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# Designing Approximations of Practice: The Case of Teacher Noticing

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## Abstract

As a core practice, teacher noticing of students' mathematical thinking is foundational to other teaching practices. Yet, this practice is difficult for preservice teachers (PSTs), particularly the component of interpreting students' thinking (e.g., Teuscher et al., 2017). We report on a study of our design of a specific approximation of teacher noticing task with the overarching goal of conceptualizing how to design approximations of practice that support PSTs' learning to notice student thinking in technology-mediated environments with a specific focus on interpreting students' mathematical thinking. Drawing on Grossman et al.'s (2009) Framework for Teaching Practice (i.e., pedagogies of practice), we provided decomposed opportunities for PSTs to engage with the practice of teacher noticing. We analyzed how our design choices led to different evidence of the PSTs' interpretations through professional development design study methods. Findings indicate that the PSTs frequently interpret what students understood. Yet, they were more challenged by interpreting what students did not yet understand. Furthermore, we found that providing lesson goals and asking the PSTs to respond to a prompt of deciding how to respond had the potential to elicit PSTs' interpretations of what the students did not yet understand. The study highlights the interplay between task design, prompt wording, and PSTs' interpretations, which emphasizes the complexity of developing teacher noticing.

Keywords: Teacher Noticing, Pedagogy of Practice, Approximation of Practice, Core Teaching Practice, Technology, Preservice Teachers

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# I. INTRODUCTION

In recent years, teacher noticing of student thinking has gained momentum in the landscape of educational research as an important teaching practice due to its relationship to other teaching practices (Thomas, 2017). In fact, Mason (2002) noted "every act of teaching depends on noticing" (p. 7). Teacher noticing of students' mathematical thinking is described as "focusing attention on and making sense of what students say or do before actually responding to them" (Jacobs & Spangler, 2017, p. 771). Jacobs et al. (2010) explain that noticing students' thinking is comprised of three interrelated skills: attending to students' strategies, interpreting their understanding, and deciding how to respond on the basis of those understandings. Jacobs and Spangler (2017) identified teacher noticing as an important core teaching practice—meaning it is "central to teaching and accessible to and learnable by novice teachers" (p. 769). As such, teacher noticing is foundational to enacting other practices like posing purposeful questions and facilitating whole class discussions. To support novices in learning core practices, Grossman et al. (2009) proposed a Framework for Teaching Practice which is commonly referred to as "pedagogies of practice" (e.g., Estapa et al., 2018). Their framework emphasized three teacher education pedagogies: *representations* of practice that make the teaching practice visible (e.g., classroom videos, student work samples), *decomposition of practice* which provides opportunities for novices to examine and identify the constituent parts of a practice, and *approximations of practice* that encompass opportunities to engage in practices that simulate the realities of the teaching practice. Grossman et al. (2009) emphasized that these three pedagogies of practice "overlap and underscore each other" (p. 2091) suggesting that they three can be used in tandem with novices as they work to develop complex teaching practices.

Research on teacher noticing of students' mathematical thinking has shown that both preservice and inservice teachers tend to struggle to fully interpret students' mathematical thinking (e.g., Callejo & Zapatera, 2017; Sánchez-Matamoros et al., 2014; Teuscher et al., 2017). This result is heightened in contexts in which students are engaged in technology-enhanced math tasks (e.g., Bailey et al., 2022; Chandler, 2017; Wilson et al., 2011). Given that teachers struggle with interpreting student thinking, we began to ponder how the design of materials that draw on Grossman et al.'s framework (2009) might support preservice teachers' (PSTs) developing practice of teacher noticing when students are engaged in technology-enhanced math tasks. Our work is in the context of a larger design-based professional development project, Preparing to Teach Mathematics with Technology - Examining Student Practice Project (PTMT-ESP), in which we have spent the last six years studying our design process as we created modules to support PSTs learning to teach secondary mathematics with technology. Within this work, we use Grossman et al.'s pedagogies of practice (i.e., representation, decomposition, and approximation of practice) to provide scaffolded opportunities for PSTs to engage with the practice of teacher noticing in technology-mediated environments. Here we report on a study of our design of a specific approximation of teacher noticing task with the overarching goal of theorizing about how to design approximations of practice that support PSTs' learning to notice student thinking in technology-mediated environments with a specific focus on interpreting students' mathematical thinking.

## II. BACKGROUND

To situate our study, we draw on two main bodies of literature. As the task under consideration is an approximation of practice, our work is grounded in pedagogies of practice (or Framework for Teaching Practice; Grossman et al., 2009). Further, the task aims to have PSTs approximate the practice of teacher noticing, so we will detail research on teacher noticing, including the decomposition of this practice.

### 1. Pedagogies of practice

Grossman et al. (2009) explained that novice teachers in the midst of developing complex teaching practices, should be provided with "opportunities to first distinguish and then to practice, the different components that go into professional work prior to integrating them fully" (pp. 2068-2069) using the three overlapping teacher education pedagogies: representations of practice, decomposition of practice, and approximations of practice The purpose of these pedagogies is to help mathematics teacher educators (MTEs) parse core teaching practices into smaller pieces that simplify complex teaching practices and assist PSTs as they focus on specific aspects of teaching practice in isolation as they work to develop more complex skills that will occur when they enact teaching practices in their own classrooms.

For PSTs new to teaching practice, representations of practice that make the teaching practice visible are needed to illustrate the practice. These include artifacts of teaching that display aspects of teaching that PSTs can visualize (e.g., classroom videos, lesson plans, clinical interviews, student work). Representations can vary in the ways they make aspects of teaching visible. For example, a lesson plan might highlight planned purposeful questions, while whole class video might highlight orchestration of a whole class discussion. Once provided with a representation of practice, the teaching practice under development is decomposed into components that illustrate the "anatomy of the practice to be learned" (Grossman et al., 2009, p. 2069). These decompositions often serve as scaffolds and require MTEs to carefully design learning experiences that build toward the complex practice that PSTs are working to develop (Tyminski et al., 2014).

Approximations of practice simulate aspects of practice (e.g., micro teaching, planning a discussion, scripting questioning). Although they "are not the real thing" (Grossman et al., 2009, p. 2078) and are proximal to actual teaching, they offer PSTs opportunities to simulate the decomposed components of practice in managed environments that may vary in their degree of authenticity (Schutz et al., 2018). Grossman et al. (2018) explained the need for approximations to "provide novices with the opportunity to enact elements of practice with a high degree of support and under conditions of reduced complexity" (p. 9). When using a pedagogies of practice approach, approximations of practice are often used alongside decompositions of practice "to support novices in distinguishing and understanding separate components before integrating them into complex [approximations of] professional practice" (Sztajn et al., 2020, p. 2). The affordances of this approach were explained by Schutz et al. (2018) who stated, "By providing structured and scaffolded ways of 'trying on' teaching, we can begin to develop novices' capabilities to simultaneously engage with content and students in ways that leverage the resources that individual children bring to their learning" (p. 62).

In the context of mathematics teacher education, many MTEs have designed learning experiences (i.e., approximations of practice) for teachers to support learning of various core teaching practices ranging from whole class discussions (e.g., Boerst et al., 2011; Tyminski et al., 2014, Webb & Wilson, 2023), tutoring (e.g., Farrell et al., 2022), teacher questioning (e.g., Billings & Swartz, 2021), selecting and sequencing student work (e.g., Ducharme et al., 2022), designing tasks (e.g., Jacobs et al., 2023), and teacher noticing of students' mathematical thinking (e.g., Estapa et al., 2018; Herbst et al., 2018; Thomas et al., 2023). For this study, teacher noticing of students' mathematical thinking when students are working in technology-mediated environments is the core teaching practice we sought to develop for PSTs.

