From an Inclination to Subtract to a Need to Divide: Exploring Student Understanding and Use of Division in Combinatorics

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In this report, we provide an initial exploration into a key but under-studied phenomenon in enumerative combinatorics – the use of division in solving counting problems. We present a case of one undergraduate student solving a combinatorics problem; this case is representative of a broader phenomenon in which students may intuitively desire to account for an overcount using subtraction, when division is a productive and useful approach. We highlight the conceptions a student demonstrated as she progressed from using subtraction to using division successfully. We frame our analysis in terms of a set-oriented perspective (Lockwood, 2014).

Keywords: Arithmetic operations, Combinatorics, Discrete mathematics, Sets, Division

When solving counting problems, we may find ourselves needing to remove some undesirable outcomes from a larger set, in order to find the cardinality of the set of outcomes in question. For example, in the problem "How many sequences of 5 digits contain at least one 9?", an efficient strategy is to count all possible 5-digit sequences and then to subtract those that do not contain a 9. This strategy is common, and can be viewed as a special case of the well-known Principle of Inclusion/Exclusion (e.g., Tucker, 2002); subtraction is a powerful tool for counting.

In some cases, however, the operation of division, and not subtraction, is most useful. In fact, division represents an important, often necessary, way to account for overcounting. Consider the Table Problem, which is the focus of our case study in this paper: "How many ways are there to arrange 10 people around a circular table?" A common efficient solution involves division – we first count the ways to arrange 10 people in a line (10!), and then we note that each of the desired circular arrangements is actually overcounted by a factor of 10 (since each linear arrangement can be rotated 10 times to yield equivalent circular arrangements). Thus, we can strategically divide the number of linear arrangements by 10 to arrive at the correct answer of 9!. In such problems, we have observed that students' initial approach tends to focus on subtraction as a way to account for overcounting. Our main goal in this paper is to consider and discuss ways in which students may progress from an initial intuitive desire to subtract to a combinatorial understanding of how and when to divide in appropriate circumstances.

There are many reasons why we might want students to develop robust, productive ways of thinking about division in combinatorics. Indeed, it occurs frequently in problems, and it is a fundamental aspect of why certain formulas (such as the binomial coefficients) work as they do. As important as division is, it has not commonly been addressed in the teaching and learning of combinatorics. We argue that better understanding division as it relates to counting could be beneficial for students. Our motivation and goal here is to provide evidence for – and to lay groundwork for – future studies to examine division in combinatorics.

In this report we focus on a case study of one undergraduate student, who solved the Table Problem and, in so doing, transitioned from an approach focused on subtraction to one that successfully leveraged division. We particularly want to highlight what conceptions about division emerged for the student that allowed her to successfully solve the counting problem (particularly when she was not able to use subtraction successfully). We attempt to answer the following research questions by examining this case: *What conceptions and ways of reasoning* *emerged for an undergraduate student as she progressed from using subtraction to using division to solve a counting problem that was designed to elicit division?*

Literature Review and Guiding Perspectives that Situate Our Work

A Set-Oriented Perspective

Lockwood (2014) introduced a set-oriented perspective as a way of thinking about counting "that involves attending to sets of outcomes as an intrinsic component of solving counting problems" (p. 31). Lockwood and others have since argued for the importance of having students connect counting to sets of outcomes in a variety of ways, connecting such a perspective to listing (Lockwood & Gibson, 2016), highlighting its importance in helping students understand and interpret counting formulas (e.g., Lockwood et al., 2015; Wasserman & Galarza, 2017; Wasserman, 2019), and emphasizing its centrality to being able to engage productively with combinatorial proof (e.g., Lockwood et al., 2020; Erickson & Lockwood, 2021a). Relatedly, Lockwood (2013) presented a model of students' combinatorial thinking that included three components: counting processes, formulas and expressions, and sets of outcomes. In this model, Lockwood emphasized the importance of sets of outcomes, suggesting affordances of having students think about ways in which their counting processes generate and organize sets of outcomes. Our findings in this paper help flesh out the relatively broad view of this set-oriented perspective initially presented by Lockwood (2014). We explore how certain ways of structuring sets of outcomes may serve to support students' reasoning about division in solving counting problems. The set-oriented perspective serves as a guiding theoretical principle, and a focus on sets of outcomes is central to how we conceive of counting.

