Chapter 9
Fourier Series for Fractals in Two Dimensions

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Dedicated to the memory of Robert Strichartz.[1]

9.1 Introduction

The past two decades have seen a flurry of interest in exploring Fourier expansions for $L^2(\mu)$ in the case that μ is in a class of fractal measures, with particular emphasis on the case when μ is singular with respect to Lebesgue measure. These studies have fallen into two groups: (i) the case of orthogonal expansions (analyzed for Cantor measures and related IFS measures) and (ii) the case of frame-like expansions. In both cases, there now exist explicit algorithms for analysis and synthesis for the corresponding Hilbert spaces $L^2(\mu)$ [22, 28]. By this we mean that there exists a sequence of frequencies $\{\lambda_n\}$ such that for every $f \in L^2(\mu)$,

$$f(x) = \sum_{n=0}^{\infty} c_n e^{2\pi i \lambda_n \cdot x}. \tag{9.1}$$

[1] The present chapter expands on a talk by the second named author at the 7th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals, June 4–8, 2022, in the memory of R. S. Strichartz.

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P. Alonso Ruiz et al. (eds.), *From Classical Analysis to Analysis on Fractals*, Applied and Numerical Harmonic Analysis, https://doi.org/10.1007/978-3-031-37800-3_9

The coefficients are given by inner products, which in many circumstances are not taken against the exponentials themselves. Moreover, the convergence may be conditional only.

Orthogonal expansions of the form in Eq. (9.1) for fractal measures were initiated by Pedersen and the second named author in [28] and have been built upon by numerous authors; we highlight just a few subsequent papers [14, 15, 32], particularly those by Strichartz [51, 52]. We note that the initial constructions of [28] were in one spatial dimension but are generalizable to higher dimensions. However, only certain fractal measures (namely spectral measures) admit an orthogonal expansion. Non-orthogonal expansions as in Eq. (9.1) for arbitrary singular measures μ were introduced by the first and third named authors in [22] using so-called effective sequences obtained via the Kaczmarz algorithm. Recent work by all three co-authors [19–21] has focused on connections between these Fourier expansions and model spaces in the form of de Branges Hilbert spaces of analytic functions. However, up to now these methods have been largely limited to the cases when the measure μ at hand has its support in one spatial dimension. Our chapter will extend the Kaczmarz Fourier expansions to measures in two spatial dimensions that satisfy a "slice-singular" condition. We will do so by developing a theory of an operator-valued Kaczmarz algorithm.

In this chapter, we present a generalized Fourier expansion for a specific family of singular measures μ admitting realizations in $\mathbb{R}^2$. The family we consider, called slice-singular measures, includes IFS measures μ, and it covers in particular a variety of Sierpinski IFS measure constructions. In this multi-variable setting, we present explicit non-orthogonal Fourier series expansion in $L^2(\mu)$. These Fourier series expansions go far beyond the related and better-known orthogonal expansions, first introduced by Jorgensen and Pedersen [28] and explored later in multiple papers by Strichartz [49–52]. Still, the Sierpinski-type class comprises only a small class of Cantor-type measures. In the present chapter, we cover $L^2(\mu)$ Fourier expansions for all slice-singular multivariable measures μ. We also stress why the orthogonality requirement for the earlier $L^2(\mu)$ Fourier expansions in the Jorgensen-Pedersen/Strichartz family is extremely restrictive.

Our main result can be loosely stated thusly:

Theorem *Suppose that μ is a slice-singular Borel probability measure on* $[0, 1]^2$. *For any $f \in L^2(\mu)$, there exists a sequence of coefficients $\{c_{nm}\}$ such that*

$$f(x, y) = \sum_{n,m=0}^{\infty} c_{nm} e^{2\pi i(nx+my)},$$

where the sum converges in norm, possibly conditionally. □

The order of the double sum depends on the slice-singular nature of the measure μ (see Theorem 9.1 and the discussion following).

The chapter is organized as follows: The remainder of this section discusses the requisite measure theory (e.g., the Rokhlin Decomposition Theorem and the

Kakutani Dichotomy Theorem) and discusses the Kaczmarz construction of Fourier series expansions for singular measures in one dimension. Section 9.2 defines slice-singular measures and derives several properties of such measures. We show that the Sierpinski gasket and carpet are both slice-singular. Section 9.3 lays out the proof of our main results concerning the construction of Fourier series expansions for slice-singular measures. Our proof utilizes the Kaczmarz algorithm for operators, described in Sect. 9.4. The proof of our main results uses the crucial result of Sect. 9.4, which is that the operators we define in Sect. 9.3 are effective.

Remark 9.1 While the present work is focused on a particular Fourier-based harmonic analysis on affine IFS-fractal structures, we stress that there are many other approaches. The following list gives a glimpse into these diverse approaches: [6, 7, 12, 13, 17, 33, 40].

9.1.1 Self-similar Fractals and Measures

Self-similar fractals are the invariant set of an iterated function system (IFS) consisting of transformations acting on Euclidean space. Well-known examples of such fractals are Cantor sets and Mandelbrot sets. Fractals are commonly studied objects in many contexts. Cantor sets, in particular, appear in the context of analysis [5, 50], number theory [8, 48], probability [4, 27, 36], geometry [41, 42], physics [1, 53, 54], and harmonic analysis [28, 44, 51].

Our main results concerning Fourier series expansions will apply to a large class of measures, those that we refer to as "slice-singular" (Definition 9.6). Notably, a common set of measures that satisfy our definition are fractal measures generated by IFSs.

Definition 9.1 Suppose $\Psi = \{\psi_1, \dots, \psi_N\}$ is a set of maps acting on a metric space (X, d). We say that $A \subset X$ is invariant for Ψ if $A = \cup_{i=1}^{N} \psi_i(A)$.

In [25], Hutchinson laid out the main relationship between fractals and iterated function systems (IFS); this relationship is the foundation of our results. Recall that $\psi : X \to X$ is Lipschitz if for all $x, y \in X$, there exists $C > 0$ such that

$$d(\psi(x), \psi(y)) \leq Cd(x, y).$$

The Lipschitz constant of ψ is the infimum of all such C. We say that ψ is a contraction if it has Lipschitz constant less than 1. Hutchinson's Theorem is as follows:

Theorem A *Let $X = (X, d)$ be a complete metric space and $\Psi = \{\psi_1, \dots, \psi_N\}$ a finite set of contraction maps on X. Then there exists a unique closed bounded set A such that*

$$A = \bigcup_{i=1}^{N} \psi_i(A).$$

Furthermore, A is compact and is the closure of the set of fixed points of finite compositions of members of Ψ. Moreover, for a closed and bounded set K, $\Psi^p(K) \to A$ in the Hausdorff metric.

Moreover, given a set of weights $w_1, \dots, w_N > 0$ such that $\sum w_j = 1$, there exists a unique Borel probability measure μ supported on A that satisfies the invariance equation

$$\int f d\mu = \sum_{j=1}^{N} w_j \int f \circ \psi_j d\mu \tag{9.2}$$

for every continuous function f. □

The measure μ is called the invariant measure for the IFS $\{\psi_1, \dots, \psi_N\}$ with weights $\{w_1, \dots, w_N\}$.

Note that in Eq. (9.2), some of the ψ_j's can be repeated, in which case we can sum the corresponding weights together to obtain the same invariance equation but with the ψ_j's all distinct. We shall use this observation in several of our examples, and unless otherwise stated, we assume this distinctiveness condition on our IFS.

Our IFSs often take the form of affine transformations acting on $\mathbb{R}^d$. We do this by specifying a scaling parameter $R > 1$ and a set of digits $B = \{b_j\}_{n=1}^{N} \subset \mathbb{R}^d$ and defining

$$\psi_j(x) = \frac{x + b_j}{R}.$$

As we will discuss in the examples in Sect. 9.2.3, the Sierpinski gasket and Sierpinski carpet both arise in this fashion.

An application of the Kakutani Dichotomy Theorem [23, 30] says that given two invariant measures of an IFS acting on the same metric space with corresponding weights, the measures are either identical or mutually singular. Consequently, if their weights are different, then the two measures are mutually singular. We formalize this statement for future reference; though Kakutani's Theorem did not concern invariant measures directly, we present the result as such:

Theorem B (Kakutani Dichotomy Theorem) *Suppose $\{\psi_1, \dots, \psi_N\}$ are distinct contractions acting on the metric space (X, d). Let μ and $\tilde{\mu}$ denote the invariant measures arising from these contractions using weights $\{w_1, \dots, w_N\}$ and $\{\tilde{w}_1, \dots, \tilde{w}_N\}$, respectively. Then:*

1. *The measures $\mu = \tilde{\mu}$ if and only if $w_j = \tilde{w}_j$ for all j.*
2. *The measures μ and $\tilde{\mu}$ are mutually singular whenever $w_j \neq \tilde{w}_j$ for some j.* □

9.1.2 Review of Fourier Series for 1D Fractals

Let $d \in \mathbb{N}$ be a number of dimensions, let $X \subseteq \mathbb{R}^d$, and let μ be a probability measure on X. By a Fourier series for $f \in L^2(\mu)$, we mean an ordered countable set of frequencies $\Lambda \subset \mathbb{R}^d$ and sequence of coefficients $\{c_\lambda\}$, where $c_\lambda \in \mathbb{C}$ for all $\lambda \in \Lambda$, such that

$$f = \sum_{\lambda \in \Lambda} c_\lambda e^{2\pi i \langle \lambda, x \rangle},$$

where the convergence occurs in the norm of $L^2(\mu)$. For concision, we will adopt the notation $e_\lambda := e^{2\pi i \langle \lambda, x \rangle}$, and we will refer to such functions as "(complex) exponential functions."

We begin by reviewing the method used in [22] to establish Fourier series for functions in $L^2(\mu)$, where μ is a one-dimensional singular probability measure on $[0, 1)$. The present work generalizes this method.

Remark 9.2 The expansions here are done with a choice of Hilbert space and inner products in the context of $L^2(\mu)$ for some Borel measure μ on $\mathbb{R}^d$ or, in our discussion of general frame expansions, an abstract Hilbert space (Definitions 9.2, 9.3, 9.4, and 9.5). In each case, the inner products occurring inside the expansions are written $\langle \cdot, \cdot \rangle$, and they will correspond to these Hilbert spaces. In Theorem D we return to the case of $L^2(\mu)$.

Such a new method was necessary due to the general insufficiency of orthonormal bases and frames to yield such series. For example, if there exists a countable set $\Lambda \subseteq \mathbb{R}$ such that $\left\{e^{2\pi i \lambda x}\right\}_{\lambda \in \Lambda}$ is an orthonormal basis of $L^2(\mu)$, then clearly

$$f = \sum_{\lambda \in \Lambda} \langle f, e_\lambda \rangle \, e^{2\pi i \lambda x}$$

for any $f \in L^2(\mu)$. The measure μ is then said to be spectral with spectrum Λ, and we have Fourier series representations.

Sometimes $L^2(\mu)$ is spectral, and sometimes it is not [28]. For example, in the case $d = 1$, $R = 4$, and $B = \{0, 2\}$, we get the IFS $\left\{\psi_0(x) = \frac{x}{4}, \psi_2(x) = \frac{x+2}{4}\right\}$, set invariant set A from Hutchinson's Theorem is the quaternary Cantor set, and the corresponding invariant measure is the quaternary Cantor measure, denoted ν_4, which is spectral. On the other hand, in the case that $d = 1$, $R = 3$, and $B = \{0, 2\}$, the set A is the famous ternary or "middle-thirds" Cantor set, and μ is the ternary Cantor measure, denoted ν_3. Jorgensen and Pedersen showed that there cannot exist three mutually orthogonal complex exponential functions in this situation, and so there cannot be an orthonormal basis of complex exponential functions. Therefore, we cannot rely on orthonormal bases of complex exponential functions for the construction of Fourier series.

Frames would be another option [16, 34]:

Definition 9.2 A sequence $\{f_n\}_{n=0}^{\infty}$ in a Hilbert space $\mathbb{H}$ is said to be *Bessel* if there exists a constant $B > 0$ such that for any $x \in \mathbb{H}$,

$$\sum_{n=0}^{\infty} |\langle x, f_n \rangle|^2 \leq B\|x\|^2. \tag{9.3}$$

This is equivalent to

$$\left\| \sum_{n=0}^{K} c_n f_n \right\| \leq \sqrt{B \sum_{n=0}^{K} |c_n|^2}$$

for any finite sequence $\{c_0, c_1, \ldots, c_K\}$ of complex numbers. The sequence is called a *frame* if in addition there exists a constant $A > 0$ such that for any $x \in \mathbb{H}$,

$$A\|x\|^2 \leq \sum_{n=0}^{\infty} |\langle x, f_n \rangle|^2 \leq B\|x\|^2. \tag{9.4}$$

If $A = B$, then the frame is said to be *tight*. If $A = B = 1$, then $\{f_n\}_{n=0}^{\infty}$ is a *Parseval frame*. The constant A is called the *lower frame bound* and the constant B is called the *upper frame bound* or *Bessel bound*.

If $\{f_n\}$ is a frame, then there exists a dual frame $\{g_n\}$ such that for all $x \in \mathbb{H}$,

$$x = \sum \langle x, g_n \rangle\, f_n = \sum \langle x, f_n \rangle\, g_n.$$

Therefore, if there were a collection of exponential functions $\left\{e_{\ell_n}\right\}$ that is a frame in $L^2(\mu)$, then taking a dual frame $\{g_n\}$, we would have

$$f = \sum_{n=0}^{\infty} \langle f, g_n \rangle\, e_{\ell_n}.$$

Unfortunately, for non-spectral singular probability measures μ, it is generally unknown whether $L^2(\mu)$ possesses a complex exponential frame. In fact, it is still an open question of whether the middle-third Cantor set possesses an exponential frame, first posed by Strichartz [49].

Moreover, the full sequence $\{e_n\}_{n=1}^{\infty}$ is not Bessel if μ is singular, and so it cannot be a frame. The fact that orthogonal bases and frames do not readily work to yield Fourier series inspired the authors in [22] to turn to effective sequences:

Definition 9.3 (Effective Sequences) Let $\{\varphi_n\}_{n=0}^{\infty}$ be a linearly dense sequence of unit vectors in a Hilbert space $\mathbb{H}$. Given any element $x \in \mathbb{H}$, we may define a sequence $\{x_n\}_{n=0}^{\infty}$ in the following manner:

$$x_0 = \langle x, \varphi_0 \rangle \varphi_0$$
$$x_n = x_{n-1} + \langle x - x_{n-1}, \varphi_n \rangle \varphi_n.$$

If $\lim_{n\to\infty} \|x - x_n\| = 0$ regardless of the choice of x, then the sequence $\{\varphi_n\}_{n=0}^{\infty}$ is said to be effective.

