given by $(g, a)(M, t) = (gMg^{-1}, a^2 t)$.

PROPOSITION 4.3

Let $\pi_{Z_{\mathbb{C}}} : \mathfrak{M}_{(\zeta_{\mathbb{R}}, Z_{\mathbb{C}})} \to \mathfrak{M}_{Z_{\mathbb{C}}}$ be the map in (2.3), and let $\mathfrak{M}'_{Z_{\mathbb{C}}} \subset \mathfrak{M}_{Z_{\mathbb{C}}}$ be its image. Assume that $\zeta = (\zeta_{\mathbb{R}}, 0)$ is generic in the sense of [25, Definition 2.10].

(1) *The fiber $\mathfrak{M}'_{\zeta_{\mathbb{C}}}$ of the projection $\mathfrak{M}'_{Z_{\mathbb{C}}} \to Z_{\mathbb{C}}$ over $\zeta_{\mathbb{C}} \in Z_{\mathbb{C}}$ is a union of strata.*

(2) *$\mathfrak{M}'_{Z_{\mathbb{C}}}$ is connected and invariant under the $(\mathrm{GL}_n(\mathbb{C}) \times \mathbb{R}^\times)$-action.*

(3) *The map $\tilde{\phi}_{n,\mathbb{C}}$ (4.7) restricts to a $(\mathrm{GL}_n(\mathbb{C}) \times \mathbb{R}^\times)$-equivariant isomorphism*

$$\phi_{n,\mathbb{C}} : \mathfrak{M}'_{Z_{\mathbb{C}}} \simeq \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n$$

of complex algebraic varieties making the following diagram commute:

$$\begin{array}{ccc} \mathfrak{M}'_{Z_{\mathbb{C}}} & \xrightarrow{\phi_{n,\mathbb{C}}} & \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n \\ \downarrow & & \downarrow \\ Z_{\mathbb{C}} & \xrightarrow{\iota_{n,\mathbb{C}}} & \mathfrak{t}_n \end{array}$$

Furthermore, the map $\phi_{n,\mathbb{C}}$ induces stratified isomorphisms between individual fibers of the projections $\mathfrak{M}'_{Z_{\mathbb{C}}} \to Z_{\mathbb{C}}$ and $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n \to \mathfrak{t}_n$. Here we equip the fibers of $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n \to \mathfrak{t}_n$ with the $\mathrm{GL}_n(\mathbb{C})$-orbit stratification.

Proof

Part (1) follows from [25, Corollary 6.11]. We prove parts (2) and (3). Since each stratum $\mathfrak{M}_{\zeta,(L)}$ is invariant under the $(\mathrm{GL}_n(\mathbb{C}) \times \mathbb{R}^\times)$-action, part (1) implies that $\mathfrak{M}'_{Z_\mathbb{C}}$ also has this property. Moreover, since the $\mathbb{R}^\times$-action on $\mathfrak{M}'_{Z_\mathbb{C}}$ is a contracting action with a unique fixed point, $\mathfrak{M}'_{Z_\mathbb{C}}$ is connected. By the result of Mirkovic and Vybornov [24, Theorem 6.1], which is a generalization of the earlier results of Kraft and Procesi [19] and Nakajima [25], the map $\phi_{n,\mathbb{C}} : \mathfrak{M}'_{Z_\mathbb{C}} \to \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n$ induces isomorphisms between individual fibers of the projections $\mathfrak{M}'_{Z_\mathbb{C}} \to Z_\mathbb{C}$, $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n \to \mathfrak{t}_n$, and hence is a bijection.[10] We claim that $\phi_{n,\mathbb{C}}$ is in fact an isomorphism of algebraic varieties. Since $\mathfrak{M}'_{Z_\mathbb{C}}$ is connected, by Zariski's main theorem, it suffices to show that $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n$ is normal.[11] For this, we observe that the input varieties $\mathfrak{g}_n$, $\mathfrak{c}_n$, $\mathfrak{t}_n$ to the fiber product $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n$ are smooth and the morphisms $\mathfrak{g}_n \to \mathfrak{c}_n \leftarrow \mathfrak{t}_n$ are flat, thus $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n$ is a complete intersection, and hence Cohen–Macaulay. On the other hand, since the restriction $(\mathfrak{g}_n)^{\mathrm{reg}} \to \mathfrak{c}_n$ of the Chevalley map to the regular locus $(\mathfrak{g}_n)^{\mathrm{reg}}$ is smooth and $\mathfrak{g}_n \setminus (\mathfrak{g}_n)^{\mathrm{reg}}$ is of codimension three (see, e.g., [16]), we conclude that $(\mathfrak{g}_n)^{\mathrm{reg}} \times_{\mathfrak{c}_n} \mathfrak{t}_n$ is smooth and $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n \setminus (\mathfrak{g}_n)^{\mathrm{reg}} \times_{\mathfrak{c}_n} \mathfrak{t}_n$ is of codimension three.[12] Thus $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n$ is Cohen–Macaulay and smooth in codimension one, and hence normal.

We claim that $\phi_{n,\mathbb{C}}$ maps each stratum $\mathfrak{M}_{\zeta_\mathbb{C},(L)}$ isomorphically to a $\mathrm{GL}_n(\mathbb{C})$-orbit. For this we observe that there are only finitely many $\mathrm{GL}_n(\mathbb{C})$-orbits on the fibers of $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n \to \mathfrak{t}_n$ and the closure of any nonclosed orbit is singular. Since each stratum $\mathfrak{M}_{\zeta_\mathbb{C},(L)}$ is smooth and connected, it follows that $\phi_{n,\mathbb{C}}(\mathfrak{M}_{\zeta_\mathbb{C},(L)})$ is a single $\mathrm{GL}_n(\mathbb{C})$-orbit. The claim follows and the proofs of (2) and (3) are complete. □

4.3. Reflection functors

Let $\mathrm{C} = (\mathrm{C}_{kl})_{1\leq k,l\leq n}$ be the Cartan matrix of type A_n. Identify $Z_\mathbb{C}$ with $\mathbb{C}^n$, and consider the reflection representation of the Weyl group W on $Z_\mathbb{C}$. For any simple reflection $s_k, k \in [1,n]$ and $\zeta_\mathbb{C} = (\zeta_1,\ldots,\zeta_n) \in Z_\mathbb{C}$, we have $s_k(\zeta_\mathbb{C}) = \zeta'_\mathbb{C}$, where $\zeta'_l = \zeta_l - \mathrm{C}_{kl}\,\zeta_k$.

In [25], Nakajima associated to each $k \in [1,n]$ a certain hyper-Kähler isometry $S_k : \mathfrak{M}_{\zeta_\mathbb{C}}(\mathbf{v},\mathbf{w}) \simeq \mathfrak{M}_{\zeta'_\mathbb{C}}(\mathbf{v}',\mathbf{w})$ called the *reflection functor*. Here $\zeta'_\mathbb{C} = s_k(\zeta_\mathbb{C})$ and $\mathbf{v}'$ is given by $v'_k = v_k - \sum_l \mathrm{C}_{kl}\, v_l + w_k$, $v'_l = v_l$ if $l \neq k$ for $\mathbf{v} = (v_1,\ldots,v_n)$, $\mathbf{w} = (w_1,\ldots,w_n)$. Moreover, it is shown in [25] that the reflection functors S_k satisfy the Coxeter relations of the Weyl group.

[10] The fibers of the projection $\mathfrak{M}'_{Z_\mathbb{C}} \to Z_\mathbb{C}$ are introduced in [24, Section 2.3.3] and are denoted there by $\mathfrak{M}^c_1(v,d)$.

[11] Indeed, Zariski's main theorem implies that there exists a factorization $\phi_{n,\mathbb{C}} : \mathfrak{M}'_{Z_\mathbb{C}} \xrightarrow{j} Z \xrightarrow{f} \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n$, where j is an open immersion and f is finite. Since $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n$ is normal, f is an isomorphism. Thus $\phi_{n,\mathbb{C}}$ is an open immersion. Since $\phi_{n,\mathbb{C}}$ is surjective, $\phi_{n,\mathbb{C}}$ is an isomorphism.

[12] Here we use the fact that for a faithfully flat morphism $f : X \to Y$ ($f : \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n \to \mathfrak{g}_n$ in our case), we have $\mathrm{codim}_Y(Z) = \mathrm{codim}_X(f^{-1}(Z))$ for any closed subset Z of Y.

In the case $\mathbf{v} = (n, n-1, \ldots, 1)$ and $\mathbf{w} = (n, 0, \ldots, 0)$, a direct calculation shows that, for $k \in [2, n]$, we have $\mathbf{v} = \mathbf{v}'$ and hence $S_k : \mathfrak{M}_{\zeta_{\mathbb{C}}}(\mathbf{v}, \mathbf{w}) \simeq \mathfrak{M}_{\zeta'_{\mathbb{C}}}(\mathbf{v}, \mathbf{w})$. Let $\mathrm{S}_n \subset \mathrm{W}$ be the subgroup generated by the simple reflections $s_2, \ldots, s_n$. As $\zeta_{\mathbb{C}}$ varies over $Z_{\mathbb{C}}$, the reflection functors $S_2, \ldots, S_n$ define a S_n-action on $\mathfrak{M}_{Z_{\mathbb{C}}} = \bigcup_{\zeta_{\mathbb{C}} \in Z_{\mathbb{C}}} \mathfrak{M}_{\zeta_{\mathbb{C}}}(\mathbf{v}, \mathbf{w})$ such that the projection map $\mathfrak{M}_{Z_{\mathbb{C}}} \to Z_{\mathbb{C}}$ is S_n-equivariant.

LEMMA 4.4
The subset $\mathfrak{M}'_{Z_{\mathbb{C}}} \subset \mathfrak{M}_{Z_{\mathbb{C}}}$ is invariant under the S_n-action and the isomorphism $\phi_{n,\mathbb{C}}$: $\mathfrak{M}'_{Z_{\mathbb{C}}} \simeq \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n$ is S_n-equivariant.