## 2. Teacher noticing as an approximation of practice

Teacher noticing is a core teaching practice (Jacobs & Spangler, 2017) focused on students' mathematical thinking that can be summarized as the attention to and interpretation of students' mathematical thinking that is used to formulate responses to students and make instructional decisions. Developing this core practice is difficult, particularly interpreting students' thinking and deciding how to respond (e.g., Callejo & Zapatera, 2017; Sánchez-Matamoros et al., 2014; Teuscher et al., 2017). Yet, recent literature reviews revealed that noticing interventions hold potential to assist PSTs in developing their teacher noticing skills (Amador et al., 2021; König et al., 2022). One method that has shown some success in supporting PSTs' noticing has been with a pedagogies of practice approach. From this lens, MTEs have supported PSTs' development of teacher noticing using a variety of representations of practice, frameworks for decomposing the practice, and different types of approximations of the practice.

Since teacher noticing of students' mathematical thinking occurs in many different classroom contexts, MTEs have chosen a variety of types of representations for teachers to consider when engaging in the practice of teacher noticing. These include but are not limited to written classroom vignettes (e.g., Dick et al., 2024), video cases of classroom vignettes either of small groups or whole class (e.g., Estapa et al., 2018; Fisher et al., 2019; Jacobs et al., 2010), student interviews (e.g., Chao et al., 2016), and animations of students engaged in mathematics (e.g., Walkoe & Levin, 2018). The selection of representations of practice is "complex and significant" (Jacobs & Spangler, 2017, p. 776). There are several important considerations when selecting a representation such as the "length of the video-recorded episode or the amount of written work to be shared, whether video will be edited or unedited, and visibility and clearly of the student mathematical thinking portrayed" (Jacobs & Spangler, 2017, p. 776). However, once representations of practice are chosen, approximations of practice can be designed to help teachers develop their teacher noticing practice.

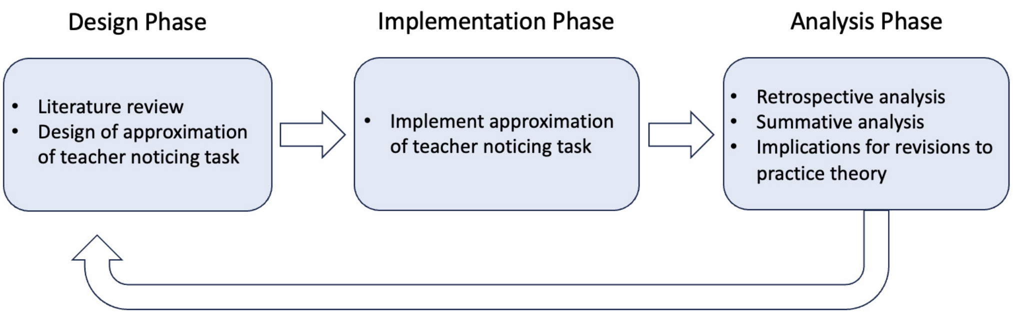
To approximate the practice of teacher noticing, various types of learning experiences have been created by MTEs. Providing prompts for PSTs to write responses is a commonly used approximation (e.g., Jacobs et al., 2010; Krupa et al., 2017; Thomas et al., 2023). Other approximations that are used in research include, but are not limited to, having PSTs create animations that incorporate their noticing (e.g., Estapa et al., 2018) or asking PSTs to rehearse leading discussions (e.g., Fernández et al., 2013). Because of the complexity of teacher noticing, researchers have employed different decompositions of practice to assist PSTs as they engage with the various components of noticing. Many researchers have adopted Jacobs and colleagues (2010) noticing framework that includes three interrelated skills: attending to students' mathematical thinking, interpreting students' mathematical understanding, and deciding how to respond as explicit decompositions of the practice of teaching noticing (e.g., Dick et al., 2024; McCulloch et al., 2023; Thomas et al., 2023) while others have structured approximation assignments for PSTs that implicitly draw upon the three component skills without actually naming them (e.g., Schack et al., 2013). Thomas et al. (2023) designed a five-lesson module for elementary PSTs that progressively nested a decomposition of noticing into Jacobs et al.'s (2010) three (i.e., attending, interpreting, deciding) components that were explicitly named for the PSTs. Other researchers have chosen to provide PSTs with frameworks for teaching practice that support their noticing approximations for specific components of noticing. For example, Ivars and colleagues (2020) provided PSTs with a learning trajectory to support their interpretations of student thinking. Similarly, Thomas and colleagues (2015) proposed integrating the 5 Practices (Smith & Stein, 2011) across the teacher noticing components as a nested decomposition.

Jacobs et al. (2010) identified teacher noticing as a learnable practice. Recent literature reviews re-emphasized this point and added that regardless of the type of decomposition employed, teacher noticing is a learnable practice (König et al., 2022; Santagata et al., 2021). Santagata and colleagues (2021) called for design-based research studies that shed light on how to improve the learning of this practice. Thus, for this paper, we have chosen to present the results of a design-based research program studying the design of one approximation of teacher noticing task across multiple semesters with different PSTs. Specifically, we consider the written decomposition of teacher noticing prompts that we provided to PSTs to elicit their noticing in which we aimed to address Jacobs' (2017) call that noted, "researchers must find ways to tease apart when teachers failed to notice something versus when they simply did not provide evidence that they had noticed it" (p. 278).

# III. METHODS

In this intrinsic case study (Yin, 2018) of the design of the approximation of teacher noticing task, we examine the ways in which our design choices elicited varying evidence of the PSTs' interpretations using professional development design study methods. Cobb et al. (2014) explain, "professional development design studies involve developing, testing, and revising conjectures about both the processes by which teachers develop increasingly sophisticated instructional practices and the means of supporting that development" (p. 491). In the context of this study, the instructional practice of focus is teacher noticing of student thinking on technology-enhanced mathematics tasks. Design studies involve cyclic phases of design, implementation, and analysis (see Figure 1; Stephan, 2015). In the following sections we describe the context in which this study took place followed by the three phases of the design cycle we went through in studying the prompts provided to the PSTs in the approximation of teacher noticing task. Ultimately, we aim to address the following research question: How does the design of a task for approximating teacher noticing of student thinking on technology-enhanced mathematics tasks support PSTs' practice of interpreting student thinking?

