Dynamic Gains for Transient-Behavior Shaping in Hybrid Dynamic Inclusions

Daniel E. Ochoa, Jorge I. Poveda

***Abstract*— This paper presents a framework that enables analytically shaping the transient behavior of nonlinear dynamical systems, including those with hybrid dynamics combining continuous-time and discrete-time evolution. Our results hinge on the interconnection of the original system with an exogenous dynamic gain system designed to induce a continuous-time deformation of hybrid time domains. Our approach provides conditions that ensure the original system's stability properties without the dynamic gain are transferable under the continuous-time deformation to the full interconnected dynamics. We develop these results by leveraging tools from hybrid dynamical systems theory, and formulating an appropriate bijective map that relates the solution sets between the original and interconnected systems. To illustrate the approach, we present applications to gradient flow systems and momentum-based optimization techniques with resets, leveraging the framework to customize convergence rates for strictly convex objective functions.**

***Index Terms*— Hybrid Systems, Nonlinear Systems, Prescribed-Time stability**

I. Introduction

Controlling convergence rates to equilibrium points or optimal solutions is critical in dynamical systems theory and optimization algorithms. In machine learning, faster convergence can accelerate training of deep neural networks, while in embedded control systems, it enables more responsive and robust performance. Recent research has explored alternative formulations to achieve faster transient performance, including momentum-based dynamics [1], [2], finite-time stability techniques [3], hyperexponential stability methods [4], and fixed-time stability formulations [5], [6] where the convergence time is uniformly bounded for all initial conditions. Within this line of work, prescribed-time stability has gained increasing attention over the past five years [7], [8], [9], [10], [11], guaranteeing convergence to the desired target set within a pre-specified time interval, regardless of the initial conditions.

State-of-the-art prescribed-time stability approaches rely on incorporating time-varying or non-Lipschitz vector fields that exhibit finite-time blow-up behavior to drive the system to equilibrium before the prescribed convergence time. A viewpoint recently explored by the authors for switched systems [12] employs this blow-up effect by formulating a cascade interconnection between an exogenous dynamical system governing the evolution of a dynamic gain and a time-invariant dynamical system with suitable stability properties for the target set. This interconnection technique has been studied in adaptive control for parameter estimation [13], but has not been comprehensively analyzed for general nonlinear and hybrid dynamical systems combining continuous and discrete behaviors.

This addresses this gap by presenting a unifying framework to analytically shape the transient behavior of hybrid dynamical systems. Our approach is based on the design of a suitable exosystem that governs the evolution of a dynamic gain. By analyzing the continuous-time deformations that the flow of this system induces on hybrid time domains, we provide sufficient conditions that ensure the original system's stability properties, without the dynamic gain, are transferable under the continuous-time deformation to the full interconnected dynamics. We develop these results by leveraging tools from hybrid dynamical systems theory [14], and formulating an appropriate bijective map that relates the solution sets between the original and cascade systems.

Our formulation enables addressing a wide array of time deformations: from simple constant rescaling of time to prescribed-time scalings. Importantly, our approach can accommodate systems with hybrid dynamics arising from logic-based switched controllers, state resets, and other discrete-time behaviors coupled with physical dynamics. The proposed techniques have applications in expediting optimization solvers and enforcing real-time performance in feedback control.

The paper is organized as follows: Section II introduces key concepts of hybrid dynamical systems and their solution sets. Section III illustrates our approach through a motivating example using a gradient flow system. We present our main theoretical result in Section IV, followed by its applications to gradient flow systems and momentum-based optimization techniques with resets for strictly convex functions in Section V. Finally, Section VI summarizes our findings and outlines directions for future research.

II. Preliminaries

Notation: We use $|\cdot|$ to denote the Euclidean norm. Given a closed set $\mathcal{A} \subset \mathbb{R}^n$ and a vector $z \in \mathbb{R}^n$, we let $|z|_{\mathcal{A}} := \min_{s\in\mathcal{A}} |z-s|$. A function $\alpha : \mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r\to 0^+} \alpha(r, s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s\to\infty} \alpha(r, s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{KLL}$ if for every $s \geq 0$, $\beta(\cdot, s, \cdot)$ and $\beta(\cdot, \cdot, s)$ belong to class $\mathcal{KL}$.

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Throughout the paper, for two (or more) vectors $u, v \in \mathbb{R}^n$, we write $(u, v) = [u^\top, v^\top]^\top$ to denote their concatenation.