Proof
We first claim that the map $\tilde{\phi}_{n,\mathbb{C}} : \mathfrak{M}_{Z_{\mathbb{C}}} \to \mathfrak{g}_n \times \mathfrak{t}_n$ (4.7) is S_n-equivariant. Recall the isomorphism $\iota_{n,\mathbb{C}} : Z_{\mathbb{C}} \simeq \mathfrak{t}_n$ in (4.8). A direct computation shows that $\iota_{n,\mathbb{C}}$ intertwines the action of s_k and the simple reflection $\sigma_{k-1,k} \in \mathrm{S}_n$ for $k \geq 2$. On the other hand, the formula for the reflection functors in [26, Section 3(i)] implies that, for any $[X, Y, x, y] \in \mathfrak{M}_{Z_{\mathbb{C}}}$, we have $S_k([X, Y, x, y]) = [\tilde{X}, \tilde{Y}, x, y]$ for $k \geq 2$. Altogether, we see that

$$\begin{aligned}\tilde{\phi}_{n,\mathbb{C}}\big(S_k\big([X, Y, x, y]\big)\big) &= \tilde{\phi}_{n,\mathbb{C}}\big([X', Y', x, y]\big) = \big(yx, \sigma_{k-1,k}\big(\iota_{n,\mathbb{C}}(\zeta_{\mathbb{C}})\big)\big) \\ &= \sigma_{k-1,k}\big(yx, \iota_{n,\mathbb{C}}(\zeta_{\mathbb{C}})\big) = \sigma_{k-1,k}\big(\tilde{\phi}_{n,\mathbb{C}}\big([X, Y, x, y]\big)\big).\end{aligned}$$

The claim follows. To complete the proof of the lemma, we need to show that $\mathfrak{M}'_{Z_{\mathbb{C}}}$ is S_n-invariant. Let $Z^0_{\mathbb{C}} \subset Z_{\mathbb{C}}$ (resp., $\mathfrak{t}^0_n \subset \mathfrak{t}_n$) be the open dense subset consisting of vectors with trivial stabilizers in S_n. The isomorphism $\phi_{n,\mathbb{C}}$ induces an isomorphism $\mathfrak{M}'_{Z^0_{\mathbb{Z}}} \simeq \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}^0_n$, where $\mathfrak{M}'_{Z^0_{\mathbb{Z}}} = \mathfrak{M}'_{Z_{\mathbb{C}}} \times_{Z_{\mathbb{C}}} Z^0_{\mathbb{C}}$, and it follows that $\mathfrak{M}'_{Z^0_{\mathbb{C}}}$ is open dense in $\mathfrak{M}'_{Z_{\mathbb{C}}}$ and the fibers of the projection $\mathfrak{M}'_{Z^0_{\mathbb{Z}}} \to Z^0_{\mathbb{C}}$ are smooth. According to [25, Theorem 4.1], the map $\pi_{Z_{\mathbb{C}}} : \mathfrak{M}_{\tilde{Z}_{\mathbb{C}}} \to \mathfrak{M}_{Z_{\mathbb{C}}}$ is an isomorphism over $\mathfrak{M}_{Z^0_{\mathbb{Z}}} = \mathfrak{M}_{Z_{\mathbb{Z}}} \times_{Z_{\mathbb{C}}} Z^0_{\mathbb{C}}$ and it follows that $\mathfrak{M}'_{Z^0_{\mathbb{Z}}} = \mathfrak{M}_{Z_{\mathbb{C}}} \times_{Z_{\mathbb{C}}} Z^0_{\mathbb{C}}$, which is S_n-invariant. On the other hand, the same argument as in the proof of [25, Theorem 4.1(1)] shows that the map $\pi_{Z_{\mathbb{C}}} : \mathfrak{M}_{(\zeta_{\mathbb{R}}, Z_{\mathbb{C}})} \to \mathfrak{M}_{Z_{\mathbb{C}}}$ is proper and hence its image $\mathfrak{M}'_{Z_{\mathbb{C}}} = \pi_{Z_{\mathbb{C}}}(\mathfrak{M}_{(\zeta_{\mathbb{R}}, Z_{\mathbb{C}})}) \subset \mathfrak{M}_{Z_{\mathbb{C}}}$ is a closed subset. Thus $\mathfrak{M}'_{Z_{\mathbb{C}}}$ is equal to the closure of $\mathfrak{M}'_{Z^0_{\mathbb{C}}}$ in $\mathfrak{M}_{Z_{\mathbb{C}}}$ and, as $\mathfrak{M}'_{Z^0_{\mathbb{C}}}$ is S_n-invariant, it implies that $\mathfrak{M}'_{Z_{\mathbb{C}}}$ is S_n-invariant. The lemma follows. □

4.4. *Involutions on the spaces of matrices with real eigenvalues*

Let $\mathfrak{M}'_Z \subset \mathfrak{M}_Z$ be the image of $\pi_Z : \mathfrak{M}_{\tilde{Z}} \to \mathfrak{M}_Z$, and consider $\mathfrak{g}_n \times_{\mathfrak{c}_n} i\mathfrak{t}_{n,\mathbb{R}}$, where $i\mathfrak{t}_{n,\mathbb{R}} \subset \mathfrak{t}_n$ is the $\mathbb{R}$-subspace consisting of diagonal matrices with pure imaginary entries. Then the isomorphisms $\phi_{n,\mathbb{C}}$ and $\iota_{n,\mathbb{C}}$ above restrict to isomorphisms

$$\mathfrak{M}'_Z \simeq \mathfrak{g}_n \times_{\mathfrak{c}_n} i\mathfrak{t}_{n,\mathbb{R}}, \qquad Z \simeq i\mathfrak{t}_{n,\mathbb{R}}. \tag{4.9}$$

Consider the family of involutions $\alpha_a : \mathfrak{M}_Z \to \mathfrak{M}_Z$ in Proposition 2.8 associated to the split conjugations in Section 3.1 and the map $\beta : \mathfrak{M}_Z \to \mathfrak{M}_Z$ in Proposition 2.9. Note that the action of $-1 \in \mathbb{R}^\times$ on $\mathfrak{M}_Z$ is trivial (it becomes the action of $1 = (-1)^2$ on $\mathfrak{g}_n \times_{\mathfrak{c}_n} i\,\mathfrak{t}_{n,\mathbb{R}}$), thus, by Proposition 2.9(1), β is an involution. Note also that the fibers of the projection $\mathfrak{M}'_Z \to Z$ are unions of strata (Proposition 4.3(1)), thus Proposition 2.8(3) and Proposition 2.9(3) imply that $\mathfrak{M}'_Z$ is invariant under the involutions α_s and β.

To relate $\mathfrak{M}'_Z$ with matrices with real eigenvalues, let us consider the composition

$$\phi_n : \mathfrak{M}'_Z \overset{(4.9)}{\simeq} \mathfrak{g}_n \times_{\mathfrak{c}_n} i\,\mathfrak{t}_{n,\mathbb{R}} \simeq \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}}, \tag{4.10}$$

$$\iota_n : Z \overset{(4.9)}{\simeq} i\,\mathfrak{t}_{n,\mathbb{R}} \simeq \mathfrak{t}_{n,\mathbb{R}}, \tag{4.11}$$

where the second isomorphisms are given by $\mathfrak{g}_n \times_{\mathfrak{c}_n} i\,\mathfrak{t}_{n,\mathbb{R}} \simeq \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}}, (x, v) \to (ix, iv)$ and $i\,\mathfrak{t}_{n,\mathbb{R}} \to \mathfrak{t}_{n,\mathbb{R}},\ v \to iv$. Note that the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{M}'_Z & \xrightarrow{\phi_n} & \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}} \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\iota_n} & \mathfrak{t}_{n,\mathbb{R}} \end{array} \tag{4.12}$$

where the vertical arrows are the natural projections.

Now the isomorphism $\phi_n : \mathfrak{M}'_Z \simeq \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}}$ gives rise to involutions on $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}}$:

$$\tilde{\alpha}_{n,s} = \phi_n \circ \alpha_s \circ \phi_n^{-1} : \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}} \to \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}}, \quad s \in \mathbb{R}, \tag{4.13}$$

$$\tilde{\beta}_n = \phi_n \circ \beta \circ \phi_n^{-1} : \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}} \to \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}}. \tag{4.14}$$

LEMMA 4.5

(1) *The involution $\tilde{\beta}_n$ is given by $\tilde{\beta}_n(M, v) = (-M^t, -v)$. In particular, $\tilde{\beta}_n$ commutes with the action of the symmetric group S_n on $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}}$.*

(2) *The involution $\tilde{\alpha}_{n,s}$ commutes with the action of the symmetric group S_n on $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}}$.*

Proof

Let $(M, v) \in \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}}$. Choose $[X, Y, x, y] \in \mathfrak{M}'_{\zeta_\mathbb{C}}$ such that

$$\phi_n\big([X, Y, x, y]\big) = i\big(yx, \iota_n(\zeta_\mathbb{C})\big) = (M, v).$$

According to Proposition 2.9, we have

$$\beta\big([X,Y,x,y]\big)=\phi(j)\circ\eta_Z\big([X,Y,x,y]\big)=[-\bar{Y}^\dagger,\bar{X}^\dagger,-\bar{y}^\dagger,\bar{x}^\dagger]\in\mathfrak{M}'_{-\zeta_{\mathbb{C}}}.$$

It follows that

$$\tilde{\beta}_n\big((M,v)\big)=\phi_n\big([-\bar{Y}^\dagger,\bar{X}^\dagger,-\bar{y}^\dagger,\bar{x}^\dagger]\big)=i\big((\bar{x}^\dagger)(-\bar{y}^\dagger),-\iota_n(\zeta_{\mathbb{C}})\big)=(-M^t,-v).$$

Part (1) follows.

According to Proposition 2.8, we have

$$\begin{aligned}\alpha_s\big([X,Y,x,y]\big)&=\big(\cos(s\pi/2)\phi(i)+\sin(s\pi/2)\phi(k)\big)\circ\eta_Z\big([X,Y,x,y]\big)\\&=[X',Y',x',y']\in\mathfrak{M}'_{\zeta_{\mathbb{C}}},\end{aligned}$$

where

$$x'=i\cos(s\pi/2)\bar{x}-i\sin(s\pi/2)\bar{y}^\dagger,\qquad y'=i\cos(s\pi/2)\bar{y}+i\sin(s\pi/2)\bar{x}^\dagger.$$

On the other hand, we have $S_k([X,Y,x,y])=[\tilde{X},\tilde{Y},x,y]$. Thus

$$\begin{aligned}\alpha_s\circ S_k\big([X,Y,x,y]\big)&=\big[(\tilde{X})',(\tilde{Y})',x',y'\big],\\ S_k\circ\alpha_s\big([X,Y,x,y]\big)&=\big[(\tilde{X'}),(\tilde{Y'}),x',y'\big].\end{aligned}\tag{4.15}$$

Since ϕ_n commutes with the S_n-action (see Lemma 4.4), we obtain

$$\begin{aligned}\tilde{\alpha}_{n,s}\circ S_k\big((M,v)\big)&=\tilde{\alpha}_{n,s}\circ S_k\circ\phi_n\big([X,Y,x,y]\big)\\&=\phi_n\circ\alpha_s\circ S_k\big([X,Y,x,y]\big)=i\big(y'x',s_k(v)\big),\\ S_k\circ\tilde{\alpha}_{n,s}\big((M,v)\big)&=S_k\circ\alpha_a\circ\phi_n\big([X,Y,x,y]\big)\\&=\phi_n\circ S_k\circ\alpha_s\big([X,Y,x,y]\big)=i\big(y'x',s_k(v)\big).\end{aligned}$$

Part (2) follows. The proof is complete. □

Let $\mathfrak{c}_{n,\mathbb{R}}\subset\mathfrak{c}_n$ be the image of the map $\mathfrak{t}_{n,\mathbb{R}}\to\mathfrak{c}_n$, and let $\mathfrak{g}'_n=\mathfrak{g}_n\times_{\mathfrak{c}_n}\mathfrak{c}_{n,\mathbb{R}}\subset\mathfrak{g}_n$. Note that both $\mathfrak{c}_{n,\mathbb{R}}$ and $\mathfrak{g}'_n$ are semialgebraic sets. We have

$$\mathfrak{g}'_n=\{x\in\mathfrak{g}_n\mid\text{eigenvalues of }x\text{ are real}\}.\tag{4.16}$$

Since the natural map $\mathfrak{g}_n\times_{\mathfrak{c}_n}\mathfrak{t}_{n,\mathbb{R}}\to\mathfrak{g}'_n=\mathfrak{g}_n\times_{\mathfrak{c}_n}\mathfrak{c}_{n,\mathbb{R}}$ is S_n-equivariant (where S_n acts trivially on $\mathfrak{g}'_n$), Lemma 4.5 implies that the involutions $\tilde{\alpha}_{n,s}$ and $\tilde{\beta}_n$ in (4.13) and (4.14) descend to a continuous family of involutions on $\mathfrak{g}'_n$:

$$\alpha_{n,s}:\mathfrak{g}'_n\to\mathfrak{g}'_n\tag{4.17}$$

compatible with projections to $\mathfrak{c}_{n,\mathbb{R}}$ and an involution

$$\beta_n : \mathfrak{g}'_n \to \mathfrak{g}'_n. \tag{4.18}$$

Moreover, β_n is equal to the restriction of the Cartan involution on $\mathfrak{g}_n$ to $\mathfrak{g}'_n$:

$$\beta_n(M) = -M^t. \tag{4.19}$$

THEOREM 4.6

The continuous one-parameter families of maps

$$\alpha_{n,s} : \mathfrak{g}'_n \longrightarrow \mathfrak{g}'_n, \quad s \in \mathbb{R},$$

satisfy the following.