**Figure 1** *Professional development assessment task design cycle (adapted from Stephan, 2015, p. 906)*

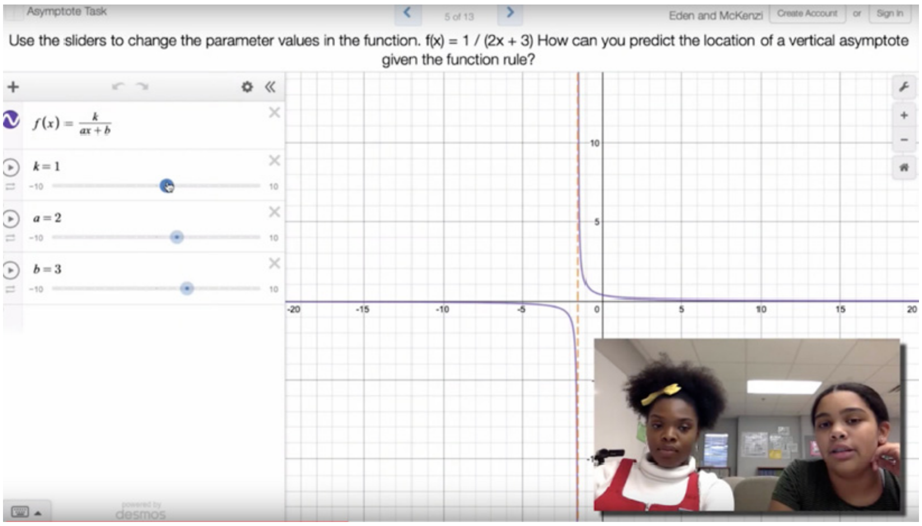



## 1. Context

The study described here is situated in a larger design based professional development project. For our work with the PTMT-ESP Project, we developed materials for PSTs' professional learning that focused on their teacher noticing of students' mathematical thinking as well as teaching practices that build on teacher noticing in the context of secondary students engaging in technology-enhanced algebra and function tasks. As designers, our team was guided by an assumption that materials grounded in representations, decompositions, and approximations of practice (Grossman et al., 2009) could support PSTs in developing these pedagogies of practice in the context of students doing math in technology-mediated environments. To that end, we designed a set of learning experiences that we believed would be effective in supporting the development of PSTs' noticing practice in the context of technology-mediated environments. We have reported on the design of the overarching instructional sequence elsewhere (e.g., Lovett et al., 2020). While we focus on the end of semester approximation of noticing task in this study, for context we will describe how the PSTs in the study were introduced to teacher noticing throughout the semester.

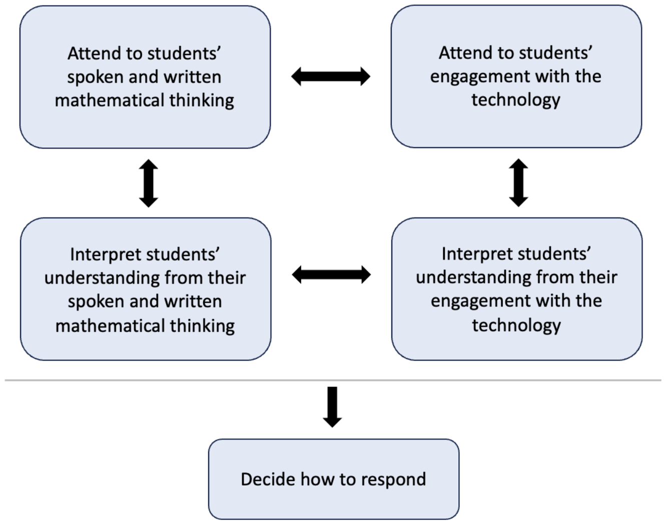
In the beginning of the semester, the PSTs had many opportunities to observe and discuss various video representations of pairs of secondary students working on technology-enhanced mathematics tasks. In-class discussions of these videos often focused on the students' mathematical thinking evidenced in the video; however, the PSTs were not formally introduced to teacher noticing until later in the semester. All of these carefully selected video representations were chosen using the project design principles (Lovett et al., 2020). Such videos included everything that the students said and did on the task and showed their written work (see Figure 2).

Figure 2 *Image of the picture in picture video clip*




Mid-semester, the PSTs were formally introduced to the practice of teacher noticing of students thinking on technology-enhanced mathematics tasks. The first activity was an explicit introduction to the Noticing in Technology-Mediated Environments (NITE) framework (Dick et al., 2021; Bailey et al., 2022; McCulloch et al., 2023) that decomposes the practice of teacher noticing of student thinking on technology-enhanced math tasks into its various components (see Figure 3). The PSTs watched carefully selected video recording representations of pairs of students engaged in a technology-enhanced mathematics task and reviewed and discussed exemplars of experienced teachers' written noticings (i.e., attend, interpret, and decide). The NITE framework is an extension of Jacobs et al.'s (2010) decomposition of teacher noticing that highlights the importance of attending to and interpreting students' technology engagement. While we acknowledge that all the components of noticing are by their nature interrelated (Jacobs et al., 2010), we have separated *attention to and interpretation of students' spoken and written responses* from *attention to and interpretation of students' engagement with the technology* to scaffold PSTs' focus on the actions students take with the technology and what students see as a result of these actions when attending to and interpreting students' mathematical thinking. Thus, the arrows in the NITE framework indicate the importance of both the horizontal coordination of attention and interpretation as well as the vertical integration of both attention and interpretation. Within the NITE decomposition, the *decide how to respond* component is separated from the other components for two reasons: (1) to indicate the importance of balancing insight gained from attending to and interpreting both students' spoken and written responses and their technology-engagement when making instructional decisions, (2) to indicate that when deciding how to respond the teacher must consider how to position the technology (or not).

Figure 3 *Teacher noticing of students' work in a technology-mediated environment (NITE) Framework (McCulloch et al., 2023)*




After the PSTs learned about the NITE framework, they had the opportunity to formally engage with various approximations of teacher noticing by completing learning activities that decomposed the practice into smaller pieces. These activities used additional video clip representations of pairs of students engaged in technology-enhanced mathematics tasks. The PSTs discussed important details to consider when attending to and interpreting the students' thinking and the students' technology engagement on these tasks, as well as possible decisions teachers might make based on their attending and interpreting. These approximation of teacher noticing tasks included prompts that decomposed noticing into the three components (i.e., attend, interpret, and decide) with the decide component further decomposed into various approximations of teaching practices for which noticing student thinking is foundational (e.g., posing purposeful questions, selecting student work, facilitating discussions, designing a next task). Moving forward we refer to the further decomposition of the decide component as nested decomposition. This nested decomposition of instructional decision making is an example of scaffolding various levels of decompositions as discussed by Sleep and Boerst (2012). In this way for the rest of the semester, the PSTs had multiple opportunities to practice the components of teacher noticing using the NITE framework as a guide.

To assess the effectiveness of our instructional sequence, we designed an approximation of practice task. This specific approximation of noticing student thinking on a technology-enhanced mathematics task was implemented both at the beginning and end of the course to capture changes in the PSTs' noticing practice. In this paper, we focus on the end of semester implementation as the PSTs had engaged with additional representations and decompositions of the practice throughout the semester. The design of this approximation of practice task is the focus of this study.

## 2. Design phase

During the initial design phase of this study, we read literature on teacher noticing of students' mathematical thinking generally, and also with a lens on the ways in which researchers elicited teacher noticing using various pedagogies of practice (Grossman, 2009). Based on our review of the literature, we decided that a good way to approximate the practice of in-the-moment noticing of students' thinking on a technology-enhanced mathematics task would be to choose a representation of practice and ask the PSTs to provide written responses to teacher noticing prompts based on the NITE framework (McCulloch et al., 2023). Specifically, the initial design of the approximation of teacher noticing task highlighted in this manuscript features a short video clip of two secondary school students engaged in a Desmos task exploring the effect of parameters on the location of the vertical asymptote of a rational function of the form $f(x)=\frac{k}{ax+b}$ (https://tinyurl.com/NoticingAssess). The Desmos task was designed to address three goals: (1) Students will recognize that a rational function of the form $f(x)=\frac{k}{ax+b}$ has a vertical asymptote at $x=-\frac{b}{a}$, (2) Students will be able to explain that the location of the vertical asymptote is found by setting the denominator equal to zero, since that is where the function is undefined, and (3) Students will be able to explain that a vertical asymptote occurs when the function is undefined (i.e., when the denominator is 0).