The above formula for producing $\{x_n\}$ is known as the Kaczmarz algorithm. The Kaczmarz sequence begins with the projection of x onto φ_0. Each next x_n is then the result of moving from x_{n-1} in the direction of φ_n to the extent that it takes us closer to x. In other words, x_n is the projection of x_{n-1} onto the affine subspace of $\mathbb{H}$ containing x parallel to the subspace $\varphi_n^{\perp}$.

In 1937, Stefan Kaczmarz [29] proved the effectivity of linearly dense periodic sequences in the finite-dimensional case. In 2001, these results were extended to infinite-dimensional Banach spaces under certain conditions by Kwapień and Mycielski [31]. These two also gave the following formula for the sequence $\{x_n\}_{n=0}^{\infty}$, which we state here for the Hilbert space setting: Define

$$\begin{aligned} g_0 &= \varphi_0 \\ g_n &= \varphi_n - \sum_{i=0}^{n-1} \langle \varphi_n, \varphi_i \rangle g_i. \end{aligned} \tag{9.5}$$

Then

$$x_n = \sum_{i=0}^{n} \langle x, g_i \rangle \varphi_i. \tag{9.6}$$

As shown by [31], and also more clearly for the Hilbert space setting by [18], we have

$$\|x\|^2 - \lim_{n\to\infty} \|x - x_n\|^2 = \sum_{n=0}^{\infty} |\langle x, g_n \rangle|^2,$$

from which it follows that $\{\varphi_n\}_{n=0}^{\infty}$ is effective if and only if

$$\sum_{n=0}^{\infty} |\langle x, g_n \rangle|^2 = \|x\|^2. \tag{9.7}$$

That is to say, $\{\varphi_n\}_{n=0}^{\infty}$ is effective if and only if the associated sequence $\{g_n\}_{n=0}^{\infty}$ is a Parseval frame. We call $\{g_n\}_{n=0}^{\infty}$ the auxiliary sequence of $\{\varphi_n\}_{n=0}^{\infty}$.

If $\{\varphi_n\}_{n=0}^{\infty}$ is effective, then (9.6) implies that for any $x \in \mathbb{H}$, $\sum_{i=0}^{\infty}\langle x, g_i\rangle\varphi_i$ converges to x in norm, and as noted, $\{g_n\}_{n=0}^{\infty}$ is a Parseval frame. This does not mean that $\{g_n\}_{n=0}^{\infty}$ and $\{\varphi_n\}_{n=0}^{\infty}$ are dual frames, since $\{\varphi_n\}_{n=0}^{\infty}$ need not even be a frame. However, $\{\varphi_n\}_{n=0}^{\infty}$ and $\{g_n\}_{n=0}^{\infty}$ are pseudo-dual in the following sense, first given by Li and Ogawa in [35]:

Definition 9.4 Let $\mathbb{H}$ be a separable Hilbert space. Two sequences $\{\varphi_n\}$ and $\{\varphi_n^{\star}\}$ in $\mathbb{H}$ form a pair of *pseudoframes* for $\mathbb{H}$ if for all $x, y \in \mathbb{H}$, $\langle x, y\rangle = \sum_n \langle x, \varphi_n^{\star}\rangle\langle\varphi_n, y\rangle$.

All frames are pseudoframes, but not the converse. Observe that if $x, y \in \mathbb{H}$ and $\{\varphi_n\}_{n=0}^{\infty}$ is effective, then

$$\begin{aligned}\langle x, y\rangle &= \left\langle \sum_{m=0}^{\infty}\langle x, g_m\rangle\varphi_m, y\right\rangle \\ &= \sum_{m=0}^{\infty}\langle x, g_m\rangle\,\langle\varphi_m, y\rangle\end{aligned}$$

and so $\{\varphi_n\}_{n=0}^{\infty}$ and $\{g_n\}_{n=0}^{\infty}$ are pseudo-dual.

Of course, since $\{g_n\}_{n=0}^{\infty}$ is a Parseval frame, it is a true dual frame for itself. We also employ the following definition:

Definition 9.5 (Dextroduality) Let $\mathcal{H}$ be a separable Hilbert space. Let $\{f_n\}$ and $\{g_n\}$ be two sequences in $\mathcal{H}$. We say that $\{g_n\}$ is dextrodual to $\{f_n\}$ if

$$\sum_{n=0}^{\infty}\langle x, g_n\rangle\, f_n = x$$

for all $x \in \mathbb{H}$.

In other words, if T_g is the analysis operator of $\{g_n\}$ and T_f^* is the synthesis operator of $\{f_n\}$, then $T_f^* T_g = I_{\mathcal{H}}$. The appearance of the synthesis operator T_g on the right side of the product is the reason for using the prefix "dextro-."

Thus, if $\{\varphi_n\}$ is effective, then $\{g_n\}$ is dextrodual to $\{\varphi_n\}$.

The aforementioned theorem of Kwapień and Mycielski in [31] demonstrates a condition under which $\{\varphi_n\}$ may be effective in an infinite-dimensional separable Hilbert space:

Theorem C (Kwapień and Mycielski) *A stationary sequence of unit vectors that is linearly dense in a Hilbert space is effective if and only if its spectral measure either coincides with the normalized Lebesgue measure or is singular with respect to Lebesgue measure.* □

By a stationary sequence, it is meant a sequence $\{\varphi_n\}$ such that $\langle\varphi_m, \varphi_n\rangle = \langle\varphi_{m+k}, \varphi_{n+k}\rangle$ for all $m, n, k \in \mathbb{N}_0$. Thus, the expression $\langle\varphi_m, \varphi_{m+k}\rangle$ depends on

k alone. Then the spectral measure of $\{\varphi\}$ is the unique positive Borel measure σ satisfying

$$\langle \varphi_m, \varphi_{m+k} \rangle = \int_0^1 e^{-2\pi i k x} \, d\sigma(x).$$

It was observed in [22] that in the case that μ is a Borel probability measure on $[0, 1)$, then the sequence $\{e_n\}_{n=0}^{\infty}$ of complex exponential functions is stationary and linearly dense in $L^2(\mu)$, and μ is itself the spectral measure of $\{e_n\}$. It follows by the theorem of Kwapień and Mycielski that $\{e_n\}$ is effective in $L^2(\mu)$ if and only if μ is Lebesgue measure or singular with respect to Lebesgue measure. Thus, the following result was obtained:

Theorem D (Herr and Weber) *If μ is a singular Borel probability measure on $[0, 1)$, then the sequence $\{e_n\}_{n=0}^{\infty}$ is effective in $L^2(\mu)$. As a consequence, any element $f \in L^2(\mu)$ possesses a Fourier series*

$$f(x) = \sum_{n=0}^{\infty} c_n e^{2\pi i n x},$$

where

$$c_n = \int_0^1 f(x) \overline{g_n(x)} \, d\mu(x)$$

and $\{g_n\}_{n=0}^{\infty}$ is the auxiliary sequence associated to $\{e_n\}_{n=0}^{\infty}$ via Equation (9.5). The sum converges in norm, and Parseval's identity $\| f \|^2 = \sum_{n=0}^{\infty} |c_n|^2$ holds. □

Thus, singular probability measures on $[0, 1)$ do yield Fourier series, and they come from performing the Kaczmarz algorithm with the sequence $\{e_n\}_{n=0}^{\infty}$. In this chapter, we turn to the problem of obtaining the same result in higher dimensions. The main obstacle to applying the Kwapień-Mycielski Theorem in the same way as before is the condition of stationarity. In $[0, 1)^d$ with μ a Borel probability measure on $[0, 1)^d$, the complex exponential functions are now of the form $e_\lambda := e^{2\pi i \langle \lambda, x \rangle}$, where $\lambda, x \in \mathbb{R}^d$. The set $\left\{e_n : n \in \mathbb{N}_0^d\right\}$ can be shown to be linearly dense in $L^2(\mu)$, but when ordered into a sequence will not be stationary. It is this issue that we address in this chapter.

The construction of Fourier Series for singular measures in 1 dimension via the Kaczmarz algorithm is enriched by its connection to the de Branges-Rovnyak subspaces of the classical Hardy space. By the Herglotz Representation Theorem, there is a 1-to-1 correspondence between the singular nonnegative Borel measures μ on $[0, 1)$ and the nonconstant inner functions b in the Hardy space H^2 given by the Poisson integral:

$$\mathrm{Re}\left(\frac{1+b(z)}{1-b(z)}\right) = \int_0^1 \frac{1-|z|^2}{\left|e^{2\pi i x}-z\right|^2}\, d\mu(x).$$

Since b is inner, the de Branges-Rovnyak subspace $\mathcal{H}(b)$ of H^2 is equal to $H^2 \ominus bH^2$ with the same norm as H^2. The normalized Cauchy transform V_μ gives a mapping of $L^2(\mu)$ onto $\mathcal{H}(b)$ via

$$V_\mu f(z) = \frac{\int_0^1 \frac{f(x)}{1-ze^{-2\pi i x}}\, d\mu(x)}{\int_0^1 \frac{1}{1-ze^{-2\pi i x}}\, d\mu(x)}.$$

In the case that μ is a probability measure (i.e., $\|\mu\| = 1$, or equivalently $b(0) = 0$), which is the case to which we restrict ourselves in this chapter, V_μ is a unitary transformation.

Suppose μ is a singular Borel probability measure on [0, 1), and $f \in L^2(\mu)$. Let $V_\mu f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then, Theorem 1.1 of [43] implies that

$$\sum_{n=0}^{\infty} a_n e^{2\pi i n x} = f(x) \tag{9.8}$$

in the $L^2(\mu)$ norm. In fact, however, this series turns out to be the same series as constructed from the Kaczmarz algorithm in Theorem D via the Kwapień-Mycielski Theorem.

9.1.3 Rokhlin Disintegration Theorem

The notion of direct integrals in the setting of analysis of operators in Hilbert spaces arises in such diverse applications as the theory of unitary representations of groups, decompositions used in the study of von Neumann algebras, mathematical physics, ergodic theory, machine learning models, statistics, and probability theory [24, 26, 38, 39, 46, 47].

Perhaps closest to our present analysis is the use of "slice decompositions" arising in Bayesian probability theory. There one studies joint distributions of systems of random variables, and then one introduces associated marginal measures and the corresponding conditional distributions. "Bayesian" derives from Thomas Bayes, who offered the first mathematical treatment of statistical data analysis, introducing what is now known as Bayesian inference.

For our purpose at hand, the readers might find it useful to think of our present direct integrals as extensions of orthogonal direct sums (the case of counting measure) to direct integrals, or alternatively, extending the notion of discrete frame expansions to their wider measurable counterparts.

Here, we stress the use of direct integrals in a very specific context: that of reducing a particular harmonic analysis in two variables to the study of one-dimensional slices, which then comprise the particular direct integral decompositions at hand. In our present application, we make precise our particular direct integrals, both for the Hilbert spaces at hand and for the resulting direct integral operators.

The Rokhlin Disintegration Theorem [46] generalizes the Fubini-Tonelli theorems for product measures. It allows for measures on product spaces to be decomposed in such a way that integration can be accomplished using iterated integrals.

Theorem E (Rokhlin Disintegration Theorem) *If μ is a Borel probability measure on $A \times B$ (with A, B metric spaces), then there exist a Borel probability measure σ on B and a parametrized family of Borel probability measures $\{\gamma^b\}_{b \in B}$ on A such that*

1. *For a Borel set $E \subset A \times B$, $\mu(E) = \int_B \gamma^b(E \cap (A \times \{b\}))\,\sigma(db)$.*
2. *For $f \in L^1(\mu)$, for σ-a.e. b, $f(\cdot, b) \in L^1(\gamma^b)$.*
3. *For $f \in L^1(\mu)$, the mapping $b \mapsto \int_B f(a,b)\gamma^b(da)$ is measurable and integrable w.r.t σ.* □

The measure σ is called the B-marginal of μ, and we refer to the $\{\gamma^b\}$ as the *slice measures*.

Note that the disintegration of μ can also be done using the A-marginal and the slices $\{\gamma_a\}_{a \in A}$ in the obvious manner. Further discussion of the Rokhlin theory can be found in [2, 3, 9, 45].

9.2 Slice-Singular Measures

In this section we present the analytic details needed for our direct integral decomposition of slice-singular measures μ in two dimensions. We shall refer to these direct integral decompositions as slice decompositions, and the measures μ considered in two dimensions are assumed to be slice-singular. Our aim is multivariable Fourier expansions for $L^2(\mu)$, generally non-orthogonal.

For a given measure μ in two dimensions, the notion "slice-singular" is made precise below; it refers to assumptions regarding both the marginal measures defined from μ and the corresponding conditional measures. The resulting decompositions for μ may be viewed as Bayes rules, but our setting is more general, and we shall refer to the general decompositions as Rokhlin-disintegration-decompositions.

If μ is supported in a subset of $\mathbb{R}^2$, then there are two marginal measures, each one a one-dimensional measure. To each of these marginal measure there is associated a one-dimensional conditional measure. Our singularity assumptions will pertain to these measures that are obtained after disintegration.

9.2.1 Projections of Invariant Measures

We let π_1 and π_2 be the projections of $\mathbb{R}^2$ onto $\mathbb{R}$ given by $(x, y) \mapsto x$ and $(x, y) \mapsto y$, respectively.

We consider a set $\{\psi_1, \ldots, \psi_N\}$ of strict contractions on $\mathbb{R}^2$. Fix nonnegative weights $\{c_1, \ldots, c_N\}$ such that $\sum c_j = 1$, and let μ be the invariant measure given by Hutchinson's Theorem.

We say that the mapping $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ is *Cartesian* if there exist mappings $\eta_1, \eta_2 : \mathbb{R} \to \mathbb{R}$ such that $\psi(x, y) = (\eta_1(x), \eta_2(y))$. This is equivalent to the condition $\eta_1 \circ \pi_1 = \pi_1 \circ \psi$ and $\eta_2 \circ \pi_2 = \pi_2 \circ \psi$.

If $\{\psi_1, \ldots, \psi_N\}$ are Cartesian, then the maps $\{\phi_1, \ldots, \phi_N\}$ given by

$$\phi_j \circ \pi_1 = \pi_1 \circ \psi_j \tag{9.9}$$

are a set of strict contractions on $\mathbb{R}$. Therefore, again by Hutchinson's Theorem, there exists a unique probability measure ρ such that for every continuous function $g : \mathbb{R} \to \mathbb{R}$,

$$\int g \; \rho(dx) = \sum_{j=1}^{N} c_j \int g \circ \phi_j \; \rho(dx).$$

Lemma 9.1 *Let $\{\psi_1, \ldots, \psi_N\}$ be a set of Cartesian strict contractions on $\mathbb{R}^2$ with invariant measure μ for the weights $\{c_1, \ldots, c_N\}$. Let μ_1 be the marginal of μ in the x-direction. Let $\{\phi_1, \ldots, \phi_N\}$ be given by Eq. (9.9). Then, for every continuous function $g : \mathbb{R} \to \mathbb{R}$,*

$$\int g \; \mu_1(dx) = \sum_{j=1}^{N} c_j \int g \circ \phi_j \; \mu_1(dx).$$

In other words, the x-marginal for μ is the (unique) invariant probability measure for the contractions $\{\phi_1, \ldots, \phi_N\}$ with weights $\{c_1, \ldots, c_N\}$.