Arithmetic Operations in Combinatorics

With the exception of multiplication, arithmetic operations have generally not been studied extensively in combinatorics education. Multiplication occurs so frequently in combinatorics that the community has developed a Multiplication Principle (MP) that describes conditions under which it is appropriate to multiply when solving a counting problem. Researchers have explored a number of ways in which the MP is presented in the teaching and learning of combinatorics, including its presentation in textbooks (e.g., Lockwood et al., 2017), students' reasoning about the MP (e.g., Lockwood & Purdy, 2020a, 2020b), and problems involving Cartesian products (e.g., Tillema, 2013). To this point, however, adequate attention has not been paid to other arithmetic operations in counting, especially subtraction and division. Lockwood and Reed (2020) describe an equivalence way of thinking in combinatorics, highlighting how equivalence relates to division in counting.

> Broadly, an equivalence way of thinking in combinatorics entails recognizing equivalence between particular outcomes, and then subsequently accounting for this equivalence. So, when employing an equivalence way of thinking, two things happen: a) one recognizes that in a given set of outcomes, there are certain outcomes that should be considered equivalent (or "the same," "duplicate," or "identical") for specified constraints in a situation or problem, and b) one understands that they can use the operation of division in order to account for the occurrence of such equivalent outcomes (Lockwood & Reed, 2020, p. 4).

Lockwood and Reed noted several places in which such equivalence and division naturally arise among topics in combinatorics. Our point here is that division in combinatorics relates to important underlying concepts, and it is worthwhile to pursue as a line of inquiry. Our data sheds light on students' conceptions of division (particularly as it relates to subtraction) that illuminate the kinds of productive meanings and ways of reasoning that have thus far been absent from the literature, and suggest a need for further exploration. We are thus motivated to think more broadly about ways in which other operations can support and augment students' combinatorial understanding beyond just multiplication.

Methods

We present an episode taken from a series of task-based clinical interviews (Hunting, 1997) exploring, among other things, students' engagement with division in counting. Nine students were recruited from a large university in the United States. The participants were a mix of six undergraduates, one early-stage graduate, and two late-stage graduate students who were enrolled in a math class and had taken (or were taking) a course that featured counting. There were no selection criteria aside from their coursework and willingness to participate. We present the work of Jillian (pseudonym), an undergraduate mathematics major, in her first interview.

Interview sessions were 90 minutes long. Because of student availability, some students participated in few sessions (1-3) while others participated in many sessions (6-9). Consistent with task-based clinical interviews (Hunting, 1997), the participants were asked to describe their work as they solved multiple counting problems, and were frequently asked hypothesis-confirming questions by the interviewer about their understandings of formulas, concepts, and strategies both during and after they solved the problems. Participants worked on an iPad to solve the problems, and their written work and gestures and utterances were recorded.

The interview protocol consisted of a diverse collection of counting problems, with many problems chosen as likely to elicit use of division as part of the problem-solving process. For instance, one might solve the *Table Problem* in at least three ways, though we hypothesized that many students would be successful by leveraging division as described in the Introduction. An alternative solution involves making an initial arbitrary choice that a single person sits first at the table. Following this, there are 9! arrangements of the remaining 9 persons around her.

Notably, one might also solve the *Table Problem* with subtraction. Beginning with the 10! linear arrangements of people, and recognizing the overcount, one might attempt to remove 9 extra outcomes for each 1 desired outcome. This leads to the insight that, if for each desired outcome there are 9 to be removed, then you solve the problem by the difference $10! - 9x$, where x is the desired number of outcomes. Since x is also the solution, you establish the equation $10! - 9x = x$ to yield the solution $\frac{10!}{10} = 9!$. As many counting problems involving division can be solved in multiple ways, we attempted to choose problems for which division would be a likely solution method. We also included other problems for which division was not a targeted solution method (e.g., problems involving sums of binomials) so students would not anticipate that division was the targeted operation of the study. Following our focus on a set-oriented perspective, other interventions were enacted to support student consideration of outcome sets during their problem solving.