The video clip is approximately 3 minutes long. In the video, the pair of students is seen dragging sliders for the parameters $k$, $a$, and $b$ while discussing their effects on the location of the vertical asymptote. The students quickly determine that $k$ does not affect the location of the vertical asymptote and focus their attention on $a$ and $b$. At one point in the video, the students set $k$ equal to zero, so that the graph is a horizontal line at $y=0$, yet because of the way the activity was designed, an asymptote is still shown on the graph. Eventually the students conjecture that the asymptote location can be found by dividing

$b$ by $a$, but they aren't sure why $b/a$ is negative. They decide that this is just "one of those weird flippy things that graphs do." The students test their conjecture with a few different values for $a$ and $b$, and then record that they would explain to a friend that the location of a vertical asymptote for a rational function of the form $f(x)=\frac{k}{ax+b}$ is $-x=b/a$.

As previously mentioned, research on teacher noticing has shown that the decomposed practice of deciding to respond is particularly challenging for PSTs (e.g., Dick et al., 2024; Jacobs et al., 2023; Krupa et al., 2017). Because of this, work within teacher noticing is often focused on the attend and interpret components with the intent being that it is important to develop these practices prior to focusing on deciding how to respond (see König et al., 2022). Based on these findings, we conjectured the same would be true for PSTs' noticing student thinking on technology-enhanced mathematics tasks. Thus, the initial design of the approximation of teacher noticing task included only attend and interpret focused instructions and prompts to accompany the carefully selected video clip representation (see Figure 4).

**Figure 4** *Initial approximation of teacher noticing task instructions and prompts*

The purpose of this assignment is to showcase your skills related to noticing student thinking in technological environments. You will watch a short video clip of a pair of students working on a technology-mediated task and respond to some prompts related to the video.

In the video you will see a pair of HS students currently enrolled in an Integrated Math 3 course working on a task in which they are being introduced to the idea of a vertical asymptote.

Before going any further, take a few moments to work through the task yourself. As you work, pay attention to the dynamic features of the task - including in the titles of the pages.

Now that you are familiar with the Making Sense of Vertical Asymptotes task you are going to analyze a video of a pair of students working on the same task. As was noted above, these students are currently enrolled in an Integrated Math 3 course. They have been working with rational functions, but have not been formally introduced to the idea of a vertical asymptote. This task serves as their introduction to vertical asymptotes.

In this video the students, Eden and McKenzie, are working on page 5 of the Desmos Asymptote Activity. Page 5 opens immediately after the definition of vertical asymptote has been provided. The students are working to determine where (if at all) a vertical asymptote will exist for a rational function with a constant numerator and linear denominator (i.e., $f(x)=\frac{k}{ax+b}$).

Watch the video clip provided in Canvas. You may watch this video as many times as you would like.

1. Attend to (i.e., describe in detail) how the students determined the location of the vertical asymptote for a rational function of the form $f(x)=\frac{k}{ax+b}$.
2. Interpret the students' current understanding of vertical asymptotes. Provide evidence from the video to support your claims.

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During the initial design phase, we asked our project advisory board for feedback on the task design and did feasibility testing with four PSTs outside of the context of a course. No changes were made based on these interactions. As previously mentioned, while we use this same approximation of noticing task structure with different video clip representations throughout the curriculum materials, for the purposes of this study we are going to focus only on the design of the specific task described above. In the findings section of this paper, we begin by reporting the findings related to this first design implementation, and then describe the subsequent iterations of design and how they were informed by our cycles of implementation and analysis.

## 3. Implementation phase

We have undergone three cycles of design for the approximation of teacher noticing task instructions and prompts, across four universities (U1-U4) in the United States (US). Universities ranged from small to large, public to private and include rural, suburban and urban from across the eastern US. The approximation of teacher noticing task was implemented in a secondary mathematics methods course for PSTs at each of the universities. Each round took place in a different semester with different PSTs. There were between three and 10 PSTs enrolled in each course section with a total of 73 PSTs (Note: there were two sections of courses for U3 in Round 1). Of those, there were complete datasets for 50 PSTs; these comprise the data corpus for the study (Table 1). Recall that in this paper, we focus on the end of semester approximation of noticing student thinking on a technology-enhanced mathematics task. The PSTs were provided either a printed or electronic version of the approximation of practice task to record their responses. All PST data were blinded and organized into a shared folder according to university and implementation phase.

Table 1 *Number of PSTs at each university for each implementation phase*

| University | Round 1 | Round 2 | Round 3 |
|---|---|---|---|
| U1 | 4 | 0 | 2 |
| U2 | 6 | 7 | 3 |
| U3 | 17 | 0 | 8 |
| U4 | 0 | 0 | 3 |

## 4. Analysis phase

After each implementation phase, the PSTs' written responses to the prompts in the task were analyzed using a rubric and codebook created by the research team. To create the rubric, we began by using a process similar to Jacobs et al. (2010), identifying the mathematically significant details present in the video clip representation. Because the video included students engaged in a technology-enhanced mathematics task, the significant mathematical details included not only what the students said and wrote, but also the ways in which the students engaged with the dynamic mathematics representations and what they saw as a result of their engagement. This work resulted in a list of what would be expected for a robust attend response. Next, based on the expectations for a robust attend response, we then described what would be included in a robust interpret response—what could be interpreted about the students' mathematical understandings based on the evidence in the video. The rubric included possible interpretations and whether or not PSTs drew upon the students' engagement with the technology as evidence of their interpretations.

The coding rubric was built in a shared spreadsheet (see Figure 5). In addition to indicating the details of PSTs' attend and interpret responses, the rubric included a check box to note if PSTs included details that were implausible or incorrect in their responses, and whether or not they provided evidence from the students' technology engagement for their interpretations. Accompanying the rubric was a codebook that detailed the level of evidence that was needed to check that an item was included in a PST's response and included both examples and counter-examples for each code. A brief summary of the four interpretation codes is included in Figure 6 which features the definition and code name for each. The first interpretation focuses on what the students understand at the point when the video clip ends: The students have identified a way to determine the location of the vertical asymptote. The next two interpretations refer to what the students have not yet provided evidence of understanding: why their procedure works and how it is related to the structure of a rational function. For brevity, we refer to these three interpretations as *students' procedure* (i.e., being able to state the location), *traditional procedure* (i.e., the location is at $x = -b/a$ because that is the result of setting the denominator equal to zero and solving), and *underlying concept* (i.e., if the denominator is zero, the function is undefined; see Figure 6). These codes are not mutually exclusive categories. In fact, an ideal interpretation would result in three codes: *students' procedure*, *traditional procedure* and *underlying concept*. For example, consider this PST response:

> One student suggested $b/a$, which they then realized must be -$b/a$ because the sign switches ⋯ [The students] understand a method to find the vertical asymptote but they are not relating it to where the function is undefined or where the denominator is equal to zero ⋯ They aren't relating the form of the functions denominator to the rule; they just came up with the rule through testing values.

This PST identified that the students understand that the location of a vertical asymptote is determined by a procedure (i.e., *students' procedure*). Additionally, the PST noted that the students have not yet shown evidence of understanding that the procedure is the result of setting the denominator equal to zero and solving (i.e., *traditional procedure*) and that the vertical asymptote is where the function is undefined (i.e., *underlying concept*).