Proof Let g be a continuous function on $\mathbb{R}$. We have that

$$\int g \circ \pi_1 \; \mu(dx\,dy) = \sum_{j=1}^{N} c_j \int g \circ \pi_1 \circ \psi_j \; \mu(dx\,dy) \tag{9.10}$$

$$= \sum_{j=1}^{N} c_j \int g \circ \phi_j \circ \pi_1 \; \mu(dx\,dy) \tag{9.11}$$

and

$$\int g \circ \pi_1 \; \mu(dx\, dy) = \int g \; \mu_1(dx).$$

Consequently, we have

$$\int g \; \mu_1(dx) = \sum_{j=1}^{N} c_j \int g \circ \phi_j \; \mu_1(dx).$$

□

Remark 9.3 Even though the contractions $\{\psi_1, \ldots, \psi_N\}$ in Lemma 9.1 may be distinct, there may be repetition in the projected contractions $\{\phi_1, \ldots, \phi_N\}$.

9.2.2 Slice-Singular Measures

Let μ be a Borel measure on $[0, 1] \times [0, 1]$. The Rokhlin decomposition states that there exists a measure μ_1 on $[0, 1]$, and for every $x \in [0, 1]$, there exists a measure ρ_x on $[0, 1]$ such that

$$\mu(dx\, dy) = \rho_x(dy)\mu_1(dx).$$

The measure μ_1 is called the *marginal* of μ (in the x-direction). It is given by

$$\mu_1(A) = \mu(\pi_1^{-1}(A))$$

for Borel sets $A \subset \mathbb{R}$. In short, $\mu_1 = \mu \circ \pi_1^{-1}$.

Likewise, the decomposition can be obtained in the other order:

$$\mu(dx\, dy) = \rho^y(dx)\mu_2(dy),$$

where $\mu_2 = \mu \circ \pi_2^{-1}$.

Definition 9.6 We say that a Borel measure on $[0, 1] \times [0, 1]$ is x-slice-singular if in the Rokhlin decomposition $\mu(dx\, dy) = \rho_x(dy)\mu_1(dx)$:

1. μ_1 is singular.
2. For μ_1 a.e. x, ρ_x is singular.

Similarly, we say that a Borel measure on $[0, 1] \times [0, 1]$ is y-slice-singular if in the Rokhlin decomposition $\mu(dx\, dy) = \rho^y(dx)\mu_2(dy)$:

1. μ_2 is singular.
2. For μ_2 a.e. y, ρ^y is singular.

We say that μ is slice-singular if it is so in either direction, and we say that μ is bi-slice-singular if it is slice-singular in both directions.

Our canonical examples of slice-singular measures correspond to affine fractals, such as the Sierpinski gasket and the Sierpinski carpet. These are generated by affine iterated function systems and generally are bi-slice-singular, as we show in Sect. 9.2.3. However, we also give an example of a measure that is slice-singular in one direction but not both in Example 3.

The following is a useful lemma for determining when a measure is slice-singular. We let λ denote Lebesgue measure on $\mathbb{R}$.

Lemma 9.2 *Suppose μ is a Borel probability measure on $[0, 1]^2$ such that both the x- and y-marginal measures are singular. Then, μ is bi-slice-singular.*

Proof Since the x-marginal measure μ_1 is singular, there exists a Borel set $A \subseteq [0, 1]$ such that $\lambda(A^C) = 0$ and $\mu_1(A) = 0$. It follows that

$$\mu(A \times [0, 1]) = \int_A \int_0^1 \rho_x(dy)\, \mu_1(dx) = 0.$$

Assume, for the sake of contradiction, that there exists a Borel set $B \subseteq [0, 1]$ of positive μ_2 measure such that for each $y \in B$, the y-marginal measure ρ^y is not purely singular, say $\rho^y = \nu^y_{\text{sing}} + \nu^y_{\text{cont}}$, where ν^y_{sing} is singular, ν^y_{cont} is absolutely continuous, and $\nu^y_{\text{cont}} \not\equiv 0$. It follows that $\nu^y_{\text{cont}}(A^C) = 0$, and so $\nu^y_{\text{cont}}(A) > 0$. Thus, $\rho^y(A) > 0$ for each $y \in B$. Therefore,

$$\mu(A \times [0, 1]) \geq \mu(A \times B) = \int_B \rho^y(A)\, \mu_2(dy) > 0,$$

which is a contradiction. By a symmetric argument, μ is also x-slice-singular, and so μ is bi-slice-singular. ◻

Lemma 9.3 *Suppose that μ is slice-singular. Then the set of exponential functions*

$$\{e^{2\pi i(nx+my)} : n, m \in \mathbb{N}_0\}$$

is dense in $L^2(\mu)$.

Proof Suppose that μ is y-slice-singular. Let $f \in L^2(\mu)$ be such that

$$\langle f(x, y), e^{2\pi i(nx+my)} \rangle_\mu = 0, \ \forall n, m \in \mathbb{N}_0.$$

For a fixed n, we have that

$$\int \left(\int f(x, y) e^{2\pi i n x} \rho^y(dx) \right) e^{2\pi i m y} \mu_2(dy) = 0.$$

Since μ_2 is singular, $\{e^{2\pi imy} : m \in \mathbb{N}_0\}$ is dense in $L^2(\mu_2)$, so we must have that

$$\int f(x, y)e^{2\pi inx}\rho^y(dx) = 0$$

for μ_2-a.e. y. It follows that for μ_2-a.e. y,

$$\int f(x, y)e^{2\pi inx}\rho^y(dx) = 0$$

for every $n \in \mathbb{N}_0$. Since ρ^y is also singular for μ_2-a.e. y, for ρ^y-a.e. x, we have $f(x, y) = 0$. It follows that $f(x, y) = 0$ for μ-a.e. (x, y). □

9.2.3 Examples

We will now give some examples. Of special interest are choices of planar measures μ that have the structure of IFS measures defined from the family of systems of 2D affine maps. We concentrate on Sierpinski constructions from the more familiar Sierpinski geometries in the plane, the Sierpinski gasket and carpet. As we will show, they are slice-singular IFS measures. In each case, using our results above applied to the particular $L^2(\mu)$-settings at hand, we can obtain corresponding non-orthogonal Fourier expansions.

Lemma 9.4 *Suppose μ is a Borel probability measure on $[0, 1]^2$ that is invariant under the reflection about the line $y = x$. Then μ is x-slice-singular if and only if it is y-slice-singular.*

Proof The proof is easy and is left as an exercise to the reader. □

Lemma 9.5 *Suppose $\Psi = \{\psi_1, \ldots, \psi_N\}$ are affine contractions of the form*

$$\psi_j(x) = \frac{x + b_j}{R}$$

with $\{b_j\} \subset \mathbb{Z}^2 \cap [0, R)^2$ and $R \nmid N$. If μ is the invariant measure for Ψ with equal weights $\frac{1}{N}$, then the marginal measures μ_1 and μ_2 of μ are singular with respect to Lebesgue measure.

Proof By Lemma 9.1, the marginal measure μ_1 is obtained by projecting Ψ onto the x-axis in $\mathbb{R}^2$. Doing so yields the IFS acting on $\mathbb{R}$ with generators

$$\phi_j(x) = \frac{x + \pi_1(b_j)}{R} \tag{9.12}$$

with weights $\dfrac{1}{N}$. If there are no repetitions among the $\{\phi_j\}$ in Eq. (9.12), we must have that $\{\pi_1(b_j)\} \subsetneq \{0, \ldots, R-1\}$, so μ_1 is singular. If there are repetitions, then we combine the generators by summing the corresponding weights to obtain μ_1 as the invariant measure for an IFS with generators

$$\tilde{\phi}_k(x) = \frac{x + c_k}{R}, \; k = 0, \ldots, K-1 \tag{9.13}$$

with weights $\{\tilde{w}_0, \ldots, \tilde{w}_{K-1}\}$. Since $R \nmid N$, we must have that these weights are not uniformly $\dfrac{1}{K}$, and so, by the Kakutani Dichotomy Theorem, μ_1 is singular. □

Example 1 Our first example is the Sierpinski gasket, which for convenience we align with the coordinate axes. The generators for the gasket are given by

$$\varphi_0(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right) \quad \varphi_1(x, y) = \left(\frac{x+1}{2}, \frac{y}{2}\right) \quad \varphi_2(x, y) = \left(\frac{x}{2}, \frac{y+1}{2}\right). \tag{9.14}$$

The measure for the gasket is obtained by using equal weights. The projection onto the x-axis yields the IFS with generators

$$\psi_0(x) = \frac{x}{2} \quad \psi_1(x) = \frac{x+1}{2} \quad \psi_2(x) = \frac{x}{2} \tag{9.15}$$

with equal weights, which is equivalent to the IFS $\{\psi_0, \psi_1\}$ with weights $(\frac{2}{3}, \frac{1}{3})$. Thus, by the Kakutani Dichotomy Theorem, the x-marginal measure for the Sierpinski gasket is singular.

It follows by symmetry that the y-marginal measure is also singular. Hence, by Lemmas 9.2 and 9.4, the Sierpinksi gasket is bi-slice-singular.

Example 2 Our second example is the Sierpinski carpet. The IFS generators are given by

$$\begin{aligned}
\varphi_0(x, y) &= \left(\frac{x}{3}, \frac{y}{3}\right) & \varphi_1(x, y) &= \left(\frac{x+1}{3}, \frac{y}{3}\right) \\
\varphi_2(x, y) &= \left(\frac{x+2}{3}, \frac{y}{3}\right) & \varphi_3(x, y) &= \left(\frac{x}{3}, \frac{y+1}{3}\right) \\
\varphi_4(x, y) &= \left(\frac{x+2}{3}, \frac{y+1}{3}\right) & \varphi_5(x, y) &= \left(\frac{x}{3}, \frac{y+2}{3}\right) \\
\varphi_6(x, y) &= \left(\frac{x+1}{3}, \frac{y+2}{3}\right) & \varphi_7(x, y) &= \left(\frac{x+2}{3}, \frac{y+2}{3}\right).
\end{aligned} \tag{9.16}$$

The projection onto the x-axis is equivalent to

$$\psi_0(x) = \frac{x}{3} \quad \psi_1(x) = \frac{x+1}{3} \quad \psi_2(x) = \frac{x+2}{3} \tag{9.17}$$

with weights $(\frac{3}{8}, \frac{2}{8}, \frac{3}{8})$. Thus, by the Kakutani Dichotomy Theorem, the x-marginal is singular. By symmetry, it now follows by Lemmas 9.2 and 9.4 that the Sierpinksi carpet is bi-slice-singular.

Remark 9.4 Observe that Examples 1 and 2 also follow immediately from Lemma 9.5.

The next example illustrates that slice-singular measures can be so in one direction but not the other.

Example 3 Consider the IFS generated by the functions

$$\begin{aligned} \varphi_0(x,y) &= \left(\frac{x}{4}, \frac{y}{4}\right) & \varphi_1(x,y) &= \left(\frac{x+1}{4}, \frac{y}{4}\right) \\ \varphi_2(x,y) &= \left(\frac{x+2}{4}, \frac{y+2}{4}\right) & \varphi_3(x,y) &= \left(\frac{x+3}{4}, \frac{y+2}{4}\right), \end{aligned} \tag{9.18}$$

and let μ be the invariant measure corresponding to equal weights. We immediately see that the projection of this IFS onto the x-axis yields the following generators:

$$\psi_0(x) = \frac{x}{4} \quad \psi_1(x) = \frac{x+1}{4} \quad \psi_2(x) = \frac{x+2}{4} \quad \psi_3(x) = \frac{x+3}{4} \tag{9.19}$$

together with equal weights. Consequently, the x-marginal of the invariant measure μ is Lebesgue measure on [0, 1], so μ is not x-slice-singular.

On the other hand, the projection onto the y-axis yields the generators

$$\gamma_0(y) = \frac{y}{4} \quad \gamma_1(y) = \frac{y}{4} \quad \gamma_2(y) = \frac{y+2}{4} \quad \gamma_3(y) = \frac{y+2}{4} \tag{9.20}$$

with equal weights (of $\frac{1}{4}$), whose invariant measure is identical to that with generators $\{\gamma_0, \gamma_2\}$ with equal weights (of $\frac{1}{2}$). Therefore, the y-marginal of μ is the (singular) spectral measure ν_4 of Jorgensen and Pedersen [28] supported on the Cantor set C_4.

We claim that for $y \in C_4$, the slice measure ρ^y is a translation of the invariant measure ν for the IFS $\{\lambda_0, \lambda_1\}$, with equal weights, where

$$\lambda_0(x) = \frac{x}{4}, \qquad \lambda_1(x) = \frac{x+1}{4}. \tag{9.21}$$

Indeed, we claim that ρ^y is ν shifted by y:

$$\int f(x)\rho^y(dx) = \int f(x-y)\nu(dx). \tag{9.22}$$

This can be seen by showing that the measure $\tilde{\mu}(dxdy) = \rho^y(dx)\nu_4(dy)$ is invariant under the IFS in Eq. (9.18). For a continuous function f, we calculate

$$\begin{aligned}
\int f\left(\frac{x}{4}, \frac{y}{4}\right)\tilde{\mu}(dxdy) &+ \int f\left(\frac{x+1}{4}, \frac{y}{4}\right)\tilde{\mu}(dxdy) \\
&= \int f\left(\frac{x}{4}, \frac{y}{4}\right)\rho^y(dx)\nu_4(dy) + \int f\left(\frac{x+1}{4}, \frac{y}{4}\right)\rho^y(dx)\nu_4(dy) \\
&= \int f\left(\frac{x-y}{4}, \frac{y}{4}\right)\nu(dx)\nu_4(dy) \\
&\quad + \int f\left(\frac{x-y+1}{4}, \frac{y}{4}\right)\nu(dx)\nu_4(dy) \\
&= 2\int f\left(x-y, \frac{y}{4}\right)\nu(dx)\nu_4(dy) \\
&= 2\int f\left(x, \frac{y}{4}\right)\rho^y(dx)\nu_4(dy).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\frac{1}{4}\Bigg[\int f\left(\frac{x}{4}, \frac{y}{4}\right)\tilde{\mu}(dxdy) &+ \int f\left(\frac{x+1}{4}, \frac{y}{4}\right)\tilde{\mu}(dxdy) \\
&+ \int f\left(\frac{x+2}{4}, \frac{y+2}{4}\right)\tilde{\mu}(dxdy) + \int f\left(\frac{x+3}{4}, \frac{y+2}{4}\right)\tilde{\mu}(dxdy)\Bigg] \\
&= \frac{1}{4}\left[2\int f\left(x, \frac{y}{4}\right)\rho^y(dx)\nu_4(dy) + 2\int f\left(x, \frac{y+2}{4}\right)\rho^y(dx)\nu_4(dy)\right] \\
&= \int f(x,y)\rho^y(dx)\nu_4(dy).
\end{aligned}$$

Therefore, the y-slice measures are singular, and so μ is y-slice-singular.