Hybrid Dynamical systems: In this paper, we study hybrid dynamical systems (HDS) aligned with the framework of [14], and described by:

$$z \in C, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} \in F(z), \tag{1a}$$

$$z \in D, \quad z^+ \in G(z), \tag{1b}$$

where $z \in \mathbb{R}^n$ is the main state, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called the flow map, $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called the jump map, $C \subset \mathbb{R}^n$ is called the flow set, and $D \subset \mathbb{R}^n$ is called the jump set. The data of system (1) is represented as $\mathcal{H} = (C, F, D, G)$. Solutions to system (1) are parameterized by a continuous-time index $t \in \mathbb{R}_{\geq 0}$, which increases continuously during flows, and a discrete-time index $j \in \mathbb{Z}_{\geq 0}$, which increases by one during jumps. Thus, solutions to (1) are defined on *hybrid time domains* (HTDs).

Definition 1 (Hybrid Time Domains): A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is called a *compact* hybrid time domain if $E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \ldots \leq t_J$. The set E is a HTD if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, \ldots, J\})$ is a compact HTD. Given a HTD E, we use $\sup_t E := \sup \{t \in \mathbb{R}_{\geq 0} : \exists\, j \in \mathbb{Z}_{\geq 0}, \text{ such that } (t, j) \in E\}$. □

Definition 2 (Hybrid Arc): A hybrid arc consists of a hybrid time domain, denoted by $\mathrm{dom}(z)$, and a function $z : \mathrm{dom}(z) \to \mathbb{R}^m$, such that $z(\cdot, j)$ is locally absolutely continuous on $I_j := \{t : (t, j) \in \mathrm{dom}(z)\}$ for each $j \in \mathbb{Z}_{\geq 0}$ such that I_j has nonempty interior. □

Solutions to a hybrid system $\mathcal{H}$ are hybrid arcs z satisfying conditions determined by the hybrid time domain $\mathrm{dom}(z)$ and the data of the HDS (C, F, D, G).

Definition 3 (Solutions to Hybrid Systems): [14, Definition 2.6] A hybrid arc z is a solution to $\mathcal{H}$ in (1) if $z(0, 0) \in \overline{C} \cup D$ and

1) For all $j \in \mathbb{Z}_{\geq 0}$ with $I_j = \{t : (t, j) \in \mathrm{dom}(z)\}$ having nonempty interior: $z(t, j) \in C$ for all $t \in \mathrm{int} I_j$, and $\dot{z}(t, j) \in F(z(t, j))$ for almost all $t \in I_j$;
2) For all $(t, j) \in \mathrm{dom}(z)$ with $(t, j + 1) \in \mathrm{dom}(z)$: $z(t, j) \in D$ and $z(t, j + 1) \in G(z(t, j))$. □

A solution z to $\mathcal{H}$ is maximal if no other solution $\tilde{z}$ exists with $\mathrm{dom}(z) \subset \mathrm{dom}(\tilde{z})$ and $z(t, j) = \tilde{z}(t, j)$ for all $(t, j) \in \mathrm{dom}(z)$. We denote by $\mathcal{S}_{\mathcal{H}}(K)$ the set of all maximal solutions z to $\mathcal{H}$ with $z(0, 0) \in K \subset \mathbb{R}^n$. For $K = \{z_0\}$, we write $\mathcal{S}_{\mathcal{H}}(z_0)$, and let $z \in \mathcal{S}_{\mathcal{H}}$ indicate that z is a maximal solution to $\mathcal{H}$ when no set K is specified. A solution is complete if its domain is unbounded. We will use the following definition to study the stability and convergence properties of systems of the form (1).

Definition 4: The closed set $\mathcal{A} \subset \mathbb{R}^n$ is said to be β-uniformly globally asymptotically stable (β-UGAS) if there exists $\beta \in \mathcal{KLL}$ such that each $z \in \mathcal{S}_{\mathcal{H}}$ satisfies

$$|z(t, j)|_{\mathcal{A}} \leq \beta(|x(0, 0)|_{\mathcal{A}}, t, j), \tag{2}$$

for all $(t, j) \in \mathrm{dom}(z)$. When $D = \emptyset$ in (1), i.e. for continuous-time systems, β-UGAS is defined with $\beta \in \mathcal{KL}$ and dropping the j-dependence in (2). □

III. Motivational Example

Consider a cost function $f(x) = \frac{x^4}{4}$, and the gradient-flow dynamics given by

$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}s} = -\nabla f(\hat{x}) = -\hat{x}^3. \tag{3}$$

The unique solution to (3) is given by

$$\hat{x}(s) = \frac{x_0}{\sqrt{2x_0^2 s + 1}} \quad \forall x_0 \in \mathbb{R},$$

which yields

$$f(x(s)) = \frac{|x_0|^4}{4\left(2x_0^2 s + 1\right)^2}, \quad \forall\, s \geq 0. \tag{4}$$

