In Defense of Transformational Activity: Analyzing Students' Productive Reasoning about Equivalence

April Paige Richardson
Oklahoma State University

John Paul Cook
Oklahoma State University

Zackery Reed
Embry-Riddle Aeronautical University Worldwide

O. Hudson Payne
Oklahoma State University

Cory Wilson
Oklahoma State University

Elise Lockwood
Oregon State University

Equivalence is a foundational idea in mathematics and a key fixture in the K-16 curriculum. There is considerable evidence, however, that students at all levels experience difficulties with it. A prevailing explanation is that students rely too much on transformations; and yet, transformational activity is absolutely essential: it is the primary means by which one generates more tractable representations that are better suited to the situation at hand. Strikingly, we found no studies that directly examine students' productive uses of transformational activity. To this end, we conducted a series of task-based interviews with undergraduate students in order to illustrate and account for productive instances of transformational activity across undergraduate mathematics. Our findings affirm a hypothesis from the literature that supplementing one's transformational activity with notions of equivalence can support productive reasoning. Additionally, we extend this idea by providing detailed analyses of what these supplementary notions of equivalence entail.

Keywords: equivalence, task-based clinical interviews, conceptual analysis, student thinking

Equivalence is one of the most fundamental notions in all of mathematics and, as such, is prevalent throughout the K-16 curriculum. Despite this importance, research suggests that students at all levels have difficulty leveraging equivalence to solve mathematical tasks (e.g., Chick, 2003; Chesney et al., 2013; Kieran, 1981). Researchers have argued that many of these difficulties are attributable to an overreliance on *transformations*–that is, operations that change a mathematical object into a new (equivalent) form. The prevailing explanation is that attending to transformations can preclude attending to underlying notions of equivalence (e.g., Alibali et al., 2007; Carpenter et al., 2003; Kieran, 1981). That is, "an overemphasis on change overshadows an emphasis on sameness" (Cook et al., 2022, p. 5).

We observe that much of the literature on transformational activity focuses on its role in students' difficulties (e.g., Cook, 2018; Pomerantsev & Korosteleva, 2003; Tall et al., 2014). And yet, transformational activity is absolutely essential in mathematics: it is the primary means by which one generates equivalent representations of an object that are more well-suited to the problem at hand. Clearly, then, transformational activity can–and should–be productive for students, but we found no empirical studies that feature and analyze productive instances of transformational activity. We infer from the literature that supplementing with other ways of reasoning about equivalence can support productive transformational activity, but what these ways of reasoning are and how they might emerge in students' activity is a question that has not yet been examined. To this end, in this study we aim to provide a "positive counterpoint" (Bagley & Rabin, 2016, p. 84) to the large body of work that primarily associates transformational activity with students' difficulties by analyzing episodes in which students use it productively. In doing so, we aim to answer the following research question: *What cross-domain ways of reasoning about equivalence do students demonstrate when engaging in productive transformational activity?*

Literature

This study addresses two gaps in the literature. First, we observe that most of the literature on students' reasoning about equivalence has taken place *within* particular mathematical contexts. At the K-12 level, for example, research has examined equivalence of fractions (e.g., Smith, 1995), numerical expressions (e.g., McNeil, 2008), algebraic expressions (e.g., Solares & Kieran, 2013), and algebraic equations (Knuth et al., 2006). At the undergraduate level, research has included, for example, examinations within the domains of combinatorial equivalence (Lockwood & Reed, 2020), isomorphism (Larsen, 2013), and modular equivalence (Smith, 2006). There is, however, a scarcity of research that has examined how students might reason about equivalence *across* various contexts.