9.3 Fourier Series for Slice-Singular Measures

In this section, we present the analytic details of the Kaczmarz algorithm that yield the generalized Fourier expansions for $L^2(\mu)$, when μ is defined in two dimensions. However, we shall present the Kaczmarz algorithm in the framework of (infinite-dimensional) Hilbert spaces and operator theory. The main features of our algorithm will be reviewed below, and we shall refer to our earlier papers, especially [19–22], for additional details. Our recent application is to slice-singular measures μ and associated multi-variable Fourier expansions for $L^2(\mu)$, generally non-orthogonal.

Our main result is the following:

Theorem 9.1 *Suppose μ is a y-slice-singular Borel probability measure on* $[0, 1)^2$. *For any $f \in L^2(\mu)$, f possesses a Fourier series expansion of the form*

$$f(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} d_{nm} e^{2\pi i(nx+my)}. \tag{9.23}$$

The series converges conditionally in norm, and the coefficients are as expressed in (9.46).

The notation presented in (9.23) is of an iterated sum and must be interpreted as such. For each fixed n, the series on m converges conditionally in norm. The series on n also converges conditionally in norm.

Note that the order of the double sum is dependent on the decomposition of the measure μ. If the measure is bi-slice-singular, then the Fourier series expansions can be obtained in either order. However, the *coefficients* still depend on the decomposition:

Corollary 9.1 *Suppose μ is a bi-slice-singular Borel probability measure on* $[0, 1)^2$. *For any $f \in L^2(\mu)$, f possesses Fourier series expansions of the form*

$$f(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{mn} e^{2\pi i(nx+my)} \tag{9.24}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} d_{nm} e^{2\pi i(nx+my)}. \tag{9.25}$$

The series converge conditionally in norm.

For the remainder of this section, we will assume that μ is y-slice-singular. Our construction is oriented in the y direction; for a measure that is x-slice-singular, our construction can be modified in the obvious manner.

Our proof of the existence of the Fourier series expansions will utilize the for operators. We will construct a sequence of operators corresponding to projections onto subspaces of $L^2(\mu)$ that are effective in the following operator-theoretic sense, which is explained in full detail in Sect. 9.4:

Definition (Effective Sequences of Operators) Let $\mathcal{H}$ be a Hilbert space, and for each $j \in \mathbb{N}_0$, let $\mathcal{H}_j$ be a Hilbert space and let $R_j : \mathcal{H} \to \mathcal{H}_j$ be a bounded surjective operator. For $x \in \mathcal{H}$, define $x_0 = R_0^* R_0 x$, and for each $j \in \mathbb{N}$, let x_j be the orthogonal projection of x_{j-1} onto the affine subspace of $\mathcal{H}$ containing x parallel to ker R_j. If for any $x \in \mathcal{H}$, $\{x_j\} \to x$, then we say the sequence of operators $\{R_j\}$ is effective. If the $\mathcal{H}_j$ are closed subspaces of $\mathcal{H}$ and the R_j are the orthogonal projections onto the $\mathcal{H}_j$, then we may also say that $\{\mathcal{H}_j\}$ is a sequence of effective subspaces of $\mathcal{H}$. □

The subspaces will have the property that functions within the subspaces have Fourier series expansions arising from the Kaczmarz algorithm. Then, we will obtain that for $f \in L^2(\mu)$, the sequence of projections can be used to reconstruct f in the form of a Fourier series, yielding a doubly indexed Fourier series expansion for f as in Eq. (9.23). The coefficients of the expansion are obtained via a careful analysis of the operator auxiliary sequence, which is defined by matrix inversion as described in Sect. 9.4.

Specifically, for μ a y-slice-singular measure, we will apply the Kaczmarz algorithm to the following sequences (indexed by $\mathbb{N}_0$) of subspaces and operators:

$$\mathcal{M}_n = \overline{span}\{e^{2\pi i(nx+my)} : m \in \mathbb{N}_0\};$$

$$P_n = \text{orthogonal projection onto } \mathcal{M}_n;$$

$$R_n : L^2(\mu) \to L^2(\mu_2) : f(x, y) \to \left(y \mapsto \int f(x, y) e^{-2\pi i n x} \rho^y(dx) \right); \tag{9.26}$$

$$S : L^2(\mu) \to L^2(\mu) : f(x, y) \mapsto e^{2\pi i x} f(x, y). \tag{9.27}$$

By the Rokhlin decomposition theorem, R_n is well-defined. Simple calculations show that

$$\mathcal{M}_n = \{e^{2\pi i n x} g(y) : g \in L^2(\mu_2)\}; \tag{9.28}$$

$$R_n^* : L^2(\mu_2) \to L^2(\mu) : g(y) \mapsto e^{2\pi i n x} g(y);$$

$$P_n = R_n^* R_n;$$

$$R_{n+j} = R_n S^{-j} \text{ for } n, j \geq 0. \tag{9.29}$$

Our aim is to show that the sequence of operators $\{R_n\}_{n=0}^{\infty}$ as defined above is effective (Def 9.8), which will then be used to show that every $f \in L^2(\mu)$ can be written as a doubly indexed Fourier series. To that end, we make the following definition:

Definition 9.7 For a sequence of operators $\{B_n\}_{n=0}^{\infty} \subset \mathcal{B}(H, K)$, we say the sequence is *stationary* if there exists a unitary $S \in \mathcal{B}(H)$ such that $B_{n+j} = B_n S^{-j}$ for $n, j \geq 0$.

We immediately see that our operators $\{R_n\}$ form a stationary sequence. Section 9.4 will be devoted to proving that the sequence $\{R_n\}$ is an effective sequence. Utilizing this fact, as articulated in Theorem 9.3, we can present the proof of Theorem 9.1.

The following lemma is an immediate consequence of Theorem D:

Lemma 9.6 *If μ is y-slice-singular, with Rokhlin decomposition $\mu(dx\,dy) = \rho^y(dx)\mu_2(dy)$, then $\{e^{2\pi imy}\}_{m=0}^{\infty}$ is an effective sequence in $L^2(\mu_2)$, and consequently for every $f \in L^2(\mu_2)$, there exists a sequence of coefficients $\{a_m\}_{m=0}^{\infty} \subset \mathbb{C}$ such that*

$$f(y) = \sum_{m=0}^{\infty} a_m e^{2\pi imy},$$

with convergence of the series in norm, given by $a_m = \langle f, g_m\rangle_{\mu_2}$, where $\{g_m\}$ is the auxiliary sequence of $\left\{e^{2\pi imy}\right\}$ in $L^2(\mu_2)$.

Proposition 9.1 *Suppose μ is y-slice-singular, with Rokhlin decomposition given by $\mu(dx\,dy) = \rho^y(dx)\mu_2(dy)$. For $f \in L^2(\mu)$, the function $P_n f$ possesses a Fourier series expansion of the form*

$$P_n f(x, y) = \sum_{m=0}^{\infty} a_{nm} e^{2\pi i(nx+my)}. \tag{9.30}$$

Proof The sequence $\{e^{2\pi i(nx+my)}\}_{m\in\mathbb{N}_0}$ is a stationary sequence with μ_2 as its spectral measure, by virtue of Lemma 9.6 and the fact that R_n^* is an isometry. The result now follows from Theorem D. □

We can say more about the nature of the Fourier series expansion in Eq. (9.30). For $f_n \in \mathcal{M}_n$, we have

$$f_n(x, y) = e^{2\pi inx}[R_n f_n](y),$$

since $P_n = R_n^* R_n$. Therefore, if $\{g_m(y)\}_{m\in\mathbb{N}_0}$ is the auxiliary sequence of $\{e^{2\pi imy}\}_{m\in\mathbb{N}_0}$ in $L^2(\mu_2)$, then by Lemma 9.6,

$$[P_n f](x, y) = e^{2\pi inx}[R_n f_n](y) \tag{9.31}$$

$$= e^{2\pi inx}\left(\sum_{m=0}^{\infty} \langle [R_n f_n](y), g_m(y)\rangle_{\mu_2}\, e^{2\pi imy}\right). \tag{9.32}$$

Thus far, we have used only the auxiliary sequence $\{g_m\}$ of $\left\{e^{2\pi imy}\right\}$ in $L^2(\mu_2)$ coming from the vector Kaczmarz algorithm used in Theorem D, but the operator Kaczmarz algorithm applied to sequence of operators also induces an auxiliary sequence of operators $\{G_n\}$. The general construction of $\{G_n\}$ is described in Sect. 9.4.1. The sequence $\{G_n\}$ is given explicitly by Eq. (9.47).

In our present setting, the auxiliary sequence of $\{R_n\}$ is concretized as follows: For each y such that ρ^y is singular, we have that $\left\{e^{2\pi inx}\right\}_{n=0}^{\infty}$ is effective in $L^2(\rho^y)$. We let $\{g_n^{(y)}\}_{n=0}^{\infty}$ denote the corresponding auxiliary sequence. When μ is y-slice-singular, this holds for μ_2 a.e. y. We first claim that for each n, $g_n^{(y)}(x)$ is a measurable function of x and y:

Lemma 9.7 *Let μ be a y-slice-singular measure on $[0, 1)^2$, and $\left\{g_n^{(y)}\right\}$ the auxiliary sequence of $L^2(\rho^y)$. Then, for each n, $g_n^{(y)}(x) : [0, 1)^2 \to \mathbb{C}$ is measurable as a function of two variables.*

Proof By [22, Prop 1], we have that $g_n^{(y)}(x) = \sum_{j=0}^{n} \alpha_{n-j}(y) e_n(x)$, where for each fixed y, the sequence $\{\alpha_k(y)\}$ is defined by

$$\frac{1}{\sum_{j=0}^{\infty} \widehat{\rho^y}(j) z^j} = \sum_{j=0}^{\infty} \alpha_j(y) z^j.$$

Now, $\widehat{\rho^y}(j) = \int_0^1 e^{2\pi i j x} \rho^y(dx)$ is a measurable function of y by the Rokhlin Disintegration Theorem. It follows that each $\alpha_j(y)$ is a measurable function of y (cf. [22, Lemma 2]). Therefore, each $g_n^{(y)}(x)$ is measurable in the variable y and continuous in the variable x and hence measurable. □

We now explicitly identify $\{G_n\}$ in our present setting:

Proposition 9.2 *We have that*

$$[G_n f](y) = \int f(x, y) \overline{g_n^{(y)}(x)} \rho^y(dx). \tag{9.33}$$

Proof In Sect. 9.4.1, Eq. (9.48), we observe that the auxiliary sequence $\{G_n\}$ uniquely solves the system

$$R_n = \sum_{j=0}^{n} R_n R_j^* G_j \tag{9.34}$$

for all $n \in \mathbb{N}_0$. Thus, we need only to show that G_j as defined in Eq. (9.33) satisfies Eq. (9.34).

Now a simple calculation shows that

$$[R_n R_j^* h](y) = \int e^{2\pi i j x} h(y) e^{-2\pi i n x} \rho^y(dx) \tag{9.35}$$

$$= \widehat{\rho^y}(n-j) h(y). \tag{9.36}$$

Thus, from [18], we have that for any fixed y,

$$\sum_{j=0}^{n} \overline{\widehat{\rho^y}(n-j)} g_j^{(y)}(x) = e^{2\pi i n x}. \tag{9.37}$$

We therefore calculate

$$\sum_{j=0}^{n}[R_n R_j^* G_j f](y) = \sum_{j=0}^{n} \widehat{\rho^y}(n-j) \int f(x,y)\overline{g_j^{(y)}(x)}\rho^y(dx) \tag{9.38}$$

$$= \int f(x,y)\overline{\sum_{j=0}^{n} \overline{\widehat{\rho^y}(n-j)} g_j^{(y)}(x)}\rho^y(dx) \tag{9.39}$$

$$= [R_n f](y). \tag{9.40}$$

Equation (9.33) now follows immediately. □

We will prove in Theorem 9.3 that the sequence $\{R_n\}$ is effective. As a consequence of this theorem, we have the following:

Proposition 9.3 *Suppose μ is y-slice-singular, with Rokhlin decomposition given by $\mu(dx\,dy) = \rho^y(dx)\mu_2(dy)$. For every $f \in L^2(\mu)$, we can express f as*

$$f(x,y) = \sum_{n=0}^{\infty}[G_n f](y)e^{2\pi i n x} \tag{9.41}$$

$$= \sum_{n=0}^{\infty}\left[\sum_{m=0}^{\infty} a_{nm}e^{2\pi i m y}\right]e^{2\pi i n x}, \tag{9.42}$$

where the sequence $\{G_n\}_{n=0}^{\infty}$ is defined by Eq. (9.33). The convergence is in norm, conditional, and order-dependent.

Proof Since the sequence of operators $\{R_n\}$ is effective, Theorem F implies that the sum $\sum_{n=0}^{\infty} R_n^* G_n$ converges in the SOT. Consequently,

$$f = \sum_{n=0}^{\infty} R_n^* G_n f \tag{9.43}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\langle [G_n f](y), g_m(y)\rangle_{\mu_2} e^{2\pi i m y}\right)e^{2\pi i n x} \tag{9.44}$$

by applying Lemma 9.6 to $G_n f$. □

Combining these results, we are ready to prove our main result:

Proof of Theorem 9.1 By combining (9.44) and (9.33), for any $f \in L^2(\mu)$, we have

$$f = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\left(\int f(x,y)\overline{g_n^{(y)}(x)g_m(y)}\,\mu(dx\,dy)\right)e^{2\pi i m y}\right)e^{2\pi i n x} \tag{9.45}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\langle f(x,y), g_n^{(y)}(x) g_m(y) \right\rangle e^{2\pi i(nx+my)}. \tag{9.46}$$

Thus, setting $d_{nm} = \left\langle f(x,y), g_n^{(y)}(x) g_m(y) \right\rangle_\mu$, Theorem 1 is established. □

The convergence in (9.46) is in norm and conditional. Therefore, we have a pseudo-duality between the sequence $\{e^{2\pi i(nx+my)}\}_{m,n=0}^{\infty}$ and $\{g_n^{(y)}(x) g_m(y)\}_{m,n=0}^{\infty}$. We will expand on this in Appendix 9.5.1.

9.4 The Kaczmarz Algorithm for Bounded Operators

The purpose of the present chapter is to establish explicit multi-variable Fourier expansions for $L^2(\mu)$ in the case of planar measures μ that are assumed to satisfy the slice-singular property (see Sect. 9.2.2). In principle (as noted in Sect. 9.1), these multi-variable Fourier series results parallel the corresponding results for the case when μ is instead assumed to be singular and supported on an interval.