Figure 5 *Example of interpret coding rubric*

| Codebook is here | | | | | | | |
|---|---|---|---|---|---|---|---|
| Files are here | PsN-S21-U1-10 | PsN-S21-U1-11 | PsN-S21-U1-12 | PsN-S21-U1-14 | PsN-S21-U2-09 | PsN-S21-U2-10 | PsN-S21-U2-11 |
| **INTERPRET** | | | | | | | |
| Interpret 1: The students understand that the location of a vertical asymptote for a rational function of the formf(x)=kax+b can be determined by x = -b/a (-x = b/a) (i.e., they have a procedure for locating the vertical asymptote) | Yes | Yes | Yes | Yes | Yes | Yes | Yes |
| Tech Evidence for Interpret 1? | ☐ | ☑ | ☑ | ☐ | ☑ | ☐ | ☑ |
| Notes | | script of what they did as tech evidence | | | | | |
| Interpret 2: The students have not connected the location of a vertical asymptote to where a rational function is undefined | No | No | No | Yes | No | No | Yes |
| Interpret 3: The students have not yet connected their rule to setting the denominator of the function equal to 0 to solve to explain why the vertical asymptote is located at x = -b/a rather than x = b/a. | No | No | No | No | No | No | Yes |
| Tech Evidence for Interpret 2 & 3? | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |
| Notes | "don't know what it actually is" | | | | | | |
| Implausible/Incorrect? | ☐ | ☐ | ☐ | ☑ | ☐ | ☐ | ☐ |
| Notes | | | | | | | |
| Incorrect -x=b/a evidence | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |
| Notes | | | | | | | |

Figure 6 *Summary of interpretation codes*

| | **Interpretations** | **Code Names** |
|---|---|---|
| Interpretations of what students did understand | The students understand that the location of a vertical asymptote for a rational function of the form $f(x)=\frac{k}{ax+b}$ can be determined by $x=-\frac{b}{a}\left(-x=\frac{b}{a}\right)$ (i.e., they have a procedure for locating the vertical asymptote; identification of a procedure) | Students' Procedure |
| Interpretations of what students *did not yet* understand | The students have not yet connected their rule to setting the denominator of the function equal to 0 to solve as an explanation why the vertical asymptote is located at $x=-\frac{b}{a}$ rather than $x=\frac{b}{a}$ (i.e., the procedure typically defined in textbooks) | Traditional Procedure |
| | The students have not yet connected the location of a vertical asymptote to where a rational function is undefined (i.e., understanding of the connection between the graph and the structure of the function ) | Underlying Concept |
| | There is an implication that the students do not yet understand something, but there is no explicit inclusion of either the traditional procedure or the underlying concept | Vague |

Using an iterative process similar to DeCuir-Gunby et al.'s (2011) process for codebook development, every member of the research team coded a subset of the PSTs' responses, and the codebook descriptions and exemplars were refined until all coders were applying the codes consistently. With the rubric and codebook complete, every PST's noticing response was coded by three members of the research team. Like Superfine et al. (2017), we read across the prompts for evidence of the PSTs' attend and interpret responses (i.e., the response did not have to be under the associated task prompt for the PST to get credit). Any discrepancies were discussed by the three coders until consensus was met; any continued disagreements were brought to the entire coding team of eight coders. After the analysis of each implementation phase, the research team met to discuss the findings and then loop back to the design phase to revise the approximation of noticing task based on the analysis. The specifics of each cycle are described in the sections that follow.

## IV. FINDINGS

The findings are organized by cycles of design (we refer to each as a round). We will begin by reporting the findings for the first round of the approximation of teacher noticing task design and how those findings informed our revisions that aimed to better capture the PSTs' developing practice of teacher noticing, specifically interpreting students' thinking. For the subsequent rounds, we will detail the specific modifications to the task instructions and prompts, detail the findings from this revised task, and identify the changes needed based on those findings given our overarching goal of supporting PSTs approximating teacher noticing of students' thinking on technology-enhanced mathematics tasks. Table 2 summarizes the elements of the task that were the focus of our design for each round. These will be described in detail in the sections that follow.

Table 2 *Approximation of practice task modifications by round*

| Round | University | N | Task Components | | | |
|---|---|---|---|---|---|---|
| | | | Attend Prompt | Interpret Prompt | Decide Prompt | Instructions: Performance Goals |
| 1 | U1, U2, U3 | 27 | X | X | | |
| 2 | U2 | 7 | X | X | X | |
| 3 | U1, U2, U3, U4 | 16 | X | X | X | X |

### 1. Round 1

The first design of the approximation of teacher noticing task, informed by the literature on teacher noticing, included an attend and an interpret prompt as described in the methods section. Recall that we identified three potential interpretations of students' thinking we would expect to see given the representation of practice video-clip that was provided as well as the accompanying evidence for these interpretations.

In Round 1, there were 27 PSTs. The overwhelming majority (96.3%) interpreted what the students understood (i.e., the students have a procedure for locating the vertical asymptote—*students' procedure*;

Table 3). A typical PST example included evidence of how the students used the parameter sliders to help them determine a procedure for the location of the vertical asymptote:

> Through trial and error with different inputs for *a* and *b*, they found that *b* divided by *a*··· yet still were confused about why the sign on the vertical asymptote was consistently the opposite···Therefore, [they] know how to find the vertical asymptote.

**Table 3** *Round 1 summary of the PST's interpretations*

| Interpretation | Number of PSTs |
|---|---|
| Students' Procedure | 26 |
| Vague | 5 |
| Traditional Procedure | 12 |
| Underlying Concept | 1 |

Over half of the PSTs in Round 1 made some kind of interpretation regarding what the students do not yet understand (N=16, 59.3%). Of these 16 PSTs, 31.5% (5 of 16) made *vague* interpretations but did not explicitly include either the *traditional procedure* or *underlying concept* interpretations. As an example of a typical *vague* interpretation, consider this PST's response: They know a method for finding the asymptote, but they don't understand what the asymptote actually is.

Three-quarters of the PSTs who interpreted what the students did not yet understand demonstrated evidence of the *traditional procedure* interpretation (12 of 16, 75%). These PSTs typically stated things like

> What they don't understand is what the asymptote actually means and how it is connected. If they had an understanding of [what] the asymptote actually means, they could maybe understand that it has to deal with when the denominator is 0.

Only one PST demonstrated evidence of both the *traditional procedure* and *underlying concept* interpretation of what students do not yet understand. This PST stated:

> This indicates that the girls understand that a vertical asymptote will be located at the -*b* value divided by the a value. They have yet to realize that this very value is the one that makes the denominator zero, as they never set the denominator equal to zero···They do not talk about the domain, which indicates that they are focused on the where, rather than the why of asymptotes currently.

This PST noted that students have not yet connected their rule to set the denominator equal to zero, and they have not yet connected the location to where the function is undefined.

One of the big findings from this round of our task implementation was that the PSTs were interpreting what the students did understand, but not consistently discussing what the students did not yet understand in their interpretations. These results prompted us to consider how revising the approximation of practice task prompts might elicit PSTs to consider what students do not yet understand as part of their interpretations. As previously mentioned, initially, the design of the approximation of practice task only included an attend and an interpret prompt which is in line with other researchers who introduced the components of attend and interpret prior to the decide component (e.g., Fisher et al., 2019) as research has shown that deciding how to respond is challenging for PSTs (e.g., Jacobs et al., 2010; Jong et al., 2021; Krupa et al., 2017). However, when we broadened our review of the literature to include other core practices, we found examples in which the decomposition of a teacher response supported teachers' interpretations of student thinking. For example, in Tyminski and colleagues' (2017) study of the use of pedagogies of practice to support PSTs' facilitations of whole-class discussions showed that when PSTs were provided sample student work and asked to select and sequence the work (i.e., decide how to respond), their interpretations of the student work improved. Based on this work and ongoing work by Dick and colleagues (2024), we conjectured that perhaps if we asked the PSTs to explicitly think about their next pedagogical moves as part of the approximation of noticing task prompts (i.e., include a decide prompt), it might lead them to write about what the students did not yet understand in their interpretations.

## 2. Round 2

In Round 2, the initial instructions and the first two prompts of the approximation of noticing task remained unchanged from Round 1 (see Figure 4). The only modification was the addition of a decide prompt which read: "Based on your responses to Q1 and Q2 above, if you were with Eden and McKenzie at the moment the video occurred, what would you do next? Why?"