To improve over the convergence rate of equation (4), we consider the case where the gradient flow dynamics in (3) are interconnected in a cascade configuration with a dynamic gain μ whose dynamics satisfy $\frac{\mathrm{d}\mu}{\mathrm{d}t} = 1$. We consider $\mu(0) = \mu_0 \in [1, \infty)$, and write the interconnected system as a HDS with flows given by:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\mu \cdot \nabla f(x), \quad \frac{\mathrm{d}\mu}{\mathrm{d}t} = 1, \quad (x, \mu) \in \mathbb{R} \times [1, \infty), \tag{5}$$

and jump set $D = \emptyset$. Since the dynamics of μ are independent of those of the state x, we can first solve for μ to obtain that $\mu(t) = t + \mu_0$, $\mu_0 \in [1, \infty)$. Replacing this result in (5) yields: $\frac{\mathrm{d}x}{\mathrm{d}t} = -(t + \mu_0)x^3$. This is a separable ODE, which results in the following solution:

$$x(t) = \frac{x_0}{\sqrt{2x_0^2\left(\frac{t^2}{2} + \mu_0 t\right) + 1}}, \tag{6}$$

for all $t \geq 0$, and leads to

$$f(x(t)) = \frac{|x_0|^4}{4\left(2x_0^2\left(\frac{t^2}{2} + \mu_0 t\right) + 1\right)^2}. \tag{7}$$

The improved convergence bound in (7) can be obtained from the bound (4) by substituting the time variable s with t and then transforming it under the diffeomorphism $\mathcal{D}_{\mu_0}(t) = \frac{(t+\mu_0)^2}{2} - \frac{\mu_0^2}{2}$. This diffeomorphism satisfies $\frac{d^2}{dt^2}\mathcal{D}_{\mu_0}(t) = 1$, which is the flow map for μ in (5).

The above motivating example illustrates that by interconnecting a gradient system with a dynamic gain, the convergence rate can be improved from $\mathcal{O}(1/t^2)$ to $\mathcal{O}(1/t^4)$. However, obtaining this result relied on having access to the closed-form solution for the state trajectory under interconnected dynamics. For general nonlinear systems where such closed-form solutions are unavailable, it remains unclear what conditions would permit the application of a similar interconnection procedure to achieve faster convergence rates. Additionally, the example only addressed a continuous-time dynamical system, while many applications involve systems with hybrid dynamics. Consequently, two fundamental questions arise:

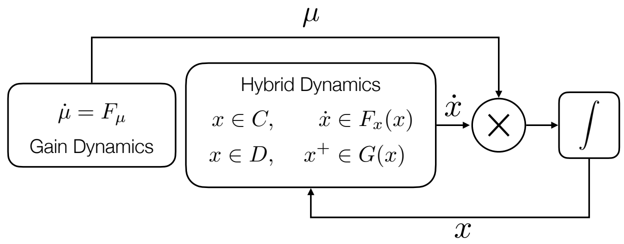

Fig. 1: Scheme of the cascade interconnection between hybrid dynamics and gain dynamics

1) What are the conditions on the original system's dynamics that allow for the application of the interconnection procedure and enable an improvement in the convergence bounds?
2) Can similar improvements be expected when interconnecting systems with hybrid dynamics?

In the following sections, we provide a unified approach to address both questions.

IV. Main Results

We consider a class of dynamical systems arising from the cascade interconnection of a target system exhibiting continuous-time and discrete-time dynamics, and a continuous-time dynamical system that governs the evolution of a dynamic gain μ. The structure of this interconnection is shown in Figure 1. We model the resulting dynamics as a hybrid dynamical system of the form (1), with state $z := (x, \mu) \in \mathbb{R}^n \times \mathcal{X}_\mu$, where $\mathcal{X}_\mu := [1, \infty)$. The hybrid dynamics are characterized by the following data:

$$z \in C_x \times \mathcal{X}_\mu, \quad \frac{\mathrm{d}z}{\mathrm{d}t} \in \mu F_x(x) \times \{F_\mu(\mu)\}, \tag{8a}$$

$$z \in D_x \times \mathcal{X}_\mu, \quad z^+ \in G(x) \times \{\mu\}, \tag{8b}$$

where $C_x, D_x \subseteq \mathbb{R}^n$. We refer to the HDS (8) as the *nominal* HDS and denote it with $\mathcal{H}$. Our main objective is to certify the stability properties of a suitable closed set $\mathcal{A} \subset \mathbb{R}^n$ under the nominal HDS $\mathcal{H}$, by deriving the stability properties from a *target* HDS $\hat{\mathcal{H}}$. This target system has state $\hat{z} = (\hat{x}, \hat{\mu})$ and is defined by the following data:

$$z \in C_x \times \mathcal{X}_\mu, \quad \frac{\mathrm{d}\hat{z}}{\mathrm{d}s} \in F_x(\hat{x}) \times \left\{\frac{1}{\hat{\mu}} F_\mu(\hat{\mu})\right\}, \tag{9a}$$

$$z \in D_x \times \mathcal{X}_\mu, \quad \hat{z}^+ \in G(\hat{x}) \times \{\hat{\mu}\}. \tag{9b}$$

To obtain our results, the following assumption will play a key role.

