What Meaning Should We Attribute to the Colon in Set-builder Notation?

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In this study, we report one group of students' efforts to create a community meaning for set-builder notation collectively. Students' ability to develop and interpret set-builder notation is essential to transition-to-proof courses. Conventionally, a colon is used in set-builder notation to (1) separate the universe of discourse from the set's defining property and (2) indicate an ordering to these components, with the universe to the left and the property to the right of the colon. We describe one normative and non-normative interpretation of this notation and how the students' individual attribution of conventional meanings for the colon to different inscriptions within the notation helped (or inhibited) them from interpreting these expressions. We report how communicative discourse between the students affected their meanings and discussions.

Keywords: Set-builder notation, set-based reasoning, symbolization, transition-to-proof

The mathematical notion of a set is a crucial component of advanced mathematics. In the context of mathematical proof, students' understanding of sets and ability to posit appropriate relationships between sets can positively influence their interpretation of mathematical statements and proofs (Dawkins, 2017; Dawkins et al., 2023; Dawkins & Roh, accepted; Hub & Dawkins, 2018). Instructors often convey information about sets visually (e.g., Euler diagrams) or symbolically (e.g., set-builder notation). Still, many students struggle to reason viably about sets through these representational mediums. For example, Eckman et al. (2023) reported that some students create oval regions in Euler diagrams to (1) *gather* elements of the universe of discourse that fulfill a particular property or (2) *distinguish* between classes of elements when comparing two equal sets. In the symbolic sense, Eckman et al. (2023) reported that students can attribute various meanings to *arbitrary particulars* in set-builder notation (i.e., ΔABC).

This paper aims to investigate students' conceptions of an additional symbolic component of set-builder notation: the colon (:). In the conventional sense, mathematicians utilize the colon (sometimes written as a vertical bar |) in set-builder notation to (1) differentiate between the universe of discourse and the property by which the elements of the universe are partitioned into a set and its complement and (2) denote an ordinality to how students are supposed to create the set (i.e., first define a universe, then sort the elements of the universe). For example, a student considering the set $S = \{x \in \mathbb{Z} : x \text{ is divisible by } 4\}$ would first construe the universe as the set of all integers and then sort these integers into two sets: the integers divisible by 4 (set S) and the integers not divisible by 4 (set S^c, or the complement of S).

We report two instances where three students attempted to interpret set-builder notation to determine the relationship between two sets. Our data stem from the fourth day of a semester-long classroom teaching experiment we conducted to investigate the affordances of set-based reasoning for students' comprehension of transition-to-proof coursework. We provide the following research question to guide our discussion: *What do students' meanings for the expressions in set-builder notation reveal about their meanings for the colon?*

Theoretical Perspective

We adopt the framework proposed by Eckman (2023) to describe students' *symbolizing activity* or the process of mental activities that entails students' creation or interpretation of a perceptible artifact (writing, drawing, gesture, verbalization) to organize, synthesize, or communicate their thinking. We use the term *symbol* to denote a personal artifact to which a student has attributed a meaning (Thompson et al., 2014) that she can re-present to herself through the artifact (c.f., Glasersfeld, 1995). We employ Eckman's (2023) framework, which involves three symbols: *personal*, *communicative*, and *conventional* expressions.

We use the following example to illustrate the difference between the three types of expressions. Suppose an instructor of a transition-to-proof course presents her students with the set $A = \{x \in \mathbb{Z} : x \text{ is a multiple of } 3\}$. We consider the set-builder notation $\{x \in \mathbb{Z} : x \text{ is a multiple of } 3\}$ to constitute a *conventional* expression because the instructor presents a perceptible artifact to the students as an authorized representative of the mathematical community. In the moment of the presentation, each student attributes a meaning (Thompson et al., 2014) to the conventional notation $\{x \in \mathbb{Z} : x \text{ is a multiple of } 3\}$ to organize or synthesize portions of their experience, forming a *personal expression*. As the students interact and negotiate a community-approved meaning, the expression $\{x \in \mathbb{Z} : x \text{ is a multiple of } 3\}$ becomes a *communicative expression*. A vital component of a *communicative expression* is that individuals must interpret and reconcile the meanings others attribute to the expression (which may or may not reflect their thinking) with their personal meanings for the symbol.

