REAL AND SYMMETRIC QUASI-MAPS

TSAO-HSIEN CHEN AND DAVID NADLER

ABSTRACT. Let $G_{\mathbb{R}}$ be a real reductive group and let X be the corresponding complex symmetric variety under the Cartan bijection. We construct a stratified homeomorphism between the based polynomial arc group of $G_{\mathbb{R}}$ and the based polynomial arc space of X. We also prove a multi-point version where we replace arcs by moduli spaces of quasi-maps from the projective line $\mathbb{P}^1$ to $G_{\mathbb{R}}$ and X. The key ingredients in the proof include: (i) a multi-point generalization of the "Gram-Schmidt" factorization of loop groups, and (ii) a nodal degeneration of moduli spaces of quasi-maps. As an application, we show that for the closures of real spherical orbits in the real affine Grassmannian, their singularities near the base point are locally homeomorphic to complex algebraic varieties.

CONTENTS

1. Introduction

1.1. **Overview.** Let G be a connected complex reductive Lie group with real form $G_\mathbb{R}$, with maximal compact $K_c \subset G_\mathbb{R}$, with complexification $K \subset G$, and corresponding complex symmetric variety $X = K\backslash G$. (See Section 2 for a collection of standard Lie theory constructions downstream of this starting point.)

This paper gives a direct geometric explanation for why much of the local spherical geometry of the real affine Grassmannian $\mathrm{Gr}_\mathbb{R} = G_\mathbb{R}((t))/G_\mathbb{R}[[t]]$ behaves like complex geometry (for example, satisfying the semisimplicity of the Decomposition Theorem as shown in [N]). For a spherical orbit $S^\lambda_\mathbb{R} = G_\mathbb{R}[[t]] \cdot \lambda \subset \mathrm{Gr}_\mathbb{R}$, we show the intersection of its closure $\overline{S^\lambda_\mathbb{R}} \subset \mathrm{Gr}_\mathbb{R}$ with the open cospherical orbit $T^0_\mathbb{R} = G_\mathbb{R}[t^{-1}] \cdot 1 \subset \mathrm{Gr}_\mathbb{R}$ is stratified homeomorphic to a complex algebraic variety. More precisely, we view the open cospherical orbit as the based polynomial arc group

$$T^0_\mathbb{R} \simeq G_\mathbb{R}[t^{-1}]_1 := \{g : \mathbb{P}^1_\mathbb{R} \setminus \{0\} \to G_\mathbb{R} \,|\, g(\infty) = 1\} \tag{1.1}$$

and construct a stratified homeomorphism

$$G_\mathbb{R}[t^{-1}]_1 \simeq X[t^{-1}]_1 \tag{1.2}$$

to the based polynomial arc space of the symmetric variety

$$X[t^{-1}]_1 := \{x : \mathbb{P}^1 \setminus \{0\} \to X \,|\, x(\infty) = 1\} \tag{1.3}$$

(In the definitions of (1.1) and (1.3), the subscript 1 conveys the condition $g(\infty) = 1$.)

Going further, our main theorem provides a generalization of the stratified homeomorphism (1.2) to spaces of maps where we allow poles at multiple points. Write $\mathcal{G}_\mathbb{R} \to \mathbb{R}^m$ for the group ind-scheme whose fiber over $(z_1, ..., z_m) \in \mathbb{R}^m$ is the group of maps $\gamma : \mathbb{P}^1_\mathbb{R} \setminus \{z_1, ..., z_m\} \to G_\mathbb{R}$ such that $\gamma(\infty) = e$. Similarly, write $\mathcal{X} \to \mathbb{R}^m$ for the ind-scheme whose fiber over $(z_1, ..., z_m) \in \mathbb{R}^m$ is the space of maps $\gamma : \mathbb{P}^1 \setminus \{z_1, ..., z_m\} \to X$ such that $\gamma(\infty) = e$.

Theorem 1.1 (See Theorem 6.9)**.** *There is a K_c-equivariant stratified homeomorphism*

$$\mathcal{G}_\mathbb{R} \simeq \mathcal{X} \tag{1.4}$$

over $\mathbb{R}^m$ that restricts to real analytic isomorphisms on spherical strata.

The construction of the stratified homeomorphism (1.4) involves two ingredients of independent interest.

(1) First, we establish a multi-point generalization of the "Gram-Schmidt" factorization of the polynomial loop group

$$G[t, t^{-1}] \simeq \Omega G_c \cdot G[t] \tag{1.5}$$

given by multiplication of the factors on the right. Here $\Omega G_c \subset G[t, t^{-1}]$ is the polynomial based loop group of the maximal compact $G_c \subset G$, i.e., the subgroup of maps that take the unit circle $S^1 \in \mathbb{A}^1_\mathbb{C}$ to $G_c \subset G$ and $1 \in S^1$ to $1 \in G$.

The following states one version of our multi-point generalization; the classical case (1.5) results from taking $S = \emptyset$, $S^+ = \{0\}$, $S^- = \{\infty\}$, and $s_0 = 1$.

Theorem 1.2 (See Theorem 3.3)**.** *Let $S^1 \subset \mathbb{A}^1_{\mathbb{C}}$ be the unit circle, $\mathbb{D}^+ \subset \mathbb{A}^1_{\mathbb{C}}$ the open unit disk, and $\mathbb{D}^- \subset \mathbb{P}^1_{\mathbb{C}}$ the complementary open disk. Let $\kappa(z) = \bar{z}^{-1}$ be the conjugation of $\mathbb{P}^1_{\mathbb{C}}$ with real form $S^1 \subset \mathbb{A}^1_{\mathbb{C}}$.*

Suppose given a finite set of points $S \subset S^1$, a finite set of points $S^+ \subset \mathbb{D}^+$, with conjugates $S^- = \kappa(S^+) \subset \mathbb{D}^-$. Set $\mathbb{S} = S \cup S^+ \cup S^-$ to be their union.

Let $G[\mathbb{P}^1 \setminus \mathbb{S}]$ be the group of polynomial maps $\mathbb{P}^1 \setminus \mathbb{S} \to G$, and $G[\mathbb{P}^1 \setminus \mathbb{S}]_c \subset G[\mathbb{P}^1 \setminus \mathbb{S}]$ the subgroup that takes $S^1 \setminus S$ to G_c.

Fix a point $s_0 \in S^1 \setminus S$, and let $G[\mathbb{P}^1 \setminus \mathbb{S}]_{c,s_0} \subset G[\mathbb{P}^1 \setminus \mathbb{S}]_c$ be the further subgroup that takes s_0 to 1.

Then multiplication provides a homeomorphism

$$G[\mathbb{P}^1 \setminus \mathbb{S}]_{c,s_0} \times G[\mathbb{P}^1 \setminus \{S^- \cup S\}] \xrightarrow{\sim} G[\mathbb{P}^1 \setminus \mathbb{S}] \tag{1.6}$$

After passing to the quotient by $G[t]$, one can interpret (1.5) as a homeomorphism $\mathrm{Gr} = G[t, t^{-1}]/G[t] \simeq \Omega G_c$ of the affine Grassmannian with the based loop group. More generally, one can similarly use (1.6) to obtain a homeomorphism of a Belinson-Drinfeld Grassmannian with the quotient of a mapping space. For our application, this is useful in it gives one access to the global mapping space via local data.

(2) Second, in Section 5, we study how maps, and more generally quasi-maps, behave under collisions of marked points (where the maps or quasi-maps are allowed to have poles) and degenerations of the domain curve itself.

Specifically, for our application, we study two such families: the bubbling of the domain curve $\mathbb{P}^1$ to the nodal curve $\mathbb{P}^1 \vee \mathbb{P}^1$ given in local coordinates by $xy = a^2$, for a parameter $a \in \mathbb{A}^1$; and the collision of distinct but Galois-conjugate marked points $x \neq \bar{x} \in \mathbb{P}^1$ of the fixed domain curve to a single Galois fixed-point $x_0 = \bar{x}_0 \in \mathbb{P}^1$.

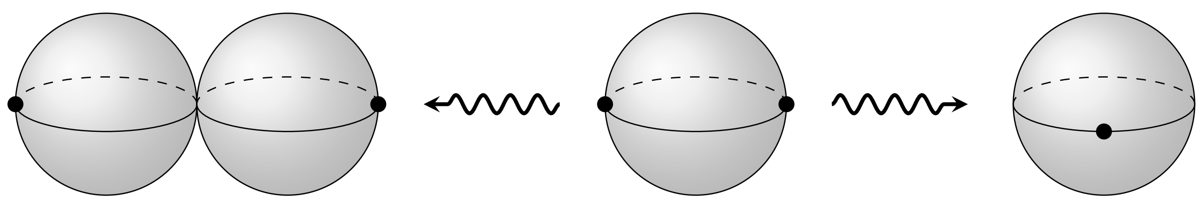

FIGURE 1. Degenerations of curve $\mathbb{P}^1$ and marked points S. Nodal degeneration of curve $\mathbb{P}^1$ to the left, collision of marked points S to the right.

A detailed study of quasi-maps under the collision of distinct but Galois-conjugate marked points can be found in [CN1]. Here let us focus on the key Galois-theoretic property of the bubbling degeneration that we exploit. We start at the parameter $a = 1$ with the conjugation of $\mathbb{P}^1$ given by $x \mapsto \bar{x}^{-1}$ in the local coordinate x. So its real form is the unit circle $S^1 \subset \mathbb{P}^1_{\mathbb{C}}$ and it exchanges the points 0 and ∞. Then we extend this over $a \neq 0$ by taking $x \mapsto \bar{a}^2\bar{x}^{-1}$. When we degenerate to $a = 0$, we find the conjugation $(x, y) \mapsto (\bar{y}, \bar{x})$ exchanging the two components of $\mathbb{P}^1 \vee \mathbb{P}^1$. Thus the Galois-action on the special fiber is almost free with the node the only real point. This allows us to describe Galois-equivariant maps, and more generally quasi-maps, on $\mathbb{P}^1 \vee \mathbb{P}^1$ in almost completely complex algebraic terms: the real structure constrains their values at the node alone.

Finally, within these families, we single out stratified subfamilies, defined over real parameters $a \in \mathbb{A}^1_{\mathbb{R}}$, that are locally constant with respect to the parameters. This provides homeomorphisms from their general fibers, which involve evident real structures, to their special fibers, where the real structures "disappear" due to the almost free nature of the Galois-action. The proof of Theorem 1.1 is an application of this idea: the general fiber is the group $\mathcal{G}_\mathbb{R}$ of real maps and the special fiber $\mathcal{X}$ is also a space of "real" maps but has a completely complex algebraic interpretation.

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2. Group data

We collect here notation and standard constructions used throughout the rest of the paper (for further discussion, see for example [CN1, N]).

2.1. **Real forms.** Let G be a connected complex reductive Lie group with Lie algebra $\mathfrak{g}$.

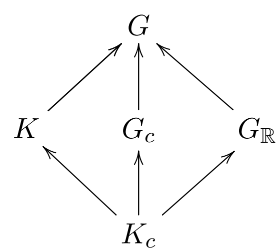
Let $G_\mathbb{R} \subset G$ be a real form, defined by a conjugation $\eta : G \to G$, with Lie algebra $\mathfrak{g}_\mathbb{R} \subset \mathfrak{g}$.

Choose a Cartan conjugation $\delta : G \to G$ that commutes with η, and let $G_c \subset G$ be the corresponding maximal compact subgroup with Lie algebra $\mathfrak{g}_c \subset \mathfrak{g}$.

Introduce the involution $\theta = \delta \circ \eta : G \to G$, and let $K \subset G$ be the fixed subgroup of θ.

One can organize the above groups into the diagram:

(2.1)



Here K_c is the fixed subgroup of θ, δ, and η together (or any two of the three) and the maximal compact subgroup of $G_\mathbb{R}$ with complexification K.

Fix a maximal δ-stable split torus $A_\mathbb{R} \subset G_\mathbb{R}$ with Lie algebra $\mathfrak{a}_\mathbb{R} \subset \mathfrak{g}_\mathbb{R}$, and complexification $A \subset G$ with Lie algebra $\mathfrak{a} \subset \mathfrak{g}$. Fix a maximal δ-stable torus $T_\mathbb{R} \subset G_\mathbb{R}$ containing $A_\mathbb{R}$ with Lie algebra $\mathfrak{t}_\mathbb{R} \subset \mathfrak{g}_\mathbb{R}$, and complexification $T \subset G$ with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$.

Fix a Borel subgroup $B \subset G$ containing T with Lie algebra $\mathfrak{b} \subset \mathfrak{g}$, and unipotent radical $U \subset B$ with Lie algebra $\mathfrak{u} \subset \mathfrak{b}$. Let $H = B/U$ be the universal Cartan with Lie algebra $\mathfrak{h} = \mathfrak{b}/\mathfrak{u}$. Note the composition $T \to B \to H$ is an isomorphism.

Let $W = N_G(\mathfrak{t})/Z_G(\mathfrak{t}) = N_G(T)/T$ denote the Weyl group, and $W_0 = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ the "baby Weyl group".

Let $\Lambda_T = \mathrm{Hom}(\mathbb{G}_m, T)$ denote the coweight lattice of T with dominant coweights $\Lambda_T^+ \subset \Lambda_T$. Let $R_G \subset \Lambda_T$ denote the coroot lattice generated by the simple coroots $\Delta_G \subset R_G$ and let $R_G^+ = R_G \cap \Lambda_T^+$. Let $\pi_1(G)$ denote the fundamental group based at the identity $e \in G$, and recall the natural isomorphism $\Lambda_T/R_G \xrightarrow{\sim} \pi_1(G)$.

Similarly, let $\Lambda_A = \mathrm{Hom}(\mathbb{G}_m, A)$ denote the coweight lattice of A with dominant coweights $\Lambda_A^+ = \Lambda_A \cap \Lambda_T^+$.

Introduce the symmetric variety $X = K\backslash G$. The map $\pi : G \to G$, $\pi(g) = \theta(g)^{-1}g$ factors through X, and descends to an isomorphism $X \xrightarrow{\sim} G^0_{sym}$, where $G_{sym} = \{g \in G \,|\, \theta(g^{-1}) = g\}$, and $G^0_{sym} \subset G_{sym}$ denotes its neutral component. Note the inclusion $A \subset G^0_{sym}$ induces a map $\Lambda_A \to \pi_1(G_{sym}) \simeq \pi_1(X)$.