The video records (iPad work and gestures and utterances) were spliced together so that we could view both the student and their work at the same time. Transcripts were made and enhanced with images, references, and comments. The research team analyzed the data for this report by searching the records for episodes where students utilized division in their solution, attending particularly to problems where subtraction was involved in an earlier solution attempt. We then reviewed the records of the episodes via conceptual analyses (Thompson, 2008) to build second-order models of the students' thinking, seeking viable explanations of the students' actions and utterances in the form of theoretical models (Steffe & Thompson, 2000; Thompson, 2008). The results that we present follow from our second-order models of Jillian's cognition.

Results

In this section, we briefly describe four episodes in Jillian's work on this problem, as we document her progress from her initial inclination to use subtraction to her use and justification of division. In each episode, we comment on her reasoning and connect it to what we think are broader important points related to student thinking on division in counting.

Episode 1: Starting from 10! with an Inclination to Subtract

Jillian had correctly solved a previous problem that asked for the number of ways to arrange 10 people in a line (the answer is 10!), and when answering the Table problem, Jillian began with that previous solution. She understood that 10! would give her too many outcomes, and she immediately demonstrated engagement with the sets of outcomes by identifying specific outcomes as "the same".

Jillian: There's going to be similar iterations, because if you have *a* through *j* [i.e. the sequence *(a,b,c,d,e,f,g,h,i,j)*], that's the same as *b* through *a* just around the circle goes *j* and then come back around [to] *a* [i.e. *(b,c,d,e,f,g,h,i,j,a)*],. So we're going to have another probably, I think, subtracting problem. So I think it would start similar to the line of 10 factorial [i.e. arranging 10 people into a line]. It'll give you all the possible ways to arrange them. Not accounting for possibilities being the same iteration around the table.

Jillian thus wanted to try to solve the problem by reducing that 10! in some way to account for duplicate rotations around the table. She went on to articulate correctly that there would be nine duplicates for every desirable arrangement, as seen in the following excerpt.

Jillian: So, for each order of 10 that you complete. There's going to be nine duplicates because each order of 10 can be shifted around the table, like there's 10 ways to express the same thing. And so, each of these 10 iterations is going to have nine duplicates.

Jillian's intuition was correct – in fact, she could even articulate what precisely would get overcounted and what she wanted to remove. However, she did not see how to use the subtraction to arrive at a solution and could not figure out what to subtract. The excerpt below shows her reasoning that because each desirable outcome had 9 duplicates, she wanted to subtract nine times the desirable (possible) outcomes (she wrote this in Figure 1).

Jillian: So there's 10 iterations. […] So each order [i.e. one desired outcome] has nine duplicates, but there's a lot of possible orders. [...] So number of possibilities [writing Figure 1a]. And then I want to take away or divide or somehow remove the nine duplicates of each of those possibilities. I think. And I believe this is a number of possibilities [circling 10!]. So how would you find the nine duplicates worth of each of those possible? […] minus 9 times "possibilities".

Figure 1a and 1b: Jillian's initial solution to the Table Problem, involving subtraction

She then clarified that she was using "possibilities" to mean two different things, and re-stated Figure 1a. as "# of possibilities and duplicates – 9 duplicates of each possibility". Jillian realized that the possible outcomes she sought was what she was trying to solve, which felt circular to her (note, she did not attempt to build on this towards the subtraction-based solution mentioned in the Methods Section). At this point, enough time passed to suggest an impasse, and the interviewer encouraged her to consider a smaller case, which we describe in Episode 2.

Episode 2: Investigating a Smaller Case, Still Focusing on Subtraction

Jillian wrote out a smaller case involving three people sitting around a table, and she wrote the outcomes in lexicographic order (Figure 2a). She noted, "So each of these [referring to *abc* and *acb*] had two duplicates, which makes sense because there's two iterations around the table."