We note that though the literature on students' transformational activity is expansive, it has almost exclusively been associated with students' difficulties (e.g., Godfrey & Thomas, 2008; Kieran, 1981; Pomerantsev & Korosteleva, 2003; Stephens, 2006). Other studies have pointed out how difficulties with transformations can constrain students' abilities to learn about subsequent ideas (e.g., Cook, 2018; Tall et al., 2014). To be clear, this body of literature establishes an important point about transformational activity: an overreliance on it can constrain students' reasoning. We do wish to call attention to the fact that even though the importance of transformational activity is difficult to understate, research that illustrates and analyzes empirical instances of productive transformational activity and what it might entail is scarce. The literature overwhelmingly focuses on students' difficulties. Our efforts here were inspired by Bagley and Rabin (2016), who, upon observing that computational activity has been oft maligned in the linear algebra literature, illustrated how it can be a very useful tool in certain situations. In the same vein, we aim to provide a "positive counterpoint" (Bagley & Rabin, 2016, p. 84) to the treatment of transformational activity in the equivalence literature.

The literature does, however, contain some provisional theoretical suggestions in this respect that shaped the current study. Alibali and colleagues (2007) argued that attending to the equivalence of equations involves "recognition that the transformation preserves the equivalence relation expressed in the first equation" (p. 223). Similarly, Harel (2008) argued that it is important for students to recognize that "algebraic expressions are not manipulated haphazardly but with the purpose of arriving at a desired form and maintaining certain properties of the expression invariant" (p. 14); other researchers have made similar recommendations (e.g., Kieran, 1981; Steinberg et al., 1991). We interpret these comments to suggest that a key component of engaging productively with transformations involves attending to the reasons that the objects being generated by the transformations are equivalent. In the current study, we examine this initial hypothesis by examining the ways of reasoning about equivalence that students demonstrate in conjunction with their transformational activity.

Theoretical Framework

We adopted Cook and colleagues' (2022) framework for analyzing students' cross-domain ways of reasoning about equivalence. The framework is a *conceptual analysis* because it articulates "what students might understand when they know a particular idea in various ways" (Thompson, 2008, p. 57). Specifically, it outlines ways of reasoning that the authors hypothesize capture meaningful aspects of equivalence as it manifests across mathematical contexts. These include:

- *Common characteristic*: involves attending to equivalence in terms of "a perceived attribute that the objects in question have in common" (Cook et al., 2022, p. 3).
- *Descriptive*: involves attending to the fact that objects "describe the same quantity or serve the same purpose with respect to a given situation" (Cook et al., 2022, p. 3).
- *Transformational*: transformational activity is defined as "a sequence of actions (either already performed or imagined) by which one object might or can be changed into another is enacted or described" (Cook et al., 2022, p. 3).

For example, consider how one might multiply the numerator and denominator of 1/2 by 3 to obtain 3/6. One might supplement this example of transformational activity by explaining that 1/2 and 3/6 are equivalent because they both correspond to the same real number: 0.5 (an example of a *common characteristic* way of reasoning). One might also reason that 1/2 and 3/6 are equivalent by imagining two different ways of shading a circle: both of these fractions correspond to the same amount of shaded area in relation to the area of the whole circle (an example of a *descriptive* way of reasoning because both fractions describe the same quantity of shaded area).

We primarily use Cook and colleagues' (2022) conceptual analysis as a lens through which to build and articulate models of students' ways of reasoning. We note that Cook and colleagues (2022) positioned their framework as their own articulation of key features of the equivalence concept that might be advantageous for students to attend to across contexts. It therefore remains unclear if and how these ideas might emerge when working with students. Put another way, the *first order model* (Steffe et al., 1983) developed by Cook and colleagues (2022) has not yet been used to construct *second order models* (Steffe et al., 1983) of students' reasoning.

Methods

In order to examine students' transformational activity and their associated ways of reasoning about equivalence, we conducted individual task-based clinical interviews (Clement, 2000) with 12 students (due to space constraints, in this proposal we focus only on two interviews). All participants had recently completed a three-course Calculus sequence; each student participated in a single interview ranging from one to two hours in length. Interviews were conducted by the second author. The tasks administered are shown in Figure 1. Students' written work was recorded on an iPad application and was synced to an audio recording.

After completing each task, students were asked to explain their reasoning, particularly how they came to (a) produce one mathematical object from another, and (b) replace one form of a mathematical object with another. For example, students who used row operations to transform the linear system in Task 2 to row echelon form were asked how the new system is related to the original and why it is an acceptable replacement for the original. Interviews were transcribed verbatim and enhanced with screenshots of the students' written work.