Consider the induced map $\pi_* : \pi_1(G) \to \pi_1(X)$. Define $\mathcal{L} \subset \Lambda_A$ to be the inverse image of $\pi_*(\pi_1(G)) \subset \pi_1(X)$ under the natural map $\Lambda_A \to \pi_1(X)$, and set $\mathcal{L}^+ = \mathcal{L} \cap \Lambda_T^+$. Note that $\mathcal{L} = \Lambda_A$ if and only if K is connected.

Example 2.1 (Complex groups)**.** An important special case is when the real form is itself a complex group. To avoid potential confusion in this case, we will use the alternative notation $\mathrm{sw} : G \times G \to G \times G$, $\mathrm{sw}(g, h) = (h, g)$ for the "swap" Cartan involution with fixed-point subgroup the diagonal $G \subset G \times G$. The corresponding conjugation is the composition $\mathrm{sw}_\delta = \mathrm{sw} \circ (\delta \times \delta)$, and the compact conjugation is the product $\delta \times \delta$. We identify the corresponding symmetric variety with the group $G \simeq G\backslash(G \times G)$ via inclusion to the left factor, which coincides with the subspace $G \simeq \{(g, h) \in G \times G \,|\, h = g^{-1}\}$.

Introduce the affine quotient

$$\mathfrak{c} = \mathfrak{t}/\!/W = \mathrm{Spec}(\mathcal{O}(\mathfrak{t})^W) \simeq \mathfrak{g}/\!/G = \mathrm{Spec}(\mathcal{O}(\mathfrak{g})^G) \tag{2.2}$$

The conjugation η descends to a real structure on $\mathfrak{c}$, and we denote its real points by $\mathfrak{c}_\mathbb{R}$. We also have the real characteristic polynomial map $\mathfrak{g}_\mathbb{R} \to \mathfrak{c}_\mathbb{R}$ from real matrices to their unordered eigenvalues.

2.2. **Loop groups.** Let $\mathcal{K} = \mathbb{C}((z))$ denote the field of Laurent series, $\mathcal{O} = \mathbb{C}[[z]]$ the ring of power series, and $\mathcal{O}^- = \mathbb{C}[z^{-1}]$ the ring of Laurent poles. Let $\mathrm{Gr} = G(\mathcal{K})/G(\mathcal{O})$ be the affine Grassmannian of G.[1] For any $g \in G(\mathcal{K})$ we denote by $[g] \in \mathrm{Gr}$ the corresponding coset.

Via the natural inclusion $\Lambda_T \to T(\mathcal{K}) \subset G(\mathcal{K})$, any coweight $\lambda \in \Lambda_T$ defines a point $[\lambda] \in \mathrm{Gr}$. For a dominant coweight $\lambda \in \Lambda_T^+$, introduce the $G(\mathcal{O})$-orbit $S^\lambda = G(\mathcal{O}) \cdot [\lambda] \subset \mathrm{Gr}$ (spherical stratum) and $G(\mathcal{O}^-)$-orbit $T^\lambda = G(\mathcal{O}^-) \cdot [\lambda] \subset \mathrm{Gr}$ (cospherical stratum). Recall the disjoint union decompositions

$$\mathrm{Gr} = \textstyle\coprod_{\lambda \in \Lambda_T^+} S^\lambda \qquad \mathrm{Gr} = \coprod_{\lambda \in \Lambda_T^+} T^\lambda \tag{2.3}$$

We can similarly repeat the above constructions over the real numbers. Let $\mathcal{K}_\mathbb{R} = \mathbb{R}((z))$ denote the field of real Laurent series, $\mathcal{O}_\mathbb{R} = \mathbb{R}[[z]]$ the ring of real power series, and $\mathcal{O}^- = \mathbb{R}[z^{-1}]$ the ring of real Laurent poles. Let $\mathrm{Gr}_\mathbb{R} = G_\mathbb{R}(\mathcal{K}_\mathbb{R})/G_\mathbb{R}(\mathcal{O}_\mathbb{R})$ be the real affine Grassmannian of the real form $G_\mathbb{R}$.

For a dominant coweight $\lambda \in \Lambda_A^+ = \Lambda_A \cap \Lambda_T^+$, set $S^\lambda_\mathbb{R} = G_\mathbb{R}(\mathcal{O}_\mathbb{R}) \cdot [\lambda] = S^\lambda \cap \mathrm{Gr}_\mathbb{R}$ (real spherical stratum) and $T^\lambda_\mathbb{R} = G_\mathbb{R}(\mathcal{O}^-_\mathbb{R}) \cdot [\lambda] = T^\lambda \cap \mathrm{Gr}_\mathbb{R}$ (real cospherical stratum). We have the disjoint union decompositions

$$\mathrm{Gr}_\mathbb{R} = \textstyle\coprod_{\lambda \in \Lambda_A^+} S^\lambda_\mathbb{R} \qquad \mathrm{Gr}_\mathbb{R} = \coprod_{\lambda \in \Lambda_A^+} T^\lambda_\mathbb{R} \tag{2.4}$$

[1]Our concerns in this paper will be exclusively topological, and we will ignore any non-reduced structure throughout.

Note that all of above constructions result from the natural conjugation $\eta_{\mathcal{K}} : G(\mathcal{K}) \to G(\mathcal{K})$, $\eta_{\mathcal{K}}(g(z)) = \eta(g(\bar{z}))$ with real form $G_{\mathbb{R}}(\mathcal{K}_{\mathbb{R}}) \subset G(\mathcal{K})$.

Now let us recall some parallel constructions where we work with the global curve $\mathbb{G}_m = \mathrm{Spec}(\mathbb{C}[z, z^{-1}]) = \mathbb{P}^1 \setminus \{0, \infty\}$ in place of the punctured disk $D^\times = \mathrm{Spec}(\mathcal{K})$, and similarly $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[z]) = \mathbb{P}^1 \setminus \{\infty\}$ in place of the disk $D = \mathrm{Spec}(\mathcal{O})$.

Introduce the polynomial loop group $LG = G(\mathbb{C}[z, z^{-1}]) \subset G(\mathcal{K})$ of maps $\mathbb{G}_m \to G$, and similarly, the polynomial arc group $L_+G = G(\mathbb{C}[z]) \subset G(\mathcal{O})$ of maps $\mathbb{A}^1 \to G$. Recall the natural map is an isomorphism $LG/L_+G \xrightarrow{\sim} G(\mathcal{K})/G(\mathcal{O}) = \mathrm{Gr}$.

We have an additional compact conjugation $\kappa : \mathbb{G}_m \to \mathbb{G}_m$, $\kappa(z) = \bar{z}^{-1}$ with real points the unit circle $S^1 \subset \mathbb{C}^\times$.[2] Note that κ does not preserve the punctured disk $D^\times \subset \mathbb{G}_m$.

We extend the conjugations $\eta, \delta : G \to G$ to conjugations $\eta, \delta : LG \to LG$ by the formulas $\eta(g)(z) = \eta(g(\kappa(z))), \delta(g)(z) = \delta(g(\kappa(z)))$. The corresponding real forms $LG_{\mathbb{R}}, LG_c \subset LG$ consist of maps $g : \mathbb{G}_m \to G$ that take $S^1 \subset \mathbb{G}_m$ respectively to $G_{\mathbb{R}}, G_c \subset G$. We denote by $\Omega G_{\mathbb{R}} \subset LG_{\mathbb{R}}$, $\Omega G_c \subset LG_c$ the based subgroups of maps that take $1 \in S^1$ to the identity $e \in G$. Note that multiplication gives isomorphisms $\Omega G_{\mathbb{R}} \times G_{\mathbb{R}} \xrightarrow{\sim} LG_{\mathbb{R}}$, $\Omega G_c \times G_c \xrightarrow{\sim} LG_c$.

3. Factorization

In this section, we record a useful extension of a well-known loop group factorization.

Recall the "Gram-Schmidt" factorization

$$\Omega G_c \times L_+G \xrightarrow{\sim} LG \tag{3.1}$$

of the polynomial loop group $LG = G(\mathbb{C}[z, z^{-1}])$, its arc subgroup $L_+G = G(\mathbb{C}[z])$, and the based loop group $\Omega G_c \subset LG$. Note that (3.1) is equivalent to the fact that the ΩG_c-action on the base point $[e] \in \mathrm{Gr} \simeq LG/L_+G$ induces a homeomorphism

$$\Omega G_c \xrightarrow{\sim} \mathrm{Gr} \tag{3.2}$$

Consider the projective line $\mathbb{P}^1 = \mathrm{Proj}(\mathbb{C}[z_0, z_1])$. Using the local coordinate $z = z_1/z_0$, we will regard the complex points of $\mathbb{P}^1$ as the Riemann sphere $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$.

Given any finite set of points $\mathbb{S} \subset \mathbb{P}^1$, we will denote by $G[\mathbb{P}^1 \setminus \mathbb{S}]$ the group ind-scheme of maps $\mathbb{P}^1 \setminus \mathbb{S} \to G$. We will only be interested in the complex points of $G[\mathbb{P}^1 \setminus \mathbb{S}]$ equipped with the classical topology, and thus ignore any non-reduced structure.

For example, if $\mathbb{S} = \emptyset$, we have $G[\mathbb{P}^1] = G$; if $\mathbb{S} = \{\infty\}$, we have $G[\mathbb{P}^1 \setminus \{\infty\}] = L_+G$; and if $\mathbb{S} = \{0, \infty\}$, we have $G[\mathbb{P}^1 \setminus \{0, \infty\}] = LG$.

For a pair $s = (s_+, s_-)$ of points $s_+, s_- \in \mathbb{P}^1$, we will write $L_sG = G[\mathbb{P}^1 \setminus \{s_+, s_-\}]$. If we choose an isomorphism $\varphi : \mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1$, $\varphi(0) = s_+$, $\varphi(\infty) = s_-$, then we obtain an isomorphism

$$\varphi^* : L_sG \xrightarrow{\sim} LG \qquad \varphi^*(g) = g \circ \varphi \tag{3.3}$$

To uniquely prescribe φ, we can further require $\varphi(p) = q$, for some $p, q \in \mathbb{P}^1$ with $p \neq 0, \infty$, $q \neq s_+, s_-$.

Lemma 3.1. *Suppose $S^+, S^- \subset \mathbb{P}^1$ are disjoint, finite sets of points with S^- non-empty.*

[2] There is another conjugation $z \mapsto -\bar{z}^{-1}$ of the curve $\mathbb{G}_m$ with empty real points, but it will not appear in the developments of this paper.

For $\mathbb{S} = S^+ \cup S^-$, the natural map is an isomorphism

$$G[\mathbb{P}^1 \setminus \mathbb{S}]/G[\mathbb{P}^1 \setminus S^-] \xrightarrow{\sim} \prod_{s \in S^+} \mathrm{Gr}_s \tag{3.4}$$

where $\mathrm{Gr}_s := G(\mathcal{K}_s)/G(\mathcal{O}_s)$ denotes the affine Grassmannian at s (which is isomorphic to the affine Grassmannian Gr by the choice of a local coordinate).

Proof. Set $U^\pm = \mathbb{P}^1 \setminus S^\pm$. Since S^- is non-empty, the restriction $\mathcal{E}|_{U^-}$ of any G-bundle $\mathcal{E}$ on $\mathbb{P}^1$ may be trivialized. Thus the left hand side classifies a trivial G-bundle $\mathcal{E}_0^+$ on U^+, a G-bundle $\mathcal{E}^-$ on U^-, and an isomorphism between them over $U^+ \cap U^-$. This is equivalent to a G-bundle $\mathcal{E}$ on $\mathbb{P}^1$ with a trivialization over U^+. It is standard that this factorizes to give the right hand side. $\square$

Fix a non-zero real number a and consider the conjugation $\kappa_a : \mathbb{P}^1 \to \mathbb{P}^1$, $\kappa(z) = \frac{1}{a^2 \bar{z}}$ with real points being the circle $S^1_{a^{-1}} = \{|z| = |a^{-1}|\}$. Let $\mathbb{D}_a^+ = \{|z| < |a^{-1}|\}$ denote the open disk, and $\mathbb{D}_a^- = \{|z| > |a^{-1}|\}$ the complementary open disk.

If a finite set of points $\mathbb{S} \subset \mathbb{P}^1$ is invariant under κ_a, then the conjugation $\delta : G \to G$ induces a conjugation $\delta : G[\mathbb{P}^1 \setminus \mathbb{S}] \to G[\mathbb{P}^1 \setminus \mathbb{S}]$, $\delta(g)(z) = \delta(g(\kappa_a(z)))$. The corresponding real points, denoted by $G[\mathbb{P}^1 \setminus \mathbb{S}]_c$, consist of maps $g : \mathbb{P}^1 \setminus \mathbb{S} \to G$ that take $S^1 \setminus (S^1 \cap \mathbb{S})$ to the maximal compact $G_c \subset G$. For example, if $\mathbb{S} = \emptyset$, we have $G[\mathbb{P}^1]_c = G_c$; if $\mathbb{S} = \{\infty\}$, we have $G[\mathbb{P}^1 \setminus \{\infty\}]_c = G_c$.

For a pair $s = (s_+, s_-)$ of points $s_+ \in \mathbb{D}_a^+$, $s_- = \kappa_a(s_+) \in \mathbb{D}_a^-$, we write $L_s G_c = G[\mathbb{P}^1 \setminus \{s_+, s_-\}]_c$. Observe that the map $\varphi : \mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1$, $\varphi(0) = s_+$, $\varphi(\infty) = s_-$, $\varphi(a^{-1}) = a^{-1}$ commutes with the conjugation κ_a. Thus the isomorphism (3.3) restricts to an isomorphism on real points

$$\varphi^* : L_s G_c \xrightarrow{\sim} LG_c \tag{3.5}$$

For any $s_0 \in S^1$, we will write $\Omega_{s,s_0} G_c \subset L_s G_c$ for the based subgroup of those maps with $g(s_0) = e$.