(1) $\alpha_{n,s}^2$ *is equal to the identity map, for all* $s \in \mathbb{R}$.

(2) *At* $s = 0$, *we have* $\alpha_{n,0}(M) = \overline{M}$.

(3) *At* $s = 1$, *we have* $\alpha_{n,1}(M) = M^t$.

(4) $\alpha_{n,s}$ *is* $\mathrm{O}_n(\mathbb{R})$*-equivariant and takes each* $\mathrm{GL}_n(\mathbb{C})$*-orbit real analytically to itself.*

(5) $\alpha_{n,s}$ *commutes both with the Cartan involution* β_n *and with the projection map* $\mathfrak{g}'_n \to \mathfrak{c}_{n,\mathbb{R}}$, *for all* $s \in \mathbb{R}$.

Proof

Part (1) follows from the construction, and part (5) follows from the commutative diagram (4.12). Let $\phi'_n : \mathfrak{M}'_Z \overset{\phi_n}{\simeq} \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}} \to \mathfrak{g}'_n$, where the last map is given by $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}} \to \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{c}_{n,\mathbb{R}} = \mathfrak{g}'_n$. Let $M \in \mathfrak{g}'_n$. Choose $[X, Y, x, y] \in \mathfrak{M}'_Z$ such that

$$M = \phi'_n\big([X, Y, x, y]\big) = iyx.$$

We have

$$\alpha_0\big([X, Y, x, y]\big) = \phi(i) \circ \eta_Z\big([X, Y, x, y]\big) = [i\overline{X}, i\overline{Y}, i\bar{x}, i\bar{y}],$$

$$\alpha_1\big([X, Y, x, y]\big) = \phi(k) \circ \eta_Z\big([X, Y, x, y]\big) = [-i\bar{Y}^\dagger, i\bar{X}^\dagger, -i\bar{y}^\dagger, i\bar{x}^\dagger].$$

It follows that

$$\alpha_{n,0}\big([X, Y, x, y]\big) = \phi'_n\big([i\overline{X}, i\overline{Y}, i\bar{x}, i\bar{y}]\big) = i(-\bar{y}\bar{x}) = \overline{M},$$

$$\begin{aligned}\alpha_{n,1}\big([X, Y, x, y]\big) &= \phi'_n\big([-i\overline{Y}^\dagger, i\overline{X}^\dagger, -i\bar{y}^\dagger, i\bar{x}^\dagger]\big)\\ &= i(\bar{x}^\dagger\bar{y}^\dagger) = i\big(\overline{(yx)}^\dagger\big) = i(yx)^t = M^t.\end{aligned}$$

Parts (2) and (3) follow.

By Proposition 4.3(3), the isomorphism $\phi_n : \mathfrak{M}'_Z \to \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}}$ maps each stratum $\mathfrak{M}_{\zeta_\mathbb{C},(L)}$ real analytically to a $\mathrm{GL}_n(\mathbb{C})$-orbit. Now part (4) follows from the fact that the involution α_s on $\mathfrak{M}'_Z$ is $\mathrm{O}_n(\mathbb{R})$-equivariant and $\mathfrak{M}_{\zeta_{\mathbb{C}},(L)}$ is invariant under α_s. □

Let $\mathfrak{g}'_{n,\mathbb{R}}$ be the space of $n \times n$ real matrices with real eigenvalues. Let $\mathfrak{p}'_n$ be the space of $n \times n$ symmetric matrices with real eigenvalues. It is clear that $\mathfrak{g}'_{n,\mathbb{R}} = (\mathfrak{g}'_n)^{\alpha_0}$ and $\mathfrak{p}'_n = (\mathfrak{g}'_n)^{\alpha_1}$.

THEOREM 4.7
There is an $(\mathrm{O}_n(\mathbb{R}) \times \mathbb{R}^\times)$*-equivariant homeomorphism*

$$\mathfrak{g}'_{n,\mathbb{R}} \xrightarrow{\sim} \mathfrak{p}'_n \tag{4.20}$$

compatible with the natural projections to $\mathfrak{c}_{n,\mathbb{R}}$*. Furthermore, the homeomorphism restricts to a real analytic isomorphism between individual* $\mathrm{GL}_n(\mathbb{R})$*-orbits and* $\mathrm{O}_n(\mathbb{C})$*-orbits.*

Proof
Consider the Lusztig stratification of $\mathfrak{g}_n$. The stratum through g with a Jordan decomposition $g = s + u$ consists of all $\mathrm{GL}_n(\mathbb{C})$-orbits through $u + Z_r(\mathfrak{l})$, where $\mathfrak{l} = Z_{\mathfrak{g}_n}(s)$ is the centralizer of s in $\mathfrak{g}_n$ and $Z_r(\mathfrak{l}) = \{x \in Z(\mathfrak{l}) \mid Z_{\mathfrak{g}_n}(x) = \mathfrak{l}\}$ is the regular part of the center $Z(\mathfrak{l})$ of $\mathfrak{l}$. It is clear that the Lusztig stratification restricts to the orbit stratifications on the fibers of the Chevalley map $\chi_n : \mathfrak{g}_n \to \mathfrak{c}_n$ and a stratification on $\mathfrak{g}'_n = \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{c}_{n,\mathbb{R}}$.

Recall the $\mathrm{U}(n)$-invariant function $\|-\| : \mathfrak{M}_Z \to \mathbb{R}_{\geq 0}$ in Example 2.13. The restriction of $\|-\|$ along the closed embedding $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n \overset{\phi_n}{\simeq} \mathfrak{M}'_Z \subset \mathfrak{M}_Z$ gives rise to a function $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n \to \mathbb{R}_{\geq 0}$. Its average with respect to the S_n-action on $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n$ defines an S_n-invariant function $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n \to \mathbb{R}_{\geq 0}$ which descends to a function $\|-\|_{\mathfrak{g}'_n} : \mathfrak{g}'_n \to \mathbb{R}_{\geq 0}$. It follows from Theorem 4.6 and the construction of $\|-\|_{\mathfrak{g}'_n}$ that the function $\|-\|_{\mathfrak{g}'_n}$ together with the real analytic map $\mathfrak{g}'_n \to \mathfrak{c}_{n,\mathbb{R}}$ and the Lusztig stratification on $\mathfrak{g}'_n$ satisfies the assumption in Lemma 2.11, and hence we obtain a stratified $\mathrm{O}_n(\mathbb{R})$-equivariant homeomorphism

$$\mathfrak{g}'_{n,\mathbb{R}} \to \mathfrak{p}'_n \tag{4.21}$$

which is real analytic on each stratum and compatible with the maps to $\mathfrak{c}_{n,\mathbb{R}}$. Since each stratum in $\mathfrak{g}'_{n,\mathbb{R}}$ (resp., $\mathfrak{p}'_n$) is a finite union of $\mathrm{GL}_n(\mathbb{R})$-orbits (resp., $\mathrm{O}_n(\mathbb{C})$-orbits) and $\mathrm{O}_n(\mathbb{R})$-acts simply transitively on connected components of each orbit, it follows that the homeomorphism (4.21) restricts to a real analytic isomorphism between individual $\mathrm{GL}_n(\mathbb{R})$-orbits and $\mathrm{O}_n(\mathbb{C})$-orbits. □

4.5. Proof of Theorem 4.2

We shall deduce Theorem 4.2 from Theorem 4.6.

Let $\mathfrak{g}$ be a simple Lie algebra of classical type with real form $\mathfrak{g}_{\mathbb{R}}$. Recall the classification of real forms of classical types.

LEMMA 4.8 ([27, Section 4])

Here is the complete list of all possible quadruples $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{k}, \eta, \theta)$ *(up to isomorphism):*

(a) $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$*:*
  - (1) $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{k} = \mathfrak{so}_n(\mathbb{C})$, $\eta(g) = \bar{g}$, $\theta(g) = -g^t$;
  - (2) $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}_m(\mathbb{H})$, $\mathfrak{k} = \mathfrak{sp}_m(\mathbb{C})$, $\eta(g) = \operatorname{Ad} S_m(\bar{g})$, $\theta(g) = -\operatorname{Ad} S_m(g^t)$ *($n = 2m$);*
  - (3) $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}_{p,n-p}$, $\mathfrak{k} = (\mathfrak{gl}_p(\mathbb{C}) \oplus \mathfrak{gl}_{n-p}(\mathbb{C})) \cap \mathfrak{g}$, $\eta(g) = -\operatorname{Ad} I_{p,n-p}(\bar{g}^t)$, $\theta(g) = \operatorname{Ad} I_{p,n-p}(g)$.

(b) $\mathfrak{g} = \mathfrak{so}_n(\mathbb{C})$*:*
  - (1) $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}_{p,n-p}$, $\mathfrak{k} = \mathfrak{so}_p(\mathbb{C}) \oplus \mathfrak{so}_{n-p}(\mathbb{C})$, $\eta(g) = \operatorname{Ad} I_{p,n-p}(\bar{g})$, $\theta(g) = \operatorname{Ad} I_{p,n-p}(g)$;
  - (2) $\mathfrak{g}_{\mathbb{R}} = \mathfrak{u}^*_m(\mathbb{H})$, $\mathfrak{k} = \mathfrak{gl}_m(\mathbb{C})$, $\eta(g) = \operatorname{Ad} S_m(\bar{g})$, $\theta(g) = \operatorname{Ad} S_m(g)$ *($n = 2m$).*

(c) $\mathfrak{g} = \mathfrak{sp}_n(\mathbb{C})$, $n = 2m$*:*
  - (1) $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}_{2m}(\mathbb{R})$, $\mathfrak{k} = \mathfrak{gl}_m(\mathbb{C})$, $\eta(g) = \bar{g}$, $\theta(g) = \operatorname{Ad} S_m(g)$;
  - (2) $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}_{p,m-p}$, $\mathfrak{k} = \mathfrak{sp}_{2p}(\mathbb{C}) \oplus \mathfrak{sp}_{2m-2p}(\mathbb{C})$, $\eta(g) = -\operatorname{Ad} K_{p,m-p}(\bar{g}^t)$, $\theta(g) = \operatorname{Ad} K_{p,m-p}(g)$.