We classified an instance of transformational activity as "productive" if the (a) the student successfully used transformations to complete the task, and (b) their answers to these follow-up questions about the objects generated by their transformations involved a description of justification of why they are equivalent. Each enhanced transcript was then independently coded by the first, fourth, and fifth authors using Clement's (2000) interpretive analysis cycles; Cook and colleagues' (2022) framework provided an initial basis for coding. Though the framework was refined as coding progressed, due to space constraints we focus here only on illustrating instances of how students productively supplemented their transformational activity with *common characteristic* and *descriptive* ways of reasoning. The codes for each transcript were then compiled, and coding discrepancies were discussed and revised until a state of negotiated agreement was reached.

Task 1:	Solve the following equation: $3(3x + 1/9) - 2(x - 1/4) = 6(x - 1/36)$
Task 2:	Solve the following system of linear equations: $x_1 + 5x_2 + 2x_3 = 8$ $2x_1 + 4x_2 + 2x_3 = 8$ $x_1 + 5x_2 + x_3 = 7$
Task 3:	Evaluate the following definite integral: $\int_0^1 2xe^{x^2} dx$
Task 4.1:	Consider how we add hours of time on a 12-hour clock. For example, 4 hours from 9:00 is 1:00. We can represent this as $4 \oplus 9 = 1$. Evaluate the following sums: a) $7 \oplus 8$; b) $9 \oplus 9$; c) $10 \oplus 5$; d) $11 \oplus 4$.
Task 4.2:	Consider now how we might multiply hours of time on a 12-hour clock. For example, $3 \odot 7 = 7 \oplus 7 \oplus 7 = 9$. Evaluate the following: a) $2 \odot 10$; b) $3 \odot 11$; c) $4 \odot 8$; d) $5 \odot 9$.
Task 4.3:	Evaluate the following: a) $7 \odot 7$; b) $11 \odot 11$; c) $9 \odot 11$; d) $8 \odot 7$; e) $10 \odot 8$.

Figure 1. Tasks administered during the interviews.

Results

Here we present the results of our analysis of the students' cross-domain ways of reasoning about equivalence. In each subsection, we begin by summarizing the student's relevant *transformational* activity to provide context. We focus on Ethan's demonstration of a *common characteristic* way of reasoning (across linear systems and the integers modulo 12) and Molly's demonstration of a *descriptive* way of reasoning (across fractions and the integers modulo 12).

Transformational Activity with a Common Characteristic Way of Reasoning

Ethan's transformational activity on Task 2 (solving a system of linear equations) and Task 4 (modular arithmetic) was supported by a *common characteristic* way of reasoning about equivalence. Using row operations to transform the augmented matrix for the given linear system into reduced row echelon form (Task 2), Ethan explained that "you do it systematically, […] you want the triangle effect…" and that "…the goal is to get it to reduced row echelon…" (see Figure 2). This action of "reduction" into reduced row echelon (RRE) form is indicative of transformational activity, whereby Ethan translated the system of equations into an augmented matrix, and further produced equivalent augmented matrices.

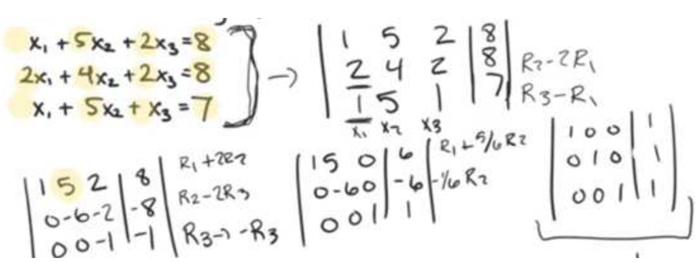

Figure 2. Ethan's written response to Task 2.