For finite subsets $\mathbb{S} \subset S^1$, we have the following:

Lemma 3.2. *For any finite subset $\mathbb{S} \subset S^1$, any element of $G[\mathbb{P}^1 \setminus \mathbb{S}]_c$ is constant, and hence $G[\mathbb{P}^1 \setminus \mathbb{S}]_c = G_c$.*

Proof. Choose an embedding $G \subset \mathrm{GL}(N)$ so that $G_c \subset U(N)$. For any $g \in G[\mathbb{P}^1 \setminus \mathbb{S}]_c$, consider a matrix entry $g_{ij} : \mathbb{P}^1 \setminus \mathbb{S} \to \mathbb{A}^1$. The restriction $g|_{S^1 \setminus \mathbb{S}}$ lands in $G_c \subset U(N)$, hence the restricted matrix entry $g_{ij}|_{S^1 \setminus \mathbb{S}}$ is bounded. Thus g_{ij} is bounded, and hence constant. $\square$

Now we are ready to state the main result of this section. The special case when $S = \emptyset$ and $S^+ = \{0\}$, so that $S^- = \{\infty\}$, recovers the "Gram-Schmidt" factorization (3.1).

Theorem 3.3. *Suppose we are given a finite set of points $S \subset S^1$, and a finite set of points $S^+ \subset \mathbb{D}^+$, with conjugates $S^- = \kappa(S^+) \subset \mathbb{D}^-$. Set $\mathbb{S} = S \cup S^+ \cup S^-$ to be their union.*

Fix a point $s_0 \in S^1 \setminus S$, and consider the kernel of the evaluation

$$G[\mathbb{P}^1 \setminus \mathbb{S}]_{c,s_0} := \ker(\mathrm{ev}_{s_0} : G[\mathbb{P}^1 \setminus \mathbb{S}]_c \longrightarrow G_c) \tag{3.6}$$

Then we have:

1) Multiplication provides a homeomorphism

$$G[\mathbb{P}^1 \setminus \mathbb{S}]_{c,s_0} \times G[\mathbb{P}^1 \setminus \{S^- \cup S\}] \xrightarrow{\sim} G[\mathbb{P}^1 \setminus \mathbb{S}] \tag{3.7}$$

2) The natural action map provides a homeomorphism

$$G[\mathbb{P}^1 \setminus \mathbb{S}]_{c,s_0} \xrightarrow{\sim} G[\mathbb{P}^1 \setminus \mathbb{S}]/G[\mathbb{P}^1 \setminus \{S^- \cup S\}] \simeq \prod_{s \in S^+} \mathrm{Gr}_s \tag{3.8}$$

3) For any ordering $S^+ = \{s_1, \dots, s_k\}$, multiplication provides a homeomorphism

$$\Omega_{s_1,s_0}G_c \times \cdots \times \Omega_{s_k,s_0}G_c \xrightarrow{\sim} G[\mathbb{P}^1 \setminus \mathbb{S}]_{c,s_0} \tag{3.9}$$

Proof. Clearly 1) and 2) are equivalent. We will prove 2) and 3) simultaneously by considering the diagram

$$\Omega_{s_1,s_0}G_c \times \cdots \times \Omega_{s_k,s_0}G_c \longrightarrow G[\mathbb{P}^1 \setminus \mathbb{S}]_{c,s_0} \longrightarrow G[\mathbb{P}^1 \setminus \mathbb{S}]/G[\mathbb{P}^1 \setminus \{S^- \cup S\}] \simeq \prod_{s \in S^+} \mathrm{Gr}_s$$

Clearly the composite map $\Omega_{s_1,s_0}G_c \times \cdots \times \Omega_{s_k,s_0}G_c \to \prod_{s\in S^+} \mathrm{Gr}_s$ is a homeomorphism by (3.2). Thus it suffices to show the map $G[\mathbb{P}^1 \setminus \mathbb{S}]_{c,s_0} \to \prod_{s\in S^+} \mathrm{Gr}_s$ is injective, i.e., the $G[\mathbb{P}^1 \setminus \mathbb{S}]_{c,s_0}$-action on the base-point $[e] \in \prod_{s\in S^+} \mathrm{Gr}_s$ is free. Any element in the stabilizer must extend across S^+, hence also across S^-, and so lie in $G[\mathbb{P}^1 \setminus S]_{c,s_0}$. Thus by Lemma 3.2, any element in the stabilizer must be constant. Since its evaluation at s_0 is trivial, the element must in fact be trivial. □

4. Quasi-maps

4.1. **Definitions.** Let B be a smooth complex base-scheme.

Let $\pi : \mathcal{Z} \to B$ be a projective family of curves, with fibers denoted $\mathcal{Z}_b = \pi^{-1}(b)$. We do not assume the total space $\mathcal{Z}$ or fibers $\mathcal{Z}_b$ are smooth. See the next section for the specific setting used in this paper.

Let $\mathrm{Bun}_G(\mathcal{Z}/B)$ denote the moduli stack of a point $b \in B$ and a G-bundle $\mathcal{E}$ on the fiber $\mathcal{Z}_b$. More precisely, an S-point consists of an S-point $a : \mathrm{Spec}S \to B$ and a G-bundle $\mathcal{E}$ on the corresponding fiber product $\mathcal{Z} \times_B \mathrm{Spec}S$. Denote by $p : \mathrm{Bun}_G(\mathcal{Z}/B) \to B$ the evident projection with fibers $p^{-1}(b) = \mathrm{Bun}_G(\mathcal{Z}_b)$.

Let $\sigma = (\sigma_1, \dots, \sigma_n) : B \to \mathcal{Z}^n$ be an ordered n-tuple of sections of π. We allow the sections to intersect or coincide, but require $\sigma(b) \in \mathcal{Z}_b^n$ to be a smooth point, for all $b \in B$.

For an affine G-variety X, let $QM_{G,X}(\mathcal{Z}/B, \sigma)$ denote the ind-stack of quasi-maps classifying a point $b \in B$, a G-bundle $\mathcal{E}$ on the fiber $\mathcal{Z}_b$, and a section

$$s : \mathcal{Z}_b \setminus \{\sigma_1(b), \dots, \sigma_n(b)\} \longrightarrow X_{\mathcal{E}} \tag{4.1}$$

to the associated X-bundle over the complement of the points $\sigma_1(b), \dots, \sigma_n(b) \in \mathcal{Z}_b$. We have the evident forgetful maps

$$q : QM_{G,X}(\mathcal{Z}/B, \sigma) \longrightarrow \mathrm{Bun}_G(\mathcal{Z}/B) \longrightarrow B \tag{4.2}$$

with fibers $q^{-1}(b) = QM_{G,X}(\mathcal{Z}_b, \sigma(b))$.

Let $\xi : B \to \mathcal{Z}$ be another section of π such that $\xi(b) \neq \sigma_i(b)$, for all $b \in B$, and $i = 1, \dots, n$.

Let $QM_{G,X}(\mathcal{Z}/B, \sigma, \xi)$ denote the ind-stack of rigidified quasimaps classifying quadruples $(b, \mathcal{E}, s, \iota)$ where $(b, \mathcal{E}, s)$ is a quasi-map as above and $\iota : \mathcal{E}_K|_{\xi(b)} \simeq K$ is a trivialization, where $\mathcal{E}_K$ is the K-reduction of $\mathcal{E}$ on $\mathcal{Z}_b \setminus \{\sigma_1(b), \dots, \sigma_n(b)\}$ given by the section s. We have the evident forgetful maps

$$r : QM_{G,X}(\mathcal{Z}/B, \sigma, \xi) \longrightarrow \mathrm{Bun}_G(\mathcal{Z}/B) \longrightarrow B \tag{4.3}$$

with fibers $r^{-1}(b) = QM_{G,X}(\mathcal{Z}_b, \sigma(b), \xi(b))$.

Suppose given conjugations $c_{\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z}$ and $c_B : B \to B$ such that $\pi \circ c_{\mathcal{Z}} = c_B \circ \pi$.

The conjugation η of G induces a conjugation $\eta : \mathrm{Bun}_G(\mathcal{Z}/B) \to \mathrm{Bun}_G(\mathcal{Z}/B)$ given by $\eta(b, \mathcal{E}) = (c_B(b), c_{\mathcal{Z}}^*\mathcal{E}_\eta)$ where we write $c_{\mathcal{Z}}^*\mathcal{E}_\eta$ for the bundle $c_{\mathcal{Z}}^*\mathcal{E}$ with its η-twisted G-action. We denote by $\mathrm{Bun}_G(\mathcal{Z}/B)_{\mathbb{R}}$ the corresponding real points.

Suppose $n = 2m$ so that we have $\sigma = (\sigma^+, \sigma^-) : B \to \mathcal{Z}^n$ with components $\sigma^\pm = (\sigma_1^\pm, \ldots, \sigma_m^\pm) : B \to \mathcal{Z}^m$. Suppose further that $c_{\mathcal{Z}} \circ \sigma_i^\pm \circ c_B = \sigma_i^\mp$, for $i = 1, \ldots, m$. Then the conjugation η of G induces a conjugation $\eta : QM_{G,X}(\mathcal{Z}/B, \sigma) \to QM_{G,X}(\mathcal{Z}/B, \sigma)$. We denote by $QM_{G,X}(\mathcal{Z}/B, \sigma)_{\mathbb{R}}$ the corresponding real points.

Suppose further that $c_{\mathcal{Z}} \circ \xi = \xi \circ c_B$. Then the conjugation η of G similarly induces a conjugation $\eta : QM_{G,X}(\mathcal{Z}/B, \sigma, \xi) \to QM_{G,X}(\mathcal{Z}/B, \sigma, \xi)$. We denote by $QM_{G,X}(\mathcal{Z}/B, \sigma, \xi)_{\mathbb{R}}$ the corresponding real points.

4.2. **Uniformizations.** Let $\mathrm{Gr}_{G,\mathcal{Z},\sigma_i}$ (resp. $\mathrm{Gr}_{G,\mathcal{Z},\sigma}$) denote the Beilinson-Drinfeld Grassmannian of a point $b \in B$, a G-bundle $\mathcal{E}$ on $\mathcal{Z}_b$, and a section $s : \mathcal{Z}_b \setminus \sigma_i(b) \to \mathcal{E}$ (resp. $s : \mathcal{Z}_b \setminus \{\sigma_1(b), \ldots, \sigma_n(b)\} \to \mathcal{E}$).

Let $G[\mathcal{Z}, \hat{\sigma}_i]$ denote the group scheme of a point $b \in B$ and a section $\hat{D}_{\sigma_i(b)} \to G$, where $\hat{D}_{\sigma_i(b)}$ is the formal disk around $\sigma_i(b)$.

Let $G[\mathcal{Z}, \sigma_i]$ (resp. $G[\mathcal{Z}, \sigma]$) denote the group ind-scheme of a point $b \in B$ and a section $s : \mathcal{Z}_b \setminus \{\sigma_i(b)\} \to G$ (resp. $s : \mathcal{Z}_b \setminus \{\sigma_1(b), \ldots, \sigma_n(b)\} \to G$).

Let $G[\mathcal{Z}, \sigma_i, \xi]$ (resp. $G[\mathcal{Z}, \sigma, \xi]$) denote the subgroup ind-scheme of $G[\mathcal{Z}, \sigma_i]$ (resp. $G[\mathcal{Z}, \sigma]$) consisting of $(b, s) \in G[\mathcal{Z}, \sigma_i]$ (resp. $(b, s) \in G[\mathcal{Z}, \sigma]$) such that $s(\xi(b)) = e$.

For any $b \in B$, we write $\mathrm{Gr}_{G,\mathcal{Z},\sigma,b}$, $G[\mathcal{Z}, \sigma, b]$, etc., for the respective fibers over b.

The conjugations $c_{\mathcal{Z}}$, c_B, η induce conjugations on $\mathrm{Gr}_{G,\mathcal{Z},\sigma}$, $G[\mathcal{Z}, \sigma]$, etc., and we denote by $\mathrm{Gr}_{G,\mathcal{Z},\sigma,\mathbb{R}}$, $G[\mathcal{Z}, \sigma]_{\mathbb{R}}$, etc., the respective real points.

For any $b \in B(\mathbb{R})$, we write $\mathrm{Gr}_{G,\mathcal{Z},\sigma,b,\mathbb{R}}$, $G[\mathcal{Z}, \sigma, b]_{\mathbb{R}}$, etc., for the respective fibers over b.

The group ind-scheme $G[\mathcal{Z}, \sigma]$ (resp. $G[\mathcal{Z}, \sigma_i]$) naturally acts on $\mathrm{Gr}_{G,\mathcal{Z},\sigma}$ (resp. $\mathrm{Gr}_{G,\mathcal{Z},\sigma_i}$) and we have uniformizations morphisms

$$G[\mathcal{Z}, \sigma_i] \backslash \mathrm{Gr}_{G,\mathcal{Z},\sigma_i} \longrightarrow \mathrm{Bun}_G(\mathcal{Z}/B) \qquad G[\mathcal{Z}, \sigma] \backslash \mathrm{Gr}_{G,\mathcal{Z},\sigma} \to \mathrm{Bun}_G(\mathcal{Z}/B) \tag{4.4}$$

$$K[\mathcal{Z}, \sigma] \backslash \mathrm{Gr}_{G,\mathcal{Z},\sigma} \longrightarrow QM_{G,X}(\mathcal{Z}/B, \sigma) \tag{4.5}$$

$$K[\mathcal{Z}, \sigma, \xi] \backslash \mathrm{Gr}_{G,\mathcal{Z},\sigma} \longrightarrow QM_{G,X}(\mathcal{Z}/B, \sigma, \xi) \tag{4.6}$$

The uniformizations are compatible with the given conjugations, hence induce uniformizations on real points:

$$G[\mathcal{Z}, \sigma_i]_{\mathbb{R}} \backslash \mathrm{Gr}_{G,\mathcal{Z},\sigma_i,\mathbb{R}} \longrightarrow \mathrm{Bun}_G(\mathcal{Z}/B)_{\mathbb{R}} \qquad G[\mathcal{Z}, \sigma]_{\mathbb{R}} \backslash \mathrm{Gr}_{G,\mathcal{Z},\sigma,\mathbb{R}} \longrightarrow \mathrm{Bun}_G(\mathcal{Z}/B)_{\mathbb{R}} \tag{4.7}$$

$$K[\mathcal{Z}, \sigma]_{\mathbb{R}} \backslash \mathrm{Gr}_{G,\mathcal{Z},\sigma,\mathbb{R}} \longrightarrow QM_{G,X}(\mathcal{Z}/B, \sigma)_{\mathbb{R}} \tag{4.8}$$

$$K[\mathcal{Z}, \sigma, \xi]_{\mathbb{R}} \backslash \mathrm{Gr}_{G,\mathcal{Z},\sigma,\mathbb{R}} \longrightarrow QM_{G,X}(\mathcal{Z}/B, \sigma, \xi)_{\mathbb{R}} \tag{4.9}$$

4.3. **Morphisms.** Let G_1 and G_2 be two reductive groups with complex conjugations η_1 and η_2 and Cartan involutions θ_1 and θ_2 respectively. Then the constructions of quasi-maps, rigidified quasi-maps, uniformization morphisms, and real forms of those are functorial with respect to homomorphisms $f : G_1 \to G_2$ that intertwine η_1, η_2 and θ_1, θ_2.