There were seven PSTs in this round. All of the PSTs (100%) interpreted what the students understood—students' procedure. These interpretations were similar to those in Round 1. In addition, 100% of the PSTs made some kind of interpretation regarding what the students do not yet understand (*vague*, *traditional procedure*, and/or *underlying concept*). Table 4 summarizes in which prompt the PSTs discussed their interpretations of what students do not yet understand.

**Table 4** *Round 2 summary of the PSTs' interpretations and in which prompt they occurred*

| Interpretation (Number of PSTs) | Prompt in which the interpretation occurred | | |
|---|---|---|---|
| | Interpret prompt only | Both the interpret and decide prompts | Decide prompt only |
| Vague (2 of 7) | 0 | 0 | 2 |
| Traditional Procedure (5 of 7) | 1 | 1 | 3 |
| Underlying Concept (4 of 7) | 0 | 0 | 4 |

Twenty nine percent of the PSTs made *vague* interpretations. These PSTs made their vague interpretations in the newly added decide prompt, meaning we would likely not have elicited evidence of these interpretations if we did not add this prompt. As part of the decide prompt, these interpretations were tied to the PSTs' envisioned next pedagogical moves. For example, one PST stated:

> I would [ask] them to answer the question 'Why does this make sense?' I think having them reason with their answer could lead to their discovery of how it relates to the graph…Before I sent them into testing new values, I would point out to them that the numbers at the top of the screen in the header move with the parameters. I would tell them that they should look at what happens to the parameters there and try to decide what is happening and why it makes sense that it is happening.

This PST noted that the students have yet to discover why the vertical asymptote occurs and that technology can help, yet the PST did not explicitly include either the *traditional procedure* or *underlying concept* in their written interpretation.

Just under three quarters of the PSTs (71.4%) demonstrated evidence of the *traditional procedure* interpretation of what students do not yet understand. Of those PSTs, 20% made their interpretation within the interpret prompt, 20% in both the interpret and decide prompts, and 60% in the decide prompt. Thus, over half of the PSTs only made this interpretation when formulating a response to deciding what to do next with the students from the video, meaning we would likely not have seen any evidence of these interpretations without the decide prompt. For example, one PST stated:

> I also might ask them to fully solve for $x$ (in other words change the equation into $x$=-$b/a$ instead of -$x$=$b/a$). From there I might ask them to try plugging in -$b/a$ for $x$ and see what happens to the equation.

When considering how to respond to the students, this PST suggested solving for $x$ and plugging in specific values, thus providing evidence of the *traditional procedure* as a means of building upon the *students' procedure* (what the students do understand).

Over half of the PSTs (57.1%) demonstrated evidence of the *underlying concept* interpretation of what students do not yet understand, and all of them made this interpretation within the decide prompt. Again, this result indicates that without the decide prompt, we would likely have no evidence that the PSTs made these interpretations. For example, one PST said they would:

> ask the students an advancing question like, 'What is the domain of your function and how does it relate to the vertical asymptote?' Another question that may advance them forwards is 'How do you calculate the domain of a function?' I chose the above approach because the students have correct understanding of where a vertical asymptote occurs, but they do not understand why. By reintroducing the domain into their thinking, they will be guided towards understanding that the points where the function does not exist are where the denominator equals 0, similar to holes in a graph for rational functions, however in this case the numerator is a constant not a function.

Of those PSTs that made the *traditional procedure* or *underlying concept* interpretations, 50% demonstrated evidence of both interpretations of what students do not yet understand.

To summarize, in Round 2 we saw a difference in the number of PSTs interpreting what students do not yet understand with the addition of the decide prompt. Specifically, we saw a jump from 44.4% to 71.4% of the PSTs that showed evidence of the *traditional procedure* interpretation. We also saw a jump from 3.7% to 57.1% of the PSTs that showed evidence of *underlying concept* interpretation.

While these increases in both the percentage of PSTs evidencing the *traditional procedure* and the *underlying concept* interpretations are promising, there was still much room for improvement. When reflecting upon the results from both Rounds 1 and 2, we wondered if part of the reason the PSTs were not considering what students did not yet understand might be because they did not know what to expect students to understand about vertical asymptotes (i.e., not familiar with standards yet) when working on a technology-enhanced math task like the one included in this representation. In fact, the National Council of Teachers of Mathematics' (2014) Mathematics Teaching Framework suggests that tasks should be selected to align with established learning and performance goals. Since the PSTs were not being asked to select the task, they possibly did not consider the goals for which it might be selected. Thus, we conjectured that including specific goals for the technology-enhanced task within the instructions for Round 3's approximation of noticing task might focus PSTs' interpretations by helping them think specifically about the teacher's intention for what the students should get out of their engagement with the task.

## 3. Round 3

Based on the results in Round 2, we made one additional modification to the task. We edited the instructions to include explicit performance goals for the activity along with the introductory text (see Figure 7) in hopes it would serve as an additional scaffold, pointing the PSTs to consider the teachers' goal prior to interpreting the students' thinking and deciding how to respond.

There were 16 PSTs in this round. All 16 PSTs interpreted what the students understood—*students' procedure*. These interpretations were similar to those that we observed in Rounds 1 and 2. Most of the PSTs made some kind of interpretation regarding what the students do not yet understand (*vague*, *traditional procedure*, and/or *underlying concept*; 81.3%). Table 5 summarizes in which prompt the PSTs wrote their interpretations of what students do not yet understand.

Of the PSTs that made some kind of interpretation regarding what the students do not yet understand, 12.5% made *vague* interpretations but did not explicitly interpret either the *traditional procedure* or *underlying concept* interpretations. Half of these PSTs made the interpretation in the decide prompt. These vague interpretations were similar to those in Round 2.

Over half of the PSTs (62.5%) demonstrated evidence of the *traditional procedure* interpretation of what students do not yet understand. Again, these interpretations were similar to those in Round 2. Of these PSTs, 80% made their interpretation in both the interpret and decide prompts. Half of the PSTs (50%) demonstrated evidence of the *underlying concept* interpretation of what students do not yet understand. Of these PSTs, 50% made their interpretation in both the interpret and decide prompts, and 37.5% made their interpretation only in the decide prompt. Again, indicating that without the decide prompt we likely would have no evidence that some of the PSTs made the *underlying concept* interpretation.

**Figure 7** *Round 3 task with newly added performance goals in bold/green*

The purpose of this assignment is to showcase your skills related to noticing student thinking in technological environments. You will watch a short video clip of a pair of students working on a technology-mediated task and respond to some prompts related to the video.

In the video you will see a pair of HS students currently enrolled in an Integrated Math 3 course working on a task in which they are being introduced to the idea of a vertical asymptote. **The performance goals for this task include:**

- **Students will recognize that a rational function of the form $f(x) = \frac{k}{ax+b}$ has a vertical asymptote at $x = -\frac{b}{a}$.**
- **Students will be able to explain that a vertical asymptote occurs when the function is undefined (i.e., when the denominator is 0).**
- **Students will be able to explain that the location of a vertical asymptote can be found by setting the denominator equal to zero, since that is where the function is undefined.**

Before going any further, take a few moments to work through the task yourself. As you work, pay attention to the dynamic features of the task - including in the titles of the pages. [Link to the Activity]

Now that you are familiar with the Making Sense of Vertical Asymptotes task you are going to analyze a video of a pair of students working on the same task. As was noted above, these students are currently enrolled in an Integrated Math 3 course. They have been working with rational functions, but have not been formally introduced to the idea of a vertical asymptote. This task serves as their introduction to vertical asymptotes.