In both cases, for the proofs, we rely on our (infinite-dimensional) Hilbert space framework for the Kaczmarz algorithm (see Sect. 9.1.2). However, the extension to the case of planar measures μ is non-trivial. It entails additional considerations involving operator theory, in particular a new analysis of infinite-by-infinite block-matrices, i.e., referring to the case of operator-valued entries (see, e.g., Theorem 9.2 below). It also entails a new framework involving general Hardy spaces (Sect. 9.4.3) relying on ideas from de Branges, especially from [11]. Since this material is not readily available, or widely known, it will be presented in the present section.

Let $\mathcal{H}$ be a Hilbert space, and for each $j \in \mathbb{N}_0$, let $\mathcal{H}_j$ be a Hilbert space (which in some situations may be closed subspaces of $\mathcal{H}$). Let $R_j : \mathcal{H} \to \mathcal{H}_j$ be bounded surjective operators. In the classical case, we typically have a sequence of vectors $\{\phi_n\} \subset \mathcal{H}$ (ultimately identified as an "effective" sequence), $\mathcal{H}_j = \mathbb{C}$ for all j, and the operators R_j are the analysis operators of the ϕ_j, that is to say, $R_j f = \langle f, \phi_j \rangle_{\mathcal{H}}$. Thus, in the classical case, $\ker R_j = \{f \in \mathcal{H} : f \perp \phi_j\}$.

Let $b_j \in \mathcal{H}_j$ for all j. The objective of the Kaczmarz algorithm is to find an $f \in \mathcal{H}$ that solves the system $R_j f = b_j$ for all j.

With the b_j's fixed, for each $j \in \mathbb{N}_0$, let $h_j \in \mathcal{H}$ be the unique solution to $R_j h_j = b_j$ lying in $(\ker R_j)^\perp$, that is, $h_j = R_j^*(R_j R_j^*)^{-1} b_j$. Let Q_j be the orthogonal projection onto $\ker R_j$.

Suppose we create such a situation as described above by fixing an $x \in \mathcal{H}$ and then letting $b_j = R_j x$. Define P_j by

$$\begin{aligned} P_j &= Q_j + h_j \\ &= Q_j + R_j^*(R_j R_j^*)^{-1} b_j \end{aligned}$$

$$= Q_j + R_j^*(R_j R_j^*)^{-1} R_j x.$$

Observe that $P_j f$ is the orthogonal projection of f onto the affine subspace of $\mathcal{H}$ containing x parallel to $\ker R_j$.

In a typical situation, we will know the b_j measurements, but we will not know the vector x by which the measurements were made. We attempt to recover x by carrying out the Kaczmarz algorithm: Define the Kaczmarz sequence $\{x_j\}$ by $x_0 = R_0^* b_0 = R_0^* R_0 x$, and for $j > 0$,

$$\begin{aligned} x_j &= P_j x_{j-1} \\ &= Q_j x_{j-1} + R_j^*(R_j R_j^*)^{-1} R_j x \\ &= Q_j x_{j-1} + R_j^*(R_j R_j^*)^{-1} R_j x_{j-1} - R_j^*(R_j R_j^*)^{-1} R_j x_{j-1} + R_j^*(R_j R_j^*)^{-1} R_j x \\ &= x_{j-1} + R_j^*(R_j R_j^*)^{-1} R_j (x - x_{j-1}) \\ &= x_{j-1} + R_j^*(R_j R_j^*)^{-1} (b_j - R_j x_{j-1}). \end{aligned}$$

Definition 9.8 If $\{x_j\} \to x$ regardless of which x was chosen, then we say the sequence of operators $\{R_j\}$ is effective. If the $\mathcal{H}_j$ are closed subspaces of $\mathcal{H}$ and the R_j are the orthogonal projections onto the $\mathcal{H}_j$, then we may also say that $\{\mathcal{H}_j\}$ is a sequence of effective subspaces of $\mathcal{H}$.

Natterer [37] introduces the Kaczmarz algorithm for bounded operators in the finite regime–there exist finitely many $\mathcal{H}_j$'s and R_j's, and he proves, in analogy to Kaczmarz's original paper [29], that the periodized sequence of R_j's is effective. We will prove an operator analogue of the Kwapień-Mycielski result [31] for stationary sequences.

Definition 9.9 Given a sequence of Hilbert spaces $\{\mathcal{H}_j\}$, we will need the following spaces:

1. $\oplus\mathcal{H}_j$ denotes the vector space of sequences of vectors whose j-th component is a vector in $\mathcal{H}_j$.
2. $\mathcal{F}(\oplus\mathcal{H}_j)$ denotes the inner-product space consisting of sequences in $\oplus\mathcal{H}_j$ with all but finitely many components equal to 0.
3. $\ell^2(\oplus\mathcal{H}_j)$ denotes the Hilbert space consisting of sequences in $\oplus\mathcal{H}_j$ that are square-summable in norm.

We will consider sequences in these spaces as column vectors.

9.4.1 The Auxiliary Sequence for the Operator Kaczmarz Algorithm

We define the auxiliary sequence $\{G_j\}$ of $\{R_j\}$ as follows: Let $M : \oplus_{k=0}^{\infty}\mathcal{H}_k \to \oplus_{j=0}^{\infty}\mathcal{H}_j$ be defined by the infinite strictly lower-triangular matrix of operators whose j, k-th entry is $R_j R_k^*$ if $j > k$ and the zero operator $0 : \mathcal{H}_k \to \mathcal{H}_j$ otherwise. (Here, the indices j and k start at 0.) Let I be the identity operator on $\oplus_{k=0}^{\infty}\mathcal{H}_k$, which can be defined by the infinite diagonal matrix whose j, j-th entry is the identity operator on $\mathcal{H}_j$.

Define $R : \mathcal{H} \to \oplus_{j=0}^{\infty}\mathcal{H}_j$ to be the 1-column matrix consisting of the R_j's. Since $(I + M)$ is lower-triangular, it is invertible. Therefore, we may define $G : \mathcal{H} \to \oplus_{j=0}^{\infty}\mathcal{H}_j$ to be the unique 1-column matrix satisfying

$$(I + M)G = R. \tag{9.47}$$

The m-th entry of G, $G_m : \mathcal{H} \to \mathcal{H}_m$, is the m-th auxiliary sequence operator. Because $(I + M)G = R$, we have

$$R_n = \sum_{j=0}^{n} R_n R_j^* G_j \tag{9.48}$$

for each $j \in \mathbb{N}_0$.

Note that there exists an operator $U : \oplus_{j=0}^{\infty}\mathcal{H}_j \to \oplus_{k=0}^{\infty}\mathcal{H}_k$ such that

$$(I + M)^{-1} = (I + U),$$

where U is realizable as an infinite strictly lower-triangular matrix whose j, k-th entry is an operator $C_{jk} : \mathcal{H}_k \to \mathcal{H}_j$ if $j > k$ and the zero operator $0 : \mathcal{H}_k \to \mathcal{H}_j$ otherwise. Then,

$$G := (I + U)R. \tag{9.49}$$

Theorem F (Natterer) $\sum_{k=0}^{n} R_k^* G_k x = x_n$.

The proof can be found in [37, Page 133].

Proposition 9.4 *For* $n \geq 1$,

$$\|x - x_{n-1}\|_{\mathcal{H}}^2 = \|x - x_n\|_{\mathcal{H}}^2 + \left\| R_n^* G_n x \right\|_{\mathcal{H}}^2.$$

Proof Recall that

$$x_n = x_{n-1} + R_n^*(R_n R_n^*)^{-1}(R_n x - R_n x_{n-1}).$$

Applying R_n to both sides,

$$R_n x_n = R_n x_{n-1} + R_n x - R_n x_{n-1} = R_n x.$$

It follows that $(x - x_n) \in \ker R_n$. Since by the Fredholm Alternative the range of R_n^* is perpendicular to $\ker R_n$, it follows that

$$\begin{aligned}\|x - x_n\|_{\mathcal{H}}^2 + \left\|R_n^* G_n x\right\|_{\mathcal{H}}^2 &= \left\|x - x_n + R_n^* G_n x\right\|_{\mathcal{H}}^2 \\ &= \|x - x_{n-1}\|_{\mathcal{H}}^2\end{aligned}$$

by Theorem F. □

Corollary 9.2 *If R_n^* is an isometry (as is the case in the classical situation when the ϕ_j are unit vectors), then $\left\|R_n^* G_n x\right\|_{\mathcal{H}}^2 = \|G_n x\|_{\mathcal{H}_n}^2$, and consequently it is also true that*

$$\|x - x_{n-1}\|_{\mathcal{H}}^2 = \|x - x_n\|_{\mathcal{H}}^2 + \|G_n x\|_{\mathcal{H}_n}^2 .$$

Corollary 9.3 *The sequence $\{R_n\}_{n=0}^{\infty}$ is effective if and only if $\sum_{n=0}^{\infty} \left\|R_n^* G_n x\right\|_{\mathcal{H}}^2 = \|x\|_{\mathcal{H}}^2$.*

Moreover, under the assumption that each R_n^* is an isometry, $\{R_n\}_{n=0}^{\infty}$ is effective if and only if $\sum_{n=0}^{\infty} \|G_n x\|_{\mathcal{H}_n}^2 = \|x\|_{\mathcal{H}}^2$.

Proof Note that $G_0 = R_0$. Since $x_0 = R_0^* R_0 x$, x_0 is in the range of R_0^* and is therefore perpendicular to $x - x_0$. So by Proposition 9.4,

$$\begin{aligned}\sum_{n=0}^{k} \left\|R_n^* G_n x\right\|_{\mathcal{H}}^2 &= \left\|R_0^* R_0 x\right\|_{\mathcal{H}}^2 + \sum_{n=1}^{k} (\|x - x_{n-1}\|_{\mathcal{H}}^2 - \|x - x_n\|_{\mathcal{H}}^2) \\ &= \|x_0\|_{\mathcal{H}}^2 + \|x - x_0\|_{\mathcal{H}}^2 - \|x - x_k\|_{\mathcal{H}}^2 \qquad (9.50) \\ &= \|x\|_{\mathcal{H}}^2 - \|x - x_k\|_{\mathcal{H}}^2 .\end{aligned}$$

Taking a limit of both sides as $k \to \infty$, the result follows. □

9.4.2 A Matrix Characterization of Effectivity

Haller and Szwarc [18] prove a general statement about when a sequence of vectors $\{\phi_j\}_{j=0}^{\infty} \subset \mathcal{H}$ is effective. Specifically, they prove that the sequence is effective if and only if the matrix $\mathscr{U}$ is a partial isometry on $\ell^2(\mathbb{N}_0)$, where

$$(I+\mathscr{U})(I+\mathscr{M})=I, \qquad \mathscr{M}_{j,k}=\begin{cases}\langle \phi_k,\phi_j\rangle, & \text{for } j>k,\\ 0, & \text{otherwise.}\end{cases}$$

We prove the analogue of the Haller-Szwarc Theorem for the Kaczmarz Algorithm for Operators. The proof proceeds nearly identically to the proof presented in [18]; we include it here for completeness.

Theorem 9.2 *Suppose every R_n^* is an isometry. Then, $\{R_j\}_{j=0}^{\infty}$ is effective if and only $\cup(\ker(R_j)^{\perp})$ is linearly dense in $\mathcal{H}$ and the matrix*

$$U=\begin{bmatrix} 0 & & & & 0\\ C_{10} & 0 & & & \\ C_{20} & C_{21} & 0 & & \\ C_{30} & C_{31} & C_{32} & 0 & \\ \vdots & & & & \ddots \end{bmatrix}$$

is a partial isometry on $\ell^2(\oplus_{j=0}^{\infty}\mathcal{H}_j)$.

To prove this theorem, we first establish some lemmas and definitions.

For a matrix A with operator entries, let $\hat{A}_n$ denote the nth principal submatrix of A, that is, the matrix A with entries changed to the zero operator in rows or columns beyond the nth.

If $A:\oplus\mathcal{H}_j\to\oplus\mathcal{H}_j$, we say that A is positive definite on $\mathcal{F}(\oplus\mathcal{H}_j)$ if $\left\langle \hat{A}_n x, x\right\rangle_{\mathcal{F}(\oplus\mathcal{H}_j)}>0$ for all $x\in\mathcal{F}(\oplus\mathcal{H}_j)$ and all $n\in\mathbb{N}_0$.

Proposition 9.5 *U is a contraction on $\ell^2(\oplus_{j=0}^{\infty}\mathcal{H}_j)$ if and only if $M+M^*+I$ is positive definite on $\mathcal{F}(\oplus_{j=0}^{\infty}\mathcal{H}_j)$.*

Proof $\hat{M}_n$ and $\hat{U}_n$ are bounded on $\ell^2(\oplus_{j=0}^{\infty}\mathcal{H}_j)$, and by assumption, $\hat{M}_n\hat{U}_n=\hat{U}_n\hat{M}_n=-\hat{U}_n-\hat{M}_n$. Assume the matrix $M+M^*+I$ is positive definite on $\mathcal{F}(\oplus_{j=0}^{\infty}\mathcal{H}_j)$. Then, the matrix $\hat{M}_n+\hat{M}_n^*+I$ corresponds to a positive bounded operator on $\ell^2(\oplus_{j=0}^{\infty}\mathcal{H}_j)$. Thus,

$$\begin{aligned}
0 &\le (\hat{U}_n^*+I)(\hat{M}_n+\hat{M}_n^*+I)(\hat{U}_n+I)\\
&=(\hat{U}_n^*+I)((\hat{M}_n+I)(\hat{U}_n+I)+\hat{M}_n^*(\hat{U}_n+I))\\
&=(\hat{U}_n^*+I)(I+\hat{M}_n^*\hat{U}_n+\hat{M}_n^*)\\
&=(\hat{U}_n^*+I)(\hat{M}_n^*+I)+(\hat{U}_n^*+I)\hat{M}_n^*\hat{U}_n\\
&=I+\hat{U}_n^*\hat{M}_n^*\hat{U}_n+\hat{M}_n^*\hat{U}_n\\
&=I+(-\hat{U}_n^*-\hat{M}_n^*)\hat{U}_n+\hat{M}_n^*\hat{U}_n^*\\
&=I-\hat{U}_n^*\hat{U}_n.
\end{aligned}$$

Hence, $\left\|\hat{U}_n\right\| \le 1$, where this denotes the operator norm of $\mathcal{B}(\ell^2(\oplus\mathcal{H}_j))$. Consequently, we obtain $\|U\| \le 1$.