5. Nodal degeneration

We invoke here the preceding constructions in a situation to be studied in the remainder of the paper.

5.1. **Universal family.** Let $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[a])$ be the affine line with coordinate a. Consider the product $\mathbb{P}^1_x \times \mathbb{P}^1_y$ with respective homogeneous coordinates $[x_0, x_1], [y_0, y_1]$, local coordinates $x = x_1/x_0, y = y_1/y_0$, and projections $p_x, p_y : \mathbb{P}^1_x \times \mathbb{P}^1_y \to \mathbb{P}^1$. For convenience, we will also set $t^+ = x^{-1}, t^- = y^{-1}$.

Introduce the surface $Z \subset \mathbb{P}^1_x \times \mathbb{P}^1_y \times \mathbb{A}^1$ cut out by $x_1 y_1 = a^2 x_0 y_0$. We will regard Z as a family of curves via the evident projection $p : Z \to \mathbb{A}^1$. We denote the fibers by $Z_a = p^{-1}(a)$, for $a \in \mathbb{A}^1$. When $a \neq 0$, projection along p_x or p_y provides an isomorphism $Z_a \simeq \mathbb{P}^1$. When $a = 0$, the image of the inclusion $Z_0 \subset \mathbb{P}^1 \times \mathbb{P}^1$ is the nodal curve

$$\mathbb{P}^1_x \vee \mathbb{P}^1_y := (\mathbb{P}^1_x \times \{0\}) \cup (\{0\} \times \mathbb{P}^1_y) \tag{5.1}$$

Equip $\mathbb{A}^1$ with the usual conjugation $c(a) = \bar{a}$ with real points $\mathbb{A}^1(\mathbb{R}) \simeq \mathbb{R}$. Equip Z with the twisted conjugation $c_Z(x, y, a) = (\bar{y}, \bar{x}, \bar{a})$. When $a \neq 0 \in \mathbb{A}^1(\mathbb{R})$, under the identification $p_x : Z \xrightarrow{\sim} \mathbb{P}^1_x$, we have $c_Z(x) = a^2/\bar{x}$, and thus $p_x : Z_a(\mathbb{R}) \xrightarrow{\sim} \mathbb{RP}^1$. When $a = 0 \in \mathbb{A}^1(\mathbb{R})$, the components of Z_0 are exchanged by c_Z, and $Z_0(\mathbb{R})$ is the single point $x = 0, y = 0$.

Note that in terms of the coordinate t^+ of $\mathbb{P}^1$ the complex conjugation is given by $c_{\mathcal{Z}}(t^+) = \frac{1}{a^2 \bar{t}^+}$. Let $\mathbb{A}^1_+ = \mathrm{Spec}(\mathbb{C}[t^+])$, $\mathbb{A}^1_- = \mathrm{Spec}(\mathbb{C}[t^-])$ be the affine lines with respective coordinates $t^+ = x^{-1}, t^- = y^{-1}$. Thus we have natural open embeddings $\mathbb{A}^1_+ \subset \mathbb{P}^1_x$, $\mathbb{A}^1_- \subset \mathbb{P}^1_y$.

Fix $n = 2m$. Consider the base scheme

$$B = \mathbb{A}^1_a \times B^+ \times B^- \qquad B^\pm = (\mathbb{A}^1_\pm)^m \tag{5.2}$$

with coordinates $(a, \mathbf{t}^+, \mathbf{t}^-)$ where $\mathbf{t}^\pm = (t^\pm_1, \ldots, t^\pm_m)$. Equip B with the twisted conjugation $c_A(a, \mathbf{t}^+, \mathbf{t}^-) = (\bar{a}, \bar{\mathbf{t}}^-, \bar{\mathbf{t}}^+)$ where $\bar{\mathbf{t}}^\pm = (\bar{t}^\pm_1, \ldots, \bar{t}^\pm_m)$. Projection provides an identification of real points

$$B(\mathbb{R}) \simeq \mathbb{A}^1(\mathbb{R}) \times B^+(\mathbb{C}) \simeq \mathbb{R} \times \mathbb{C}^m \tag{5.3}$$

Consider the family of curves

$$p : \mathcal{Z} = Z \times_{\mathbb{A}^1} B \longrightarrow B \tag{5.4}$$

with the tautological sections

$$\sigma = (\sigma^+, \sigma^-) : B \longrightarrow \mathcal{Z}^{2m} \tag{5.5}$$

$$\sigma^+_i(a, \mathbf{t}^+, \mathbf{t}^-) = (a, [t^+_i, 1], [1, t^+_i a^2], \mathbf{t}^+, \mathbf{t}^-) \qquad i = 1, \ldots, m \tag{5.6}$$

$$\sigma^-_i(a, \mathbf{t}^+, \mathbf{t}^-) = (a, [1, t^-_i a^2], [t^-_i, 1], \mathbf{t}^+, \mathbf{t}^-) \qquad i = 1, \ldots, m \tag{5.7}$$

Equip $\mathcal{Z}$ with the twisted conjugation $c_{\mathcal{Z}}(a, x, y, \mathbf{t}^+, \mathbf{t}^-) = (\bar{a}, \bar{y}, \bar{x}, \bar{\mathbf{t}}^-, \bar{\mathbf{t}}^+)$. Projection provides an identification of real points

$$\mathcal{Z}(\mathbb{R}) \simeq Z(\mathbb{R}) \times B^+(\mathbb{C}) \simeq Z(\mathbb{R}) \times \mathbb{C}^m \tag{5.8}$$

Introduce the canonical section

$$\xi : B \longrightarrow \mathcal{Z} \qquad \xi(a, \mathbf{t}^+, \mathbf{t}^-) = (a, [1, a], [1, a], \mathbf{t}^+, \mathbf{t}^-) \tag{5.9}$$

We denote by B' the open subset of B consisting of (a, t^+, t^-) with $a \neq 0$ and we define $\mathcal{Z}' := \mathcal{Z} \times_B B'$.

Note that $\xi(a, \mathbf{t}^+, \mathbf{t}^-) = \sigma_i^\pm(a, \mathbf{t}^+, \mathbf{t}^-)$ if and only if $t_i^\pm = a^{-1}$. Denote by $B_\circ \subset B$ the open complement of such coincidences. Introduce the base-change

$$p : \mathcal{Z}_\circ = \mathcal{Z} \times_B B_\circ \longrightarrow B_\circ \tag{5.10}$$

and note, by construction, the tautological sections

$$\sigma = (\sigma^+, \sigma^-) : B_\circ \longrightarrow \mathcal{Z}_\circ^{2m} \tag{5.11}$$

do not intersect the canonical section

$$\xi : B_\circ \longrightarrow \mathcal{Z}_\circ \tag{5.12}$$

With the above choices fixed, we will study the real points of the ind-stacks of quasimaps with their natural projections

$$q : QM_{G,X}(\mathcal{Z}/B, \sigma)_\mathbb{R} \longrightarrow B(\mathbb{R}) \tag{5.13}$$

$$r : QM_{G,X}(\mathcal{Z}_\circ/B_\circ, \sigma, \xi)_\mathbb{R} \longrightarrow B_\circ(\mathbb{R}) \tag{5.14}$$

5.2. **Complex groups.** We specialize here our prior constructions to the distinguished case of complex groups.

Recall $\delta = \theta \circ \eta = \eta \circ \theta$ denotes the Cartan conjugation of G with compact real form G_c. Equip $G \times G$ with the swap involution $\mathrm{sw}(g, h) = (h, g)$ and the conjugation $\mathrm{sw}_\delta(g, h) = (\delta(h), \delta(g))$. The fixed-point subgroup of sw is the diagonal $G \subset G \times G$, and the corresponding symmetric space is isomorphic to the group $G\backslash(G \times G) \simeq G$.

Lemma 5.1. *The uniformization morphisms* (4.8), (4.9) *are isomorphisms:*

$$G[\mathcal{Z}', \sigma]_\mathbb{R}\backslash \mathrm{Gr}_{G\times G, \mathcal{Z}', \sigma, \mathbb{R}} \times_{B'} B'(\mathbb{R}) \xrightarrow{\sim} QM_{G\times G, G}(\mathcal{Z}'/B', \sigma)_\mathbb{R} \tag{5.15}$$

$$G[\mathcal{Z}'_\circ, \sigma, \xi]_\mathbb{R}\backslash \mathrm{Gr}_{G\times G, \mathcal{Z}', \sigma, \mathbb{R}} \times_{B'} B'_\circ(\mathbb{R}) \xrightarrow{\sim} QM_{G\times G, G}(\mathcal{Z}'_\circ/B'_\circ, \sigma, \xi)_\mathbb{R} \tag{5.16}$$

Proof. Note that the above uniformization morphisms over a base point $b \in B'$ (resp. $b \in B'_\circ$) are the (multipoint version) of the real or complex uniformization morphisms in [CN1, Section 6.3]. Since the fixed-point subgroup $G \subset G \times G$ is connected and $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), G \times G)$ is trivial for the Galois-action given by sw_δ, it follows from [CN1, Remark 5.8 and Lemma 6.1] that any real bundles on the curve $\mathcal{Z}_b(\mathbb{R}) \simeq \mathbb{RP}^1$ (associated to the real form sw_δ of $G \times G$) admit either real or complex uniformizations. The lemma follows by standard arguments. □

Next, the projection maps $\mathrm{pr}_1, \mathrm{pr}_2 : G \times G \to G$, $\mathrm{pr}_i(g_1, g_2) = g_i$, provide an isomorphism

$$\mathrm{Gr}_{G\times G, \mathcal{Z}, \sigma} \xrightarrow{\sim} \mathrm{Gr}_{G, \mathcal{Z}, \sigma} \times_B \mathrm{Gr}_{G, \mathcal{Z}, \sigma} \tag{5.17}$$

The conjugations $c_\mathcal{Z}$ of $\mathcal{Z}$ and sw_δ of $G \times G$ together induce a conjugation of $\mathrm{Gr}_{G\times G, \mathcal{Z}, \sigma}$ which, under the isomorphism (5.17), is given by the map on pairs of bundles with sections

$$(\mathcal{E}, s, \mathcal{E}', s') \longmapsto (c_\mathcal{Z}^* \mathcal{E}'_\delta, c_\mathcal{Z}^*(s'), c_\mathcal{Z}^* \mathcal{E}_\delta, c_\mathcal{Z}^*(s)) \tag{5.18}$$

Thus the isomorphism (5.17) followed by pr_1 provides an isomorphism

$$\mathrm{Gr}_{G\times G,\mathcal{Z},\sigma,\mathbb{R}} \xrightarrow{\sim} \mathrm{Gr}_{G,\mathcal{Z},\sigma} \times_B B(\mathbb{R}) \tag{5.19}$$

of real analytic spaces over $B(\mathbb{R}) \simeq \mathbb{P}^1(\mathbb{R}) \times_{\mathbb{P}^1(\mathbb{C})} B^+(\mathbb{C})$.

Thus the preceding lemma has the following consequence.

Corollary 5.2. *There are natural isomorphisms*

$$G[\mathcal{Z}',\sigma]_{\mathbb{R}}\backslash(\mathrm{Gr}_{G,\mathcal{Z}',\sigma} \times_{B'} B'(\mathbb{R})) \xrightarrow{\sim} QM_{G\times G,G}(\mathcal{Z}'/B',\sigma)_{\mathbb{R}} \tag{5.20}$$

$$G[\mathcal{Z}'_\circ,\sigma,\xi]_{\mathbb{R}}\backslash(\mathrm{Gr}_{G,\mathcal{Z}'_\circ,\sigma} \times_{B'_\circ} B'_\circ(\mathbb{R})) \xrightarrow{\sim} QM_{G\times G,G}(\mathcal{Z}'_\circ/B'_\circ,\sigma,\xi)_{\mathbb{R}} \tag{5.21}$$

Next, consider the natural map

$$\mathrm{Gr}_{G,\mathcal{Z},\sigma^+} \longrightarrow \mathrm{Gr}_{G,\mathcal{Z},\sigma} \tag{5.22}$$

given by restricting a trivialization of a G-bundle defined away from the section σ^+ to the complement of both sections σ^+ and σ^-.

We will find open loci where the map (5.22) induces an isomorphism.