Assumption 1: a) For every $\hat{\mu}_0 \in \mathcal{X}_\mu$, there exists a unique and complete solution to the *gain ODE* $\frac{\mathrm{d}\hat{\mu}}{\mathrm{d}s} = \frac{1}{\hat{\mu}} F_\mu(\hat{\mu})$.
b) For each $c \in \mathcal{X}_\mu$ there exists non-decreasing diffeomorphism $\mathcal{D}_c : \mathcal{T}_c \to \mathbb{R}_{\geq 0}$, where $\mathcal{T}_c \subseteq \mathbb{R}_{\geq 0}$ and $\min \mathcal{T}_c = 0$, satisfying $\mathcal{D}_c(0) = 0$ for all $c \in \mathcal{X}_\mu$ and:

$$\frac{d}{dt}\mathcal{D}_{\mu_0}(t) = \left(\hat{\mu} \circ \mathcal{D}_{\mu_0}\right)(t), \quad \forall t \in \mathcal{T}_{\mu_0}, \ \mu_0 \in \mathcal{X}_\mu, \tag{10}$$

where $\hat{\mu}$ is the unique solution to the gain ODE. □

Assumption 1 restricts the type of dynamic gains we consider for our approach to those compatible with suitable deformations of the continuous-time domain, as characterized by the parameterized diffeomorphisms $\mathcal{D}_c$. Below, we present several examples illustrating different dynamic gains, their associated flow-maps F_μ, and the respective diffeomorphisms $\mathcal{D}_c$ satisfying Assumption 1.

Example 1: a) *Linear*: Let, $F_\mu(\mu) = 0$. Then, the diffeomorphism $\mathcal{D}_c(t) := ct$, with $c \in \mathcal{X}_\mu$, satisfies the matching equation (10) with $\mathcal{T}_c = \mathbb{R}_{\geq 0}$. This type of transformation appears in the singular-perturbation literature where constant parameters are employed to induce sufficient time-scale separations in multi-time scale systems; see [15, Ch. 11].

b) *Monomial*: Let $F_\mu(\mu) = (p-1)\mu^{\frac{p-2}{p-1}}$, $p > 1$. The unique solution to the gain ODE $\frac{\mathrm{d}\hat{\mu}}{\mathrm{d}s} = (p-1)\hat{\mu}^{\frac{1}{1-p}}$, $\hat{\mu} \in \mathcal{X}_\mu$, is given by

$$\hat{\mu}(s) = \left(\mu_0^{\frac{p}{p-1}} + s \cdot p\right)^{\frac{p-1}{p}}, \ \hat{\mu}_0 \in \mathcal{X}_\mu.$$

Then the map $\mathcal{D}_c : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$:

$$\mathcal{D}_c(t) := \frac{\left(t + c^{\frac{1}{p-1}}\right)^p - c^{\frac{p}{p-1}}}{p}, \ c \in \mathcal{X}_\mu,$$

satisfies the matching equation (10) with $\mathcal{T}_c := \mathbb{R}_{\geq 0}$.

c) *Exponential*: Let $F_\mu(\mu) = \mu$. The unique solution to the gain ODE $\dot{\hat{\mu}} = 1$, $\hat{\mu} \in \mathcal{X}_\mu$ is given by: $\hat{u}(s) = s + u_0$. Then, the map $\mathcal{D}_c : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined as

$$\mathcal{D}_c(t) := ce^t - c, \ c \in \mathcal{X}_\mu,$$

satisfies the matching equitation (10) with $\mathcal{T}_c = \mathbb{R}_{\geq 0}$.

d) *Prescribed-Time*: Let $F_\mu(\mu) = \frac{\mu^2}{\Upsilon}$, $\Upsilon > 0$. The unique solution to the gain ODE $\frac{\mathrm{d}\hat{\mu}}{\mathrm{d}s} = \hat{\mu}/\Upsilon$ is given by

$$\hat{\mu}(s) = \hat{\mu}_0 e^{\frac{s}{\Upsilon}}.$$

Then, the map $\mathcal{D}_c : \mathcal{T}_c := [0, \Upsilon/c) \to \mathbb{R}_{\geq 0}$ defined as

$$\mathcal{D}_c(t) := \Upsilon \ln\left(\frac{\Upsilon}{\Upsilon - ct}\right), \ c \in \mathcal{X}_\mu,$$

satisfies the matching equation (10) with $\mathcal{T}_c := [0, \Upsilon/c)$. dynamic gains characterized by a finite-escape time $T_{\mu_0} := \Upsilon/\mu_0 > 0$ which evolve according to the flow-map $F_\mu(\mu) = \mu^2/\Upsilon$ have recently gained widespread adoption in the context of prescribed-time regulation of dynamical systems [7]. In this context, the term T_{μ_0} receives the name of the *prescribed-time*.