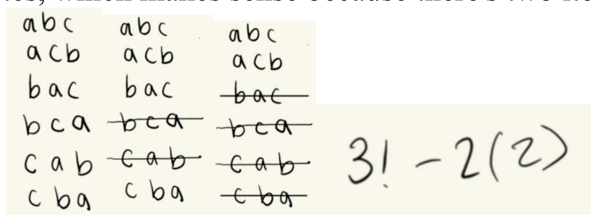

Figure 2a - 2d: Jillian's set of outcomes for the 3-person case

Jillian then proceeded to cross off some outcomes from her list. She noted that *bca* and *cab* were duplicates of *abc*, and so she crossed those off first in Figure 2b. Then, she noted that *cba* and *bac* were duplicates of *acb*, so she crossed them off next in Figure 2c. As she did this, she wrote the expression "3! – 2(2)" in Figure 2d, and she related that expression to her previous expression "10! – 9(possible)" in the larger case. Her language in the excerpt below shows her relating the expression "3! – 2(2)" to her process of crossing out the outcomes from her list; the bolded language summarizes her understanding of that equation. In this way, the smaller case and the set of outcomes helped her confirm that her expression was correct. We want to highlight that the expression here and the process by which she crossed out outcomes is reflected both in how the outcomes are listed, which aligns with Lockwood's (2013) model.

Jillian: Six original options minus […] and then instead of multiplying by nine [in the larger case] I will by two of each of those original options. By the previous logic, there would be three factorial, or six, so I'm happy with that of without not accounting for order, we're going to have the same number of options. And then instead of minus 9 times the actual amount of possibilities [i.e. desired possibilities and duplicates], I'd say minus 2 times the actual amount of possibilities, because **there's two duplicate iterations when you have three people and there's two legitimate possibilities that aren't duplicates. And so, the [expression] makes sense.**

Notably, Jillian realized that in this case she knew there were 2 possibilities (because she had actually written them out and counted them), but she wasn't sure how to get the answer in the bigger larger 10-person case. We infer that at this point the smaller case served to support her in confirming the formula could make sense if she knew what the number of possibilities were, but it did not help her actually solve the problem for the larger case. The interviewer let her wait and think, and we interpret that she had come to an impasse and was not sure how to proceed.

Episode 3: A Different Structure on the Set of Outcomes in the Smaller Case

We had hypothesized that an alternative way of organizing outcomes could help to motivate the use of division. So, once Jillian seemed unsure of how to proceed, the interviewer intervened by writing the outcomes in a different way (Figure 3a). He wrote the outcomes in two columns, with each equivalent rotation of an outcome in its same column.

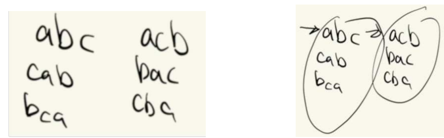

Figure 3a and 3b: The interviewer's, and Jillian's circling of two groups of 3

Jillian noted that she had two groups of three, and she realized (because she knew that the correct answer was 2) that to get the answer of 2 she would need to divide the 3! by 3. She circled the two groups and said, "Yeah, three factorial, you just needed to split it into groups. But it wouldn't be divided by two, by three." However, she continued to reason about the formula and the outcomes and to move back and forth between the smaller 3-person case and the original 10-person case. After some time, the interviewer re-stated her current strategy, as a means of confirming her current reasoning about the problem.

Interviewer: So, you're thinking we can make our two [solution to the 3-case] in this case out of three factorial divided by three. But you're noticing that your instinct is to - instead of divide - subtract by two, because there are two duplicates.

Jillian: Yes, yeah. Because you should be able to split it […] divide by. Oh. I guess in this case, it would be divided by ten [$\frac{10!}{10}$] and the other one too, divided by three [$\frac{3!}{3}$]. Because I'm dividing it into groups of three, the original and its two duplicates to see how many groups there are because that basically doesn't count the duplicates. It just counts how many original generators do we have, like for these different groups

We interpret that Jillian realized that although the desirable outcomes each had *two* duplicates, she could divide by *three* because the groups of three include the desirable outcome and its two duplicates. She also referred to generators, which the interviewer asked about and we discuss in the next episode. She then related this insight about two duplicates and a group of size three to the original 10-person case, and she said, "So wouldn't that work if we divided this [10!] by 10 because it would split it into […] 10, because we know each of these possibilities has 9 duplicates. So, if we split it into groups of 10, we should account for […] the 9 in addition to all of the original one."