First, consider the restriction of (5.22) to the generic family

$$\mathrm{Gr}_{G,\mathcal{Z}',\sigma^+} \longrightarrow \mathrm{Gr}_{G,\mathcal{Z}',\sigma} \tag{5.23}$$

Proposition 5.3. *The map* (5.23) *induces isomorphisms*

$$G_c\backslash\mathrm{Gr}_{G,\mathcal{Z}',\sigma^+} \times_{B'} B'(\mathbb{R}) \xrightarrow{\sim} G[\mathcal{Z}',\sigma]_{\mathbb{R}}\backslash(\mathrm{Gr}_{G,\mathcal{Z}',\sigma} \times_{B'} B'(\mathbb{R})) \tag{5.24}$$

$$\mathrm{Gr}_{G,\mathcal{Z}'_\circ,\sigma^+} \times_{B'_\circ} B'_\circ(\mathbb{R}) \xrightarrow{\sim} G[\mathcal{Z}'_\circ,\sigma,\xi]_{\mathbb{R}}\backslash(\mathrm{Gr}_{G,\mathcal{Z}'_\circ,\sigma} \times_{B'_\circ} B'_\circ(\mathbb{R})) \tag{5.25}$$

Proof. It suffices to establish the second with its natural G_c-equivariance then glue to obtain the first. Let $b = (a, t_1^+, ..., t_n^+) \in B'_\circ(\mathbb{R})$ and let $z_1, ..., z_k \in \mathbb{C}$ be k-distinct points such that $\{z_1, .., z_k\} = \{t_1^+, ..., t_n^+\}$. It follows from the standard factorization property of Beilinson-Drinfeld Grassmannian and Theorem 3.3 that, over the based point b, the second map above can be identified with the map

$$\prod_{i=1}^k \mathrm{Gr}_{z_i} \to \prod_{i=1}^k (\mathrm{Gr}_{z_i} \times \mathrm{Gr}_{\frac{1}{a^2\bar{z}_i}}) \to \prod_{i=1}^k \Omega_{z_i,a^{-1}} G_c\backslash(\mathrm{Gr}_{z_i} \times \mathrm{Gr}_{\frac{1}{a^2\bar{z}_i}}) \tag{5.26}$$

where the first map is the left copy embedding sending γ to (γ, e), where e is the base point of $\mathrm{Gr}_{\frac{1}{a^2\bar{z}_i}}$, and the second map is the natural quotient map. (Note that in terms of the local coordinate z_i the complex conjugation on $\mathcal{Z}_b \simeq \mathbb{P}^1_{z_i}$ is given by $z_i \to \frac{1}{a^2\bar{z}_i}$.) Thus the assertion follows from the fact that the based loop group $\Omega_{z_i,a^{-1}} G_c$ acts freely on Gr_{z_i} and $\mathrm{Gr}_{\frac{1}{a^2\bar{z}_i}}$. ☐

Corollary 5.4. *There are natural isomorphisms*

$$G_c\backslash\mathrm{Gr}_{G,\mathcal{Z}',\sigma^+} \times_{B'} B'(\mathbb{R}) \xrightarrow{\sim} QM_{G\times G,G}(\mathcal{Z}'/B',\sigma)_{\mathbb{R}} \tag{5.27}$$

$$\mathrm{Gr}_{G,\mathcal{Z}'_\circ,\sigma^+} \times_{B'_\circ} B'_\circ(\mathbb{R}) \xrightarrow{\sim} QM_{G\times G,G}(\mathcal{Z}'_\circ/B'_\circ,\sigma,\xi)_{\mathbb{R}} \tag{5.28}$$

Next, let $\mathrm{Bun}^0_G(\mathcal{Z}/B) \subset \mathrm{Bun}_G(\mathcal{Z}/B)$ denote the open sub-stack of a point $b \in B$ and a trivializable G-bundle on $\mathcal{Z}_b$. Denote by

$$QM^0_{G\times G,G}(\mathcal{Z}/B,\sigma)_{\mathbb{R}} \simeq G_c\backslash \mathrm{Gr}^0_{G,\mathcal{Z},\sigma^+} \qquad QM^0_{G\times G,G}(\mathcal{Z}_\circ/B_\circ,\sigma,\xi)_{\mathbb{R}} \simeq \mathrm{Gr}^0_{G,\mathcal{Z}^+_\circ,\sigma^+} \tag{5.29}$$

the base-changes to $\mathrm{Bun}^0_G(\mathcal{Z}/B) \subset \mathrm{Bun}_G(\mathcal{Z}/B)$. Consider the restriction of (5.22) to the trivial bundle locus

$$\mathrm{Gr}^0_{G,\mathcal{Z},\sigma^+} \longrightarrow \mathrm{Gr}^0_{G,\mathcal{Z},\sigma} \tag{5.30}$$

Proposition 5.5. *The map* (5.30) *induces isomorphisms*

$$G_c\backslash \mathrm{Gr}^0_{G,\mathcal{Z},\sigma^+} \times_B B(\mathbb{R}) \xrightarrow{\sim} G[\mathcal{Z},\sigma]_{\mathbb{R}}\backslash(\mathrm{Gr}^0_{G,\mathcal{Z},\sigma} \times_B B(\mathbb{R})) \tag{5.31}$$

$$\mathrm{Gr}^0_{G,\mathcal{Z}_\circ,\sigma^+} \times_{B_\circ} B_\circ(\mathbb{R}) \xrightarrow{\sim} G[\mathcal{Z}_\circ,\sigma,\xi]_{\mathbb{R}}\backslash(\mathrm{Gr}^0_{G,\mathcal{Z}_\circ,\sigma} \times_{B_\circ} B_\circ(\mathbb{R})) \tag{5.32}$$

Proof. By Proposition 5.3, it suffices to establish the maps are isomorphisms at the special fiber $b = (a, t_1^+, ..., t_n^+) \in B(\mathbb{R})$ with $a = 0$. Moreover, it suffices to establish the second with its natural G_c-equivariance then glue to obtain the first.

Now it is elementary to see we have a natural isomorphism at the special fiber

$$\mathrm{Gr}_{G,\mathcal{Z}_\circ,\sigma} \times_{B_\circ} B_\circ(\mathbb{R})|_b \xrightarrow{\sim} \mathrm{Gr}_{G,\mathcal{Z}_\circ,\sigma^+,b} \times \mathrm{Gr}_{G,\mathcal{Z}_\circ,\sigma^-,b} \tag{5.33}$$

And similarly, we have a natural isomorphism of groups at the special fiber

$$G[\mathcal{Z}_\circ,\sigma,\xi]_{\mathbb{R}}|_b \xrightarrow{\sim} \Delta_{\mathbb{R}} \subset G[\mathbb{P}^1,\sigma^+,\xi,b] \times G[\mathbb{P}^1,\sigma^-,\xi,b] \tag{5.34}$$

where $\Delta_{\mathbb{R}}$ denotes the conjugate diagonal. Thus the assertion follows from the facts that the isomorphism (5.33) restricts to an isomorphism

$$\mathrm{Gr}^0_{G,\mathcal{Z}_\circ,\sigma} \times_{B_\circ} B_\circ(\mathbb{R})|_b \xrightarrow{\sim} \mathrm{Gr}^0_{G,\mathcal{Z}_\circ,\sigma^+,b} \times \mathrm{Gr}^0_{G,\mathcal{Z}_\circ,\sigma^-,b} \tag{5.35}$$

and the action of $G[\mathbb{P}^1,\sigma^+,\xi,b]$ on $\mathrm{Gr}^0_{G,\mathcal{Z}_\circ,\sigma^+,b}$ is free and transitive. $\square$

Corollary 5.6. *There are natural isomorphisms*

$$G_c\backslash \mathrm{Gr}^0_{G,\mathcal{Z},\sigma^+} \times_B B(\mathbb{R}) \xrightarrow{\sim} QM^0_{G\times G,G}(\mathcal{Z}/B,\sigma)_{\mathbb{R}} \tag{5.36}$$

$$\mathrm{Gr}^0_{G,\mathcal{Z}_\circ,\sigma^+} \times_{B_\circ} B_\circ(\mathbb{R}) \xrightarrow{\sim} QM^0_{G\times G,G}(\mathcal{Z}_\circ/B_\circ,\sigma,\xi)_{\mathbb{R}} \tag{5.37}$$

Remark 5.7. The isomorphisms of Cor. 5.4 and 5.37 coincide on the intersections of their domains.

5.3. **Stratification.** Recall the Beilinson-Drinfeld Grassmannian $\mathrm{Gr}^{(m)} \to (\mathbb{P}^1)^m$ of a G-bundle $\mathcal{E}$ on $\mathbb{P}^1$, a point $(z_1, ..., z_m) \in (\mathbb{P}^1)^m$, and a section $s : \mathbb{P}^1 \setminus \{z_1, ..., z_m\} \to \mathcal{E}$. We denote by $\mathrm{Gr}^{(m),0} \subset \mathrm{Gr}^{(m)}$ the open subset consisting of $(\mathcal{E}, z_1, ..., z_m, s)$ such that $\mathcal{E}$ is trivializable.

First, let us stratify the base $(\mathbb{P}^1)^m$ by coincidences among points. For any partition $\mathfrak{p}$ of the set $\{1, \ldots, m\}$, denote by $(\mathbb{P}^1)^{\mathfrak{p}} \subset (\mathbb{P}^1)^m$ the locus where $z_i = z_j$ if and only if i and j are in the same part of $\mathfrak{p}$. This provides a Whitney stratification of $(\mathbb{P}^1)^m$.

Next, for any partition $\mathfrak{p}$ of the set $\{1, \ldots, m\}$, and any map $\lambda_{\mathfrak{p}} : \mathfrak{p} \to \Lambda_T^+$, denote by $\mathcal{S}^{\lambda_{\mathfrak{p}}} \subset \mathrm{Gr}^{(m)}$ (resp. $\mathcal{S}^{\lambda_{\mathfrak{p}},0} \subset \mathrm{Gr}^{(m),0}$) the spherical stratum of a G-bundle $\mathcal{E}$ (resp. a trivializable G-bundle $\mathcal{E}$), a point $(z_1, ..., z_m) \in (\mathbb{P}^1)^{\mathfrak{p}}$, and a section $s : \mathbb{P}^1 \setminus \{z_1, \ldots, z_m\} \to \mathcal{E}$ of modification type $\lambda_{\mathfrak{p}}$. This provides a Whitney stratification of $\mathrm{Gr}^{(m)}$ (resp. $\mathrm{Gr}^{(m),0}$) compatible with that of $(\mathbb{P}^1)^m$.

Note that the coordinate x^{-1} provides an isomorphism

$$(\mathbb{R} \times \mathrm{Gr}^{(m)}) \times_{\mathbb{R}\times(\mathbb{P}^1)^m} B(\mathbb{R}) \simeq \mathrm{Gr}_{G,\mathcal{Z},\sigma^+} \times_B B(\mathbb{R}) \tag{5.38}$$

here we identify $B(\mathbb{R}) \overset{(5.3)}{\simeq} \mathbb{R}\times\mathbb{C}^m \subset \mathbb{R}\times(\mathbb{P}^1)^m$. The isomorphism above restricts to an isomorphism

$$(\mathbb{R} \times \mathrm{Gr}^{(m),0}) \times_{\mathbb{R}\times(\mathbb{P}^1)^m} B(\mathbb{R}) \simeq \mathrm{Gr}^0_{G,\mathcal{Z},\sigma^+} \times_B B(\mathbb{R}) \tag{5.39}$$

between the corresponding open loci and, by Corollary 5.4 and 5.6, we obtain:

$$(\mathbb{R} \times \mathrm{Gr}^{(m)}) \times_{\mathbb{R}\times(\mathbb{P}^1)^m} B'_\circ(\mathbb{R}) \simeq QM_{G\times G,G}(\mathcal{Z}'_\circ/B'_\circ, \sigma, \xi)_{\mathbb{R}}, \tag{5.40}$$

$$(\mathbb{R} \times \mathrm{Gr}^{(m),0}) \times_{\mathbb{R}\times(\mathbb{P}^1)^m} B_\circ(\mathbb{R}) \simeq QM^0_{G\times G,G}(\mathcal{Z}_\circ/B_\circ, \sigma, \xi)_{\mathbb{R}}. \tag{5.41}$$

Let us equip $\mathbb{R} \times \mathrm{Gr}^{(m)}$ (resp. $\mathbb{R} \times \mathrm{Gr}^{(m),0}$) with the product stratification $\{\mathbb{R} \times \mathcal{S}^{\lambda_{\mathfrak{p}}}\}$ (resp. $\{\mathbb{R} \times \mathcal{S}^{\lambda_{\mathfrak{p}},0}\}$) and transport the spherical strata $\mathbb{R} \times \mathcal{S}^{\lambda_{\mathfrak{p}}} \subset \mathbb{R} \times \mathrm{Gr}^{(m)}$ (resp. $\mathbb{R} \times \mathcal{S}^{\lambda_{\mathfrak{p}},0} \subset \mathbb{R} \times \mathrm{Gr}^{(m),0}$) across the isomorphisms (5.40) and (5.41) and denote the resulting strata by

$$\mathcal{S}^{\lambda_{\mathfrak{p}}}_{\mathcal{Z},\mathbb{R}} \subset QM_{G\times G,G}(\mathcal{Z}'_\circ/B'_\circ, \sigma, \xi)_{\mathbb{R}} \qquad (\text{resp. } \mathcal{S}^{\lambda_{\mathfrak{p}},0}_{\mathcal{Z},\mathbb{R}} \subset QM^0_{G\times G,G}(\mathcal{Z}_\circ/B_\circ, \sigma, \xi)_{\mathbb{R}}). \tag{5.42}$$

Note that $\mathcal{S}^{\lambda_{\mathfrak{p}},0}_{\mathcal{Z},\mathbb{R}}$ and $\mathcal{S}^{\lambda_{\mathfrak{p}},0}_{\mathcal{Z},\mathbb{R}}$ are non-empty if and only if the total coweight $|\lambda_{\mathfrak{p}}| \in \Lambda_T^+$, given by summing the values of $\lambda_{\mathfrak{p}}$ over the parts of $\mathfrak{p}$, in fact lies in $R_G^+ \subset \Lambda_T^+$.

5.4. **Symmetry.** We continue here with the distinguished case of complex groups. We will describe an involution of quasi-maps.

Definition 5.8 (Swap involution $\delta_{\mathcal{Z}}$)**.** The conjugation $\delta \times \delta$ of $G \times G$ commutes with the conjugation sw_δ. Hence together with the conjugation $c_{\mathcal{Z}}$ of $\mathcal{Z}$ it induces a fiberwise involution of $QM_{G\times G,G}(\mathcal{Z}/B, \sigma)_{\mathbb{R}}$ and $QM_{G\times G,G}(\mathcal{Z}_\circ/B_\circ, \sigma, \xi)_{\mathbb{R}}$ denoted by $\delta_{\mathcal{Z}}$.