The interviewer then asked how the solution to the system represented by the RRE augmented matrix related to the system given in the task. Ethan's response suggested a *common characteristic* way of reasoning:

Ethan: This [RRE] helps complete the original task because […] if you plug these values in [1, 1, 1], they [the equations] would all come out as true. Like $1 + 5 + 2$, that equals 8. […] And then, if you went into that for every single one, you would see that the values would be true. […]

Interviewer: So this solution $x_1 = 1, x_2 = 1, x_3 = 1$ is not just a solution to this last augmented matrix with the 0s and 1s?

Ethan: No. […] It's a solution for all of them throughout the whole time. If you went back through and made equations with these variables, they would all end up being true. So […] every single step of the way, […] these values would make those equations true.

Interviewer: So, […] even though you're changing-

Ethan: The coefficients?

Interviewer: You're changing the coefficients […],

Ethan: Right. […] So, even though you're changing the system, the values, the corresponding values will still be the same.

Ethan identified a shared attribute among the objects he produced–i.e., the solution of $(x_1, x_2, x_3) = (1,1,1)$–indicating a *common characteristic* way of reasoning about equivalence. More specifically, Ethan was aware that the result of his transformational activity preserved an attribute of the objects he was transforming.

Ethan's initial activity in response to Task 4 (modular arithmetic) centered on multiplying the two integers using the typical multiplication and then repeatedly adding or subtracting 12 until he obtained an integer less than 12 (see Figure 3).

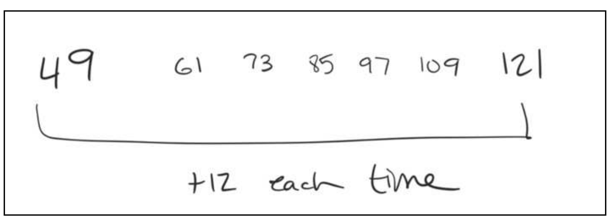

Figure 3. Part of Ethan's written response to Task 4.3.

On Task 4.2, after pointing out that $3 \odot 7$ and $5 \odot 9$ are both 9, the interviewer asked Ethan about the relationship between 21 and 45. Ethan explained that "if you add 12 to 21 you get 33" and "if you add 12 to 33 you get 45." He concluded, "so they are equivalent." On Task 4.3, similarly relating 49 and 121, Ethan said, "we have 49. We want to get to 121, and then […] I'm trying to add 12 to each interval, […] you're adding 12 each time" (see Figure 3). Ethan's transformational activity entailed adding 12 to one integer to produce another equivalent integer.

Ethan eventually pointed out a connection between division and his strategy of repeatedly adding/subtracting 12. He explained that "the reason that it's okay to subtract 12 is because that's basically what you do in division. […] It was how many times can 12 actually go into it, and it came out as, hey, let me just keep subtracting off of it." Shortly thereafter, the interviewer asked Ethan if he could characterize the integers that are equivalent to 4. He initially used his strategy of repeatedly adding 12, yielding 16, 28, and 40, before pivoting to the notion of remainder he had previously described:

Ethan: Their corresponding remainder will always be the original number you were wanting.
Interviewer: And in this case that's …
Ethan: Is 4.
Interviewer: I see. Okay. Even though you're adding 12 repeatedly?
Ethan: Correct. The remainder will always stay constant at that number.

Ethan demonstrated a *common characteristic* way of reasoning by identifying a shared attribute of the equivalent objects he was producing: their remainder after division by 12. Again, Ethan demonstrated awareness that his transformational activity (in this case, adding/subtracting 12) preserved the common characteristic (remainder after division by 12) of the objects in question.

A Descriptive Way of Reasoning about Equivalence

Molly demonstrated evidence of pairing transformational activity with a *descriptive* way of reasoning on Tasks 1 and 4. In her response to Task 1 (solving a linear equation), Molly transformed each fraction into an equivalent fraction with denominator of 36 in order to combine like terms (see Figure 4). The interviewer prompted Molly to explain why she made such replacements:

Interviewer: You rewrote […] -2/4 as -18/36. Can you talk about what you see as the relationship between those two?
Molly: Yeah. So, um, if you multiply -2/4 by 9/9 […] you get -18/36, but it doesn't change the value of the fraction. The fraction stays the same. It's just being written in a different way.
Interviewer: Okay. So, when you say, "The value of the fraction," what do you mean?
Molly: Um ... I mean, it's like if you take a circle and you cut it into 4 and shade 2, and then you cut another circle into 36 parts and shade 18, you'll see the exact same amount is shaded on both circles. […] So, that's how I know that's gonna be the same value.