In this video the students, Eden and McKenzie, are working on page 5 of the Desmos Asymptote Activity. Page 5 opens immediately after the definition of vertical asymptote has been provided. The students are working to determine where (if at all) a vertical asymptote will exist for a rational function with a constant numerator and linear denominator (i.e., $f(x) = \frac{k}{ax+b}$).

Watch the video clip provided in Canvas. You may watch this video as many times as you would like.

1. Attend to (i.e., describe in detail) how the students determined the location of the vertical asymptote for a rational function of the form $f(x) = \frac{k}{ax+b}$.
2. Interpret the students' current understanding of vertical asymptotes. Provide evidence from the video to support your claims.
3. Based on your responses to Q1 and Q2 above, if you were with Eden and McKenzie at the moment the video occurred, what would you do next? Why?

**Table 5** *Round 3 summary of the PSTs' interpretations and in which prompt they occurred*

| Interpretation (Number of PSTs) | Prompt in which the interpretation occurred | | |
|---|---|---|---|
| | Interpret prompt only | Both the interpret and decide prompts | Decide prompt only |
| Vague (2 of 16) | 1 | 0 | 1 |
| Traditional Procedure (10 of 16) | 2 | 8 | 0 |
| Underlying Concept (8 of 16) | 1 | 4 | 3 |

## 4. Looking across rounds

Looking at the PSTs' responses on the approximation of teacher noticing task across the three rounds revealed some trends. With respect to their written interpretations of what students did understand, the PSTs consistently interpreted that the students had a procedure for finding the location of the vertical asymptote (Table 6). With respect to interpretations of what students did not yet understand, we saw a jump in the percentage of PSTs making *traditional procedure* and *underlying concept* interpretations from Round 1 to Round 2 when we included the decide prompt as part of the task. While there was a slight decrease in the percentage of PSTs making *traditional procedure* and *underlying concept* interpretations from Round 2 to Round 3, the percentages are still markedly improved compared to Round 1. Once the decide prompt was introduced in Round 2, there was an immediate increase in the percentage of PSTs making interpretations of what students did not yet understand in the decide prompt.

Table 6 *The PSTs' interpretations by round*

| | Interpretation | Round 1 (N=27) | Round 2 (N=7) | Round 3 (N=16) |
|---|---|---|---|---|
| What the students did understand | Students' Procedure | 96.3% (26) | 100% (7) | 100% (16) |
| What the students did not yet understand | Any interpretations of what the students did not yet understand | 59.3% (N=16) | 100% (N=7) | 81.3% (N=13) |
| | Vague | 18.5% (N=5) | 28.6% (N=2) | 12.5% (N=2) |
| | Traditional Procedure | 44.4% (N=12) | 71.4% (N=5) | 62.5% (N=10) |
| | Underlying Concept | 3.7% (N=1) | 57.1% (N=4) | 50% (N=8) |

In Round 3, there were less *vague* interpretations (Table 7) and more PSTs made the *traditional procedure* interpretation in the interpret prompt rather than the decide prompt when we added the specific performance goals as part of the task instructions (Table 7). We saw a similar trend but to a lesser extent with the *underlying concept* interpretations from Round 2 to Round 3. After the decide prompt was added in Round 2, we saw more PSTs making *underlying concept* interpretations within the decide prompt. However, once the performance goals were added, more PSTs made the *underlying concept* interpretation in the interpret prompt.

Table 7 *In which prompt the PSTs made their interpretations*

| | Location | Round 2 (N=5*) | Round 3 (N=10*) |
|---|---|---|---|
| *Traditional Procedure* Interpretation | Interpret prompt only | 20% (N=1) | 80% (N=8) |
| | Both the interpret and decide prompt | 20% (N=1) | 20% (N=2) |
| | Decide prompt only | 60% (N=3) | 0% (N=0) |
| *Underlying Concept* Interpretation | Interpret prompt only | 0% (N=0) | 50% (N=4) |
| | Both the interpret and decide prompt | 0% (N=0) | 12.5% (N=1) |
| | Decide prompt only | 100% (N=4) | 37.5% (N=3) |

Note: * indicates the number of PSTs in the Round that made a procedural interpretation, not the total N

# V. DISCUSSION

To provide PSTs with opportunities to develop the core teaching practice of teacher noticing, we designed materials using the lens of pedagogies of practice to represent, decompose, and approximate the practice of teacher noticing in technology-mediated environments. Using design-based research methodologies, we aimed to conceptualize how to best design an approximation of practice task that supports PSTs' learning to notice student thinking in technology-mediated environments with a specific focus on interpreting students' mathematical thinking. We initially found that the PSTs were good at interpreting what students did understand, as evidenced by including instances of *students' procedure* as part of their written interpretations. However, the PSTs were not consistently discussing what the students did not yet understand in their interpretations which led us to change our design in the following rounds. We made two important design decisions aimed to further elicit interpretations of students' thinking within the PSTs' approximation of the practice of teacher noticing. First, including prompts for all three components as the decomposition of teacher noticing (i.e., including decide) was essential in eliciting interpretations of what students did not yet understand. Second, the addition of performance goals resulted in more interpretations within the interpret prompt and less *vague* interpretations overall. In the following paragraphs, we will discuss these results based on our design decisions. In the sections that follow we will briefly describe the of this study, implications for task design, and implications for MTEs.

## 1. Limitations of the study

There are several important limitations of this study that should be taken into account when considering the findings. Most prominent are issues with Round 2 regarding sample size, course type, and university. Round 2 had a small sample of PSTs (N=7) who were all from one course at the same university; thus, this particular round lacked the diversity afforded by larger samples across multiple universities that were seen in Rounds 1 and 3. Given the small size and lack of diversity, it is tenuous to compare across all three rounds, yet we believe that despite this limitation, the findings from Round 2 do indicate important implications for the design choices made in that round that continued in Round 3. We were further limited by the nature of using a written approximation of practice since that means we only have access to what the PSTs articulate in writing in response to the prompts. Thus, it is possible that the PSTs attended to or interpreted students' mathematical thinking that they did not write in their responses. Another limitation involves the window (Sherin et al., 2009) afforded by our selected artifact. Our selected representation of practice (the picture in picture video of students engaged in the technology-enhanced task) affords a wide window into students' thinking, particularly it includes a window into what students do not yet understand. The moments of confusion within the video clip resulted in evidence of the students meeting one performance goal and no evidence in the clip of meeting the other two performance goals (see Figure 7 for a list of goals). Other representations (i.e., video clips) might yield different results. It is important for MTEs to consider the window afforded by the representation when designing approximations of teacher noticing tasks since the goal is to elicit as much as possible from PSTs who are developing this practice.

## 2. Implications for task design

The initial design of the approximation of noticing task revealed that the PSTs consistently interpreted what the students understood. Across all three rounds, this finding remained true with most PSTs interpreting the *students' procedure*—over 96% of interpretations per round included that the students had a procedure that correctly identified the location of the vertical asymptote. Previous research on teacher noticing revealed that PSTs struggle with interpreting students' understandings (see König et al., 2022; Stahnke et al., 2016), which contrasts with our findings. Thus, separating the analysis of interpretation of what students did understand and what students did not yet understand might better provide insight into PSTs' interpretations of student thinking.