To leverage the deformations of time-domains induced by the dynamic gains satisfying Assumption 1, we use the following assumption on the target system $\hat{\mathcal{H}}$.

Assumption 2: Every solution $\hat{z} \in \mathcal{S}_{\hat{\mathcal{H}}}$ satisfies $\sup_t \mathrm{dom}(\hat{z}) = \infty$. □

Now, by exploiting the cascade interconnection structure of the HDS $\mathcal{H}$, we can obtain the following result.

Lemma 1: Suppose Assumptions 1 and 2 are satisfied, and let $\mathbb{D}_c := \mathcal{D}_c \times \mathrm{id}_{\mathbb{Z}_{\geq 0}}$ for all $c \in \mathcal{X}_\mu$. Then, the map

$$\mathcal{W}: \quad \mathcal{S}_{\hat{\mathcal{H}}} \longrightarrow \mathcal{S}_{\mathcal{H}} \tag{11a}$$

$$\hat{z} = (\hat{x}, \hat{\mu}) \longmapsto \hat{z} \circ \mathbb{D}_{\hat{\mu}(0,0)}, \tag{11b}$$

is a bijection between maximal solution sets. Moreover, $\mathrm{dom}\,(\mathcal{W}(\hat{z})) = \mathbb{D}^{-1}_{\hat{\mu}(0,0)}\,(\mathrm{dom}(\hat{z}))$ for every $\hat{z} \in \mathcal{S}_{\hat{\mathcal{H}}}$. □

We are now ready to present our main result, which establishes a connection between the stability properties of the target hybrid dynamical system $\hat{\mathcal{H}}$ and the nominal hybrid dynamical system $\mathcal{H}$.

Theorem 1: Let $\mathcal{A}_0 := \mathcal{A} \times \mathcal{X}_\mu$, with $\mathcal{A} \subset \mathbb{R}^n$ a closed set, and suppose that Assumptions 1 and 2 are satisfied. Suppose that $\mathcal{A}_0$ is β-UGAS for $\hat{\mathcal{H}}$. Then, for every $z_0 = (x_0, \mu_0) \in (\overline{C}_x \cup D_x) \times \mathcal{X}_\mu$, and every $z \in \mathcal{S}_{\mathcal{H}}(z_0)$ it follows that

$$|z(t,j)|_{\mathcal{A}_0} \leq \beta(|z_0|_{\mathcal{A}_0}, \mathcal{D}_{\mu_0}(t), j),$$

for all $(t,j) \in \mathrm{dom}(z)$. □

Remark 1: When the function $\beta \in \mathcal{KL}$ takes an exponential form, and the dynamics of the gain follow the structure given in Examples 1a)-c), Theorem 1 certifies that the nominal system exhibits hyperexponential stability properties in the sense defined by [4]. If the gain dynamics are specified as in Example 1d), then Theorem 1 recovers the *prescribed-time stability results via flows* studied in [12] for hybrid systems, and in [16] for ODEs.

V. Applications

This section showcases Theorem 1 through two optimization examples.

1) Acceleration of Gradient Flows: Let f be a strictly convex, radially unbounded, and continuously differentiable function. Consider the nominal HDS denoted as $\mathcal{H}_g$, with continuous-time dynamics

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\mu \nabla f(x), \quad \frac{\mathrm{d}\mu}{\mathrm{d}t} = F_\mu(\mu), \;\; (x,\mu) \in \mathbb{R}^n \times \mathcal{X}_\mu, \tag{12}$$

where we assumed that the map F_μ satisfies Assumption 1, and with no discrete-time evolution, i.e., with $D = \emptyset$. Under these dynamics, we establish the stability of the compact set $\mathcal{A} := \{x^\star\}$, where $x^\star$ is the unique minimizer of f. To this end, we begin by considering the target HDS $\hat{\mathcal{H}}_g$ with continuous-time evolution:

$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}s} = -\nabla f(\hat{x}), \quad \frac{\mathrm{d}\hat{\mu}}{\mathrm{d}s} = \frac{1}{\hat{\mu}} F_\mu(\hat{\mu}), \quad \hat{z} = (\hat{x}, \hat{\mu}) \in \mathbb{R}^n \times \mathcal{X}_\mu,$$

and no discrete-time dynamics, and assume that the ODE $\frac{\mathrm{d}\hat{\mu}}{\mathrm{d}s}$ satisfies Assumption 1. Next, we augment the system with a timer state $\hat{\tau}$ with dynamics $\frac{\mathrm{d}\hat{\tau}}{\mathrm{d}s} = 1$, $\hat{\tau} \in \mathbb{R}_{\geq 0}$, and denote the resulting HDS with $\hat{\mathcal{H}}_\tau$. We study the stability properties of the set $\mathcal{A}_0 \times \mathbb{R}_{\geq 0}$, $\mathcal{A}_0 := \mathcal{A} \times \mathcal{X}_\mu$, by considering the candidate Lyapunov function

$$V(\hat{z}, \hat{\tau}) = \hat{\tau}(f(\hat{x}) - f^\star) + \frac{1}{2}|\hat{x}|^2_{\mathcal{A}},$$

where $f^\star := f(x^\star)$. The continuous-time derivative of V along the trajectories of $\hat{\mathcal{H}}_\tau$ is given by:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s} V(\hat{z}, \hat{\tau}) = &-\big(f^\star - [f(\hat{x}) + \langle \nabla f(\hat{x}), x^\star - \hat{x}\rangle]\,\big) \\ &- \hat{\tau}|\nabla f(\hat{x})|^2. \end{aligned} \tag{13}$$

Letting $d_f(x^\star, x) := f^\star - [f(\hat{x}) + \langle \nabla f(\hat{x}), x^\star - \hat{x}\rangle]$, by the strict convexity of f, it follows that $d_f(x^\star, x) > 0$ for all $\hat{x} \in \mathbb{R}^n \setminus \mathcal{A}$, and that $d_f(x^\star, x) = 0$ and $\nabla f(\hat{x}) = 0$ if and only if $\hat{x} \in \mathcal{A}$. From (13), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}s} V(\hat{z}, \hat{\tau}) = -\rho_c\,(|\hat{x}|_{\mathcal{A}}) = -\rho_c(|(\hat{z}, \hat{\tau})|_{\mathcal{A}_0 \times \mathbb{R}_{\geq 0}}), \tag{14}$$

for some $\rho_c \in \mathcal{PD}$, and all $(\hat{z}, \hat{\tau}) \in \mathbb{R}^n \times \mathcal{X}_\mu \times \mathbb{R}_{\geq 0}$, where we used the fact that $|(\hat{\mu}, \hat{\tau})|_{\mathcal{X}_\mu \times \mathbb{R}_{\geq 0}} = 0$ for all $(\hat{\mu}, \hat{\tau}) \in \mathcal{X}_\mu \times \mathbb{R}_{\geq 0}$. By [17, Theorem 3.19.3a)], this implies that there exists $\beta_g \in \mathcal{KL}$ such that $\mathcal{A}_0 \times \mathbb{R}_{\geq 0}$ is β_g-UGAS for $\hat{\mathcal{H}}_\tau$. Additionally, since for every solution $(\hat{z}, \hat{\tau}) \in \mathcal{S}_{\hat{\mathcal{H}}_\tau}$ it follows that $|\hat{\tau}(s)|_{\mathbb{R}_{\geq 0}} = 0$ for all $s \in \mathrm{dom}(\hat{z}, \hat{\tau}) = \mathbb{R}_{\geq 0}$, we obtain that $\mathcal{A}_0$ is β_g-UGAS for $\hat{\mathcal{H}}_g$. Additionally, for any such solution $(\hat{z}, \hat{\tau})$, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}s} V(\hat{z}(s), \hat{\tau}(s)) = -\rho_c\,(|\hat{x}(s)|_{\mathcal{A}})\,,$$

for every $s \in \mathbb{R}_{\geq 0}$. Integrating both sides and using the definition of V yields

$$\hat{\tau}(s)(f(\hat{x}(s)) - f^\star) + \frac{1}{2}|\hat{x}(s)|^2_{\mathcal{A}} = V_0 - \int_0^s \rho_c(|x(\tilde{s})|_{\mathcal{A}})d\tilde{s},$$

where $V_0 := V(z_0, \tau_0)$ and $(z_0, \tau_0) := (\hat{z}(0), \hat{\tau}(0))$. Then, using the positive definiteness of ρ_c together with the fact that $\hat{\tau}(s) = \tau_0 + s$, we obtain the following bound:

$$f(\hat{x}(s)) - f^\star \leq \frac{V_0}{s + \hat{\tau}_0} \leq \frac{V_0}{s}, \tag{15}$$

for any $s \in \mathbb{R}_{\geq 0}$. Using these results, together with Lemma 1 and Theorem 1, we obtain the following proposition.

