STUDENT USE OF THREE REPRESENTATIONS OF SET RELATIONSHIPS TO REASON ABOUT LOGIC IN UNDERGRADUATE TRANSITION TO PROOF COURSES

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In transition to proof courses for undergraduates, we conducted teaching experiments supporting students to learn logic and proofs rooted in set-based meanings. We invited students to reason about sets using three representational systems: set notation (including symbolic expressions and set-builder notation), mathematical statements (largely in English), and Euler diagrams. In this report, we share evidence regarding how these three representations provided students with tools for reasoning and communicating about set relationships to explore the logic of statements. By analyzing student responses to tasks that asked them to translate between the representational systems, we gain insight into the accessibility and productivity of these tools for such instruction.

Keywords: Logic, multiple representations, Euler diagrams, undergraduate

Introduction

Using multiple representations to support student reasoning and problem solving has long been acknowledged as a cross-cutting theme in mathematics education (e.g., NCTM, 2000). Working within and across representations is often a productive means of supporting student reasoning and promoting communication in the classroom. In the realm of mathematical logic, there is a long tradition of developing various visual and symbolic representation systems (e.g., Venn diagrams, Euler diagrams, truth tables, logical calculus), which would suggest this is a ripe space for using visual and symbolic representations to support student learning. However, in our experience, the use of spatial representations such as Euler or Venn diagrams to teach undergraduate transition to proof (TTP) students is rare (see Dawkins et al., 2022). One explanation for this is that diagrammatic representations of logic generally rely on set relationships (Mineshima et al., 2012), but common approaches to teaching logic in undergraduate TTP courses generally base logical concepts on truth-values rather than sets (Dawkins et al., 2022). This is the case despite a body of evidence supporting the power of visual representations for student reasoning in logic (e.g., Stenning, 2002; Sato & Mineshima, 2015).

Based on a series of experiments (e.g., Dawkins & Cook, 2017; Dawkins & Roh, 2024; Dawkins et al., 2023; Eckman et al., 2023) involving a cycle of modeling student reasoning about logic and task design to support learning of logic, our team has developed a teaching sequence to foster learning of logic using set relationships. We use three primary representation systems (Goldin, 1998) to engage students in reasoning about set relationships: set theoretic symbols including set-builder notation, mathematical statements (rendered in English), and Euler diagrams. Figure 1 portrays how a subset relationship between two properties P and Q might be alternatively portrayed in the three representations. As Thompson (1994) explained regarding different representations of a function, we should not assume that students always see all of these as different representations of the same underlying object, even if that is our goal.

Set-theoretic symbolic representation

$$A \subseteq B$$

where $A = \{x \in S: x \text{ has property } P\}$
and $B = \{x \in S: x \text{ has property } Q\}$

Mathematical statement representation

Given any $x \in S$, if x has property P, then x has property Q.

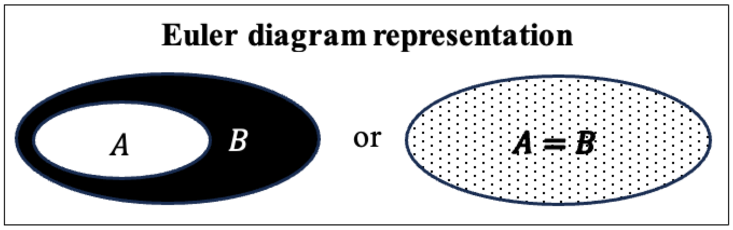


Figure 1: Expressions of a subset relationship in the three representations

As with much teaching using various representations, in our teaching experiments our research team did not want these representations to be the focus of instruction during most of the unit in which we taught logic. Rather, after they were introduced, we wanted them to be means by which students could reason and communicate about set relationships and the logic of statements. In two undergraduate TTP whole-class teaching experiments, we encouraged students to communicate within and across the three representational systems to learn set relationships, logic of statements, and proof techniques. We assigned a number of tasks in which students translated between the representations or generated new objects in one or more representations. In this report, we share our analysis of student work on such tasks to consider whether these three representations served as *accessible* and *mathematically productive* ways for students to reason about and communicate about set relationships. In particular, we share whether student use of these representations was normative – meaning the claims students made were mathematically accurate – and whether they were consistent – meaning a student's various claims for related tasks agreed, even if the interpretations or claims were non-normative. Our goals in this analysis are twofold: 1) to investigate the efficacy of teaching logic using these three representations for supporting student inquiry (as evidenced by the conditions in the previous sentence) and 2) pending positive evidence, to portray the potential of using these representations for instruction on logic and proof techniques (as it stands in contrast to common practice).