Eckman (2023) described a *relational meaning* as one that students might attribute to their personal expressions. This paper focuses on relational meaning students may attribute to the colon in set-builder notation, which instructors conventionally use to express a relationship between the universe of discourse and the defining property for a set. There are two components to a relational meaning: *connector* and *comparator*. A *connector-oriented* meaning refers to students' conception of a relationship between two expressions and attributing this relationship to a symbol separating the two expressions. For example, the normative connector-oriented meaning an instructor may attempt to convey through the colon in the expression $\{x \in \mathbb{Z} : x \text{ is a multiple of } 3\}$ would be that the set of integers, $\mathbb{Z}$, is the universe of discourse and "multiples of 3" is the property defining the elements in the set. A *comparator-oriented* meaning refers to (1) a comparison action the student attributes to the symbol separating the two related expressions and (2) an ordering in which this comparison must occur. For example, an instructor might portray the colon as denoting an ordered process by which the set is created: (1) define the universe as the set of integers, $\mathbb{Z}$, and (2) use the property "x is a multiple of 3" to separate the elements of the universe into set A and its complement. The instructor would expect her students to use the notation (generally) and the colon (specifically) as *communicative expressions* to convey to others their images of the relationships between the integers and the multiples of three.

Mathematicians often attribute multiple meanings to a mathematical expression (Gray & Tall, 1994). In this sense, we consider a conventional meaning for the relational inscription (:) in set-builder notation to include viable *connector-oriented* and *comparator-oriented* meanings. We further expect that students possessing these meanings can re-present them through their *personal* and *communicative* expressions. In the results section of this paper, we address how the students' various comparator-oriented meanings for the components of set-builder notation facilitated or hindered their construction of *communicative expressions*.

Methodology

The data we present in this paper come from an ongoing project to investigate how set-based reasoning might help students to access transition-to-proof coursework (Dawkins et al., 2023; Eckman et al., 2023; Roh et al., 2023; Ruiz et al., 2023; Tucci et al., 2023). Specifically, we report data from the fourth day of a semester-long constructivist teaching experiment (Steffe & Thompson, 2000) during the Fall 2022 semester at a large public university in the United States. During this session, the students worked in groups to determine relationships between sets defined using set-builder notation. The instructor had presented the conventional meaning of set-builder notation immediately before this activity, and we consider the students' group work to constitute their attempt to co-construct set-builder notation as a communicative expression. We focus on one group of three male students, Enrique, Simón, and Juan. The second author was the instructor for the course, and the first author served as the discussion facilitator.

We collected students' work through audio recordings, photographs of students' notes, and pictures of collective whiteboard work. Our analysis first consisted of identifying moments in the data when the students disagreed about the meaning of an expression and needed to negotiate a collective interpretation. We analyzed these key moments using the principles of grounded theory (Strauss & Corbin, 1998). For example, we initially modeled the coevolution of the students' meanings for the set-builder notation during the critical moments (i.e., open coding). After generating a set of initial codes, we attempted to coordinate our codes into an overarching idea, which we determined to be students' attribution of meaning to the colon inscription (i.e., axial coding). In the results section of this paper, we describe how students' relational meanings they attributed (or did not attribute) to the colon inscription allowed them to interpret set-builder notation appropriately or, in other instances, led to cognitive conflict.

Results

The results section comprises two subsections. First, we provide an example of Enrique, Simón, and Juan's productive reasoning about set-builder notation and the relationship between two sets. Second, we share an example where each group member interpreted an instance of set-builder notation differently. In the discussion section, we describe how the students might have leveraged the non-normative interpretations for components of the notation they exhibited to make the seemingly "correct" interpretations we describe in the first subsection.

Distinct Meanings for Set-builder Notation Producing a Conventional Interpretation

At one point during the class, the students compared sets $A = \{x \in \mathbb{Z} : x \text{ is a multiple of } 3\}$ and $F = \{x \in \mathbb{Z} : x^2 - 1 \text{ is not a multiple of } 3\}$. The mathematical relationship between these sets is $A = F$ (i.e., both sets contain, and only contain, the multiples of 3).

During this comparison exercise, Simón and Enrique quickly interpreted the elements of set F and posited a relationship between sets A and F:

Simón Ok, so they're saying $x^2 - 1$ leaves a remainder of either 1 or 2 (*unintelligible*). So either x is a multiple of 3 itself, or, no, I think that's the only option, x has to be like a multiple of 3 because ,like, yeah (…) Ok, so I guess it's the same thing [i.e., set A and set F are identical].

Eckman Juan is looking confused.

Juan I don't understand how you got there.

Simón Because it's like, x, so, like if x were a multiple of 3, then this [$x^2 - 1$] won't be [a multiple of 3], like this $x^2 - 1$.