Here $S_m = \begin{pmatrix} 0 & -Id_m \\ Id_m & 0 \end{pmatrix}$, $I_{p,n-p} = \begin{pmatrix} Id_p & 0 \\ 0 & -Id_{n-p} \end{pmatrix}$, *and* $K_{p,m-p} = \begin{pmatrix} I_{p,m-p} & 0 \\ 0 & I_{p,m-p} \end{pmatrix}$.

Consider the commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\iota_{\mathfrak{g}}} & \mathfrak{g}_n \\ \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \chi_n} \\ \mathfrak{c} & \xrightarrow{\iota_{\mathfrak{c}}} & \mathfrak{c}_n \end{array} \tag{4.22}$$

where $\iota_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}_n$ is the natural embedding and $\iota_{\mathfrak{c}} : \mathfrak{c} = \mathfrak{g}//G \to \mathfrak{c}_n = \mathfrak{g}_n//\mathrm{GL}_n(\mathbb{C})$. We have the following explicit description of χ and $\iota_{\mathfrak{c}}$. For any $M \in \mathfrak{g}_n$, let

$$T^n + c_1 T^{n-1} + c_2 T^{n-1} + \cdots + c_n$$

be the characteristic polynomial of M. In the case $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, we have $c_1 = 0$ and one can identify $\mathfrak{c}$ with $\mathbb{C}^{n-1}$ so that

$$\chi(M) = (c_2, c_3, \ldots, c_n),$$

$$\iota_{\mathfrak{c}}(c_1, \ldots, c_n) = (0, c_2, \ldots, c_n).$$

In the case $\mathfrak{g} = \mathfrak{sp}_{\mathfrak{n}}(\mathbb{C})$ or $\mathfrak{so}_n(\mathbb{C})$, we have $c_1 = c_3 = \cdots = 0$ and one can choose an identification of $\mathfrak{c} = \mathbb{C}^{[n/2]}$ such that $\chi : \mathfrak{g} \to \mathfrak{c} = \mathbb{C}^{[n/2]}$ is given by

$$\chi(M) = (c_2, c_4, \ldots, c_n) \quad \text{if } \mathfrak{g} = \mathfrak{sp}_n(\mathbb{C}),$$
$$\chi(M) = (c_2, c_4, \ldots, c_{n-1}) \quad \text{if } \mathfrak{g} = \mathfrak{so}_n(\mathbb{C})\ n = 2m+1,$$
$$\chi(M) = (c_2, c_4, \ldots, c_{n-2}, \tilde{c}_n) \quad \text{if } \mathfrak{g} = \mathfrak{so}_n(\mathbb{C})\ n = 2m,$$

where $\tilde{c}_n = \mathrm{Pf}(M)$ is the Pfaffian of M satisfying $\mathrm{Pf}(M)^2 = \det(M) = c_n$, and the map $\iota_{\mathfrak{c}}$ is given by

$$\begin{aligned} \iota_{\mathfrak{c}}(c_2, c_4, \ldots, c_{n-1}) &= (0, c_2, 0, c_4, \ldots, 0, c_{n-1}) \quad \text{if } \mathfrak{g} = \mathfrak{sp}_n(\mathbb{C}), \\ \iota_{\mathfrak{c}}(c_2, c_4, \ldots, c_{n-1}) &= (0, c_2, 0, c_4, \ldots, 0, c_{n-1}) \\ &\qquad \text{if } \mathfrak{g} = \mathfrak{so}_n(\mathbb{C})\ n = 2m+1, l = m, \\ \iota_{\mathfrak{c}}(c_2, c_4, \ldots, c_{n-2}, \tilde{c}_n) &= (0, c_2, 0, c_4, \ldots, 0, \tilde{c}_n^2) \\ &\qquad \text{if } \mathfrak{g} = \mathfrak{so}_n(\mathbb{C})\ n = 2m, l = m. \end{aligned} \tag{4.23}$$

Remark 4.9
It follows that the map $\iota_{\mathfrak{c}} : \mathfrak{c} \to \mathfrak{c}_n$ is a closed embedding except in the case $\mathfrak{g} = \mathfrak{so}_n$, $n = 2m$.

Recall the semialgebraic sets $\mathfrak{c}_{\mathfrak{p},\mathbb{R}} \subset \mathfrak{c}$ and $\mathfrak{g}' = \mathfrak{g} \times_{\mathfrak{c}} \mathfrak{c}_{\mathfrak{p},\mathbb{R}} \subset \mathfrak{g}$ introduced in (4.1). Since for any $x \in \mathfrak{g}'$ the eigenvalues of ad_x are real, the embedding $\mathfrak{g}' \to \mathfrak{g}_n$ factors through $\mathfrak{g}' \to \mathfrak{g}_n' \subset \mathfrak{g}_n$ and diagram (4.22) restricts to a diagram

$$\begin{array}{ccc} \mathfrak{g}' & \xrightarrow{\iota_{\mathfrak{g}}} & \mathfrak{g}_n' \\ \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \chi_n} \\ \mathfrak{c}_{\mathfrak{p},\mathbb{R}} & \xrightarrow{\iota_{\mathfrak{c}}} & \mathfrak{c}_{n,\mathbb{R}} \end{array} \tag{4.24}$$

LEMMA 4.10
The compositions

$$\alpha_{n,s} \circ \beta_n \circ \theta : \mathfrak{g}_n' \to \mathfrak{g}_n', \quad s \in \mathbb{R}, \tag{4.25}$$

are involutions. Moreover, the subspace $\mathfrak{g}' \subset \mathfrak{g}_n'$ is invariant under the involutions in (4.25).

Proof
Note that Proposition 2.7 implies that $\alpha_{n,a} \circ \beta_n = \beta_n \circ \alpha_{n,a}$. On the other hand, since

the elements $S_m, I_{p,n-p}, K_{p,m-p} \in \mathrm{O}_n(\mathbb{R})$, Proposition 4.6(4) implies that the involutions $\operatorname{Ad} S_m$, $\operatorname{Ad} I_{p,n-p}$, and $\operatorname{Ad} K_{p,m-p}$ on $\mathfrak{g}_n'$ commute with both $\alpha_{n,s}$ and β_n. Now a direct computation, using the formula of θ in Lemma 4.8, shows that the compositions

$$\alpha_{n,s} \circ \beta_n \circ \theta : \mathfrak{g}_n' \to \mathfrak{g}_n', \quad s \in \mathbb{R},$$

are involutions.

Consider the involution σ on $\mathfrak{g}_n$ such that $(\mathfrak{g}_n)^\sigma = \mathfrak{g}$, that is, σ is given by $\sigma = \beta_n$ if $\mathfrak{g} = \mathfrak{so}_n(\mathbb{C})$ and $\sigma = \operatorname{Ad}(S_m) \circ \beta_n$ if $\mathfrak{g} = \mathfrak{sp}_n(\mathbb{C})$. Since the map (4.25) commutes with the involution σ, the σ-fixed points $(\mathfrak{g}_n')^\sigma$ is invariant under the map (4.25). The lemma follows. $\square$

The diagram (4.24) implies that $\mathfrak{g}'$ is equal to the base change

$$\mathfrak{g}' = (\mathfrak{g}_n')^\sigma \times_{\mathfrak{c}_{n,\mathbb{R}}} \iota_{\mathfrak{c}}(\mathfrak{c}_{\mathfrak{p},\mathbb{R}}) \tag{4.26}$$

of $(\mathfrak{g}_n')^\sigma$ to the subspace $\iota_{\mathfrak{c}}(\mathfrak{c}_{\mathfrak{p},\mathbb{R}}) \subset \mathfrak{c}_{n,\mathbb{R}}$ and hence the maps (4.25) restrict to a family of involutions

$$\alpha_s : \mathfrak{g}' \to \mathfrak{g}', \quad s \in \mathbb{R}. \tag{4.27}$$

LEMMA 4.11
The map α_s above satisfies properties (1)–(5) in Theorem 4.2.

Proof
Properties (1), (2), (3) of $\alpha_{n,s}$ in Theorem 4.6 immediately imply that α_s satisfies properties (1), (2), (3) in Theorem 4.2. Property (4) follows from the fact that the intersection of an adjoint orbit of $\mathfrak{g}_n$ with $\mathfrak{g}$ is a finite disjoint union of G-orbits and each G-orbit is a connected component. We now check property (5). We need to show that α_s preserves the fibers of $\chi : \mathfrak{g}' \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}}$. Assume that $\mathfrak{g}$ is not of type D. Then by Remark 4.9, the map $\mathfrak{c}_{\mathfrak{p},\mathbb{R}} \to \mathfrak{c}_{n,\mathbb{R}}$ is a closed embedding and property (5) follows from the one for $\alpha_{n,s}$. Assume that $\mathfrak{g} = \mathfrak{so}_n$, $n = 2m$. Then from the diagram (4.24), we see that the involution α_s preserves the fibers of $\iota_{\mathfrak{c}} \circ \chi : \mathfrak{g}' \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}} \to \mathfrak{c}_{n,\mathbb{R}}$. Let $c = (c_2, c_4, \ldots, \tilde{c}_n) \in \mathfrak{c}_{\mathfrak{p},\mathbb{R}}$. According to (4.23), if $\tilde{c}_n = 0$, then $\chi^{-1}(c) = (\iota_{\mathfrak{c}} \circ \chi)^{-1}(\iota_{\mathfrak{c}}(c))$, and if $\tilde{c}_n = 0$, then $(\iota_{\mathfrak{c}} \circ \chi)^{-1}(\iota_{\mathfrak{c}}(c)) = \chi^{-1}(c) \sqcup \chi^{-1}(c')$, where $c' = (c_2, c_4, \ldots, c_{n-2}, -\tilde{c}_n)$. In the first case, $\chi^{-1}(c)$ is equal to a fiber of $\iota_{\mathfrak{c}} \circ \chi$ and hence is invariant under α_s. Consider the second case. Since $\chi^{-1}(c)$ contains a vector in $\mathfrak{a}_{\mathbb{R}}$ and $\alpha_0(M) = M$ for $M \in \mathfrak{a}_{\mathbb{R}}$, it follows that $\alpha_0(\chi^{-1}(c)) = \chi^{-1}(c)$. Since $\chi^{-1}(c)$ and $\chi^{-1}(c')$ are connected components of $(\iota_{\mathfrak{c}} \circ \chi)^{-1}(\iota_{\mathfrak{c}}(c))$, we must have $\alpha_s(\chi^{-1}(c)) = \chi^{-1}(c)$ for all $s \in \mathbb{R}$. We are done. $\square$

This finishes the proof of Theorem 4.2.