Proposition 1: There exists $\beta_g \in \mathcal{KL}$ such that for any $(x_0, \mu_0) \in \mathbb{R}^n \times \mathcal{X}_\mu$ and any solution (x, μ) to the accelerated gradient flow dynamics (12), the following bound holds

$$|x(t) - x^\star| \leq \beta_g(|x_0 - x^\star|, \mathcal{D}_{\mu_0}(t)), \tag{16}$$

for all $t \in \mathrm{dom}((x, \mu))$, where $\mathcal{D}_{\mu_0}$ is as defined in Assumption 1. Moreover, for any such solution, the sub-optimality measure $f(\cdot) - f^\star$ satisfies

$$f(x(t)) - f^\star \leq \frac{c_0}{\mathcal{D}_{\mu_0}(t)}, \tag{17}$$

where $c_0 \in \mathbb{R}_{\geq 0}$ is a constant that depends on the initial condition x_0. □

Remark 2: Proposition 1 extends the results of the example in Section III to the case of strictly convex functions, and arbitrary dynamic gains satisfying Assumption 1. In particular, if we let $F_\mu(\mu) = \frac{\mu^2}{\Upsilon}$ (Example 1-d)), the dynamics described in (12) achieve convergence of the function $f(x(t))$ to the optimal value $f^\star$ within prescribed-time $T_{\mu_0} = \mu_0/\Upsilon$, adjustable to the user preference by the choice of $\Upsilon > 0$. □

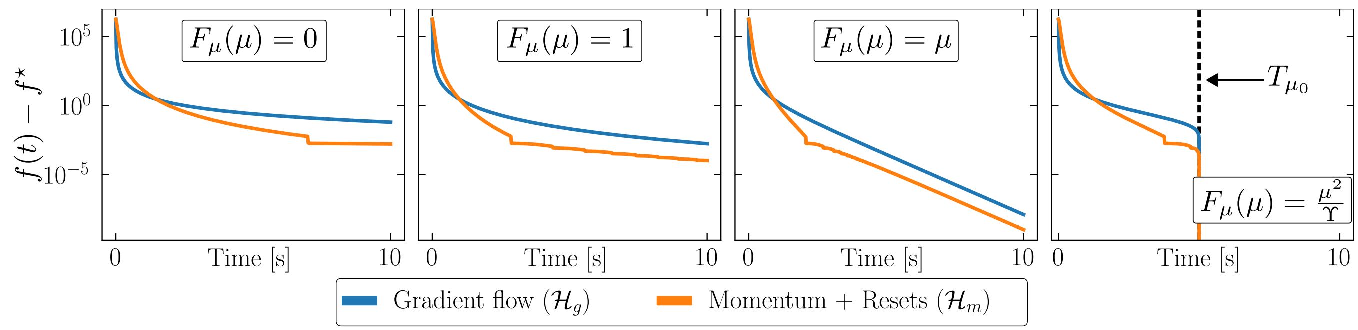

Fig. 2: Trajectories of solutions to the gradient-flow dynamics with dynamic gain $\mathcal{H}_g$ and the momentum-based dynamics with resets $\mathcal{H}_m$, under different parameterizations of the gain flow maps F_μ.

2) Acceleration of Hybrid Dynamics with Momentum: When the convex function f has low curvature, the dynamics in (12) might still suffer from poor transient performance. When $\mu \equiv 1$ and $F_\mu(\mu)=0$, this issue has been addressed in the existing literature by the use of momentum-based dynamics; see [1],[2],[18]. Inspired by these approaches, we consider a momentum-based nominal HDS, denoted as $\mathcal{H}_m$, with state $z := (x,\mu)$, where $x := (x_1,x_2,x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times [\underline{T},\overline{T}]$, flow-map given by [1]:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = \mu\cdot F_m(x) := \mu\begin{pmatrix}\frac{2}{x_3}(x_2-x_1)\\ -2x_3\nabla f(x_1)\\ \frac{1}{2}\end{pmatrix},\ \dot{\mu}=F_\mu(\mu),$$

and flow-set defined as $C_m \times \mathcal{X}_\mu$, where the map F_μ is assumed to satisfy Assumption 1, and where $C_m := \mathbb{R}^n \times \mathbb{R}^n \times [\underline{T},\overline{T}]$, with $\overline{T} > \underline{T} > 0$ being tunable parameters. The discrete-time dynamics of $\mathcal{H}_m$ evolve according to

$$x^+ = G_m(x) := \begin{pmatrix}x_1\\ x_1\\ \underline{T}\end{pmatrix},\quad \mu^+=\mu, \tag{18}$$