Conversely, suppose $\|U\| \le 1$, where this is the norm of $\mathcal{B}(\ell^2(\oplus\mathcal{H}_j))$. Then, $\left\|\hat{U}_n\right\| \le 1$ for all n, and so $I - \hat{U}_n^*\hat{U}_n \ge 0$ on $\ell^2(\oplus\mathcal{H}_j)$. Therefore,

$$
\begin{aligned}
0 &\le (\hat{M}_n^* + I)(I - \hat{U}_n^*\hat{U}_n)(\hat{M}_n + I) \\
&= (\hat{M}_n^* + I)(\hat{M}_n + I - \hat{U}_n^*\hat{U}_n\hat{M}_n - \hat{U}_n^*\hat{U}_n) \\
&= (\hat{M}_n^* + I)(\hat{M}_n + I - \hat{U}_n^*(-\hat{U}_n - \hat{M}_n) - \hat{U}_n^*\hat{U}_n) \\
&= (\hat{M}_n^* + I)(\hat{M}_n + I + \hat{U}_n^*\hat{M}_n) \\
&= \hat{M}_n^*\hat{M}_n + \hat{M}_n^* + \hat{M}_n^*\hat{U}_n^*\hat{M}_n + \hat{M}_n + I + \hat{U}_n^*\hat{M}_n \\
&= \hat{M}_n^*\hat{M}_n + \hat{M}_n^* + (-\hat{U}_n^* - \hat{M}_n^*)\hat{M}_n + \hat{M}_n + I + \hat{U}_n^*\hat{M}_n \\
&= \hat{M}_n + \hat{M}_n^* + I.
\end{aligned}
$$

It follows that $M + M^* + I$ is positive definite.

□

Lemma 9.8 *Suppose every R_n^* is an isometry. Then, $\sum_{n=0}^{\infty} G_n^*G_n = I_{\mathcal{H}}$ in the weak operator topology, if and only if $\left\{R_j\right\}_{j=0}^{\infty}$ is effective.*

Proof Suppose $\sum_{n=0}^{\infty} G_n^*G_n = I_{\mathcal{H}}$ in the weak operator topology. Let $x \in \mathcal{H}$. Observe that

$$
\begin{aligned}
\sum_{n=0}^{\infty} \left\|R_n^*G_n x\right\|_{\mathcal{H}}^2 &= \sum_{n=0}^{\infty} \|G_n x\|_{\mathcal{H}_n}^2 \\
&= \sum_{n=0}^{\infty} \langle G_n x, G_n x\rangle_{\mathcal{H}_n} \\
&= \sum_{n=0}^{\infty} \left\langle G_n^*G_n x, x\right\rangle_{\mathcal{H}} \\
&= \lim_{k\to\infty} \left\langle \sum_{n=0}^{k} G_n^*G_n x, x \right\rangle_{\mathcal{H}} \\
&= \langle I x, x\rangle_{\mathcal{H}} \\
&= \|x\|_{\mathcal{H}}^2 .
\end{aligned}
$$

Therefore, by Corollary 9.3, the Kaczmarz sequence $x_n \to x$. Since x was arbitrary, it follows that $\{R_j\}$ is effective.

Conversely, suppose $\{R_j\}_{j=0}^{\infty}$ is effective. Let $x \in \mathcal{H}$. Then, the Kaczmarz sequence $x_n \to x$. Because of (9.50), we may define $T : \mathcal{H} \to \ell^2(\oplus \mathcal{H}_n)$ by

$$Tx = \begin{bmatrix} G_0 x \\ G_1 x \\ G_2 x \\ \vdots \end{bmatrix}. \tag{9.51}$$

(Indeed, by (9.50), it follows that $\|T\|_{\ell^2(\oplus \mathcal{H}_j)} \leq 1$ regardless of whether $\{R_j\}_{j=0}^{\infty}$ is effective.) By Corollary 9.3, since $\{R_j\}_{j=0}^{\infty}$ is effective, we have

$$\begin{aligned} \|Tx\|_{\ell^2(\oplus \mathcal{H}_j)} &= \sqrt{\sum_{n=0}^{\infty} \|G_n x\|_{\mathcal{H}_n}^2} \\ &= \|x\|_{\mathcal{H}}. \end{aligned}$$

Thus, T is an isometry. Let $y \in \mathcal{H}$. Then,

$$\begin{aligned} \langle x, y \rangle_{\mathcal{H}} &= \langle Tx, Ty \rangle_{\ell^2(\oplus \mathcal{H}_j)} \\ &= \sum_{n=0}^{\infty} \langle G_n x, G_n y \rangle_{\mathcal{H}_n} \\ &= \sum_{n=0}^{\infty} \langle G_n^* G_n x, y \rangle_{\mathcal{H}} \\ &= \lim_{k \to \infty} \left\langle \sum_{n=0}^{k} G_n^* G_n x, y \right\rangle_{\mathcal{H}}. \end{aligned} \tag{9.52}$$

Since x and y were arbitrary, it follows that $\sum_{n=0}^{\infty} G_n^* G_n = I_{\mathcal{H}}$ in the weak operator topology. □

Suppose $\{R_j\}$ is effective. Then, by Lemma 9.8, $\sum_{n=0}^{\infty} G_n^* G_n = I_{\mathcal{H}}$ in the weak operator topology. It follows that for any $j, k \in \mathbb{N}_0$,

$$R_j R_k^* = \sum_{n=0}^{\infty} R_j G_n^* G_n R_k^* \tag{9.53}$$

in the weak operator topology.

For each j, define $D_j : \mathcal{H}_j \to \oplus_{k=0}^{\infty}\mathcal{H}_k$ to be the 1-column matrix whose nth entry is the zero operator $0 : \mathcal{H}_j \to \mathcal{H}_n$ if $n \neq j$ and the identity operator $I_{\mathcal{H}_j}$ if $n = j$.

Lemma 9.9

$$R_j G_n^* = D_j^*(UM^* + M^* + I)^* D_n.$$

Proof Set $C_{nn} = M_{nn} = I_{\mathcal{H}_n}$. Recall that since $G = (I + U)R$, we have

$$G_n = \sum_{k=0}^{n} C_{nk} R_k.$$

Therefore,

$$\begin{aligned}
R_j G_n^* &= R_j \left(\sum_{k=0}^{n} R_k^* C_{nk}^* \right) \\
&= \sum_{k=0}^{n} R_j R_k^* C_{nk}^* \\
&= \begin{cases} \sum_{k=0}^{n} M_{jk} C_{nk}^* & \text{if } j > n \\ \sum_{k=0}^{j-1} M_{jk} C_{nk}^* + \sum_{k=j}^{n} R_j R_k^* C_{nk}^* & \text{if } j \leq n \end{cases} \\
&= \begin{cases} \sum_{k=0}^{n} M_{jk} C_{nk}^* & \text{if } j > n \\ \sum_{k=0}^{j-1} M_{jk} C_{nk}^* + \sum_{k=j}^{n} \left(R_k R_j^* \right)^* C_{nk}^* & \text{if } j \leq n \end{cases} \\
&= \begin{cases} \sum_{k=0}^{n} M_{jk} C_{nk}^* & \text{if } j > n \\ \sum_{k=0}^{j-1} M_{jk} C_{nk}^* + \sum_{k=j}^{n} \left(M_{kj} \right)^* C_{nk}^* & \text{if } j \leq n \end{cases} \\
&= \begin{cases} \sum_{k=0}^{n} M_{jk} C_{nk}^* & \text{if } j > n \\ \sum_{k=0}^{j-1} M_{jk} C_{nk}^* + \left(\sum_{k=j}^{n} C_{nk} M_{kj} \right)^* & \text{if } j \leq n. \end{cases}
\end{aligned}$$

Since $(I + U)(I + M) = I$, for $j \leq n$, we get

$$\sum_{k=j}^{n} C_{nk} M_{kj} = D_{jn}.$$

Therefore,

$$
\begin{aligned}
R_j G_n^* &= \begin{cases} \sum_{k=0}^{n} M_{jk} C_{nk}^* & \text{if } j > n \\ \sum_{k=0}^{j-1} M_{jk} C_{nk}^* + D_{jn}^* & \text{if } j \leq n \end{cases} \\
&= \begin{cases} D_j^* M(U^* + I) D_n & \text{if } j > n \\ D_j^* M U^* D_n + D_j^* D_n & \text{if } j \leq n \end{cases} \\
&= \begin{cases} D_j^* M(U^* + I) D_n + D_j^* D_n & \text{if } j > n \\ D_j^* M U^* D_n + D_j^* D_n + D_j^* M D_n & \text{if } j \leq n \end{cases}
\end{aligned}
$$

because $D_j^* D_n : \mathcal{H}_n \to \mathcal{H}_j$ is the zero operator when $j \neq n$ and because $D_j^* M D_n : \mathcal{H}_n \to \mathcal{H}_j$ is the zero operator when $j \leq n$. Therefore,

$$
\begin{aligned}
R_j G_n^* &= D_j^* \left(M U^* + M + I \right) D_n \\
&= D_j^* \left(U M^* + M^* + I \right)^* D_n .
\end{aligned}
$$

□

Lemma 9.10 $\sum_{n=0}^{\infty} D_n D_n^* = I$ *in the weak operator topology.*

Proof Let $x, y \in \ell^2(\oplus \mathcal{H}_j)$. We have

$$
\begin{aligned}
\lim_{k \to \infty} \left\langle \sum_{n=0}^{k} D_n D_n^* x, y \right\rangle_{\ell^2(\oplus \mathcal{H}_j)} &= \lim_{k \to \infty} \sum_{n=0}^{k} \left\langle D_n D_n^* x, y \right\rangle_{\ell^2(\oplus \mathcal{H}_j)} \\
&= \lim_{k \to \infty} \sum_{n=0}^{k} \left\langle D_n^* x, D_n^* y \right\rangle_{\mathcal{H}_n} \\
&=: \langle x, y \rangle_{\ell^2(\oplus \mathcal{H}_j)} .
\end{aligned}
$$

□

Proof of Theorem 9.2 Let $A = U M^* + M^* + I$. Let $j, k \in \mathbb{N}_0$, and let $x \in \mathcal{H}_k$ and $y \in \mathcal{H}_j$. By applying Lemmas 9.9 and 9.10 and Eq. (9.53), we get

$$
\begin{aligned}
\left\langle (R_j R_k^* - D_j^* A^* A D_k) x, y \right\rangle_{\mathcal{H}_j} &= \lim_{N \to \infty} \left\langle \left(R_j R_k^* - \sum_{n=0}^{N} D_j^* A^* D_n D_n^* A D_k \right) x, y \right\rangle_{\mathcal{H}_j} \\
&= \lim_{N \to \infty} \left\langle \left(R_j R_k^* - \sum_{n=0}^{N} R_j G_n^* G_n R_k^* \right) x, y \right\rangle_{\mathcal{H}_j} \\
&= \left\langle \left(R_j R_k^* - R_j R_k^* \right) x, y \right\rangle_{\mathcal{H}_j} \\
&= 0.
\end{aligned}
$$

Since y was arbitrary, it follows that

$$R_j R_k^* = D_j^* A^* A D_k.$$

We claim that

$$D_j^* M U^* U M^* D_k = D_j^* M M^* D_k.$$

By the relation $MU = -U - M$, we have

$$\begin{aligned}
A^* A &= (UM^* + M^* + I)^*(UM^* + M^* + I) \\
&= (MU^* + M + I)(UM^* + M^* + I) \\
&= MU^*UM^* + MU^*M^* + MU^* + MUM^* + MM^* \\
&\quad + M + UM^* + M^* + I \\
&= MU^*UM^* + MU^*M^* + MU^* - UM^* - MM^* + MM^* \\
&\quad + M + UM^* + M^* + I \\
&= MU^*UM^* + MU^*M^* + MU^* + M + M^* + I
\end{aligned}$$

If $j > k$, then $D_j^* D_k = 0$ and $D_j^* M^* D_k = 0$, so $D_j^* M D_k + D_j^* M^* D_k + D_j D_k = D_j^* M D_k = R_j R_k^*$.

If $j < k$, then $D_j^* D_k = 0$ and $D_j^* M D_k = 0$, so $D_j^* M D_k + D_j^* M^* D_k + D_j D_k = D_j^* M^* D_k = (R_k R_j^*)^* = R_j R_k^*$.

If $j = k$, then $D_j^* M D_k = 0$ and $D_j^* M^* D_k = 0$, so $D_j^* M D_k + D_j^* M^* D_k + D_j D_k = D_j^* D_j = I = R_j R_j^*$.

Hence,

$$\begin{aligned}
& R_j R_k^* = D_j^* A^* A D_k = \\
& D_j^* M U^* U M^* D_k + D_j^* M U^* M^* D_k + D_j^* M U^* D_k \\
& \quad + D_j^* M D_k + D_j^* M^* D_k + D_j^* D_k \\
& = D_j^* M U^* U M^* D_k + D_j^* M U^* M^* D_k + D_j^* M U^* D_k + R_j R_k^*.
\end{aligned}$$

Therefore,

$$\begin{aligned}
D_j^* M U^* U M^* D_k &= D_j^*(-MU^*M^* - MU^*) D_k \\
&= D_j^*(M(-U^*M^* - U^*)) D_k \qquad (9.54) \\
&= D_j^* M M^* D_k.
\end{aligned}$$

Recall that $\mathcal{F}(\oplus\mathcal{H}_j)$ is the subspace of $\oplus\mathcal{H}_j$ consisting of elements whose entries are zero in all but finitely many components, i.e., $x \in \oplus\mathcal{H}_j$ such that $D_n^* D_n x \neq 0$ for only finitely many $n \in \mathbb{N}_0$. Now, define

$$\mathbb{H} = \overline{M^*\mathcal{F}(\oplus\mathcal{H}_j)}.$$

Let $x, y \in \mathcal{F}(\oplus\mathcal{H}_j)$. This means there exist $x_0, x_1, \ldots, x_N$ and $y_0, y_1, \ldots, y_N$, where $x_j, y_j \in \mathcal{H}_j$, such that $x = \sum_{n=0}^{N} D_n x_n$ and $y = \sum_{n=0}^{N} D_n y_n$. By Eq. (9.54), we then have

$$\begin{aligned}
\left\langle UM^*x, UM^*y \right\rangle_{\ell^2(\oplus\mathcal{H}_j)} &= \left\langle UM^* \sum_{n=0}^{N} D_n x_n, UM^* \sum_{m=0}^{N} D_m y_m \right\rangle_{\ell^2(\oplus\mathcal{H}_j)} \\
&= \sum_{n=0}^{N} \sum_{m=0}^{N} \left\langle D_m^* M U^* U M^* D_n x_n, y_m \right\rangle_{\mathcal{H}_m} \\
&= \sum_{n=0}^{N} \sum_{m=0}^{N} \left\langle D_m^* M M^* D_n x_n, y_m \right\rangle_{\mathcal{H}_m} \\
&= \left\langle M^*x, M^*y \right\rangle_{\ell^2(\oplus\mathcal{H}_j)} .
\end{aligned}$$

This establishes that U is isometric on $M^*\mathcal{F}(\oplus\mathcal{H}_j)$. Because U is represented by a matrix, it must be the unique bounded extension of its restriction to $M^*\mathcal{F}(\oplus\mathcal{H}_j)$, and hence U is isometric on $\mathbb{H} = \overline{M^*\mathcal{F}(\oplus\mathcal{H}_j)}$.