4.6. Proof of Theorem 4.1

The proof is similar to that of Theorem 4.7. Since $\mathfrak{g} = (\mathfrak{g}_n)^\sigma$ is the fixed-point subspace of the involution σ on $\mathfrak{g}_n$ and the strata of the Lusztig stratification of $\mathfrak{g}_n$ are invariant under σ (the strata are invariant under the adjoint action and transpose), we obtain a stratification of $\mathfrak{g}$ given by the σ-fixed points of the strata. The stratification on $\mathfrak{g}$ induces a stratification on $\mathfrak{g}' = \mathfrak{g} \times_{\mathfrak{c}} \mathfrak{c}_{\mathfrak{p},\mathbb{R}}$. Moreover, the intersection of each stratum with the fibers of $\mathfrak{g}' \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}}$, if nonempty, is a finite union of G-orbits.

Let $\|-\|_{\mathfrak{g}'} : \mathfrak{g}' \to \mathbb{R}_{\geq 0}$ be the restriction of the function $\|-\|_{\mathfrak{g}_n'}$ to $\mathfrak{g}' \subset \mathfrak{g}_n'$ in the proof of Theorem 4.7. It follows from Theorem 4.2 and the construction of the function $\|-\|_{\mathfrak{g}_n'}$ that the real analytic map $\mathfrak{g}' \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}}$ together with the stratification of $\mathfrak{g}'$ described above and the function $\|-\|_{\mathfrak{g}'}$ satisfies the assumption in Lemma 2.11, and hence we obtain a stratified $K_\mathbb{R}$-equivariant homeomorphism

$$\mathfrak{g}_\mathbb{R}' = (\mathfrak{g}')^{\alpha_0} \to \mathfrak{p}' = (\mathfrak{g}')^{\alpha_1} \tag{4.28}$$

which is real analytic on each stratum and compatible with the maps to $\mathfrak{c}_{\mathfrak{p},\mathbb{R}}$. Since each stratum in $\mathfrak{g}_\mathbb{R}'$ (resp., $\mathfrak{p}'$)is a finite union of $G_\mathbb{R}$-orbits (resp., K-orbits) and $K_\mathbb{R}$-acts simply transitively on connected components of each orbits, it follows that the homeomorphism (4.28) restricts to a real analytic isomorphism between individual $G_\mathbb{R}$-orbits and K-orbits. The proof of Theorem 4.1 is complete.

5. Real and symmetric Springer theory

5.1. The real Grothendieck–Springer map

Let $A_\mathbb{R} = \exp \mathfrak{a}_\mathbb{R}$, which is a closed, connected, abelian, diagonalizable subgroup of $G_\mathbb{R}$. Let $(\Phi, \mathfrak{a}_\mathbb{R}^*)$ be the root system (possibly nonreduced) of $(\mathfrak{g}_\mathbb{R}, \mathfrak{a}_\mathbb{R})$. For each $\alpha \in \Phi$, we denote by $\mathfrak{g}_{\mathbb{R},\alpha} \subset \mathfrak{g}_\mathbb{R}$ the corresponding α-eigenspace. Choose a system of simple roots $\Delta = \{\alpha_1, \dots, \alpha_r\} \subset \Phi$, and denote by Φ^+ (resp., Φ^-) the corresponding set of positive roots (resp., negative roots). We have the decomposition

$$\mathfrak{g}_\mathbb{R} = \mathfrak{m}_\mathbb{R} \oplus \mathfrak{a}_\mathbb{R} \oplus \mathfrak{n}_\mathbb{R} \oplus \bar{\mathfrak{n}}_\mathbb{R},$$

where $\mathfrak{m}_\mathbb{R} = Z_{\mathfrak{k}_\mathbb{R}}(\mathfrak{a}_\mathbb{R})$, $\mathfrak{n}_\mathbb{R} = \oplus_{\alpha\in\Phi^+} \mathfrak{g}_{\mathbb{R},\alpha}$, $\bar{\mathfrak{n}}_\mathbb{R} = \oplus_{\alpha\in\Phi^-} \mathfrak{g}_{\mathbb{R},\alpha}$.

Let $\mathfrak{b}_\mathbb{R} = \mathfrak{m}_\mathbb{R} \oplus \mathfrak{a}_\mathbb{R} \oplus \mathfrak{n}_\mathbb{R}$ be a minimal parabolic subalgebra of $\mathfrak{g}_\mathbb{R}$, and we denote by $B_\mathbb{R} = M_\mathbb{R} A_\mathbb{R} N_\mathbb{R}$ the corresponding minimal parabolic subgroup, where $N_\mathbb{R} = \exp(\mathfrak{n}_\mathbb{R})$ and $M_\mathbb{R} = Z_{K_\mathbb{R}}(A_\mathbb{R})$ is a group (possible not connected) with Lie algebra $\mathfrak{m}_\mathbb{R}$. We write $F = \pi_0(M_\mathbb{R})$.

An element $x \in \mathfrak{g}_\mathbb{R}$ is called *semisimple* (resp., *nilpotent*) if ad_x is diagonalizable over $\mathbb{C}$ (resp., *nilpotent*). An element $x \in \mathfrak{g}_\mathbb{R}$ is called *hyperbolic* (resp., *elliptic*) if it is semisimple and the eigenvalues of ad_x are real (resp., purely imaginary). For any $x \in \mathfrak{g}_{\mathbb{R}}$, we have the Jordan decomposition $x = x_e + x_h + x_n$, where x_e is elliptic, x_h is hyperbolic, x_n is nilpotent, and the three elements x_e, x_h, x_n commute.

Consider the adjoint action of $G_{\mathbb{R}}$ on $\mathfrak{g}_{\mathbb{R}}$. By a result of Richardson and Slodowy [28], there exists a semialgebraic set $\mathfrak{g}_{\mathbb{R}}//G_{\mathbb{R}}$ whose points are the semisimple $G_{\mathbb{R}}$-orbits on $\mathfrak{g}_{\mathbb{R}}$. Furthermore, there are maps $\chi_{\mathbb{R}} : \mathfrak{g}_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}}//G_{\mathbb{R}}$ and $\mathfrak{g}_{\mathbb{R}}//G_{\mathbb{R}} \to \mathfrak{c}$ such that the restriction of the Chevalley map $\chi : \mathfrak{g} \to \mathfrak{c}$ to $\mathfrak{g}_{\mathbb{R}}$ factors as

$$\begin{array}{ccc} \mathfrak{g}_{\mathbb{R}} & \longrightarrow & \mathfrak{g} \\ \downarrow{\scriptstyle \chi_{\mathbb{R}}} & & \downarrow{\scriptstyle \chi} \\ \mathfrak{g}_{\mathbb{R}}//G_{\mathbb{R}} & \longrightarrow & \mathfrak{c} \end{array}$$

For any $x \in \mathfrak{g}_{\mathbb{R}}$, its image $\chi_{\mathbb{R}}(x)$ is given by the $G_{\mathbb{R}}$-orbit through the semisimple part $x_e + x_h$ of x. We also have an embedding $\mathfrak{a}_{\mathbb{R}}//\mathrm{W} \to \mathfrak{g}_{\mathbb{R}}//G_{\mathbb{R}}$, whose image consists of hyperbolic $G_{\mathbb{R}}$-orbits in $\mathfrak{g}_{\mathbb{R}}$, such that the restriction of $\chi_{\mathbb{R}}$ to $\mathfrak{a}_{\mathbb{R}}$ factors as $\mathfrak{a}_{\mathbb{R}} \to \mathfrak{a}_{\mathbb{R}}//\mathrm{W} \to \mathfrak{g}_{\mathbb{R}}//G_{\mathbb{R}}$.

Recall the subspace $\mathfrak{g}'_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ consisting of elements in $\mathfrak{g}_{\mathbb{R}}$ with hyperbolic semisimple parts (4.2). By a result of Kostant [17, Proposition 2.4], any hyperbolic element x in $\mathfrak{g}_{\mathbb{R}}$ is conjugate to an element in $\mathfrak{a}_{\mathbb{R}}$. Moreover, the set of elements in $\mathfrak{a}_{\mathbb{R}}$ which are conjugate to x is a single W-orbit. It follows that the embedding $\mathfrak{g}'_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}}$ factors through an isomorphism

$$\mathfrak{g}'_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R}} \times_{\mathfrak{g}_{\mathbb{R}}//G_{\mathbb{R}}} \mathfrak{a}_{\mathbb{R}}//\mathrm{W}. \tag{5.1}$$

In particular, we have a natural projection map

$$\mathfrak{g}'_{\mathbb{R}} \to \mathfrak{a}_{\mathbb{R}}//\mathrm{W} \tag{5.2}$$

such that the composition $\mathfrak{g}'_{\mathbb{R}} \to \mathfrak{a}_{\mathbb{R}}//\mathrm{W} \to \mathfrak{c}$ is equal to the map $\mathfrak{g}'_{\mathbb{R}} \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}} \subset \mathfrak{c}$ in (4.4).

Introduce the *real Grothendieck–Springer map*

$$\widetilde{\mathfrak{g}}_{\mathbb{R}} = G_{\mathbb{R}} \times^{B_{\mathbb{R}}} \mathfrak{b}_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}}, \quad (g, v) \to \mathrm{Ad}_g(v). \tag{5.3}$$

Note that unlike the complex case, the real Grothendieck–Springer map (5.3) in general is not surjective. Consider the base change of the real Grothendieck–Springer map to $\mathfrak{g}'_{\mathbb{R}}$:

$$\widetilde{\mathfrak{g}}'_{\mathbb{R}} \to \mathfrak{g}'_{\mathbb{R}}, \tag{5.4}$$

where $\widetilde{\mathfrak{g}}'_{\mathbb{R}} = \widetilde{\mathfrak{g}}_{\mathbb{R}} \times_{\mathfrak{g}_{\mathbb{R}}} \mathfrak{g}'_{\mathbb{R}}$. By [17, Proposition 2.5], an element $x \in \mathfrak{g}_{\mathbb{R}}$ is in $\mathfrak{g}'_{\mathbb{R}}$ if and only if it is conjugate to an element in $\mathfrak{a}_{\mathbb{R}} + \mathfrak{n}_{\mathbb{R}}$.[13] It follows that

$$\widetilde{\mathfrak{g}}'_{\mathbb{R}} = G_{\mathbb{R}} \times^{B_{\mathbb{R}}} (\mathfrak{a}_{\mathbb{R}} + \mathfrak{n}_{\mathbb{R}})$$

and the map (5.4) is surjective. Moreover, we have the commutative diagram

$$\begin{CD} \widetilde{\mathfrak{g}}'_{\mathbb{R}} @>>> \mathfrak{g}'_{\mathbb{R}} \\ @VVV @VVV \\ \mathfrak{a}_{\mathbb{R}} @>>> \mathfrak{a}_{\mathbb{R}}//\mathrm{W} \end{CD} \tag{5.5}$$

where the map $\widetilde{\mathfrak{g}}'_{\mathbb{R}} \to \mathfrak{a}_{\mathbb{R}}$ is given by $(g, v = v_a + v_n) \to v_a$.