Again, Jillian articulated that the division by 10 made sense because each possibility has 9 duplicates, making a group of size 10. The interviewer wanted to ensure he correctly interpreted her reasoning, and so he asked for additional justification, which we discuss in Episode 4.

Episode 4: Ultimately Justifying the Division

In this final episode, we document Jillian's understanding of why division made sense. The exchange below highlights how she came to talk about the groups of 3 (in the smaller case) or 10 (in the larger case) as including the desirable outcome and its duplicates.

Jillian: Yeah. So over here [smaller case] […] I'm counting *abc* and *acb* [draws arrows in Figure 3b] as our like "generators" [air quotes], any one of these [group members] could be a generator […]

Interviewer: And so, by generator there, you mean […] *abc* you said could be swapped out for any of the others. So, what makes that, like, a generator? Or what does generator mean.

Jillian: Yeah. I guess it's almost like if there was an operation that rotated them or something like it's operating on itself over and over again […]

Interviewer: [After confirming that Jillian was not formally referring to algebraic groups] So for you, maybe in layman's terms we would say you've got *abc* but you could generate any of the other duplicates from *abc*.

Jillian: Yes, yes, yes, yes, yes. It's like, it's kids or something. You have *abc* and then you have all of the family of *abc*. And I know that there's going to be in this [larger] case, there's going to be nine children of every parent. If we were to split it into groups of ten, even if the kids got all jumbled up and they weren't the right groups of ten, we still should account for the fact that each one is multiplied by ten different iterations.

Interviewer: So, we take the one generator, as you've been calling it, and you multiply by ten. That gives you like the entire collection of iterations.

Jillian: Yes. And so, if we divide by ten, we should get the number of generators.

Jillian could reason about the whole set of outcomes being split up into groups, and in this way she understood that the division was giving her the number of generators. We infer that she understood that each group would have one generator, and so the division would yield the number of desirable groups and, ultimately, of desirable outcomes.

Discussion and Conclusion

We highlight a couple of points of discussion here and articulate potential implications. In accounting for Jillian's transition from subtraction to division, we think that a key understanding was focusing not just on the number of duplicates (2 duplicates or 9 duplicates in the respective cases), but thinking of the entire sets of equivalent outcomes, including the desirable outcomes and the duplicates (a total of 3 or 10 in the respective cases); we call these the equivalence classes. Transitioning from a focus on a 1:2 or a 1:9 ratio and instead thinking of 3 or 10 seemed important in understanding the appropriate use of division. An implication then is that while subtraction of duplicates is useful, it is valuable to think of those duplicates not in a ratio to the desirable outcome but as part of a set (or equivalence class) with the desirable outcome itself. Jillian's notion of a generator was one useful way to think about this.

Another observation is that reasoning about sets of outcomes and a set-oriented perspective was useful, but certain ways of structuring the sets of outcomes might be associated with different solution strategies. The common lexicographic ordering of outcomes first accompanied Jillian's duplicate-removal strategy, whereas grouping the outcomes into equivalence classes provided a way for Jillian to focus on the 3 and the 10 (the sizes of the groups, rather than the sizes of the groups of duplicates). This re-orientation towards division gives insight into more nuances about *sets of outcomes* and how they might relate to *processes* and *formulas* that may be particularly suggestive of operations. Indeed, we feel that the listing and expression in Figures 2a-2d highlight Jillian's movement between components of Lockwood's (2013) model, and reinforce the currently underexplored idea that different lists of outcomes may be suggestive of different *processes* and *expressions*.

A final point and potential implication is that there are natural connections between counting problems and equivalence, and problems that focus on division in counting may offer rich opportunities and contexts in which to explore such connections. We believe that there is much more work to be done to investigate students' reasoning about division in combinatorics, and we hope that researchers will undertake systematic explorations into how students use and come to understand division in combinatorics.

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