Enrique Right, because x^2 would also be a multiple of 3, but if you subtract 1, then it's no longer a multiple of 3.

(omitted dialogue)

Simón Or, I guess, it's like you could factor $x^2 - 1$ as $(x - 1)$ and $(x + 1)$.

Enrique Right, since you have a plus 1 and minus 1. Like, the thing about the second number [i.e., the expression "$x^2 - 1$ not a multiple of 3"] is it [i.e., x] has to be a multiple of 3, so it's got to be a multiple of 3 in order for it to [work].

Juan Oh yeah, that makes sense. So they would be the same set.

In this excerpt, Simón's order of reasoning indicates that he considered the multiples of 3 (i.e., set A) to be a subset of all integers x with the property "$x^2 - 1$ is not a multiple of 3" (i.e., set F). In contrast, Enrique's order of reasoning indicates that he considered the integers fulfilling the property "$x^2 - 1$ is not a multiple of 3" (i.e., set F) to be a subset of the multiples of 3 (i.e., set A). Collectively, Simón's reasoning that $A \subseteq F$ and Enrique's reasoning that $F \subseteq A$ satisfy the conditions to show that $A = F$. However, there was no indication from this excerpt that (1) Enrique and Simón recognized the subtle (to them) difference in their thinking or (2) either student attributed both conceptions to their communicative expression $F = \{x \in \mathbb{Z} : x^2 - 1 \text{ is not a multiple of 3}\}$. Juan's final comment indicates the possibility that he considered both qualifications (i.e., $A \subseteq F$ and $F \subseteq A$) when positing the elements of set F.

We purposefully made no direct reference to the colon (:) in this subsection. Instead, we chose to describe the possibility that Enrique, Juan, and Simón agreed on a collective meaning for the elements of the set F while maintaining distinct personal meanings for the set-builder notation denoting these elements. In the following subsection, we describe how these students' differences in *relational* meanings for their communicative expressions might be attributed to their meanings for the colon (:) inscription.

Distinct Meanings Producing a Mathematically Incorrect Interpretation

Immediately prior to comparing sets A and F, the students compared the sets $A = \{x \in \mathbb{Z} : x \text{ is a multiple of 3}\}$ and $E = \{2x \in \mathbb{Z} : x \text{ is a prime number}\}$. The mathematical relationship between these two sets is that $E \subset A$ (i.e., both sets share only the number 6).

The students spent most of their discussion negotiating a meaning for the elements of set E. Similar to the comparison between sets A and F, each student consecutively expressed a different personal meaning for the expression $\{2x \in \mathbb{Z} : x \text{ is a prime number}\}$:

Enrique To answer [the question] is anything in both sets, maybe there's nothing in E. Because you can't have a prime number that's a multiple of 2.

Simón Well, E, to me E was either the set of primes or the set of all the primes times two.

Juan I think [the set E is] all the prime numbers multiplied by two.

In contrast with their comparison between sets A and E, the students recognized the differences in their thinking this time. Enrique first claimed that set E is empty because (to him) no integers exist that are simultaneously even and prime (in actuality, the number 2 satisfies both conditions). Simón countered that the set E constituted one of two options (between which he could not decide): (1) the set of all primes or (2) the set of all primes multiplied by 2. Finally, Juan posited that the set E contains only the doubles of all primes (the normative interpretation). In the following subsections, we describe how each student's responses were influenced by their *comparator-oriented* meanings for the relationship between the expressions $2x \in \mathbb{Z}$ and "x is a prime number," which mathematicians conventionally attribute to the colon (:) inscription.

Enrique: The colon does not matter—the universe is determined by x. Enrique talked the least during the negotiation of meaning for the expression $\{2x \in \mathbb{Z} : x \text{ is a prime number}\}$ and did not explicitly agree to Simón and Juan's final decision for the meaning of this notation. Still, Enrique's comments about his meaning for set E were relatively consistent, as evidenced by the following statements he made at various times during this discussion:

Enrique Right. But, if you were to like plug in, I don't think this, it can't even exist.

Enrique No, because then if three equals x, then you'd have six [for $2x$] and six is not a prime number.

Enrique $2x$ is an element of the integers such that x is a prime number. Could you, does that even exist?

Enrique To answer [the question] is anything in both sets, maybe there's nothing in E. Because you can't have a prime number that's a multiple of 2.