Consider the *real Springer map*

$$\pi_{\mathbb{R}} : \widetilde{\mathcal{N}}_{\mathbb{R}} = G_{\mathbb{R}} \times^{B_{\mathbb{R}}} \mathfrak{n}_{\mathbb{R}} \to \mathcal{N}_{\mathbb{R}}. \tag{5.6}$$

We have the following Cartesian diagrams:

$$\begin{CD} \widetilde{\mathcal{N}}_{\mathbb{R}} @>>> \widetilde{\mathfrak{g}}'_{\mathbb{R}} @>>> \widetilde{\mathfrak{g}}_{\mathbb{R}} \\ @VVV @VVV @VVV \\ \mathcal{N}_{\mathbb{R}} @>>> \mathfrak{g}'_{\mathbb{R}} @>>> \mathfrak{g}_{\mathbb{R}} \end{CD} \tag{5.7}$$

Since (5.4) is surjective, the real Springer map (5.6) is also surjective.

LEMMA 5.1
We have a $K_{\mathbb{R}}$-equivariant isomorphism $\widetilde{\mathfrak{g}}'_{\mathbb{R}} \simeq \widetilde{\mathcal{N}}_{\mathbb{R}} \times \mathfrak{a}_{\mathbb{R}}$ commuting with projections to $\mathfrak{a}_{\mathbb{R}}$.

Proof
The Iwasawa decomposition $G_{\mathbb{R}} = K_{\mathbb{R}} A_{\mathbb{R}} N_{\mathbb{R}}$ gives rise to $K_{\mathbb{R}}$-equivariant isomorphism

$$\widetilde{\mathfrak{g}}'_{\mathbb{R}} = G_{\mathbb{R}} \times^{B_{\mathbb{R}}} (\mathfrak{a}_{\mathbb{R}} + \mathfrak{n}_{\mathbb{R}}) \simeq K_{\mathbb{R}} \times^{M_{\mathbb{R}}} (\mathfrak{a}_{\mathbb{R}} + \mathfrak{n}_{\mathbb{R}}).$$

Since $M_{\mathbb{R}}$ acts trivially on $\mathfrak{a}_{\mathbb{R}}$, we obtain

$$\widetilde{\mathfrak{g}}'_{\mathbb{R}} \simeq (K_{\mathbb{R}} \times^{M_{\mathbb{R}}} \mathfrak{n}_{\mathbb{R}}) \times \mathfrak{a}_{\mathbb{R}}.$$

[13] In [17, Proposition 2.5], the claim is proved in the setting of the adjoint action of $G_{\mathbb{R}}$ on $G_{\mathbb{R}}$. But the same argument works in the case of the adjoint action of $G_{\mathbb{R}}$ on the Lie algebra $\mathfrak{g}_{\mathbb{R}}$.

On the other hand, we have

$$\widetilde{\mathcal{N}}_{\mathbb{R}} = G_{\mathbb{R}} \times^{B_{\mathbb{R}}} \mathfrak{n}_{\mathbb{R}} \simeq K_{\mathbb{R}} \times^{M_{\mathbb{R}}} \mathfrak{n}_{\mathbb{R}}.$$

Combining the isomorphisms above, we get the desired $K_{\mathbb{R}}$-equivariant trivialization

$$\widetilde{\mathfrak{g}}'_{\mathbb{R}} \simeq \widetilde{\mathcal{N}}_{\mathbb{R}} \times \mathfrak{a}_{\mathbb{R}}$$

commuting with projections to $\mathfrak{a}_{\mathbb{R}}$. The proof is complete. □

5.2. Sheaves of real nearby cycles

Fix a point $a_{\mathbb{R}} \in \mathfrak{a}_{\mathbb{R}}^{\mathrm{rs}}$ with image $\xi_{\mathbb{R}} \in \mathfrak{a}_{\mathbb{R}}//\mathrm{W}$. Let $\mathcal{O}_{\xi_{\mathbb{R}}}$ be the semisimple $G_{\mathbb{R}}$-orbit through $a_{\mathbb{R}}$. The centralizer $Z_{G_{\mathbb{R}}}(a_{\mathbb{R}})$ is isomorphic to $M_{\mathbb{R}}A_{\mathbb{R}}$, and it follows that the $G_{\mathbb{R}}$-equivariant fundamental group of $\mathcal{O}_{\xi_{\mathbb{R}}}$ is isomorphic to $\pi_0(M_{\mathbb{R}}A_{\mathbb{R}}) \simeq \pi_0(M_{\mathbb{R}}) = F$. For any one-dimensional character χ of F, we denote by $\mathcal{L}_{\mathbb{R},\chi}$ the $G_{\mathbb{R}}$-equivariant perverse sheaf on $\mathcal{O}_{\xi_{\mathbb{R}}}$ corresponding to χ (note that $\mathcal{L}_{\mathbb{R},\chi}$ is a local system up to shifts).

Consider the path $\gamma_{\mathbb{R}} : [0,1] \to \mathfrak{a}_{\mathbb{R}}//\mathrm{W}$ given by $\gamma_{\mathbb{R}}(s) = s\xi_{\mathbb{R}}$, and denote by

$$\mathcal{Z}_{\mathbb{R}} = \mathfrak{g}'_{\mathbb{R}} \times_{\mathfrak{a}_{\mathbb{R}}//\mathrm{W}} [0,1]$$

the base change of $\mathfrak{g}'_{\mathbb{R}} \to \mathfrak{a}_{\mathbb{R}}//\mathrm{W}$ (5.2) along $\gamma_{\mathbb{R}}$. Note that $\gamma_{\mathbb{R}}$ is an embedding and hence $\mathcal{Z}_{\mathbb{R}}$ is closed subvariety of $\mathfrak{g}'_{\mathbb{R}}$. The fibers of the natural projection $f : \mathcal{Z}_{\mathbb{R}} \to [0,1]$ over 0 and 1 are isomorphic to the nilpotent cone $\mathcal{N}_{\mathbb{R}}$ in $\mathfrak{g}_{\mathbb{R}}$ and semisimple orbit $\mathcal{O}_{\xi_{\mathbb{R}}}$, respectively. Moreover, the $\mathbb{R}_{>0}$-action on $\mathfrak{g}'_{\mathbb{R}}$ induces a trivialization

$$\mathcal{O}_{\xi_{\mathbb{R}}} \times (0,1] \simeq \mathcal{Z}_{\mathbb{R}}|_{(0,1]} \quad (g,s) \to (sg,s). \tag{5.8}$$

Consider the following diagram

$$\begin{array}{ccccc} \mathcal{O}_{\xi_{\mathbb{R}}} \times (0,1] \simeq \mathcal{Z}_{\mathbb{R}}|_{(0,1]} & \xrightarrow{u} & \mathcal{Z}_{\mathbb{R}} & \xleftarrow{v} & \mathcal{N}_{\mathbb{R}} \\ \downarrow & & \downarrow f & & \downarrow \\ (0,1] & \longrightarrow & [0,1] & \longleftarrow & \{0\} \end{array} \tag{5.9}$$

where u and v are the natural embeddings. Note that all the varieties in the diagram above carry natural $G_{\mathbb{R}}$-actions and that all the maps between them are $G_{\mathbb{R}}$-equivariant. Define the nearby cycles functor:

$$\Psi_{\mathbb{R}} : D_{G_{\mathbb{R}}}(\mathcal{O}_{\xi_{\mathbb{R}}}) \to D_{G_{\mathbb{R}}}(\mathcal{N}_{\mathbb{R}}), \qquad \Psi_{\mathbb{R}}(\mathcal{F}) = \psi_f(\mathcal{F} \boxtimes \mathbb{C}_{(0,1]}) = v^* u_*(\mathcal{F} \boxtimes \mathbb{C}_{(0,1]}). \tag{5.10}$$

For any character χ of F, consider the sheaf of nearby cycles with coefficient $\mathcal{L}_\chi$,

$$\mathcal{F}_{\mathbb{R},\chi} = \Psi_{\mathbb{R}}(\mathcal{L}_\chi). \tag{5.11}$$

We will call $\Psi_{\mathbb{R}}$ the *real nearby cycles functor* and $\mathcal{F}_{\mathbb{R},\chi}$ the *sheaf of real nearby cycles*.

We shall give a formula of the nearby cycle sheaves in terms of the real Springer map $\pi_{\mathbb{R}} : \widetilde{\mathcal{N}}_{\mathbb{R}} \to \mathcal{N}_{\mathbb{R}}$ (5.6). Since the $G_{\mathbb{R}}$-equivariant fundamental group of $G_{\mathbb{R}}/B_{\mathbb{R}}$, and hence that of $\widetilde{\mathcal{N}}_{\mathbb{R}}$, is isomorphic to $\pi_0(B_{\mathbb{R}}) = \pi_0(M_{\mathbb{R}}) = F$, any character χ of F gives rise to a $G_{\mathbb{R}}$-equivariant perverse sheaf $\widetilde{\mathcal{L}}_\chi$ on $\widetilde{\mathcal{N}}_{\mathbb{R}}$. Introduce the *real Springer sheaf*

$$\mathcal{S}_{\mathbb{R},\chi} = (\pi_{\mathbb{R}})_! \widetilde{\mathcal{L}}_\chi. \tag{5.12}$$

THEOREM 5.2
We have $\mathcal{F}_{\mathbb{R},\chi} \simeq \mathcal{S}_{\mathbb{R},\chi}$.

Proof
Consider the path $\tilde{\gamma}_{\mathbb{R}} : [0,1] \to \mathfrak{a}_{\mathbb{R}}$ given by $\tilde{\gamma}_{\mathbb{R}}(s) = s(a_{\mathbb{R}})$, and let

$$\widetilde{\mathcal{Z}}_{\mathbb{R}} = \tilde{\mathfrak{g}}_{\mathbb{R}} \times_{\mathfrak{a}_{\mathbb{R}}} [0,1]$$

be the base change of the map $\tilde{\mathfrak{g}}_{\mathbb{R}} \to \mathfrak{a}_{\mathbb{R}}$ along the path $\tilde{\gamma}_{\mathbb{R}}$. The fibers of the projection $\tilde{f} : \widetilde{\mathcal{Z}}_{\mathbb{R}} \to [0,1]$ over 0 and 1 are given by $\widetilde{\mathcal{N}}_{\mathbb{R}}$ and $\mathcal{O}_{\xi_{\mathbb{R}}}$, respectively. Moreover, there is a trivialization

$$\widetilde{\mathcal{Z}}_{\mathbb{R}}|_{(0,1]} \simeq \mathcal{O}_{\xi_{\mathbb{R}}} \times (0,1], \qquad \big((g,v),s\big) \to \big(\mathrm{Ad}_g(s^{-1}v), s\big). \tag{5.13}$$

It follows that the real Grothendieck–Springer map $\widetilde{\mathfrak{g}}_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}}$ restricts to a map $\tau_{\mathbb{R}} : \widetilde{\mathcal{Z}}_{\mathbb{R}} \to \mathcal{Z}_{\mathbb{R}}$ which is an isomorphism over $\mathcal{Z}_{\mathbb{R}}|_{(0,1]}$. Consider the following commutative diagram

$$\begin{array}{ccccccc}
\mathcal{O}_{\xi_{\mathbb{R}}} \times (0,1] & \xrightarrow{(5.13)} & \widetilde{\mathcal{Z}}_{\mathbb{R}}|_{(0,1]} & \xrightarrow{\tilde{u}} & \widetilde{\mathcal{Z}}_{\mathbb{R}} & \xleftarrow{\tilde{v}} & \widetilde{\mathcal{N}}_{\mathbb{R}} \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \tau_{\mathbb{R}}} & & \downarrow{\scriptstyle \tau_{\mathbb{R}}} & & \downarrow{\scriptstyle \pi_{\mathbb{R}}} \\
\mathcal{O}_{\xi_{\mathbb{R}}} \times (0,1] & \xrightarrow{(5.8)} & \mathcal{Z}_{\mathbb{R}}|_{(0,1]} & \xrightarrow{u} & \mathcal{Z}_{\mathbb{R}} & \xleftarrow{v} & \mathcal{N}_{\mathbb{R}} \\
& \searrow & \downarrow & & \downarrow & & \downarrow \\
& & (0,1] & \longrightarrow & [0,1] & \longleftarrow & \{0\}
\end{array} \tag{5.14}$$