Relevant Literature

This section reviews literature relevant to our project before briefly reviewing our own line of research that informed the instructional approach employed in the teaching experiments.

Sets, diagrams, and the teaching and learning of logic

While Venn diagrams are perhaps the most well-known visual system for representing and reasoning about logic, a variety of such systems were developed, primarily in the 19th century. Research on how people reason with and learn from such systems is much more recent. Sato et al. (2010) compared how people solve syllogism tasks using a few different representation systems (see also Bronkhorst et al., 2022). In particular, they compared verbal solution methods (no diagrams), Venn diagrams (in which regions always overlap and regions are shaded), and

Euler diagrams (in which possibilities are displayed by the overlap/non-overlap of regions, as in Figure 1), each with a short period of training in each diagram system. They found that the college student participants performed better with diagrams than with only verbal representations and performed better with Euler diagrams than with Venn diagrams. Those authors explain the value of diagrams by claiming, "we may plausibly assume that the semantic primitives of quantificational sentences in natural language are *relations* between sets, and that people's inferences with quantified constructions are sensitive to such a relational structure" (Sato et al., 2011, p. 2183). They assume that treating statements as relations among sets (as is portrayed in Venn and Euler diagrams) rather than relations quantified over ranges of individual objects (as is done in most standard treatments of logic, such as truth tables) is more consistent with natural language. Sato et al. (2010) further claims that Euler diagrams fostered better performance since they are in some sense "self-guiding" (p. 20) for minimally trained learners. We do not endorse such an interpretation of representational transparency as though learners must not engage in some constructive process of making meaning of the diagrams, but the evidence suggests that students find Euler diagrams easier to use with minimal training nevertheless.

Mathematics education studies of sets, diagrams, and logic

Deloustal-Jorrand (2002, 2004) provides a strong antecedent to the present work as she argued that student understanding of conditional statements should be built upon three viewpoints: formal logic (truth table definition and quantification), sets (represented by spatial diagrams), and implication (the conclusion can be inferred from the hypothesis). She hypothesized that "it is necessary to know and establish links between these three points of view on the implication for a good apprehension and a correct use of it" (Deloustal-Jorrand, 2002, p. 4). Similarly, Durand-Guerrier et al. (2012) emphasized that logic must be taught with attention to semantic and syntactic aspects. While they do not endorse spatial diagrams in particular, such diagrams are classically viewed as a representation of the semantics of statements which supplement the syntax of formal statements or symbolic expressions. These authors support the claim that students should reason about sets, likely expressed through spatial diagrams, to learn about the logic of statements and proofs – however, how students bridge between these representations has been explored less. In a forthcoming paper, Antonides et al. (in press) explore how students link spatial and logical structures, recognizing the challenge and opportunity posed by operating across these representational systems[1].

Our approach to teaching logic is consistent with these other studies, but only indirectly drew upon them. Our focus on sets arose from observations of productive student reasoning about mathematical statements (Dawkins, 2017). This led us to depart from the common truth conditions for statements defined by truth tables, and to adopt truth conditions based on sets. Specifically, the truth of a conditional corresponds to a subset relationship between the truth sets of the two predicates, as portrayed in Figure 1. As we explored how to use sets to support students to reason about the logic of statements and proofs (e.g., Dawkins et al., 2023; Dawkins & Roh, 2024), we saw the need to more directly teach basic set theory beforehand (Eckman et al., 2023, provides examples motivating this need for instruction). The three-representation approach investigated in this report arose as a tool for enacting this teaching sequence. Two other

[1] Both Sato et al. (2010) and Antonides et al. (in press) use Euler diagrams in which regions do not have existential significance. This means that in Figure 1, the left-hand diagram could represent both the case where $A \subset B$ and $A = B$. We do not adopt those conventions, but rather use the two diagrams in Figure 1 to separately express the two cases. This creates a two-to-one mapping between the diagrammatic representation and the symbolic and sentential representations in the way we teach, as portrayed in Figure 1.

aspects of our approach worth mentioning are that 1) we focus on sets defined by mathematical properties, not arbitrary sets such as $\{1, \pi, \text{Ford Taurus}\}$ and 2) we focus on mathematical statements rather than everyday, nonsense, or purely symbolic statements.

Methods

As part of a larger project investigating student abstraction of logic (NSF DUE #1954768 and #1954613), we conducted two whole-class teaching experiments (Steffe & Thompson, 2000) in undergraduate TTP courses. These courses occurred at two large Southwestern, public universities and were taught by the first and second authors. The data gathered consisted of videos of all class meetings in which sets, logic, and proof techniques were covered, student homework and exams, group interactions in target small groups, task-based interviews on logic with members of the target groups both before and after instruction, and pre- and post- logic assessments delivered online. Consistent with teaching experiment methodology, outside observers were present at all class meetings and the research teams at the two sites met weekly to discuss and conduct iterative analysis and planning.