These excerpts indicate that to Enrique, membership in set E, as defined by the expression $\{2x \in \mathbb{Z} : x \text{ is a prime number}\}$, required an integer first to satisfy the property "x is a prime number" and then satisfy the property $2x \in \mathbb{Z}$ (where $2x$ is also prime). In other words, Enrique first selected the set of prime numbers, P, as the universe of discourse. Then, he attempted to classify primes whose double was also prime as the elements of set E. Conventionally, we might write Enrique's definition for set E as $\{x \in P : x \text{ is a multiple of } 2\}$. When he could find no integers that satisfied his personal meaning for set E, he claimed the set was empty.

Enrique also stated that "the colon [in the set-builder notation] is like a subset." His comment emerged in response to Simón reviewing an example of set-builder notation he had written down in his notes. Enrique's meaning for E and his subset comment about the colon implies that he attributed his personal *comparator-oriented* meaning for the set-builder notation for E to the inscription x. In effect, Enrique considered x in relation to the universe of discourse and the colon as a synonym for "subset," giving no indication that he considered the colon to be more than a dividing symbol between two expressions (e.g., $x \in P, x$ is a multiple of 2) whose subset-relationship could be utilized to define the elements of a set.

Simón: Does x or the colon relate to the universe? Simón often led group discussions and frequently presented conventional interpretations of sets and relationships. Simón's initial conception of set E was that it contained "the set of primes." After realizing that neither Enrique nor Juan agreed with his personal meaning for the expression $\{2x \in \mathbb{Z} : x \text{ is a prime number}\}$, Simón began to wonder whether the set E contained the set of primes or the set of primes doubled. At this point of the discussion, the group facilitator intervened and asked the students to consider whether individual integers were elements of set E.

Eckman So I think in this case it might be nice to just pick some numbers and say like, "Oh, [Is] five in E? Is eight in E?" and see if you can figure it out that way.

Simón Okay, so … like the number 5. So 5 is a prime number, and 10 is an element of the integers?

Enrique Is that what it's saying that?

Simón But so then is 5 or 10 the number that is in the set [E]?

Simón's words indicate that to him, membership in E, as defined by the expression $\{2x \in \mathbb{Z} : x \text{ is a prime number}\}$, required an integer to satisfy the properties "x is a prime number" and $2x \in \mathbb{Z}$. Unlike Enrique, who envisioned $E = \{x \in P : x \text{ is a multiple of } 2\}$, Simón considered two distinct sets: $E_1 = \{x \in P : 2x \text{ is an integer}\}$ and $E_2 = \{2x \text{ is an integer} : x \in P\}$. For set E_1, Simón considered the set of primes to be the universe of discourse and claimed E contained all primes whose doubles were integers. This was a different approach than Enrique, who insisted that the double of a prime must also be prime to merit inclusion in set E. For set E_2, Simón considered the universe to constitute the set of even integers, and the elements of set E to comprise those even numbers whose value, when divided by two, is prime.

Although Simón did not explicitly describe his meaning for the colon (:) in the expression for set E, we infer, based on his remarks, that he recognized the order of comparison mattered for the expressions $2x \in \mathbb{Z}$ and "x is a prime number." At the beginning of the discussion, Simón appeared to attribute this ordinality to the inscription x (similar to Enrique). His later cognitive confusion emerged from considering whether to attribute the ordinality to the positions of the expressions (and the colon dividing them) or the variable x. In this case, Simón's *comparator-oriented* meaning for the inscription (:) was in development from a simple connector to an indication of an ordered process and relationship.

Juan: The colon divides the universe (left) from the property (right). Juan actively participated in group discussions when he agreed with the claims of his fellow students and quietly interjected or listened if he did not agree with or understand others' comments. Throughout the discussion about sets A and E, Juan insisted (quietly at first, then more rigorously later) that the set E contained the set of all primes doubled (the normative interpretation of set E). After Simón attempted to discern whether 5 or 10 was an element of E, Juan took command of the discussion:

Juan	Is the variable x the number that's in the set [E]? I don't think it is.
(*omitted dialogue*)	
Juan	Yeah, so what I'm reading here in my notes for set-builder notation is that where the $2x$ is in the general form [of set-builder notation], there's an $f(x)$ that represents a format by which every element of the set can be represented.
Enrique	Oh, that's useful.
Juan	So, every element of the set [E] is 2 times a prime number.

In this excerpt, Juan pinpointed what he considered the central point of conflict in the discussion: whether the integers represented by x were given inclusion in set E or the integers represented by $2x$ were given inclusion in the set. After referring to his notes on the general form of set-builder notation that he had taken during direct instruction, Juan stated that the expression $2x \in \mathbb{Z}$ was the arbitrary "format" (i.e., universe) by which to define the set and concluded that "2 times a prime number" was the way to denote the elements in set E.