Several researchers have suggested that the decide component of teacher noticing is challenging (e.g., Jacobs et al., 2023; Jong et al., 2021; Krupa et al., 2017), and subsequently some researchers have focused on attend and interpret (and not decide) when introducing PSTs to the practice of teacher noticing in an attempt to simplify the approximation of practice (e.g., Fisher et al., 2019). We believed the same as evidenced by our design decision in Round 1 to have only an attend and interpret prompt. Yet, our findings suggest that including the more challenging component of deciding how to respond can actually be helpful and support PSTs in articulating their interpretations of what students do not yet understand. We discovered that interpreting what students *did not yet* understand is potentially more difficult or at least not as intuitive for PSTs. We conjectured that by asking the PSTs to reflect upon their next pedagogical moves, the PSTs would be more likely to consider what the students were still grappling with; thus, they had a reason to go beyond what is already known about the students' thinking (i.e., what was seen in the video-clip representation) when deciding how to respond. In Round 2, when we added the decide prompt, more than half of the PSTs interpreted the *traditional procedure* within the decide prompt, and all PSTs interpreted the *underlying concept* within the decide prompt. Thus, without including this prompt, we would have missed the PSTs' interpretations of what the students did not yet understand. We posit that it might not have been that they didn't notice it, but they did not see a reason to articulate their noticing of what the students did not yet understand because the wording of the task prompts did not elicit this aspect of the PSTs' interpretations. As shown by Estapa and colleagues (2018), the focus of PSTs' noticing and the way they articulated their noticing changed between different representations of practice. In our study, PSTs articulated their noticing based on how we decomposed the approximation of noticing prompts (i.e., asking them to decide based on their attend and interpret responses). Both highlight the influence of carefully choosing a pedagogies of practice approach to designing tasks to develop PSTs' teacher noticing skills.

## 3. Implications for mathematics teacher educators

Our findings related to PSTs' interpretations when considering how to respond corroborate similar findings in other studies on approximation of practice design. In their study of elementary PSTs' approximation of organizing a classroom discussion, Tyminski et al. (2014) found that most of the PSTs were interpreting students' strategies while in the midst of considering their discussion goals. Thus, when PSTs are asked to make pedagogical decisions, there is a distinct need to interpret students' strategies, and Tyminski et al.'s study demonstrated that PSTs interpret well when the approximation of practice includes pedagogical decisions. This result implies that when MTEs are designing approximations of practice tasks, they should consider if the PSTs are asked to make pedagogical decisions since such decisions can result in more depth of interpretations of student thinking.

When considering the third component of noticing (i.e., deciding), research has shown that teachers do not always ground their decisions on their interpretations of students' thinking (e.g., Jacobs et al., 2023). Jacobs et al. described some teachers' decisions as having focused on their own preferred strategy instead of utilizing the interpretation of the students' thinking to further the students' understanding. While we did not analyze the quality of PSTs' decisions, our findings do suggest that the wording of the decide prompt can support PSTs in using their interpretations to inform their pedagogical decisions. We intentionally asked PSTs to refer back to their attend and interpret responses when articulating their decision (see Figure 7). Future studies are needed to better understand how to best support PSTs' development of robust decisions that rely on interpretations of students' thinking.

Furthermore, results of this study suggest that both MTEs and researchers who are focused on PSTs' interpretations when noticing student thinking need to consider PSTs' responses to both interpret and decide prompts. Not only does including a decide prompt seem to elicit PSTs interpretations of what students did not yet understand, we also saw evidence of interwoven responses across prompts (e.g., interpretations in the decide prompt and not the interpret prompt). This finding seems to contradict what some other studies have found. Thomas et al. (2023) reported that the elementary preservice teachers in their study "on occasion, addressed other components within a single response (e.g., proposing a decision within a response to an interpreting prompt)" (p. 367). In contrast, our study revealed that many PSTs addressed their interpretations in their response to the decide prompt. We hypothesize that both choice of representation of practice (i.e., artifacts) and decomposition of practice (i.e., prompt wording and prompts based on a provided decomposed framework) play an important role in approximating the component of interpreting within the practice of teacher noticing.

Since teacher noticing is a practice that is foundational to other teaching practices, it is often introduced in early courses for PSTs. Thus, it is not uncommon for PSTs to be less familiar with local mathematics standards and be early in their learning about the complex ways in which students are expected to understand procedures and their underlying concepts—meaning that learning and/or performance goals for a particular task might not be obvious to many PSTs. The findings of this study suggest that the inclusion of specific performance goals can support PSTs' approximation of teacher noticing. In Round 3, with the introduction of the performance goals, PSTs interpreted the *traditional procedure* more frequently in the interpret prompt. The increase of the *traditional procedure* within the interpret prompt was likely related to the inclusion of the performance goals (as that was the only modification in that Round). If PSTs explicitly know what the students were expected to achieve within the lesson, then they have a guide for what evidence to look for when approximating the practice and are less likely to provide vague interpretations. However, while we saw an increase in frequency of the *traditional procedure* in the interpret prompt when the goals were added, we did not see a parallel shift for the *underlying concept* suggesting that even with the addition of goals, it is important to keep the decide prompt and that providing goals as a scaffold is not enough to prompt PSTs full articulations of their interpretations of students' thinking. Reflecting upon the iterative design changes, we saw evidence that PSTs were interpreting what the students did not yet understand, which could indicate that the PSTs were interpreting this all along, yet our prompts failed to elicit such interpretations.

## VI. CONCLUSION

The goal of professional development design studies is to build an understanding of how teachers, in this case PSTs in the context of their teacher preparation coursework, develop sophisticated practices as well as how to support that development (Cobb et al., 2014). Through our three cycles of design, we were able to do just that with respect to teacher noticing. In the initial design of our approximation of teacher noticing task, we drew on the recent reviews of studies on teacher noticing to justify our decision to include only an attend prompt and an interpret prompt (Amador et al., 2021; König et al., 2022). Yet, as our study revealed, adding a decide prompt and performance goals influenced what the PSTs articulated in their interpretations of student thinking and in which prompt they did so. Our results further point to Thomas' (2017) call to "[understand the] relationship among the component processes of teacher noticing"

(p. 510). Thomas et al. (2023) indicated that the nature of the task can "[limit] somewhat, the nuance and complexity with which we might consider the interplay of professional noticing components as participants are responding to the prompts" (p. 8) We want to extend this notion and suggest that the complex interplay between the decomposed components when approximating the practice of teacher noticing are also related to and potentially constrained by the nature of the prompts and information provided as a scaffold to help the PSTs approximate the practice of teaching that occurs before the approximation of teacher noticing.

There are a number of important implications for MTEs that arose from this study. First, interpreting what students did not yet understand is more difficult than interpreting what they did understand. The inclusion of targeted goals within a representation of practice can help PSTs explicitly focus on what students should learn from a task—including conceptual understandings. Second, PSTs need to be challenged to adopt a teacher mindset and consider their pedagogical decisions to elicit their interpretations of what students do not yet understand. Third, tasks which decompose the practice of noticing might guide students to consider different types of evidence (i.e., spoken, written, technology engagement) as well as what students do and do not yet understand. Last, our focus was specifically on teacher noticing when students are engaged in technology-enhanced mathematics tasks; thus, we used a framework specific to this context (i.e., NITE; McCulloch et al., 2023) to support the decomposition of this practice. However, what we have learned about supporting PSTs' development of more sophisticated teacher noticing practice is relevant regardless of context; a different context would require using an aligned representation of practice and decomposition framework. Like Mason (2002), we feel that teacher noticing underlies all other content specific teaching practices, and as such understanding how to better support PSTs' development of this practice is crucial. We echo Cobb et al.'s (2014) call that sharing studies of how such supports have been designed is "urgently needed" as they "make critical contributions to the development and refinement of practice-specific theories" (p. 499). Our hope is that others will join us in continuing to develop and refine theories of PSTs' development of teacher noticing through a pedagogies of practice approach.

## CONFLICTS OF INTEREST

No potential conflict of interest relevant to this article was reported.

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