In the remainder of this section, we will give concrete descriptions of how the involution $\delta_{\mathcal{Z}}$ act at the generic and special fibers in terms of our prior uniformizations.

5.4.1. *Generic fiber.* Let $b = (a \neq 0, t_1^+, ..., t_m^+) \in B'_\circ(\mathbb{R})$. Let $z_1, ..., z_k \in \mathbb{C}$ be the distinct k points such that there is an equality of sets $\{z_1, ..., z_k\} = \{t_1^+, ..., t_m^+\}$ and consider the isomorphism

$$v_b : \mathrm{Gr}_{z_1} \times \cdots \times \mathrm{Gr}_{z_k} \simeq (\mathbb{R} \times \mathrm{Gr}^{(m)})|_b \overset{(5.38)}{\simeq} \mathrm{Gr}_{G,\mathcal{Z},\sigma^+,b} \overset{(5.28)}{\simeq} QM_{G\times G,G}(\mathcal{Z}_\circ/B_\circ, \sigma, \xi, b)_{\mathbb{R}}. \tag{5.43}$$

Here the first map is the factorization isomorphism. For $i = 1, \ldots, k$, we define

$$v_i : \mathrm{Gr}_{z_i} \hookrightarrow \mathrm{Gr}_{z_1} \times \cdots \times \mathrm{Gr}_{z_k} \overset{v_b}{\simeq} QM_{G\times G,G}(\mathcal{Z}_\circ/B_\circ, \sigma, \xi, b)_{\mathbb{R}}$$

Proposition 5.9. *For $i = 1, \ldots, k$, the involution $\delta_{\mathcal{Z}}$ satisfies:*

(1) When $z_i \in S^1_{a^{-1}} = \{|z| = |a^{-1}|\} \subset \mathbb{P}^1$, we have

$$\delta_{\mathcal{Z}} \circ v_i = v_i \circ \delta \tag{5.44}$$

where δ is the conjugation on Gr_{z_i} induced by the conjugations $c(z) = \frac{1}{a^2\bar{z}}$ on $\mathbb{P}^1$ and δ on G.

(2) When $z_i\in\mathbb{C}\setminus S^1_{a^{-1}}$, we have

$$\delta_{\mathcal{Z}}\circ v_i=v_i\circ\mathrm{inv} \tag{5.45}$$

where $\mathrm{inv}(g)(z)=g(z)^{-1}$ is the group-inverse on $\Omega_{z_i,a^{-1}}G_c\simeq\mathrm{Gr}_{z_i}$.

Proof. We first note that the isomorphism

$$\prod_{i=1}^k\Omega_{z_i,a^{-1}}G_c\backslash(\mathrm{Gr}_{z_i}\times\mathrm{Gr}_{c(z_i)})\simeq G[\mathcal{Z}'_\circ,\sigma,\xi,b]_{\mathbb{R}}\backslash\mathrm{Gr}_{G,\mathcal{Z}'_\circ,\sigma,b}\simeq QM_{G\times G,G}(\mathcal{Z}_\circ/B_\circ,\sigma,\xi,b)_{\mathbb{R}}$$

intertwines the conjugation $\delta_{\mathcal{Z}}$ with the one $\delta'_{\mathcal{Z}}$ on the product $\prod_{i=1}^k\Omega_{z_i,a^{-1}}G_c\backslash(\mathrm{Gr}_{z_i}\times\mathrm{Gr}_{c(z_i)})$ given by

$$\delta'_{\mathcal{Z}}(\gamma_{i,1},\gamma_{i,2})=(\gamma'_{i,1},\gamma'_{i,2}),\quad i=1,...,k$$

where $\gamma'_{i,j}=\delta(\gamma_{i,j})$, $j=1,2$, if $z_i=c(z_i)\in S^1_{a^{-1}}$, otherwise $\gamma'_{i,1}=\delta(\gamma_{i,2})$ and $\gamma'_{i,2}=\delta(\gamma_{i,1})$ if $z_i\neq c(z_i)$. Here δ is the map $\delta:\mathrm{Gr}_{z_i}\to\mathrm{Gr}_{c(z_i)}$ induced by the complex conjugation $c(z)=\frac{1}{a^2\bar{z}}$ on $\mathbb{P}^1$ and δ on G.

On the other hand, a direct computation shows that the isomorphism

$$\prod_{i=1}^k\mathrm{Gr}_{z_i}\simeq\prod_{i=1}^k\Omega_{z_i,a^{-1}}G_c\backslash(\mathrm{Gr}_{z_i}\times\mathrm{Gr}_{c(z_i)})$$

induced by the left copy embedding $\gamma\to(\gamma,e)$ intertwines the map $\delta'_{\mathcal{Z}}$ above with the one $\delta''_{\mathcal{Z}}$ on Gr_{z_i} given by $\delta''_{\mathcal{Z}}(\gamma)=\delta(\gamma)$ when $z_i=c(z_i)$ and $\delta''_{\mathcal{Z}}(\gamma)=\gamma^{-1}$ is the group-inverse on $\Omega_{z_i,a^{-1}}G_c\simeq\mathrm{Gr}_{z_i}$ when $z_i\neq c(z_i)$.

To deduce the proposition, we note that the map v_b in (5.43) is equal to the composition of the above two isomorphisms. □

5.4.2. *Special fiber.* Let $b=(0,t_1^+,...,t_m^+)\in B_\circ(\mathbb{R})$. Consider the map

$$v_b:G[\mathbb{P}^1\setminus\{t_1^+,...,t_m^+\}]_{\infty\to e}\simeq(\mathbb{R}\times\mathrm{Gr}^{(m),0})|_b\overset{(5.39)}{\simeq}\mathrm{Gr}^0_{G,\mathcal{Z},\sigma^+,b}\overset{(5.37)}{\simeq}QM^0_{G\times G,G}(\mathcal{Z}_\circ/B_\circ,\sigma,\xi,b)_{\mathbb{R}}. \tag{5.46}$$

Here and in what follows, the subscropt $\infty\to e$ conveys the condition $g(\infty)=e$ on a mapping.

Proposition 5.10. *The involution $\delta_{\mathcal{Z}}$ satisfies*

$$\delta_{\mathcal{Z}}\circ v_b=v_b\circ\mathrm{inv}$$

here $\mathrm{inv}(g(z))=g(z)^{-1}$ is the group inverse on $G[\mathbb{P}^1\setminus\{t_1^+,...,t_m^+\}]_{\infty\to e}$.

Proof. We first note that the isomorphism

$$G[\mathbb{P}^1\backslash\{t_1^+,...,t_m^+\}]_{\infty\to e}\backslash(\mathrm{Gr}^0_{G,\mathcal{Z},\sigma^+,b}\times\mathrm{Gr}^0_{G,\mathcal{Z},\sigma^-,b})\simeq G[\mathcal{Z}'_\circ,\sigma,\xi,b]_{\mathbb{R}}\backslash\mathrm{Gr}^0_{G,\mathcal{Z}'_\circ,\sigma,b}\simeq QM^0_{G\times G,G}(\mathcal{Z}_\circ/B_\circ,\sigma,\xi,b)_{\mathbb{R}}$$

intertwines the involution $\delta_{\mathcal{Z}}$ with the one $\delta'_{\mathcal{Z}}$ on $G[\mathbb{P}^1\setminus\{t_1^+,...,t_m^+\}]_{\infty\to e}\backslash(\mathrm{Gr}^0_{G,\mathcal{Z},\sigma^+,b}\times\mathrm{Gr}^0_{G,\mathcal{Z},\sigma^-,b})$ given by $\delta'_{\mathcal{Z}}(\gamma_1,\gamma_2)=(\delta(\gamma_2),\delta(\gamma_1))$ (note that the group $G[\mathbb{P}^1\setminus\{t_1^+,...,t_m^+\}]_{\infty\to e}$ acts on the product via the conjugate diagonal embedding $\gamma\to(\gamma,\delta(\gamma))$). On the other hand, the isomorphism

$$G[\mathbb{P}^1\setminus\{t_1^+,...,t_m^+\}]_{\infty\to e}\simeq\mathrm{Gr}^0_{G,\mathcal{Z},\sigma^+,b}\simeq G[\mathbb{P}^1\setminus\{t_1^+,...,t_m^+\}]_{\infty\to e}\backslash(\mathrm{Gr}^0_{G,\mathcal{Z},\sigma^+,b}\times\mathrm{Gr}^0_{G,\mathcal{Z},\sigma^-,b})$$

where the fist map is the action map and the second map is induced by the left copy embedding intertwines the involution $\delta'_{\mathcal{Z}}$ with the group inverse on $G[\mathbb{P}^1\setminus\{t_1^+,...,t_m^+\}]_{\infty\to e}$. To deduce the proposition, we observe that the map v_b is equal to the composition of the above two isomorphisms. $\square$

5.5. **Compatibility with stratifications.** We have the following compatibility of the involution $\delta_{\mathcal{Z}}$ with stratifications. Let $w_0 \in W$ denote the longest element of the Weyl group. For any partition $\mathfrak{p}$ of the set $\{1, \ldots, m\}$, and map $\lambda_{\mathfrak{p}} : \mathfrak{p} \to \Lambda_T^+$, we set $-w_0(\lambda_{\mathfrak{p}}) = -w_0 \circ \lambda_{\mathfrak{p}}$.

Lemma 5.11. *(1) The involution $\delta_{\mathcal{Z}}$ preserves the open subspace*

$$QM^0_{G\times G,G}(\mathcal{Z}_\circ/B_\circ, \sigma, \xi)_{\mathbb{R}} \subset QM_{G\times G,G}(\mathcal{Z}_\circ/B_\circ, \sigma, \xi)_{\mathbb{R}} \tag{5.47}$$

(2) The involution $\delta_{\mathcal{Z}}$ restricts to a map on spherical strata

$$\mathcal{S}^{\lambda_{\mathfrak{p}}}_{\mathcal{Z},\mathbb{R}} \longrightarrow \mathcal{S}^{-w_0(\lambda_{\mathfrak{p}})}_{\mathcal{Z},\mathbb{R}} \tag{5.48}$$

within $QM_{G\times G,G}(\mathcal{Z}'_\circ/B'_\circ, \sigma, \xi)_{\mathbb{R}}$, and also the spherical strata

$$\mathcal{S}^{\lambda_{\mathfrak{p}},0}_{\mathcal{Z},\mathbb{R}} \longrightarrow \mathcal{S}^{-w_0(\lambda_{\mathfrak{p}}),0}_{\mathcal{Z},\mathbb{R}} \tag{5.49}$$

within $QM^0_{G\times G,G}(\mathcal{Z}_\circ/B_\circ, \sigma, \xi)_{\mathbb{R}}$.

Proof. The claim follows from the facts that the conjugation δ of Gr maps S^λ (resp. Gr^0) to $S^{-w_0(\lambda)}$ (resp. Gr^0), and the involution inv of ΩG_c maps S^λ to $S^{-w_0(\lambda)}$. $\square$

5.6. **The involution $\eta_{\mathcal{Z}}$.** Recall θ denotes the Cartan involution of G with fixed-point subgroup K, and $\delta = \theta \circ \eta = \eta \circ \theta$ the Cartan conjugation of G with compact real form G_c. Since η and θ commute with δ, the conjugation $\eta \times \eta$ and involution $\theta \times \theta$ of $G \times G$ define involutions of the real moduli $QM_{G\times G,G}(\mathcal{Z}_\circ/B_\circ, \sigma, \xi)_{\mathbb{R}}$ of rigidified quasi-maps which we denote respectively by $\eta_{\mathcal{Z}}$ and $\theta_{\mathcal{Z}}$.

Propositions 5.9 and 5.10, and Lemma 5.11 immediately imply:

Proposition 5.12. *The involution $\eta_{\mathcal{Z}}$ on $QM_{G\times G,G}(\mathcal{Z}_\circ/B_\circ, \sigma, \xi)_{\mathbb{R}}$ satisfies:*

1. *The involution commutes with the natural G_c-action and preserves the open subfamilies $QM_{G\times G,G}(\mathcal{Z}'_\circ/B'_\circ, \sigma, \xi)_{\mathbb{R}}$ and $QM^0_{G\times G,G}(\mathcal{Z}_\circ/B_\circ, \sigma, \xi)_{\mathbb{R}}$ and their spherical stratifications $\{\mathcal{S}^{\lambda_{\mathfrak{p}}}_{\mathcal{Z},\mathbb{R}}\}$ and $\{\mathcal{S}^{\lambda_{\mathfrak{p}},0}_{\mathcal{Z},\mathbb{R}}\}$.*
2. *At $b = (a \neq 0, t_1^+, ..., t_m^+) \in B_\circ(\mathbb{R})$, the isomorphism*
$$v_b : \mathrm{Gr}_{z_1} \times \cdots \times \mathrm{Gr}_{z_k} \xrightarrow{\sim} QM_{G\times G,G}(\mathcal{Z}_\circ/B_\circ, \sigma, \xi, b)_{\mathbb{R}}$$
of (5.43) intertwines the involution $\delta_{\mathcal{Z}}$ with the involution on $\mathrm{Gr}_{z_1} \times \cdots \times \mathrm{Gr}_{z_k}$ given by $(\gamma_1, ..., \gamma_k) \to (\gamma'_1, ..., \gamma'_k)$, where $\gamma'_i = \eta(\gamma_i)$ if $z_i \in S^1_{a^{-1}}$ and η is the conjugation on Gr_{z_i} induced by $c(z) = \frac{1}{a^2\bar{z}}$ on $\mathbb{P}^1$ and η on G, otherwise $\gamma'_i = \mathrm{inv} \circ\theta(\gamma_i)$ if $z_i \in \mathbb{C} \setminus S^1_{a^{-1}}$ and $\mathrm{inv} \circ\theta$ *is the involution on $\Omega_{z_i,a^{-1}} G_c \simeq \mathrm{Gr}_{z_i}$.*
3. *At $b = (0, t_1^+, , , .t_m^+) \in B_\circ(\mathbb{R})$, the isomorphism*
$$v_b :\ G[\mathbb{P}^1 \setminus \{t_1^+, , , .t_m^+\}]_{\infty\to e} \xrightarrow{\sim} QM^0_{G\times G,G}(\mathcal{Z}_\circ/B_\circ, \sigma, \xi, b)_{\mathbb{R}}$$
of (5.46) intertwines the involution $\eta_{\mathcal{Z}}$ with the involution $\mathrm{inv} \circ\theta(\gamma) = \theta(\gamma)^{-1}$ *on $G[\mathbb{P}^1 \setminus \{t_1^+, , , .t_m^+\}]_{\infty\to e}$.*

6. Stratified homeomorphisms

6.1. **Trivializations.** Set $\mathbb{B} = [0,1] \times \mathbb{R}^m$ and $\mathbb{B}' = (0,1] \times \mathbb{R}^m$.