The interviewer then prompted Molly to draw a picture to accompany her explanation, noting that she could choose another fraction to compare to 2/4 instead of 18/36 for ease of drawing (see Figure 4).

Molly: Like, if you do a circle that has 4, and you shade that, and then you have a circle that has 8, the same amount will be shaded in.
Interviewer: I see. And so, the relationship between 2/4 and 4/8 you're saying is similar to the relationship between 2/4 and 18/36 in here?
Molly: Yeah.

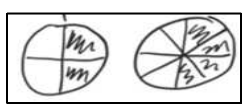

Figure 4. Part of Molly's written response to Task 1 to explain why 2/4 is equivalent to 4/8.

Molly's statement that "if you multiply -2/4 by 9/9 […] you get -18/36, but it doesn't change the value of the fraction" indicates transformational activity because she described a process ("multiply […] by 9/9") by which one fraction (-2/4) could be transformed into another (-18/36).

This transformational activity was supported by a *descriptive* way of reasoning when explaining *why* the transformation was valid. In particular, when describing why these fractions had the same value, Molly appealed to two circles having the same shaded region. This interpretation is further evidenced by Molly's drawing (see Figure 4) to show that 2/4 and 4/8 had the same value, explaining that “the same amount [i.e., the same area] will be shaded in” on both circles.

On Task 4, Molly produced integers she identified as “related” by repeatedly subtracting 12 until obtaining an integer between 1 and 12. Molly wrote“ $n = N + x(12)$”, where N represented the integer she was starting with, $x(12)$ signified adding a multiple of 12, and n represented the resulting integer. That is, Molly’s transformational activity entailed a procedure by which one object could be obtained from a “related” one. The interviewer asked Molly to describe how she would use this process to determine if two given integers were equivalent:

Interviewer: So if I give you... two integers, let’s say, um... 412 and 378... and asked you if they are related in the same way that you’ve been talking about these other numbers, how would you go about figuring that out?

Molly: … I would subtract both by, um, a multiple of 12, and I’ll get a number. And then I’ll see if n is equal. And if n is equal then I’ll know that they, um... are related.

Molly then enacted her procedure of subtracting multiples of 12 from 412 and 378 until she obtained the numbers 4 and 6, respectively.

Interviewer: Okay. Now, what are 4 and 6? ... how are you thinking about those...

Molly: These are what their position on the clock would be.

Interviewer: Okay.

Molly: Um, and since 4 and 6 are not the same position, then, um, 412, um, and 378 are not related.

Here, Molly viewed the resulting objects of her transformational activity in *descriptive* terms: “what their position on the clock would be.” Molly was able to judge the equivalence of 412 and 378 by reasoning that since the objects resulting from her transformational activity did not serve the same purpose with regards to telling time (i.e., their positions on a 12-hour clock), the objects were not equivalent.

Discussion

The analysis demonstrated here contributes to the literature in two ways. First, we have expanded the theoretical scope of the framework that we used to execute these objectives. That is, we used Cook and colleagues’ (2022) first-order conceptual analysis of equivalence to construct second-order models of students’ reasoning, demonstrating that these ways of reasoning can, in fact, account for students’ reasoning. Second, our analysis establishes a counterpoint to well-documented difficulties associated with transformational activity in the literature. Previously, analyses of productive instances of students’ transformational activity across domains had not yet been documented. In doing so, we affirmed a provisional hypothesis we inferred from the literature (that a key to reasoning productively with transformations includes supplementing with notions of how the objects being generated by these transformations are equivalent), and also explicated what these notions might entail (e.g., demonstrating *common characteristic* and *descriptive* ways of reasoning).

Acknowledgments

This project was funded by NSF Grant DUE #2055590.

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