All students were invited to provide informed consent to participate in the study. The data analyzed in this report is limited to homework and exam work from students who opted into the study (13 students in each class), which represents 87% and 100% of the students in the two classes. This portion of the data was deemed most appropriate for analyzing all participating students' use of the three representations. The homework and exam tasks were not the same across the two classes, though some tasks were shared. This creates an asymmetry in the available data at the two sites. While this might be a problem if we tried to make claims about student learning over the course of the teaching sequence, our goals in this report are more modest. We want to consider the extent to which the three representations provided accessible and mathematically productive ways for students to reason and communicate about set relationships. We provide an example to illustrate what we mean below.

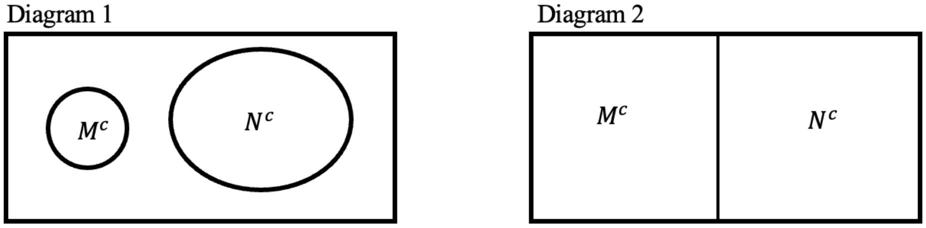
To answer this question, the third and fourth authors analyzed all homework and exam tasks from the portions of the courses on sets, logic, and proof techniques. We looked for all the tasks that invited students to operate between representations, often providing input in one and asking students to respond in one or both of the others (see Fig 2 for an example). The research team then selected a subset of these tasks for student response analysis, giving preference to those tasks used in both classes. The third and fourth authors then analyzed all consenting student responses to these tasks. Responses were coded for whether the response 1) made normative claims in the representation, 2) was internally consistent, and 3) exhibited any recurring feature observed in other responses and salient to translation between representations. As allowed by the structure of each task, we attended to whether students were internally consistent in the claims they made about sets across the representations. In other words, we sought to discern whether they used different representations as ways to express the same underlying relationship. Though many of the tasks in the courses were in a particular mathematical context (e.g., quadrilaterals), many of the tasks we analyzed dealt with arbitrary sets and properties. Such tasks were assigned to support student abstraction. These tasks are useful for this study to see how students related the three representations without underlying reference to the specifics of some underlying mathematical context.

Consider the task in Figure 2 to illustrate our coding. The normative responses for the first question were 11a – true-true, 11b – false-true, and 11c – false-true. To us, the three statements in question 12 corresponded to the set relationships in question 11: 11a~12b, 11b~12c, and

11c~12a. Since we know one of the two diagrams is the case, but we are not sure what is the precise state of affairs between these properties, the normative answers to question 12 are that statement b must be true while statements a and c may be true or may be false. Even if students did not give those normative answers to these questions, their answers to question 11 may be consistent with their answers to question 12. We interpret this to mean that they linked the statement and the set relationship normatively, but they might have read the diagram non-normatively. This is an example of what we would have called consistent, though not normative responses. These codes were then tallied to provide descriptive summaries that allowed us to survey student use of the three representations, as we shall share in the following sections. Since it is not our goal to compare the two classes, we aggregate all of the codes across the two classes for tasks used at both sites.

Assume that for the sets M and N, we know that one of the following two set diagrams is the case, but we are not sure which.

Diagram 1 Diagram 2



11) Determine whether the following set relationships is true or false in each of the two diagrams (in other words, give two answers, one for Diagram 1 and one for Diagram 2).
    a. $M^c \subseteq N$
    b. $M^c = N$
    c. $N \cap M = \emptyset$

12) For each of the following statements, circle the best explanation of its truth value based on the diagrams above.
    a. There exists some $x \in S$ such that x is in M and x is in N.
       i. Must be true. ii. May be true or may be false. iii. Must be false.
    b. Given any $x \in S$, if x is in M^c, then x is in N.
       i. Must be true. ii. May be true or may be false. iii. Must be false.
    c. Given any $x \in S$, x is in M^c if and only if x is in N.
       i. Must be true. ii. May be true or may be false. iii. Must be false.