Definition 6.1. An embedding

$$\zeta : \mathbb{B} = [0,1] \times \mathbb{R}^m \longrightarrow B_\circ(\mathbb{R}) \ \subset \mathbb{R} \times \mathbb{C}^m$$

is called *admissible* if it is given by $\zeta(a, z_1, ..., z_m) = (a, f_a(z_1), ..., f_a(z_m))$ where $f : [0,1] \to \mathrm{Aut}(\mathbb{P}^1)$ is a real analytic map satisfying $f_0(z) = z$ and $f_1(z) = \frac{z-i}{z+i}$.

Remark 6.2. Each admissible ζ defines a one-parameter family of embeddings $\zeta_a := \zeta|_{a \times \mathbb{R}^m} : \mathbb{R}^m \to \mathbb{C}^m$, $a \in [0,1]$ satisfying $\zeta_0(\mathbb{R}^m) = \mathbb{R}^m$ and $\zeta_1(\mathbb{R}^m) = (S^1 \setminus \{1\})^m$.

Example 6.3. Consider the map $f : [0,1] \to \mathrm{Aut}(\mathbb{P}^1)$ given by $f_a(z) = \frac{z-ai}{az+ai+(1-a)}$. A direct computation shows that $f_a(z) \neq a^{-1}$ for all $a \in \mathbb{R}$, and $f_0(z) = z$, $f_1(z) = \frac{z-i}{z+i}$, and hence the corresponding embedding $\zeta : \mathbb{B} \to B_\circ(\mathbb{R})$ is admissible.

Let $\zeta : \mathbb{B} \to B_\circ(\mathbb{R})$ be an admissible embedding. Consider the following base changes

$$QM_{G\times G,G}(\mathcal{Z}/\mathbb{B}, \sigma, \xi)_\mathbb{R} := QM_{G\times G,G}(\mathcal{Z}_\circ/B_\circ, \sigma, \xi)_\mathbb{R} \times_{B_\circ(\mathbb{R})} \mathbb{B} \longrightarrow \mathbb{B}$$

$$QM^0_{G\times G,G}(\mathcal{Z}/\mathbb{B}, \sigma, \xi)_\mathbb{R} := QM^0_{G\times G,G}(\mathcal{Z}_\circ/B_\circ, \sigma, \xi)_\mathbb{R} \times_{B_\circ(\mathbb{R})} \mathbb{B} \longrightarrow \mathbb{B}$$

$$QM_{G\times G,G}(\mathcal{Z}'/\mathbb{B}', \sigma, \xi)_\mathbb{R} := QM_{G\times G,G}(\mathcal{Z}'_\circ/B'_\circ, \sigma, \xi)_\mathbb{R} \times_{B'_\circ(\mathbb{R})} \mathbb{B}' \longrightarrow \mathbb{B}'$$

Then (5.40) and (5.41) restrict to isomorphisms

$$(\mathbb{R} \times \mathrm{Gr}^{(m)}) \times_{\mathbb{R}\times(\mathbb{P}^1)^m} \mathbb{B}' \xrightarrow{\sim} QM_{G\times G,G}(\mathcal{Z}'/\mathbb{B}', \sigma, \xi)_\mathbb{R} \ , \tag{6.1}$$

$$(\mathbb{R} \times \mathrm{Gr}^{(m),0}) \times_{\mathbb{R}\times(\mathbb{P}^1)^m} \mathbb{B} \xrightarrow{\sim} QM^0_{G\times G,G}(\mathcal{Z}/\mathbb{B}, \sigma, \xi)_\mathbb{R} \ . \tag{6.2}$$

Note that there is an isomorphism over $[0,1]$

$$[0,1] \times (\mathrm{Gr}^{(m)} \times_{(\mathbb{P}^1)^m} \mathbb{R}^m) \xrightarrow{\sim} (\mathbb{R} \times \mathrm{Gr}^{(m)}) \times_{\mathbb{R}\times(\mathbb{P}^1)^m} \mathbb{B}$$

given by

$$(a, (\mathcal{E}, z_1, ..., z_m, s)) = (a, (f_a)_*\mathcal{E}, f_a(z_1), ..., f_a(z_m), (f_a)_* s)$$

and in view of (6.1), (6.2), we obtain

Proposition 6.4. *Each admissible embedding $\zeta : \mathbb{B} \to B_\circ(\mathbb{R})$ induces isomorphisms:*

$$(0,1] \times (\mathrm{Gr}^{(m)} \times_{(\mathbb{P}^1)^m} \mathbb{R}^m) \xrightarrow{\sim} QM_{G\times G,G}(\mathcal{Z}'/\mathbb{B}', \sigma, \xi)_\mathbb{R} \ , \tag{6.3}$$

$$[0,1] \times (\mathrm{Gr}^{(m),0} \times_{(\mathbb{P}^1)^m} \mathbb{R}^m) \xrightarrow{\sim} QM^0_{G\times G,G}(\mathcal{Z}/\mathbb{B}, \sigma, \xi)_\mathbb{R} \ . \tag{6.4}$$

Note that the isomorphisms above coincide on the intersections of their domains.

6.2. **Families of involutions.** Let $p : \mathcal{G} \to \mathbb{R}^m$ be the group ind-scheme over $\mathbb{R}^m$ whose fiber over $(z_1, ..., z_m) \in \mathbb{R}^m$ is the group ind-scheme $G[\mathbb{P}^1 \setminus \{z_1, ..., z_m\}]_{\infty \to e}$ of maps $\gamma : \mathbb{P}^1 \setminus \{z_1, ..., z_m\} \to G$ such that $\gamma(\infty) = e$. The acton of $\mathcal{G}$ on the base point of $\mathrm{Gr}^{(m)} \times_{(\mathbb{P}^1)^m} \mathbb{R}^m$ defines an isomorphism

$$\mathcal{G} \xrightarrow{\sim} \mathrm{Gr}^{(m),0} \times_{(\mathbb{P}^1)^m} \mathbb{R}^m \tag{6.5}$$

Consider the transported strata $\mathcal{G}^{\lambda_\mathfrak{p}} := \mathcal{S}^{\lambda_\mathfrak{p},0} \cap \mathcal{G}$ of $\mathcal{G}$ and equip $[0,1] \times \mathcal{G}$ with the product stratification $\{[0,1] \times \mathcal{G}^{\lambda_\mathfrak{p}}\}$. By Proposition 6.4, we obtain a stratified isomorphism

$$p_\zeta : \ [0,1] \times \mathcal{G} \xrightarrow{\sim} QM^0_{G\times G,G}(\mathcal{Z}/\mathbb{B}, \sigma, \xi)_\mathbb{R}. \tag{6.6}$$

Recall the involution $\eta_\mathcal{Z}$ on $QM^0_{G\times G,G}(\mathcal{Z}_\circ/B_\circ, \sigma, \xi)_\mathbb{R}$ of Section 5.6. Via the isomorphism p_ζ, the involution $\eta_\mathcal{Z}$ gives rise to a family of involutions

$$\alpha_a : \mathcal{G} \longrightarrow \mathcal{G}, \quad a \in [0,1], \tag{6.7}$$

where $\alpha_a(\gamma) := p_\zeta^{-1} \circ \eta_\mathcal{Z} \circ p_\zeta(\{a\} \times \gamma)$. The following proposition follows from Proposition 5.12:

Theorem 6.5. *The family of involutions $\alpha_a : \mathcal{G} \longrightarrow \mathcal{G},\ a \in [0,1]$ satisfy the following:*

(1) We have $q \circ \alpha_a = q : \mathcal{G} \to \mathbb{R}^m$.
(2) α_a is G_c-equivariant and preserves the stratification $\{\mathcal{G}^{\lambda_\mathfrak{p}}\}$.
(3) At $a = 0$, we have $\alpha_0(\gamma(z)) = \theta(\gamma(z))^{-1}$.
(4) At $a = 1$, we have $\alpha_1(\gamma(z)) = \eta(\gamma(\bar{z}))$.

Proof. Part (1), (2), (3) is clear. Part (4) follows from the fact that the automorphism $f_1(z) = \frac{z-i}{z+i}$ satisfies $f_1(\bar{z}) = \frac{\bar{z}-i}{\bar{z}+i} = (\overline{f_1(z)})^{-1}$. □

6.3. **Trivializations of fixed-points.** Our aim is to trivialize the fixed-point of the family involution α_a. To that end, we will invoke the following lemma:

Lemma 6.6. *Let $I \subset \mathbb{R}$ be an interval. Let $M \to I$ and $N \to I$ be two stratified real analytic submersions of real analytic Whitney stratified ind-varieties M and N (where I is equipped with the trivial stratification). Let $f : M \to N$ be a Thom map.*

(1) Assume there is a compact group $H \times \mathbb{Z}/2$ acting real analytically on M such that the action preserves the stratifications and f is $H \times \mathbb{Z}/2$-invariant (where $H \times \mathbb{Z}/2$ acts trivially on N). Then the $\mathbb{Z}/2$-fixed-point ind-variety $M^{\mathbb{Z}/2}$ is Whitney stratified by the fixed-points of the strata and the induced map $f^{\mathbb{Z}/2} : M^{\mathbb{Z}/2} \to N$ is an H-equivariant Thom map.
(2) Assume further that f is ind-proper and there is an H-equivariant stratified trivialization of $f : M \to N$ over I, that is, there are stratified preserving homeomorphisms h_M and h_N fitting into a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{h_M} & I \times M_0 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id} \times f_0} \\ N & \xrightarrow{h_N} & I \times N_0 \end{array}$$

that are real analytic on each stratum. Then there is an H-equivariant stratified trivialization of $f^{\mathbb{Z}/2} : M^{\mathbb{Z}/2} \to N$ that is real analytic on each stratum.

Proof. Part (1) is proved in [N, Lemma 4.5.1]. For part (2), the H-equivariant stratified trivialization of $f : M \to N$ provides a horizontal lift of the coordinate vector field ∂_t on I to a continuous H-invariant vector field v on M that is tangent to and real analytic along each stratum. Let w be the average of v with respect to the $\mathbb{Z}/2\mathbb{Z}$-action. As f is ind-proper and the $\mathbb{Z}/2\mathbb{Z}$-action is real analytic, the vector field w is complete and the integral curves of w define an H-equivariant stratified trivialization of $f^{\mathbb{Z}/2} : M^{\mathbb{Z}/2} \to N$ over $[0,1]$ that is real analytic along each stratum. □

Now let us apply the above lemma to the map

$$QM_{G\times G,G}(\mathcal{Z}'/\mathbb{B}',\sigma,\xi)_{\mathbb{R}} \longrightarrow \mathbb{B}'$$

$$(\text{resp. } QM^0_{G\times G,G}(\mathcal{Z}/\mathbb{B},\sigma,\xi)_{\mathbb{R}} \longrightarrow \mathbb{B})$$

with the stratifications $\{\mathcal{S}^{\lambda_\mathfrak{p}}_{\mathcal{Z},\mathbb{R},\mathbb{B}'}\}$ and $\{(\mathbb{B}')^\mathfrak{p} = (0,1]\times\mathbb{R}^\mathfrak{p}\}$ (resp. $\{\mathcal{S}^{\lambda_\mathfrak{p},0}_{\mathcal{Z},\mathbb{R},\mathbb{B}}\}$ and $\{(\mathbb{B}^\mathfrak{p} = [0,1]\times\mathbb{R}^\mathfrak{p}\}$). We will consider the $H\times\mathbb{Z}/2$-action given by $K_c\times\langle\eta_\mathcal{Z}\rangle$.

Proposition 6.7. *(1) There is a K_c-equivariant topological trivialization of the fixed-points of $\eta_\mathcal{Z}$ of the map*

$$QM_{G\times G,G}(\mathcal{Z}'/\mathbb{B}',\sigma,\xi)_{\mathbb{R}} \longrightarrow \mathbb{B}'$$

over $I = (0,1]$.

(2) There is a K_c-equivariant topological trivialization of the fixed-points of $\eta_\mathcal{Z}$ of the map

$$QM^0_{G\times G,G}(\mathcal{Z}/\mathbb{B},\sigma,\xi)_{\mathbb{R}} \longrightarrow \mathbb{B}$$

over $I = [0,1]$.

Proof. For part (1), by Proposition 6.4, there is a K_c-equivariant stratified trivialization of

$$QM_{G\times G,G}(\mathcal{Z}'/\mathbb{B}',\sigma,\xi)_{\mathbb{R}} \longrightarrow \mathbb{B}' = (0,1]\times\mathbb{R}^m \tag{6.8}$$

Applying Lemma 6.6 with $H\times\mathbb{Z}/2 = K_c\times\langle\eta_\mathcal{Z}\rangle$, we obtain part (1).

For part (2), by Proposition 6.4, there is a K_c-equivariant stratified trivialization of

$$QM^0_{G\times G,G}(\mathcal{Z}/\mathbb{B},\sigma,\xi)_{\mathbb{R}} \longrightarrow \mathbb{B} = [0,1]\times\mathbb{R}^m \tag{6.9}$$

Following the proof of Lemma 6.6, consider the averaged vector field w with respect to the $\mathbb{Z}/2\mathbb{Z}$-action given by $\langle\eta_\mathcal{Z}\rangle$. We claim that w is complete, hence the integral curves of w provide the desired trivialization.