It suffices to show that U vanishes on $\mathbb{H}^\perp$. To this end, observe that the matrices U^* and M^* leave the subspace $\mathcal{F}(\oplus\mathcal{H}_j)$ invariant. We have

$$\begin{aligned}
M^*(U^* + I) &= M^*U^* + M^* \\
&= -U^*.
\end{aligned}$$

Therefore,

$$\begin{aligned}
U^*(\mathcal{F}(\oplus\mathcal{H}_j)) &= -M^*(U^* + I)(\mathcal{F}(\oplus\mathcal{H}_j)) \\
&\subseteq -M^*(\mathcal{F}(\oplus\mathcal{H}_j)) \\
&\subseteq \mathbb{H}.
\end{aligned}$$

Thus, by the Fredholm Alternative,

$$\begin{aligned}
\mathbb{H}^\perp &\subseteq (U^*(\mathcal{F}(\oplus\mathcal{H}_j)))^\perp \\
&\subseteq (U^*M^*\mathcal{F}(\oplus\mathcal{H}_j))^\perp
\end{aligned}$$

$$
\begin{aligned}
&= (\mathrm{ran}_{M^*\mathcal{F}(\oplus\mathcal{H}_j)} U^*)^\perp \\
&= \ker_{M^*\mathcal{F}(\oplus\mathcal{H}_j)} U \\
&\subseteq \ker_{\mathbb{H}} U.
\end{aligned}
$$

Thus, U is a partial isometry.

Conversely, let U be a partial isometry on $\ell^2(\oplus\mathcal{H}_j)$. Hence, U is isometric on

$$
\begin{aligned}
(\ker_{\ell^2(\oplus\mathcal{H}_j)} U)^\perp &= \overline{\mathrm{ran}_{\ell^2(\oplus\mathcal{H}_j)} U^*} \\
&= \overline{\mathrm{ran}_{\mathcal{F}(\oplus\mathcal{H}_j)} U^*}.
\end{aligned}
$$

The formula $U^*(M^* + I) = -M^*$ implies that U is isometric on $M^*(\mathcal{F}(\oplus\mathcal{H}_j))$. This is equivalent to

$$\langle UM^*x, UM^*y\rangle_{\ell^2(\oplus\mathcal{H}_j)} = \langle M^*x, M^*y\rangle_{\ell^2(\oplus\mathcal{H}_j)}$$

for $x, y \in \mathcal{F}(\oplus\mathcal{H}_j)$. Tracking backward, the proof of the first part implies the formula (9.53). That is to say, we obtain

$$R_j R_k^* = \sum_{n=0}^{\infty} R_j G_n^* G_n R_k^*$$

in the weak operator topology of $B(\mathcal{H}_k, \mathcal{H}_j)$.

Let $T : \mathcal{H} \to \ell^2(\oplus\mathcal{H}_j)$ be as in (9.51), and note that $\|T\| \le 1$. By (9.52), $\sum_{n=0}^{\infty} G_n^* G_n = T^*T$ in the weak operator topology of $B(\mathcal{H})$.

Let $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_N$ and $\hat{y}_0, \hat{y}_1, \ldots, \hat{y}_N$ be such that $\hat{x}_j, \hat{y}_j \in \mathcal{H}_j$. Then,

$$
\begin{aligned}
\left\langle T^*T \sum_{k=0}^{N} R_k^*\hat{x}_k, \sum_{j=0}^{N} R_k^*\hat{y}_k \right\rangle_{\mathcal{H}} &= \lim_{m\to\infty} \left\langle \sum_{n=0}^{m} G_n^* G_n \sum_{k=0}^{N} R_k^*\hat{x}_k, \sum_{j=0}^{N} R_j^*\hat{y}_k \right\rangle_{\mathcal{H}} \\
&= \lim_{m\to\infty} \sum_{n=0}^{m}\sum_{j=0}^{N}\sum_{k=0}^{N} \left\langle G_n^* G_n R_k^*\hat{x}_k, R_j^*\hat{y}_j \right\rangle_{\mathcal{H}} \\
&= \lim_{m\to\infty} \sum_{n=0}^{m}\sum_{j=0}^{N}\sum_{k=0}^{N} \left\langle R_j G_n^* G_n R_k^*\hat{x}_k, \hat{y}_j \right\rangle_{\mathcal{H}_j} \\
&= \sum_{j=0}^{N}\sum_{k=0}^{N} \lim_{m\to\infty} \sum_{n=0}^{m} \left\langle R_j G_n^* G_n R_k^*\hat{x}_k, \hat{y}_j \right\rangle_{\mathcal{H}_j}
\end{aligned}
$$

$$= \sum_{j=0}^{N} \sum_{k=0}^{N} \langle R_j R_k^* \hat{x}_k, \hat{y}_j \rangle_{\mathcal{H}_j}$$

$$= \left\langle \sum_{k=0}^{N} R_k^* \hat{x}_k, \sum_{j=0}^{N} R_j^* \hat{y}_j \right\rangle_{\mathcal{H}} .$$

Then, by the assumption that $\cup(\ker R_j)^{\perp}$ is linearly dense in $\mathcal{H}$ and the fact that T^*T is a bounded operator, we have that $T^*T = I$. It follows that $\sum_{n=0}^{\infty} G_n^* G_n = I$ in the weak operator topology. Therefore, $\{R_j\}_{j=0}^{\infty}$ is effective by Lemma 9.8. □

9.4.3 Stationary Sequences of Operators

Recall from Eq. (9.29) that $R_{n+j} S^{-j} = R_n$. This provides an operator analogue of a stationary sequence of vectors. We want to show that for a slice-singular measure, the stationary sequence of operators $\{R_n\}$ is effective. We need to show that the operators satisfy Theorem 9.2. First, consider $(I + M)$, which becomes

$$I + M = \begin{bmatrix} I & & & 0 \\ R_0 S^{-1} R_0^* & I & & \\ R_0 S^{-2} R_0^* & R_0 S^{-1} R_0^* & I & \\ R_0 S^{-3} R_0^* & R_0 S^{-2} R_0^* & R_0 S^{-1} R_0^* & \ddots \\ \vdots & & \vdots & \ddots \end{bmatrix} . \tag{9.55}$$

Note that in our notation, $\mathcal{H}_j = \mathcal{H}_0 = L^2(\mu_2)$ (or $L^2(\mu_1)$, depending on the direction of the slice-singularity of μ). We will think of $\mathcal{H}_0$ as a subspace of $L^2(\mu)$ via its image under R_0^*.

We write the formal inverse of $I + M$ as

$$I + U = \begin{bmatrix} I & & & 0 \\ A_1 & I & & \\ A_2 & A_1 & I & \\ A_3 & A_2 & A_1 & \ddots \\ \vdots & & \vdots & \ddots \end{bmatrix} . \tag{9.56}$$

By Theorem 9.2, the stationary sequence of operators $\{R_n\}_{n=0}^{\infty}$ is effective if and only if U is a partial isometry on $\ell^2(\oplus\mathcal{H}_0)$. We will show that if the unitary S has a spectral representation that corresponds to a slice-singular measure, then the matrix is an isometry.

Inspired by [11, Lemma 2], we will define $B(z) \in \mathcal{B}(\mathcal{H}_0)$ by

$$(I - B(z))^{-1} = \sum_{n=0}^{\infty} R_0 S^{-n} R_0^* z^n \tag{9.57}$$

so that $B(z) = \sum_{n=1}^{\infty} A_n z^n$. In this form, $B(z)$ also acts on the space

$$\mathcal{H}_0(z) = \left\{ \sum_{n=0}^{\infty} f_n z^n \,\middle|\, f_n \in \mathcal{H}_0,\ \| \sum_{n=0}^{\infty} f_n z^n \|^2 = \sum_n \| f_n \|^2 \right\}.$$

From [11, Lemma 2], we have that there exists an operator-valued function $B(z)$ that satisfies the equation

$$\left\langle \frac{I + B(z)}{I - B(z)} a, c \right\rangle_{\mathcal{H}_0} = \left\langle \frac{I + zS^*}{I - zS^*} a, c \right\rangle_{L^2(\mu)} \tag{9.58}$$

for $a, c \in \mathcal{H}_0$. We claim that the $B(z)$ as defined in Eqs. (9.57) and (9.58) coincide.

As calculated in de Branges's proof of [11, Lemma 2], the power series expansion of the RHS of Eq. (9.58) can be written as

$$\langle I + 2zS^{-1} + 2z^2 S^{-2} \dots a, c \rangle,$$

which we write as

$$\langle a, c \rangle + 2z \langle R_0 S^{-1} R_0^* a, c \rangle + 2z^2 \langle R_0 S^{-2} R_0^* a, c \rangle + \cdots$$

or

$$2\langle S_+(z) a, c \rangle - \langle a, c \rangle = \langle (2S_+(z) - I) a, c \rangle,$$

with $S_+(z) = \sum_{n=0}^{\infty} R_0 S^{-n} R_0^* z^n$.

Formally, then, we have that

$$\frac{I + B(z)}{I - B(z)} = 2S_+(z) - I.$$

Solving for $B(z)$ (as in the scalar case), we obtain

$$B(z) = I - [S_+(z)]^{-1},$$

from which our claim follows.

Now, we want to calculate explicitly the action of $B(z)$ on $\mathcal{H}_0(z)$. We suppose that μ is y-slice-singular and write $\mu(dx\, dy) = \rho^y(dx)\mu_2(dy)$ as before. We let

$b_y(z)$ be the inner function associated with the singular measure ρ^y according to the Herglotz representation:

$$\frac{1+b_y(z)}{1-b_y(z)} = \int \frac{e^{2\pi i x}+z}{e^{2\pi i x}-z}\rho^y(dx).$$

Note that for a fixed z, by the Rokhlin Disintegration Theorem, $b_y(z)$ is a measurable function in y.

Consider $a = a(y)$ and $c = c(y)$ as elements of $\mathcal{H}_0 = L^2(\mu_2)$. By the functional calculus, the action of $(I+zS^*)(I-zS^*)^{-1}$ on $L^2(\mu)$ corresponds to multiplication by $(1+ze^{-2\pi i x})(1-ze^{-2\pi i x})^{-1}$. Therefore, we have

$$\begin{aligned}
\left\langle \frac{1+zS^*}{1-zS^*}a, c\right\rangle_{L^2(\mu)} &= \int \frac{1+ze^{-2\pi i x}}{1-ze^{-2\pi i x}} a(y)\overline{c(y)}\mu(dx\,dy) \\
&= \int\int \frac{1+ze^{-2\pi i x}}{1-ze^{-2\pi i x}}\rho^y(dx)\; a(y)\overline{c(y)}\;\mu_2(dy) \\
&= \int \frac{1+b_y(z)}{1-b_y(z)} a(y)\overline{c(y)}\mu_2(dy) \\
&= \left\langle \frac{1+B(z)}{1-B(z)}a, c\right\rangle_{\mathcal{H}_0}.
\end{aligned} \tag{9.59}$$

Again, by the functional calculus, we have that the action of $B(z)$ on $\mathcal{H}_0$ is given by

$$[B(z)a](y) = b_y(z)a(y).$$

Consider $F \in \mathcal{H}_0(z)$, which we write as

$$F(z) = \sum_{n=0}^{\infty} f_n z^n$$

$$F_y(z) = \sum_{n=0}^{\infty} f_n(y) z^n.$$

The action of $B(z)$ on $\mathcal{H}_0(z)$ can then be written as

$$\begin{aligned}
[BF]_y(z) &= B(z)F_y(z) \\
&= b_y(z)F_y(z).
\end{aligned}$$

For μ_2 a.e. y, $F_y \in H^2(\mathbb{D})$, and $\|b_y F_y\|_{H^2} = \|F_y\|_{H^2}$. Moreover,

$$\|F\|^2_{\mathcal{H}_0(z)} = \int \|F_y\|^2_{H^2}\mu_2(dy),$$

so we obtain that

$$\|BF\|_{\mathcal{H}_0(z)} = \|F\|_{\mathcal{H}_0(z)}. \tag{9.60}$$

We now have that B, whose matrix representation is given by U in Eq. (9.56), is an isometry on $\ell^2(\oplus_{n=0}^{\infty}\mathcal{H}_0)$. We therefore obtain the following theorem.

Theorem 9.3 *Suppose μ is a Borel probability measure on $[0,1]^2$ and is y-slice-singular. The operators $\{R_n\}$ as defined in Eq. (9.26) are effective in $L^2(\mu)$.*

9.5 Appendix

9.5.1 Duality

In Sect. 9.3, we derived our multi-variable $L^2(\mu)$ Fourier expansions with the use of some lemmas from the theory of Parseval frames in Hilbert spaces (see especially Sects. 9.1.2 and 9.4.1). The purpose of the present section is to present certain needed parts of frame theory.