Consider the nearby cycles functor

$$\widetilde{\Psi}_{\mathbb{R}} : D_{G_{\mathbb{R}}}(\mathcal{O}_{\xi_{\mathbb{R}}}) \to D_{G_{\mathbb{R}}}(\widetilde{\mathcal{N}}_{\mathbb{R}}), \qquad \widetilde{\Psi}_{\mathbb{R}}(\mathcal{F}) = \tilde{v}^* \tilde{u}_*(\mathcal{F} \boxtimes \mathbb{C}_{(0,1]}).$$

Since $\tau_{\mathbb{R}}$ is proper and $(\tau_{\mathbb{R}})_!(\mathcal{F} \boxtimes \mathbb{C}_{(0,1]}) \simeq \mathcal{F} \boxtimes \mathbb{C}_{(0,1]}$, proper base change for nearby cycles functors implies that there is a canonical isomorphism

$$\begin{aligned}(\pi_{\mathbb{R}})_! \widetilde{\Psi}_{\mathbb{R}}(\mathcal{F}) = (\pi_{\mathbb{R}})_! \psi_{\tilde{f}}(\mathcal{F} \boxtimes \mathbb{C}_{(0,1]}) &\simeq \psi_f\big((\tau_{\mathbb{R}})_!(\mathcal{F} \boxtimes \mathbb{C}_{(0,1]})\big) \\ &\simeq \psi_f(\mathcal{F} \boxtimes \mathbb{C}_{(0,1]}) = \Psi_{\mathbb{R}}(\mathcal{F}). \end{aligned} \tag{5.15}$$

On the other hand, the $K_{\mathbb{R}}$-equivariant trivialization in Lemma 5.1 gives rise to a $K_{\mathbb{R}}$-equivariant isomorphism

$$\widetilde{\mathcal{Z}}_{\mathbb{R}} \simeq \widetilde{\mathcal{N}}_{\mathbb{R}} \times [0,1] \tag{5.16}$$

commuting with projections to $[0,1]$. In addition, there exists a $K_{\mathbb{R}}$-equivariant isomorphism $q : \widetilde{\mathcal{N}}_{\mathbb{R}} \simeq \mathcal{O}_{\xi_{\mathbb{R}}}$ such that $q^*\mathcal{L}_\chi \simeq \widetilde{\mathcal{L}}_\chi$ and making the following diagram commute:

$$\begin{array}{ccc} \mathcal{Z}_{\mathbb{R}}|_{(0,1]} & \xrightarrow{(5.16)} & \widetilde{\mathcal{N}}_{\mathbb{R}} \times (0,1] \\ \big\downarrow \mathrm{id} & & \big\downarrow q\times\mathrm{id} \\ \widetilde{\mathcal{Z}}_{\mathbb{R}}|_{(0,1]} & \xrightarrow{(5.13)} & \mathcal{O}_{\xi_{\mathbb{R}}} \times (0,1] \end{array}$$

It follows that

$$\widetilde{\Psi}_{\mathbb{R}}(\mathcal{L}_\chi) \simeq \psi_{\tilde{f}}(\mathcal{L}_\chi \boxtimes \mathbb{C}_{(0,1]}) \simeq \psi_{\tilde{f}}(\widetilde{\mathcal{L}}_\chi \boxtimes \mathbb{C}_{(0,1]}) \simeq \widetilde{\mathcal{L}}_\chi \tag{5.17}$$

as objects in $D_{K_{\mathbb{R}}}(\widetilde{\mathcal{N}}_{\mathbb{R}})$. Since $D_{G_{\mathbb{R}}}(\widetilde{\mathcal{N}}_{\mathbb{R}}) \subset D_{K_{\mathbb{R}}}(\widetilde{\mathcal{N}}_{\mathbb{R}})$ is a full subcategory (as $G_{\mathbb{R}}/K_{\mathbb{R}}$ is contractible), we conclude that

$$\mathcal{S}_{\mathbb{R},\chi} = (\pi_{\mathbb{R}})_! \widetilde{\mathcal{L}}_\chi \overset{(5.17)}{\simeq} (\pi_{\mathbb{R}})_! \widetilde{\Psi}_{\mathbb{R}}(\mathcal{L}_\chi) \overset{(5.15)}{\simeq} \Psi_{\mathbb{R}}(\mathcal{L}_\chi) = \mathcal{F}_{\mathbb{R},\chi} \in D_{G_{\mathbb{R}}}(\mathcal{N}_{\mathbb{R}}).$$

The proof is complete. □

5.3. *Sheaves of symmetric nearby cycles*

The discussion in the previous subsection has a counterpart in the setting of symmetric space. Recall the subspace $\mathfrak{p}' \subset \mathfrak{p}$ consisting of elements x in $\mathfrak{p}$ such that the eigenvalues of ad_x are real. In [18], Kostant and Rallis proved that for any such x, its semisimple part $x_s \in \mathfrak{p}$ is conjugate to an element in $\mathfrak{a}_{\mathbb{R}}$, moreover, the set of elements in $\mathfrak{a}_{\mathbb{R}}$ which are conjugate to x_s is a single W-orbit. It follows that the subspace $\mathfrak{p}'$ is equal to the base change

$$\mathfrak{p}' = \mathfrak{p} \times_{\mathfrak{c}_{\mathfrak{p}}} \mathfrak{a}_{\mathbb{R}}//\mathrm{W}$$

of $\chi_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{c}_{\mathfrak{p}}$ along $\mathfrak{a}_{\mathbb{R}}//\mathrm{W} \subset \mathfrak{c}_{\mathfrak{p}}$.

Consider $a_{\mathfrak{p}} \in \mathfrak{a}_{\mathbb{R}}^{\mathrm{rs}}$ with image $\xi_{\mathfrak{p}} \in \mathfrak{a}_{\mathbb{R}}//\mathrm{W}$. Let $\mathcal{O}_{\xi_{\mathfrak{p}}}$ be the K-orbit through $a_{\mathfrak{p}}$. We have $Z_K(a_{\mathfrak{p}}) = MA$, and it follows that the K-equivariant fundamental group of $\mathcal{O}_{\xi_{\mathfrak{p}}}$ is isomorphic to $\pi_0(Z_K(a_{\mathfrak{p}})) = \pi_0(MA) = \pi_0(M) = F$. For any character χ of F, we denote by $\mathcal{L}_{\mathfrak{p},\chi}$ the corresponding K-equivariant perverse sheaf on $\mathcal{O}_{\xi_{\mathfrak{p}}}$ (note that $\mathcal{L}_{\mathfrak{p},\chi}$ is a local system up to shifts). Consider the path $\gamma_{\mathfrak{p}} : [0,1] \to \mathfrak{a}_{\mathbb{R}}//\mathrm{W}$ given by $\gamma_{\mathfrak{p}}(s) = s\xi_{\mathfrak{p}}$, and define

$$\mathcal{Z}_{\mathfrak{p}} = \mathfrak{p}' \times_{\mathfrak{a}_{\mathbb{R}}//\mathrm{W}} [0,1].$$

The fibers of the natural projection $f_{\mathfrak{p}} : \mathcal{Z}_{\mathfrak{p}} \to [0,1]$ over 0 and 1 are isomorphic to the nilpotent cone $\mathcal{N}_{\mathfrak{p}}$ in $\mathfrak{p}$ and the K-orbit $\mathcal{O}_{\mathfrak{p}}$. Moreover, the $\mathbb{R}_{>0}$-action on $\mathfrak{p}'$ induces a trivialization

$$\mathcal{O}_{\xi_{\mathfrak{p}}} \times (0,1] \simeq \mathcal{Z}_{\mathfrak{p}}|_{(0,1]}, \qquad (g,s) \to (sg,s). \tag{5.18}$$

Consider the following diagram

$$\begin{array}{ccccc} \mathcal{O}_{\xi_{\mathfrak{p}}} \times (0,1] \simeq \mathcal{Z}_{\mathfrak{p}}|_{(0,1]} & \xrightarrow{u} & \mathcal{Z}_{\mathfrak{p}} & \xleftarrow{v} & \mathcal{N}_{\mathfrak{p}} \\ \downarrow & & \downarrow{\scriptstyle f_{\mathfrak{p}}} & & \downarrow \\ (0,1] & \longrightarrow & [0,1] & \longleftarrow & \{0\} \end{array} \tag{5.19}$$

where u and v are the natural embeddings. Note that all the varieties in the diagram above carry natural K-actions and all the maps between them are K-equivariant. Introduce the nearby cycles functor:

$$\Psi_{\mathfrak{p}} : D_K(\mathcal{O}_{\xi_{\mathfrak{p}}}) \to D_K(\mathcal{N}_{\mathfrak{p}}), \qquad \Psi_{\mathfrak{p}}(\mathcal{F}) = \psi_{f_{\mathfrak{p}}}(\mathcal{F} \boxtimes \mathbb{C}_{(0,1]}) = v^* u_*(\mathcal{F} \boxtimes \mathbb{C}_{(0,1]}). \tag{5.20}$$

For any character χ of F, consider the nearby cycles sheaf with coefficient $\mathcal{L}_{\mathfrak{p},\chi}$,

$$\mathcal{F}_{\mathfrak{p},\chi} = \Psi_{\mathfrak{p}}(\mathcal{L}_{\mathfrak{p},\chi}) \tag{5.21}$$

We will call $\Psi_{\mathfrak{p}}$ the *symmetric nearby cycles functor* and $\mathcal{F}_{\mathfrak{p},\chi}$ the *sheaf of symmetric nearby cycles*.