whenever $z \in D_m \times \mathcal{X}_\mu$, where $D_m := \mathbb{R}^n \times \mathbb{R}^n \times \{\overline{T}\}$. The HDS $\mathcal{H}_m$ with $\mu \equiv 1$ and $F_\mu(\mu)$ was introduced in [18, Eq. (8)] as an alternative to overcome the lack of uniformity in the convergence properties of Nesterov's ODE [1], which precludes obtaining β-UGAS bounds of the from (4); see [19, Remark. 2]. Indeed, using [18, Theorem 3.1], we can obtain the β-UGAS of the set

$$\mathcal{A}_0 := \mathcal{A}\times\mathcal{X}_\mu, \text{where } \mathcal{A} := \{x^\star\}\times\{x^\star\}\times[\underline{T},\overline{T}],$$

for the target HDS $\hat{\mathcal{H}}_m$, which is of the from in (9) with $F_x = F_m$, $C_x = C_m$, and $D_x = D_m$. This fact, together with Lemma 1 and Theorem 1, allows us to obtain the following proposition.

Proposition 2: There exists a function $\beta_m \in \mathcal{KLL}$ such that for any $(x_0,\mu_0) \in \mathbb{R}^n \times \mathcal{X}_\mu$ and any solution (x,μ) to the HDS $\mathcal{H}_m$,

$$|x|_\mathcal{A} \le \beta_m(|x_0|_\mathcal{A}, \mathcal{D}_{\mu_0}(t), j), \tag{19}$$

for all $(t,j) \in \text{dom}((x,\mu))$, where $\mathcal{D}_{\mu_0}$ is as defined in Assumption 1. Moreover, any such solution induces the bound

$$f(x_1(t,j)) - f^\star \le \frac{c_j}{(\mathcal{D}_{\mu_0}(t)-\underline{t}_j)^2}, \tag{20}$$

where $\underline{t}_j := \min\{t : (t,j)\in \text{dom}((x,\mu))\}$, and $\{c_j\}_{j=1}^\infty$ is a sequence of monotonously decreasing positive constants.□

3) Numerical Example: To numerically verify Propositions 1 and 2, we consider the cost function $f(x)=\frac{1}{400}|x-x^\star|^4$, where $x^\star=(\pi,2\pi,3\pi)$. We simulate the trajectories resulting from the HDS $\mathcal{H}_g$ and $\mathcal{H}_m$, using realizations of each gain flow map presented in Example 1, and using the initial conditions $x_0 = (100,100,100)$ and $\mu_0 = 1$. The momentum-based system $\mathcal{H}_m$ is implemented with reset parameters $\underline{T}=0.01$ and $\overline{T}=3.5$, and initial conditions $x_1(0,0)=x_2(0,0)=x_0$, $x_3(0,0)=\underline{T}$, and $\mu_0=1$.

The resulting trajectories of the suboptimality measure $f(x(t))-f^*$ are shown in Figure 2, verifying the theoretical bounds derived in Propositions 2 and 1. As illustrated, incorporating momentum and resets can improve convergence to the cost function minimizer compared to the gradient flow, regardless of the chosen dynamics for the gain. However, suitable reset parameters $\underline{T}$ and $\overline{T}$ must be selected in advance. As the time deformation induced by the gain flow map transitions from constant rescaling (leftmost plot) to prescribed-time scaling (rightmost plot), the frequency of resets naturally increases. This occurs because the state x_3 responsible for triggering resets is scaled by the dynamic gain, while the reset threshold $\overline{T}$ remains constant. This empirical evidence indicates that tuning reset or switching parameters with dynamic gains generally requires accounting for the time-domain deformation induced by the gain flow mappings. Recent work by the authors in [12] explored this direction by introducing blow-up average-activation time and blow-up average-dwell time conditions for prescribed-time regulation of switched systems. These conditions account for the finite escape times of the dynamic gains employed in prescribed-time stability approaches for switched systems.

VI. Conclusions

This paper introduces a framework for modulating the transient behavior of nonlinear dynamical systems, including hybrid systems with both continuous-time and discrete-time dynamics. Our approach involves interconnecting the original system with an exogenous dynamic gain, which generates

[1]Here $\dot{x}_i$ stands for $\frac{dx_i}{dt}$.

appropriate continuous-time deformations of hybrid time domains. We establish sufficient conditions for preserving stability properties through these time deformations, thus enabling a spectrum of transient behaviors ranging from constant to prescribed-time scalings. The proposed approach has potential applications in enhancing the performance of optimization algorithms and improving real-time control systems. Future research directions include extending this framework to systems exhibiting input-to-state, local, semiglobal, and practical stability properties. Other directions include endowing the dynamic gains μ with different discrete-time evolutions.

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