Proposition 9.6 *If $\{g_n\}$ is a Parseval frame in $L^2(\mu)$, then $\|g_n\|_\mu \leq 1$.*

Proof For any $k \in \mathbb{N}_0$,

$$\|g_k\|_\mu^2 = \sum_n \left|\langle g_k, g_n\rangle_\mu\right|^2 = \|g_k\|_\mu^4 + \sum_{n\neq k} \left|\langle g_k, g_n\rangle_\mu\right|^2.$$

Therefore,

$$\|g_k\|_\mu^2 (1 - \|g_k\|_\mu^2) = \sum_{n\neq k} \left|\langle g_k, g_n\rangle_\mu\right|^2.$$

It follows that $\|g_k\|_\mu \leq 1$, or else the left side above would be negative, contrary to the right side being nonnegative. □

Proposition 9.7 *If μ disintegrates as $\rho^y(dx)\,\mu_2(dy)$, then $g_n^{(y)}(x)g_m(y) \in L^2(\mu)$.*

Proof

$$\begin{aligned}\int\int \left|g_n^{(y)}(x)g_m(y)\right|^2 d\rho_x(y)\,\mu_2(dy) &= \int |g_m(y)|^2 \int \left|g_n^{(y)}(x)\right|^2 \rho^y(dx)\,\mu_2(dy)\\ &= \int |g_m(y)|^2 \left\|g_n^{(x)}(\cdot)\right\|_{\rho_x}^2 \mu_2(dy)\\ &\leq \int |g_m(y)|^2\ \mu_2(dy)\end{aligned}$$

$$= \|g_m(\cdot)\|^2_{\mu_1}$$
$$< \infty.$$

□

Proposition 9.8 *If $F(x, y) \in L^2(\mu)$, then $\int F(\cdot, y)\overline{g_n^{(\cdot)}(y)}\,\rho^y(dx) \in L^2(\mu_2)$.*

Proof

$$\int \left| \int F(x, y)\overline{g_n^{(y)}(x)}\,\rho^y(dx) \right|^2 \mu_2(dy)$$
$$\leq \int \left(\int \left| F(x, y)\overline{g_n^{(y)}(x)} \right| \rho^y(dx) \right)^2 \mu_2(dy)$$
$$\leq \int \left(\|F(x, \cdot)\|_{\rho_x} \left\| g_n^{(x)}(\cdot) \right\|_{\rho_x} \right)^2 \mu_2(dy)$$
$$\leq \int \|F(x, \cdot)\|^2_{\rho_x}\ \mu_2(dy)$$
$$= \int \int |F(x, y)|^2\ \rho^y(dx)\,\mu_2(dy)$$
$$= \|F(x, y)\|^2_\mu < \infty.$$

□

Let μ be a y-slice-singular measure on $[0, 1)^2$. Let $\mathbf{n} = (m, n)$. Define $g_{\mathbf{n}}(x, y) = g_m(y)g_n^{(y)}(x)$, where $\{g_m\}$ is the auxiliary sequence of the marginal measure μ_2, and for μ_2-almost-every y, $g_n^{(y)}$ is the auxiliary sequence of the slice measure ρ^y. Also, define $e_{\mathbf{n}}(x, y) := e^{2\pi i m y}e^{2\pi i n x}$. (Note: for an x-slice-singular measure, we would define $g_{\mathbf{n}}(x, y) = g_m(x)g_n^{(x)}(y)$ and $e_{\mathbf{n}}(x, y) := e^{2\pi i m x}e^{2\pi i n y}$.)

Theorem 9.4 *If μ is y-slice-singular, then $\{g_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}_0^2}$ is a Parseval frame in $L^2(\mu)$ in the y-x order.*

Let $F \in L^2(\mu)$. Then,

$$\sum_n \sum_m \left| \left\langle F, g_m(y)g_n^{(y)}(x) \right\rangle_\mu \right|^2$$
$$= \sum_n \sum_m \left| \int \int F(x, y)\overline{g_m(y)}\overline{g_n^{(y)}(x)}\,\rho^y(dx)\,\mu_2(dy) \right|^2$$
$$= \sum_n \sum_m \left| \int \overline{g_m(y)} \int F(x, y)\overline{g_n^{(y)}(x)}\,\rho^y(dx)\,\mu_2(dy) \right|^2$$

$$= \sum_n \sum_m \left| \left\langle \int F(x, y) \overline{g_n^{(y)}(x)} \, \rho^y(dx), g_m(y) \right\rangle_{\mu_2} \right|^2 \qquad \text{[By Proposition 9.8]}$$

$$= \sum_n \left\| \int F(x, y) \overline{g_n^{(y)}(x)} \, \rho^y(dx) \right\|_{\mu_2}^2 \text{ [Because } \{g_m\} \text{ is a Parseval frame in } L^2(\mu_2)\text{]}$$

$$= \sum_n \int \left| \int F(x, y) \overline{g_n^{(y)}(x)} \, \rho^y(dx) \right|^2 d\mu_2(y)$$

$$= \int \sum_n \left| \int F(x, y) \overline{g_n^{(y)}(x)} \, \rho^y(dx) \right|^2 d\mu_2(y)$$

$$= \int \sum_n \left| \left\langle F(x, y), g_n^{(y)}(x) \right\rangle_{\rho^y} \right|^2 d\mu_2(y)$$

$$= \int \|F(x, y)\|_{\rho^y}^2 \, d\mu_2(y) \text{ [Because } \left\{ g_n^{(y)} \right\} \text{ is a Parseval frame in } L^2(\rho^y)\text{]}$$

$$= \int \int |F(x, y)|^2 \, \rho^y(dx) \, d\mu_2(dy)$$

$$= \|F\|_\mu^2 .$$

Theorem 9.5 *If μ is y-slice-singular, then $\{g_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{N}_0^2}$ is dextrodual to $\{e_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{N}_0^2}$ in $L^2(\mu)$ in the y-x order.*

Proof

$$\sum_n \sum_m \langle F, g_{\mathbf{n}} \rangle_\mu \, e_{\mathbf{n}}(\tilde{x}, \tilde{y})$$

$$= \sum_n \sum_m \left(\int \int F(x, y) \overline{g_m(y)} \overline{g_n^{(y)}(x)} \, \rho^y(dx) \, \mu_2(dy) \right) e^{2\pi i m \tilde{y}} e^{2\pi i n \tilde{x}}$$

$$= \sum_n \sum_m \left(\int \int F(x, y) \overline{g_n^{(y)}(x)} \, \rho^y(dx) \, e^{2\pi i n \tilde{x}} \overline{g_m(y)} \, \mu_2(dy) \right) e^{2\pi i m \tilde{y}}$$

$$= \sum_m \left(\int \left(\sum_n \int F(x, y) \overline{g_n^{(y)}(x)} \, \rho^y(dx) \, e^{2\pi i n \tilde{x}} \right) \overline{g_m(y)} \, \mu_2(dy) \right) e^{2\pi i m \tilde{y}}$$

$$= \sum_m \left(\int \left(\sum_n \left\langle F(x, y), g_n^{(y)}(x) \right\rangle_{\rho^y} e_n(\tilde{x}) \right) \overline{g_m(y)} \, \mu_2(dy) \right) e^{2\pi i m \tilde{y}}$$

$$= \sum_m \left(\int F(\tilde{x}, y) \overline{g_m(y)} \, \mu_2(dy) \right) e^{2\pi i m \tilde{y}}$$

$$= \sum_m \langle F(\tilde{x}, y), g_m(y) \rangle_{\mu_2} e_m(\tilde{y})$$
$$= F(\tilde{x}, \tilde{y}).$$

□

9.5.2 Direct Integrals

The proof of Theorem 9.3 can be simplified by using the formalism of direct-integral theory. Doing so also reveals additional structure of the Kaczmarz algorithm in terms of the action of $B(z)$ on $\mathcal{H}_0$ in analogy to the one-dimensional version as we presented in Eq. (9.8), namely in terms of the Clark theory [22]. We avoided the direct-integral theory in our initial proof for the benefit of the reader who is not familiar with the subject.

For the y-slice-singular measure μ, we can obtain a direct-integral decomposition of our space as

$$L^2(\mu) = \int^{\oplus} L^2(\rho^y)\, \mu_2(dy).$$

In the context of the Kaczmarz algorithm, the subspace $\mathcal{H}_0 = L^2(\mu_2)$ then yields a similar direct integral decomposition

$$\mathcal{H}_0(z) = \int^{\oplus} H^2_y(\mathbb{D})\, \mu_2(dy).$$

Here, $H^2_y(\mathbb{D}) = H^2(\mathbb{D})$; we simply use the subscript y to indicate the fibers of the decomposition are indexed by y.

Now, the action of the operator $B(z)$ has a particularly simple form: for $F = \int^{\oplus} f_y\, \mu_2(dy) \in \mathcal{H}_0(z)$,

$$BF = \int^{\oplus} b_y f_y\, \mu_2(dy).$$

From here, we immediately see that B is an isometry. However, we see even more, since we obtain that the range of B is given by

$$B\left(\mathcal{H}_0(z)\right) = \int^{\oplus} \mathcal{H}(b_y)\, \mu_2(dy),$$

where $\mathcal{H}(b_y)$ is the de Branges-Rovnyak space $H^2 \ominus b_y H^2$.

Moreover, we see the dilation theory of de Branges [11] as well as a significant part of the Clark theory on model subspaces and the Normalized Cauchy Transform [10].

Indeed, recall that in the one-variable case, the Normalized Cauchy Transform acts as a unitary operator from $L^2(\nu)$ to $\mathcal{H}(b)$, where ν is a singular measure on $\mathbb{T}$ and $\mathcal{H}(b) \subset H^2(\mathbb{D})$ is the model subspace corresponding to the inner function b. Via the direct integral theory, we can identify $L^2(\mu) \simeq \int^{\oplus} \mathcal{H}(b_y)\ \mu_2(dy)$ via the operator $\int^{\oplus} V_{\rho^y}\ \mu_2(dy)$ acting on $\int^{\oplus} L^2(\rho^y)\ \mu_2(dy)$.

Regarding the dilation theory of de Branges, the direct integral decomposition also provides a concrete representation of the dilation spaces that de Branges constructs, e.g., [11, Lemmas 4 and 11]. In fact, in de Branges' notation, we have $\mathscr{C} = L^2(\mu_2)$ and $\mathscr{C}(z) = \int^{\oplus} H_y^2(\mathbb{D})\mu_2(dy)$. Then, the operator $B(z)$ arising from the Herglotz representation decomposes the dilation space $\int H_y^2(\mathbb{D})\mu_2(dy)$ into

$$B(\mathscr{C}(z)) = \int^{\oplus} \mathcal{H}(b_y)\ \mu_2(dy) \simeq \int^{\oplus} L^2(\rho^y)\ \mu_2(dy) = L^2(\mu).$$

This space de Branges refers to as $\mathcal{H}(B)$, the operator-valued analogy of the model space $\mathcal{H}(b)$. In this view, we can extend the Normalized Cauchy Transform to act on $L^2(\mu)$ by the mapping

$$f \mapsto \sum_n \sum_m \langle f(x,y), g_n^{(y)}(x) g_m(y)\rangle z_1^n z_2^m, \tag{9.61}$$

which is a function in $H^2(\mathbb{D}^2)$. This mapping is an isometry (as a consequence of Theorem 9.4), and its image is a vector-valued analog of the model space $\mathcal{H}(b)$. Moreover, this mapping can be represented as an iteration of one-variable Normalized Cauchy Transforms, as follows:

Let $V_{\mu_2} : L^2(\mu_2) \to H^2$ and $V_{\rho^y} : L^2(\rho^y) \to H^2$ be the normalized Cauchy transforms of μ_2 and ρ^y, respectively. Then, for an $f(x,y) \in L^2(\mu)$, $f(\cdot, y) \in L^2(\rho^y)$ for μ_2-almost-every $y \in [0,1]$.

Then, for $f(x,y) \in L^2(\mu)$, consider $V_{\mu_2}[V_{\rho^y}[f(\cdot,y)](z_2)](z_1)$, where first V_{ρ^y} acts on $f(x,y)$ as a function of x, returning for μ_2-almost-every $y \in [0,1]$ a function $F_y(z_2) := V_{\rho^y}[f(\cdot,y)](z_2) \in H^2$. Then, regarding z_2 as fixed, V_{μ_2} acts on $F_y(z_2)$ as a function of y, returning for each fixed $z_2 \in \mathbb{D}$ a function $G(z_1,z_2) := V_{\mu_2}[F_y(z_2)](z_1) \in H^2$ as a function of z_1. We will now verify that $G(z_1,z_2)$ is the function returned by the mapping (9.61).

Recall that for a measure ν on $[0,1]$, the normalized Cauchy transform $V_\nu f : L^2(\nu) \to H^2(\mathbb{D})$ is given by

$$V_\nu f(z) = \frac{\int_0^1 \frac{f(x)}{1-ze^{-2\pi i x}}\ \nu(dx)}{\int_0^1 \frac{1}{1-ze^{-2\pi i x}}\ \nu(dx)}.$$

In [22, Proposition 1], it was proved that if ν is singular, then $V_\nu f(z) = \sum_{n=0}^{\infty} \langle f, g_n \rangle z^n$, where $\{g_n\}$ is the auxiliary sequence of $\{e_n\}$ in $L^2(\nu)$. Therefore, we have

$$
\begin{aligned}
&V_{\mu_2}[V_{\rho^y}[f(\cdot, y)](z_2)](z_1) \\
&\quad = V_{\mu_2}[\sum_{n=0}^{\infty} \left\langle f(x,y), g_n^{(y)}(x) \right\rangle_{\rho^y(x)} z_2^n](z_1) \\
&\quad = \sum_{m=0}^{\infty} \left\langle \sum_{n=0}^{\infty} \left\langle f(x,y), g_n^{(y)}(x) \right\rangle_{\rho^y(x)} z_2^n, g_m(y) \right\rangle_{\mu_2(y)} z_1^m \\
&\quad = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\langle \int f(x,y) \overline{g_n^{(y)}(x)}\, \rho^y(dx) z_2^n, g_m(y) \right\rangle_{\mu_2(y)} z_1^m \\
&\quad = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int \left(\int f(x,y) \overline{g_n^{(y)}(x)}\, \rho^y(dx) z_2^n \right) \overline{g_m(y)}\, \mu_2(dy) z_1^m \\
&\quad = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int \int f(x,y) \overline{g_n^{(y)}(x) g_m(y)}\, \rho^{(y)}(dx)\, \mu_2(dy) z_1^m z_2^n \\
&\quad = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int f(x,y) \overline{g_n^{(y)}(x) g_m(y)}\, \mu(dx\, dy) z_1^m z_2^n \\
&\quad = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\langle f(x,y), g_n^{(y)}(x) g_m(y) \right\rangle_{\mu} z_1^m z_2^n.
\end{aligned}
$$

We will expound on these connections more in a subsequent paper. In addition, this dilated view also presents the opportunity for analyzing the boundary representations of subspaces of $H^2(\mathbb{D}^2)$ as we did in [20, 21]; this too will be expounded upon in a subsequent paper.

9.5.3 Higher Dimensions

Our results have concerned two-dimensional slice-singular measures. Much of our work can be extended easily to higher dimensions once we have a definition of slice-singular.

We define a d-dimensional slice-singular measure by the $d-1$-dimensional case as follows. A positive Borel measure μ in $\mathbb{R}^d$ is said to be x_j-slice-singular if its marginal measure $\mu \circ \pi_j^{-1}$ is singular and the corresponding conditional measures are $d-1$ slice-singular. We then say that μ is slice-singular if it is x_j-slice-singular for some $j \in \{1, \ldots, d\}$. Recall that when μ is a measure on $\mathbb{R}^d$, then each of the indexed systems of conditional measures $\mu(\cdot|\pi_j = a)$, $a \in \mathbb{R}$, is supported in a "hyperplane," or rather subspace, and so is a measure in $d-1$ dimensions. The conditional measures may be viewed as Borel measures on $\mathbb{R}^{d-1}$, allowing us to define slice-singular by induction.

Our results in Sects. 9.2.1 and 9.2.2 extend naturally in the higher dimensional case. The results in Sect. 9.4 are dimension independent. The one result that is not immediate in higher dimensions is Theorem 9.3, particularly the question of whether the operator $B(z)$ is an isometry on $\mathcal{H}_0(z)$. We will address this issue in a subsequent paper.

Acknowledgments Eric S. Weber was supported in part by the National Science Foundation and the National Geospatial Intelligence Agency under award #1830254.

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