Figure 2: Translation between representations task from a midterm exam

Results

In this section, we present our findings from analyzing student responses to three tasks (though each is multi-part). The first two tasks were given in both classes. The third (in Figure 2) was only given at one site. While we identified other tasks relevant to our investigation, space does not permit us to report more in this conference paper. When possible, we share instances of interesting student thinking to support the claim that the three representations were productive for student inquiry into logic.

Task 1: Building sets containing the given set

Figure 3 presents Task 1 in which students were asked to build sets containing the given set. This was both an opportunity to use set-builder notation and to think about how properties influence the membership of a set. While this task did not include either statements or Euler diagrams, we consider the coordination of properties and sets of objects another key aspect of constructing set relationships. Table 1 presents the results of the coding analysis. Most students used set builder notation as intended. Furthermore, more than 81.7% of the responses identified a superset of the given set (even if equal to the given set). Set D proved to be the most challenging for many students because there are no familiar sets of quadrilaterals that contain all trapezoids. Student responses tended to lean heavily on familiar conditions (i.e., those taught in school) to construct their supersets. One interesting pattern in some of the non-normative responses is illustrated by the student whose response for set D was the set of quadrilaterals in which all four sides are parallel. This produces a subset of D, but the student was likely thinking about how having two parallel sides is "contained in" having four parallel sides. This way of reasoning arises on other tasks, such as when students think the set of equilateral triangles contains the set of isosceles (defined inclusively) since three equal sides contains two.

Let $\mathbb{T}$ denote the set of all triangles. $\mathbb{Q}u$ is the set of all quadrilaterals.
$\mathbb{Z}$ is the set of integers. $\mathbb{N}$ is the set of natural numbers, which is $\{1, 2, 3, 4, \dots\}$.

For each of the following sets, use set-builder notation to construct another set that contains the given set. In other words, the set given should be a subset of the set you define. (My answers are that $A, B \subseteq \mathbb{Z}$, $C \subseteq \mathbb{T}$, and $D \subseteq \mathbb{Q}u$, so you may not use those answers.) Each task is worth ½ point.

$A = \{x \in \mathbb{Z}: x\ is\ a\ multiple\ of\ 2\ and\ a\ multiple\ of\ 11\}$.
$B = \{x \in \mathbb{Z}: when\ x\ is\ divided\ by\ 4, it\ has\ a\ remainder\ of\ 2\}$.
$C = \{\Delta XYZ \in \mathbb{T}: \Delta XYZ\ is\ obtuse\}$.
$D = \{LMNO \in \mathbb{Q}u: at\ least\ two\ sides\ of\ LMNO\ are\ parallel\}$.

Figure 3: Task 1, which invites students to use set-builder notation

Table 1: Features of student responses to Task 1 (n=26)

Response Feature	Set A	Set B	Set C	Set D	**Total**
Used set builder notation	73%	81%	88%	85%	**81.7%**
Did not use set builder notation	27%	19%	12%	12%	**17.3%**
Superset	62%	81%	70%	23%	**58.7%**
Equivalent Set	31%	8%	27%	27%	**23.1%**
Subset or Incorrect Set	8%	12%	4%	50%	**18.3%**

Task 2: Constructing diagrams and new set relationships from them

Task 2 (see Figure 4) invited students to translate a set expression into Euler diagrams and then to use those diagrams to produce new set expressions. We had noted in our previous experiments that students often struggled to think about complement sets as the inside of regions in Euler diagrams. For this reason, we purposefully asked them to draw diagrams where the given information contained a complement and chose to provide them a diagram where the complement was inside of an oval region. There are two normative diagrams for this arrangement similar to those in Figure 1, but a range of other diagrams may be drawn if students represent S as the inside of a region or if they imagine any sets to be empty or universal. Accordingly, as displayed in Table 2, some students produced three diagrams. About one quarter of students could only produce one normative diagram.

Task 2 (2 pts): Assume for the sets S, T, which are in the universal set Ω, that we know $S^c \subseteq T$

a) Draw as many different diagrams as you can portraying the possible relationships between S and T.
b) Identify 2 other set relationships that must also be true among the sets S, T, S^c, T^c.
c) Suppose another student drew their diagram as shown to the right. Are your answers to part b still true in this diagram? How could you shade the diagram to show how your set relationships are true in this diagram?