To prove the claim, observe that, over the open locus $\mathbb{B}'\subset\mathbb{B}$ in the base, the K_c-equivariant trivialization in (6.9) extends to a K_c-equivariant trivialization of the ind-proper family in (6.8).

Thus for any $b = (a,z_1,...,z_m)\in\mathbb{B}'$, any integral curve $p(t)$ for w with initial point $p(a)\in QM^0_{G\times G,G}(\mathcal{Z}'/\mathbb{B}',\sigma,\xi,b)_{\mathbb{R}}$ exists for $t\geq a$. Together with the local existence of integral curves with initial point in the special fiber $QM^0_{G\times G,G}(\mathcal{Z}'/\mathbb{B}',\sigma,\xi,b)_{\mathbb{R}}$, $b=(0,z_1,...,z_m)$, this implies $p(t)$ exists for all $t\in[0,1]$. Hence w is complete and we have proved the claim. □

6.4. **Real and symmetric spherical strata.** Let us summarize here the results obtained by the preceding considerations.

Recall we write $\mathcal{G} \simeq \mathrm{Gr}^{(m),0} \times_{(\mathbb{P}^1)^m} \mathbb{R}^m \to \mathbb{R}^m$ for the group ind-scheme over $\mathbb{R}^m$ whose fiber over $(z_1, ..., z_m) \in \mathbb{R}^m$ is the group of maps $\gamma : \mathbb{P}^1 \setminus \{z_1, ..., z_m\} \to G$ such that $\gamma(\infty) = e$. Recall the transported spherical strata $\mathcal{G}^{\lambda_\mathfrak{p}} = \mathcal{G} \cap \mathcal{S}^{\lambda_\mathfrak{p}}$.

The isomorphism $p_\zeta : [0,1] \times \mathcal{G} \simeq QM^0_{G\times G,G}(\mathcal{Z}/\mathbb{B}, \sigma, \xi)_\mathbb{R}$ in (6.6) together with Theorem 6.5 and Proposition 6.7 immediately imply:

Theorem 6.8. *There is a K_c-equivariant stratified homeomorphism between the fixed-points of η and $\mathrm{inv} \circ \theta$ on $\mathcal{G}$ compatibile with projections to $\mathbb{R}^m$. The homeomorphism restricts to a K_c-equivariant real analytic isomorphism between the fixed-points of η and $\mathrm{inv} \circ \theta$ on $\mathcal{G}^{\lambda_\mathfrak{p}}$.*

Observe that the fixed-points $\mathcal{G}^\eta$ coincide with the group ind-scheme $\mathcal{G}_\mathbb{R} \to \mathbb{R}^m$ of a point $(z_1, ..., z_m) \in \mathbb{R}^m$ and a map $\gamma : \mathbb{P}^1 \setminus \{z_1, ..., z_m\} \to G$, such that $\gamma(\mathbb{P}^1(\mathbb{R}) \setminus \{z_1, ..., z_m\}) \subset G_\mathbb{R}$ and $\gamma(\infty) = e$. Denote by $\mathcal{G}^{\lambda_\mathfrak{p}}_\mathbb{R} = \mathcal{G}^{\lambda_\mathfrak{p}} \cap \mathcal{G}_\mathbb{R}$ its spherical strata. Similarly, observe that the fixed-points $(\mathcal{G})^{\mathrm{inv}\circ\theta}$ coincides with the space $\mathcal{X} \to \mathbb{R}^m$ of a point $(z_1, ..., z_m) \in \mathbb{R}^m$ and a map $\gamma : \mathbb{P}^1 \setminus \{z_1, ..., z_m\} \to X \subset G$ such that $\gamma(\infty) = e$. Denote by $\mathcal{X}^{\lambda_\mathfrak{p}} = \mathcal{G}^{\lambda_\mathfrak{p}} \cap \mathcal{X}$ its spherical strata.

We can restate the above theorem in the form:

Theorem 6.9. *There is a K_c-equivariant stratified homeomorphism*

$$\mathcal{G}_\mathbb{R} \xrightarrow{\sim} \mathcal{X} \tag{6.10}$$

fitting into the diagram

$$\begin{array}{ccc} \mathcal{G}_\mathbb{R} & \xrightarrow{\sim} & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathbb{R}^m & \xrightarrow{\mathrm{id}} & \mathbb{R}^m \end{array}$$

that restricts to real analytic isomorphisms on strata

$$\mathcal{G}^{\lambda_\mathfrak{p}}_\mathbb{R} \xrightarrow{\sim} \mathcal{X}^{\lambda_\mathfrak{p}} \tag{6.11}$$

where $\lambda_\mathfrak{p} : \mathfrak{p} \to \mathcal{L}^+$ (for the subset $\mathcal{L}^+ \subset \Lambda_A^+$ defined in Section 2.1) with $|\lambda_\mathfrak{p}| \in R_G^+$.

Remark 6.10. Since the stratum $\mathcal{G}^{\lambda_\mathfrak{p}}_\mathbb{R} = S^{\lambda_\mathfrak{p}}_\mathbb{R} \cap \mathcal{G}_\mathbb{R}$ is the intersection of the real spherical stratum $S^{\lambda_\mathfrak{p}}_\mathbb{R}$ in the real Beilinson-Drinfeld grassmannian $\mathrm{Gr}^{(m)}_\mathbb{R}$ with the open cospherical stratum $\mathcal{G}_\mathbb{R}$, the above theorem implies that the singularities of the closures of the real spherical strata $\overline{S^{\lambda_\mathfrak{p}}_\mathbb{R}}$ are *locally* homeomorphic to complex algebraic varieties. However, one should note that the closures $\overline{S^{\lambda_\mathfrak{p}}_\mathbb{R}}$ are not in general *globally* homeomorphic to complex projective varieties. For example, for the rank one group $G_\mathbb{R} = \mathrm{SL}_2(\mathbb{H})$, and the generator $\lambda \in \Lambda_A^+$ (i.e., $m = 1$ and $\lambda = \lambda^\mathfrak{p} : \mathfrak{p} = \{1\} \to \Lambda_A^+$), one finds that $\overline{S^\lambda_\mathbb{R}} \subset \mathrm{Gr}_\mathbb{R}$ is the one-point compactification of the cotangent bundle $T^*\mathbb{HP}^1$ of the quaternionic projective line. Thus its intersection cohomology Poincaré polynomial is $1 + t^4 + t^8$ so does not satisfy the Hard Lefschetz Theorem.

6.5. **Kostant-Sekiguchi homeomorphism for GL_n.** Here we explain how the prior construction recover Theorem 1.1 and Theorem 1.2 in [CN2] but by a completely different argument.[3]

[3]The argument in [CN2] used quiver varieties and hyperkähler rotations.

Consider the case when $G = \mathrm{GL}_n(\mathbb{C})$ with real form $G_{\mathbb{R}} = \mathrm{GL}_n(\mathbb{R})$. We have $K = \mathrm{O}_n(\mathbb{C})$ and $K_c = \mathrm{O}_n(\mathbb{R})$ the complex and real orthogonal groups respectively. The conjugation η and involution θ on $\mathrm{GL}_n(\mathbb{C})$ are given by $\eta(M) = \overline{M}$ and $\theta(M) = (M^t)^{-1}$. Let $\mathfrak{g}_n = \mathfrak{gl}_n(\mathbb{C})$ be the Lie algebra of $G = \mathrm{GL}_n(\mathbb{C})$, $\mathfrak{t}_n \subset \mathfrak{g}_n$ the subspace of diagonal matrices, and $\mathfrak{c}_n = \mathfrak{t}_n/\!/\,\mathrm{S}_n$ the quotient of $\mathfrak{t}_n$ by the symmetric group S_n on n letters. Let $\chi : \mathfrak{g}_n \to \mathfrak{g}_n/\!/\,\mathrm{GL}_n(\mathbb{C}) \simeq \mathfrak{c}_n$ be the Chevalley map.

As observed by G. Lusztig and B.C. Ngo, for any $(M, z_1, ..., z_n) \in \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n(\mathbb{R})$, the formula $\gamma := \mathrm{id} - z^{-1}M$ defines a map from $\gamma : \mathbb{P}^1 \setminus \{z_1, .., z_n\} \to \mathrm{GL}_n(\mathbb{C})$ satisfying $\gamma(\infty) = \mathrm{id}$. Indeed, the collection $\{z_1, .., z_n\}$ is the set of eigenvalues of M and hence the matrix $\mathrm{id} - z^{-1}M$ is invertible for $z \in \mathbb{P}^1 \setminus \{z_1, .., z_n\}$. Thus we have an embedding

$$\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n(\mathbb{R}) \longrightarrow \mathcal{G} \simeq \mathrm{Gr}^{(n),0} \times_{\mathbb{P}^1} \mathbb{R}^n$$

given by

$$(M, z_1, ..., z_n) \to (1 - z^{-1}M, z_1, ..., z_n)$$

compatible with the natural projection maps to $\mathfrak{t}_n(\mathbb{R}) \simeq \mathbb{R}^n$. Moreover, by [Ngo, Lemma 2.3.1], the image of $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n(\mathbb{R})$ in $\mathcal{G}$ is a union of strata and the restriction of those strata to the fibers of the projection $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n(\mathbb{R}) \to \mathfrak{t}_n(\mathbb{R})$ are unions of orbits under the natural adjoint action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n(\mathbb{R})$. Thus the family of involutions in Theorem 6.5 restricts to a family of involutions

$$\alpha_a : \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n(\mathbb{R}) \to \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n(\mathbb{R}), \quad a \in [0,1] \tag{6.12}$$

satisfying the following properties:

Theorem 6.11. *The family of involutions* $\alpha_a : \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n(\mathbb{R}) \to \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n(\mathbb{R})$, $a \in [0,1]$ *satisfy the following:*

1. *We have* $\mathrm{pr} \circ \alpha_a = \mathrm{pr} : \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n(\mathbb{R}) \to \mathfrak{t}_n(\mathbb{R})$, *for all* $a \in [0,1]$.
2. α_a *is* $\mathrm{O}_n(\mathbb{R})$*-equivariant and takes a* $\mathrm{GL}_n(\mathbb{C})$*-orbit real analytically to a* $\mathrm{GL}_n(\mathbb{C})$*-orbit.*
3. *At* $a = 0$, *we have* $\alpha_0(M, z_1, ...z_n) = (\overline{M}, z_1, ..., z_n)$.
4. *At* $a = 1$, *we have* $\alpha_\infty(M, z_1, ...z_n) = (M^t, z_1, ...z_n)$.

Proof. Only part (3) and (4) require proof, and they follow from the following identities: for $\gamma(z) = \mathrm{id} - z^{-1}M$ we have $\alpha_0(\gamma(z)) = \overline{(\gamma(\bar{z}))} = \overline{(\mathrm{id} - \bar{z}^{-1}M)} = \mathrm{id} - z^{-1}\overline{M}$ and $\alpha_1(\gamma(z)) = \theta(\gamma(z))^{-1} = (\mathrm{id} - z^{-1}M)^t = \mathrm{id} - z^{-1}M^t$. □

Let $\mathfrak{g}_n(\mathbb{R})$ be the space of real $n \times n$-matrices and let $\mathfrak{g}_n^{\mathrm{sym}}$ be the space of $n \times n$ complex symmetric matrices. Theorem 6.9 implies the following result which can be viewed as a lift of the well-known Kostant-Sekiguchi bijection between real and symmetric nilpotent orbits to a stratified homeomorphism:

Theorem 6.12. *There is an* $\mathrm{O}_n(\mathbb{R})$*-equivariant homeomorphism*

$$\mathfrak{g}_n(\mathbb{R}) \times_{\mathfrak{c}_n} \mathfrak{t}_n(\mathbb{R}) \xrightarrow{\sim} \mathfrak{g}_n^{\mathrm{sym}} \times_{\mathfrak{c}_n} \mathfrak{t}_n(\mathbb{R}) \tag{6.13}$$

fitting into the diagram

$$\begin{array}{ccc} \mathfrak{g}_n(\mathbb{R}) \times_{\mathfrak{c}_n} \mathfrak{t}_n(\mathbb{R}) & \xrightarrow{\sim} & \mathfrak{g}_n^{\mathrm{sym}} \times_{\mathfrak{c}_n} \mathfrak{t}_n(\mathbb{R}) \\ \downarrow & & \downarrow \\ \mathfrak{t}_n(\mathbb{R}) & \xrightarrow{\mathrm{id}} & \mathfrak{t}_n(\mathbb{R}) \end{array}$$

that restricts to real analytic isomorphisms between $\mathrm{GL}_n(\mathbb{R})$ *and* $\mathrm{O}_n(\mathbb{C})$ *adjoint orbits on* $\mathfrak{g}_n(\mathbb{R}) \times_{\mathfrak{c}_n} \mathfrak{t}_n(\mathbb{R})$ *and* $\mathfrak{g}_n^{\mathrm{sym}} \times_{\mathfrak{c}_n} \mathfrak{t}_n(\mathbb{R})$.

Remark 6.13. It follows from the construction that the family of involutions in (6.7) are in fact equivariant under the natural S_n-action on $\mathcal{G} \simeq \mathrm{Gr}^{(m),0} \times_{(\mathbb{P}^1)^m} \mathbb{R}^m$. Thus the involutions in (6.12) are also equivariant under the natural S_n-action on $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n(\mathbb{R})$. Hence Theorem 6.11 and Theorem 6.12 have a direct analogy for the quotient $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n(\mathbb{R})/\!/\, \mathrm{S}_n$.

On the other hand, the quotient $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n(\mathbb{R})/\!/\, \mathrm{S}_n$ is isomorphic to the subset $\mathfrak{g}_n' \subset \mathfrak{g}_n$ consisting of matrices with real eigenvalues. Thus in this way we recover Theorem 1.1 and Theorem 1.2 in [CN2], but by a completely different argument.

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School of Mathematics, University of Minnesota, Minneapolis, Vincent Hall, MN, 55455

Email address: `chenth@umn.edu`

Department of Mathematics, UC Berkeley, Evans Hall, Berkeley, CA 94720

Email address: `nadler@math.berkeley.edu`