Although Juan used his written notes to justify his claim about membership in set E, he repeatedly commented throughout the discussion that the elements of set E were the set of primes multiplied by 2. His consistent comments imply his *comparator-oriented* meaning for the colon (:) included a distinct and consistent ordering. This ordering included a notion that the expression that comes to the left of the colon constitutes the universe, and the expression following the colon constitutes the defining property for determining membership in a set. Juan's description of set-builder notation in arbitrary form (e.g., the first expression is $f(x)$) could also indicate that he was beginning to construct a general form of set-builder notation, which has been called a *personal expression template* (Eckman, 2023) or a *symbolic form* (e.g., Jones, 2013).

Discussion

The purpose of this paper was to provide insight into the question: *What do students' meanings for the expressions in set-builder notation reveal about their meanings for the colon?* In this discussion, we address three distinct ideas: (1) the three types of comparator-oriented meanings revealed by our data, (2) how Simón and Enrique might have leveraged their meanings for set E to make their mathematically correct interpretation of set F, and (3) the relevance of this paper in the context of research literature and student instruction.

We have described three relational *comparator-oriented* meanings that Juan, Simón, and Enrique attributed to the expressions $\{2x \in \mathbb{Z} : x \text{ is a prime number}\}$ and $\{x \in \mathbb{Z} : x^2 - 1 \text{ is not a multiple of } 3\}$. All three students appeared to attribute viable *connector-oriented* meanings to both expressions, perceiving that the expression on one side of the colon referred to the universe of discourse and that the other expression referred to the property by which elements of the set are identified.

However, the portion of the expression to which each student wished to attribute a *comparator-oriented* meaning differed. For instance, Enrique (and, at times, Simón) attributed a *comparator-oriented* meaning to the inscription x, which they considered to relate to the universe of discourse. Consequently, in these moments, the students merely attributed a *connector-oriented* meaning to the colon to divide two connected (to them) ideas. As the discussion progressed, Simón's *comparator-oriented* meaning developed so that he discerned two distinct ways to describe the elements of set E. In this moment, Simón began attributing a *comparator-oriented* meaning to the colon. Still, his meaning was tenuous and only achieved equal status with his prior meaning for x. Finally, Juan's intervention and description of his *comparator-oriented* meaning resulted in the group attributing exactly one order to the notation, indicating the relationship between a universe of discourse and a set defined within that universe.

Simón agreed, and Enrique did not disagree, with the Juan-proposed meaning for the communicative expression $E = \{2x \in \mathbb{Z} : x \text{ is a prime number}\}$. Still, it is possible that Simón and Enrique continued to leverage their personal meanings for x as an indicator of the universe to relate the two expressions for the set $F = \{x \in \mathbb{Z} : x^2 - 1 \text{ is not a multiple of } 3\}$. For instance, these students might have (1) identified the expression containing x ($x \in \mathbb{Z}$) to define the universe of discourse and (2) selected elements in the domain of discourse that satisfied the other expression (i.e., $x^2 - 1$ is not a multiple of 3) to define the elements of F.

This study furthers previously reported research on students' understanding of sets, set-builder notation, and symbolization. For example, our report provides additional insight into how students might interpret set-builder notation previously reported by (Eckman et al., 2023). Additionally, our explanation of *comparator-oriented* meanings including an ordered component adds to the examples and constructs proposed by (Eckman, 2023). This study also supports (to some extent) that conventional meanings for mathematical topics and symbols are not merely transmitted to individuals. Instead, the individuals must construct their personal meanings and attribute them to a conventional symbol to achieve a normative meaning for a mathematical topic. The students' active participation in the group discussion seemed to serve as a catalyst to bridge their personal meaning and conventional meaning by engaging in community efforts to build a collective meaning for the notation as a communicative expression.

This study also informs efforts to improve the instruction of sets and set-based reasoning in the context of transition-to-proof courses. For example, our results provide further insight into how students come to interpret set-builder notation, which can further inform teaching-oriented research projects related to set-based reasoning (Dawkins & Roh, accepted; Hub & Dawkins, 2018; Roh et al., 2023). Specifically, we describe a potential idiosyncratic student interpretation of set-builder notation, students' attribution (or misattribution) of *comparator-oriented* meanings in the context of set-builder notations. We encourage transition-to-proof instructors to explicitly address the comparison order between expressions in set-builder notation and the inscription (i.e., the colon :) to which this meaning should be attributed.

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