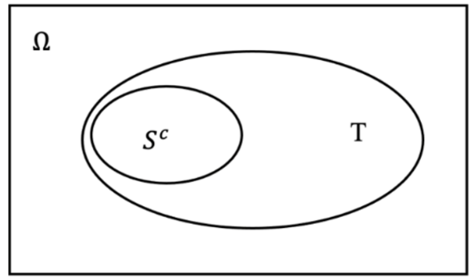


Figure 4: Task 2, which asks students to connect set notation and Euler diagrams

Table 2: Features of student responses to Task 2

Number of Normative Diagrams Produced		Set Relation Normativity		Internal Consistency	
Three	11.5%	Normative set relations	53.8%	Fully internally consistent	57.7%
Two	57.7%	Non-normative set relations	46.2%	At least one consistent	88.5%
One	26.9%				

The intended answers to part b (based on what was discussed in the classes) were that $T^c \subseteq S$ and that $S^c \cap T^c = \emptyset$ since these are both logically equivalent to the given condition, though other statements were possible. As the third column of Table 2 displays, 88.5% of students generated a set relation that was consistent with at least one of their diagrams. We interpret this as evidence of some fluency between the representations. The lower performance indicated in other cells points to the challenge of this task caused both by the presence of the complement and the many-to-one pairing between diagrams and set relationships. Less than 60% of student responses presented a set relation that matched all of their diagrams (according to our interpretation thereof). Since student diagrams were either non-normative or did not capture all of the possibilities, this meant that almost half of their given set relationships were non-normative (not necessarily true given $S^c \subseteq T$).

We want to highlight two types of responses reflected in this data. One student only drew a diagram in which S and T are complements of one another. This is consistent with the given information but does not show all possibilities. As a result, her first set relation, $S \cap T = \emptyset$, is consistent with her diagram, but is not normative since it is untrue when $S^c \subset T$. Her other set relation, $S \cup S^c = \Omega$, is a tautology. It is consistent with the given information, her diagram, and is normative, but it would be true of any set and does not depend upon the given information. A second type of response we noticed was when students generated other types of equivalent conditions, such as $S^c \cap T = S^c$. We had not taught these other conditions equivalent to a subset relation, so we infer that the Euler diagrams supported students in noticing new, normative relationships. Both these equivalent conditions and the tautologies suggest that the Euler diagrams were productive for students' ability to identify new set relationships.

Task 3: Evaluating set relationships and statements from given Euler diagrams

Task 3 appeared in Figure 2. As with Task 2, we purposefully introduced complement sets into this task to see if students could reason about them as sets much like any other set. Table 3 below presents how frequently student responses were normative and consistent with corresponding other responses. The codes group the parts of question 11 and question 12 that correspond according to normative logic. The data show that students frequently gave normative answers to most tasks, though responses to 12c were often neither normative nor consistent. We interpret this to suggest that the language of "if and only if" was still not being coordinated with the other representations in the manner intended. On the other tasks, students were consistent with their diagrams on nearly 4 out of 5 responses, suggesting relatively strong fluency between the representations.

Table 3: Student responses to Task 3

Problem Pair 1				Problem Pair 2				Problem Pair 3			
	D1	D2	Claim		D1	D2	Claim		D1	D2	Claim
Normative	100%	79%	86%	Normative	93%	93%	50%	Normative	71%	93%	79%
Non-Normative	0%	21%	14%	Non-Normative	7%	7%	50%	Non-Normative	29%	7%	21%
Internally Consistent		79%		Internally Consistent		36%		Internally Consistent		79%	

Discussion

This paper analyzed how students in two undergraduate TTP courses, which were designed to foster set-based meanings for logic and proof, used the three representations of set relationships to reason about and communicate about logic. This was done by analyzing student responses to tasks that particularly asked them to operate within and across the representations. Student responses were coded both in terms of whether they were, first, normatively correct and, second, internally consistent with students' responses to other parts of the task. Third, we also noted when students gave responses that were mathematically accurate even if logically less interesting, such as tautologies. While we are pleased by normative responses, we take consistent and tautological responses as supporting evidence that the three representations were productive for student reasoning and communicating.

Much of the data shows that a majority, but not all, student responses were normative and/or consistent. We draw two implications from this. We claim that operating in the representations was accessible and productive for students since most of them were able to demonstrate productive reasoning about set relationships on these tasks. By reasoning within and across the representations, we hoped students could construct and abstract set relationships both between particular properties and arbitrary properties. The second implication we draw is that reasoning between representations is non-trivial. By contrast, Sato et al. (2010) claim that Euler diagrams are self-guiding and certain inferences can be read directly from diagrams. We agree that they are facilitating, but we claim learning to operate in these representations is a meaningful accomplishment. Further, our tasks reveal how task features such as complement sets, coordinating different cases, and needing to generate non-familiar sets all increase the challenge of student reasoning about logic. We hope that future work will continue to explore student learning in this arena and future instruction will seek to make use of these three representations to support student progress.

Acknowledgements

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