Recall the $K_{\mathbb{R}}$-equivariant stratified homeomorphism

$$\mathfrak{g}'_{\mathbb{R}} \simeq \mathfrak{p}' \tag{5.22}$$

in Theorem (4.1). Since the homeomorphism (5.22) commutes with projection to $\mathfrak{c}_{\mathfrak{p},\mathbb{R}}$ and the natural map $\mathfrak{a}_{\mathbb{R}}//\mathrm{W} \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}}$ is a finite map,[14] for any $\xi_{\mathbb{R}} \in \mathfrak{a}_{\mathbb{R}}^{\mathrm{rs}}/\mathrm{W}$ there exists

[14]Recall that $\mathfrak{c}_{\mathfrak{p},\mathbb{R}}$ is by definition the image of the map $\mathfrak{a}_{\mathbb{R}} \to \mathfrak{a}//\mathrm{W} = \mathfrak{c}_{\mathfrak{p}} \to \mathfrak{c}$. Since the latter map $\mathfrak{c}_{\mathfrak{p}} \to \mathfrak{c}$ is in general not a closed embedding, the map $\mathfrak{a}_{\mathbb{R}}//\mathrm{W} \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}}$ is not a closed embedding in general.

a unique $\xi_{\mathfrak{p}} \in \mathfrak{a}_{\mathbb{R}}^{\mathrm{rs}}//W$ such that (5.22) restricts to a $K_{\mathbb{R}}$-equivariant real analytic isomorphism between individual fibers

$$\mathcal{O}_{\xi_{\mathbb{R}}} \simeq \mathcal{O}_{\xi_{\mathfrak{p}}}.$$

Since (5.22) is $\mathbb{R}_{>0}$-equivariant, the isomorphism above and the trivializations (5.8) and (5.18) imply that (5.22) induces a $(K_{\mathbb{R}} \times \mathbb{R}_{>0})$-equivariant homeomorphism

$$\mathcal{Z}_{\mathbb{R}} \simeq \mathcal{Z}_{\mathfrak{p}} \tag{5.23}$$

commuting with projections to $[0,1]$. The homeomorphism above gives rise to a canonical commutative square of functors

$$\begin{array}{ccc} D_{G_{\mathbb{R}}}(\mathcal{O}_{\xi_{\mathbb{R}}}) & \xrightarrow{\Psi_{\mathbb{R}}} & D_{G_{\mathbb{R}}}(\mathcal{N}_{\mathbb{R}}) \\ \downarrow & & \downarrow \\ D_K(\mathcal{O}_{\xi_{\mathfrak{p}}}) & \xrightarrow{\Psi_{\mathfrak{p}}} & D_K(\mathcal{N}_{\mathfrak{p}}) \end{array} \tag{5.24}$$

where the upper and lower arrows are the real and symmetric nearby cycles, respectively, and the vertical arrows are the equivalences in (1.17). Since the equivalence $D_{G_{\mathbb{R}}}(\mathcal{O}_{\xi_{\mathbb{R}}}) \simeq D_K(\mathcal{O}_{\xi_{\mathfrak{p}}})$ maps $\mathcal{L}_{\mathbb{R},\chi}$ to $\mathcal{L}_{\mathfrak{p},\chi}$, the diagram (5.24) and Theorem 5.2 imply the following.

THEOREM 5.3
Assume that $\mathfrak{g}$ is of classical type. Under the equivalence $D_K(\mathcal{N}_{\mathfrak{p}}) \simeq D_{G_{\mathbb{R}}}(\mathcal{N}_{\mathbb{R}})$ in (1.17), the sheaf of symmetric nearby cycles $\mathcal{F}_{\mathfrak{p},\chi}$ becomes the sheaf of real nearby cycles $\mathcal{F}_{\mathbb{R},\chi}$, which is also isomorphic to the real Springer sheaf $\mathcal{S}_{\mathbb{R},\chi}$. In particular, the real Springer sheaf $\mathcal{S}_{\mathbb{R},\chi}$ is a perverse sheaf.

Remark 5.4
The fact that the real Springer sheaf is perverse implies that the real Springer map $\pi_{\mathbb{R}}$ is *cohomologically semismall*, that is, it is a proper stratified map $f : X \to Y$ with real even-dimensional stratum and smooth X, such that $f_*\mathbb{C}$ is a perverse sheaf up to shifts. If f is in fact complex algebraic, then we know that cohomologically semismall implies semismall (and vice versa). On the other hand, it is interesting to note that it is not the case when f is only real analytic; for example, the projection map $f : \mathbb{RP}^2 \to \mathrm{pt}$ from the real projective plane to a point is cohomologically semismall, but not semismall.

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References

[1] D. BARBASCH and M. R. SEPANSKI, *Closure ordering and the Kostant-Sekiguchi correspondence*, Proc. Amer. Math. Soc. **126** (1998), no. 1, 311–317. MR 1422847. DOI 10.1090/S0002-9939-98-04090-8. *(1631)*

[2] G. BELLAMY and T. SCHEDLER, *Symplectic resolutions of quiver varieties*, Selecta Math. (N.S.) **27** (2021), no. 3, art. ID 36. MR 4265941. DOI 10.1007/s00029-021-00647-0. *(1649)*

[3] R. BIELAWSKI, "Lie groups, Nahm's equations and hyper-Kähler manifolds" in *Algebraic Groups*, Universitätsverlag Göttingen, Göttingen, 2007, 1–17. MR 2309936. *(1631)*

[4] O. BIQUARD, *Extended correspondence of Kostant-Sekiguchi-Vergne*, preprint, 2009, http://webusers.imj-prg.fr/~olivier.biquard/eksv2.pdf. *(1631)*

[5] T.-H. CHEN and D. NADLER, *Real and symmetric quasi-maps*, preprint, arXiv:1805.06564v3 [math.RT]. *(1635)*

[6] ———, *Real Steinberg varieties and Hecke algebras at roots of unity*, in preparation. *(1634, 1635)*

[7] T.-H. CHEN, K. VILONEN, and T. XUE, *Springer correspondence for the split symmetric pair in type A*, Compos. Math. **154** (2018), no. 11, 2403–2425. MR 3867304. DOI 10.1112/s0010437x18007443. *(1634)*

[8] M. GRINBERG, *A generalization of Springer theory using nearby cycles*, Represent. Theory **2** (1998), 410–431. MR 1657203. DOI 10.1090/S1088-4165-98-00053-3. *(1633, 1634)*

[9] ———, *Errata and notes on the paper "A generalization of Springer theory using nearby cycles"*, preprint, arXiv:2002.11568v1 [math.AG]. *(1633)*

[10] M. GRINBERG, K. VILONEN, and T. XUE, *Nearby cycle sheaves for symmetric pairs*, Amer. J. Math. **145** (2023), no. 1, 1–63. MR 4545842. DOI 10.1353/ajm.2023.0000. *(1633)*

[11] A. HENDERSON, *Fourier transform, parabolic induction, and nilpotent orbits*, Transform. Groups **6** (2001), no. 4, 353–370. MR 1870052. DOI 10.1007/BF01237252. *(1625, 1631)*

[12] N. HITCHIN, *Hyper-Kähler manifolds*, Astérisque **206** (1992), 137–166, Séminaire Bourbaki 1991/1992, no. 748. MR 1206066. *(1636)*

[13] N. HITCHIN, A. KARLHEDE, U. LINDSTRÖM, and M. ROČEK, *Hyper-Kähler metrics and supersymmetry*, Comm. Math. Phys. **108** (1987), no. 4, 535–589. MR 0877637. *(1636)*

[14] G. KEMPF and L. NESS, "The length of vectors in representation spaces" in *Algebraic Geometry (Copenhagen, 1978)*, Lecture Notes in Math. **732**, Springer, Berlin, 1979, 233–243. MR 0555701. *(1628)*

[15] F. KOBAK and A. SWAN, *Classical nilpotent orbits as hyper-Kähler quotients*, Internat. J. Math. **7** (1996), no. 2, 193–210. MR 1382722. DOI 10.1142/S0129167X96000116. *(1625, 1629)*

[16] B. KOSTANT, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963), no. 3, 327–404. MR 0158024. DOI 10.2307/2373130. *(1631, 1653)*

[17] ———, *On convexity, the Weyl group and the Iwasawa decomposition*, Ann. Sci. Éc. Norm Supér. (4) **6** (1973) no. 4, 413–455 MR 0364552. *(1663, 1664)*

[18] B. KOSTANT and S. RALLIS, *Orbits and representations associated with symmetric spaces*, Amer. J. Math. **93** (1971), no. 3, 753–809. MR 0311837. DOI 10.2307/2373470. *(1633, 1667)*

[19] H. KRAFT and C. PROCESI, *Closures of conjugacy classes of matrices are normal*, Invent. Math. **53** (1979), no. 3, 227–247. MR 0549399. DOI 10.1007/BF01389764. *(1625, 1629, 1653)*

[20] P. B. KRONHEIMER, *Instantons and the geometry of the nilpotent variety*, J. Differential Geom. **32** (1990), no. 2, 473–490. MR 1072915. DOI 10.4310/jdg/1214445316. *(1630)*

[21] G. LUSZTIG and Z. YUN, *$\mathbf{Z}/m$-graded Lie algebras and perverse sheaves, I*, Represent. Theory **21** (2017), 277–321. MR 3697026. DOI 10.1090/ert/500. *(1633)*

[22] A. MAFFEI, *Quiver varieties of type A*, Comment. Math. Helv. **80** (2005), no. 1, 1–27. MR 2130242. DOI 10.4171/CMH/1. *(1625, 1629)*

[23] J. MATHER, *Notes on topological stability*, Bull. Amer. Math. Soc. (N.S.) **49** (2012), no. 4, 475–506 MR 2958928. DOI 10.1090/S0273-0979-2012-01383-6. *(1646)*

[24] I. MIRKOVIC and M. VYBORNOV, *Comparison of quiver varieties, loop Grassmannians and nilpotent cones in type A*, with appendix "Explicit isomorphism of quivers and loop Grassmannians" by V. Krylov, Adv. Math. **407** (2022), no. 108397. MR 4452665. DOI 10.1016/j.aim.2022.108397. *(1625, 1630, 1653)*

[25] H. NAKAJIMA, *Instantons on ALE space, quiver varieties, and Kac-Moody algebras*, Duke Math. J. **76** (1994), no. 2, 365–416. MR 1302318. DOI 10.1215/S0012-7094-94-07613-8. *(1625, 1629, 1647, 1652, 1653, 1654)*

[26] ———, *Reflection functors for quiver varieties and Weyl group action*, Math. Ann. **327** (2003), no. 4, 671–721. MR 2023313. DOI 10.1007/s00208-003-0467-0. *(1654)*

[27] A. L. ONISHCHIK and E. B. VINBERG, eds., *Lie Groups and Lie Algebras, III: Structure of Lie Groups and Lie Algebras*, Encyclopaedia Math. Sci., Springer, Berlin, 1994. *(1659)*

[28] R. W. RICHARDSON and P. J. SLODOWY, *Minimum vectors for real reductive algebraic groups*, J. Lond. Math. Soc. (2) **42** (1990), no. 3, 409–429. MR 1087217. DOI 10.1112/jlms/s2-42.3.409. *(1663)*

[29] W. SCHMID and K. VILONEN, *On the geometry of nilpotent orbits*, Asian J. Math. **3** (1999), no. 1, 233–274. MR 1701929. DOI 10.4310/AJM.1999.v3.n1.a10. *(1631)*

[30] J. SEKIGUCHI, *Remarks on nilpotent orbits of a symmetric pair*, J. Math. Soc. Japan **39** (1987), no. 1, 127–138. MR 0867991. DOI 10.2969/jmsj/03910127. *(1630)*

[31] M. VERGNE, *Instantons et correspondance de Kostant-Sekiguchi*, C. R. Math. Acad. Sci. Paris. **320** (1995), no. 8, 901–906. MR 1328708. *(1630, 1631)*

[32] K. VILONEN and T. XUE, *Character sheaves for symmetric pairs*, Trans. Amer. Math. Soc. **376** (2023), no. 2, 837–853. MR 4531663. DOI 10.1090/tran/8825. *(1634)*

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