HOW DO INSTRUCTORS DESCRIBE STUDENTS' MATHEMATICAL WORK AND OPPORTUNITY TO LEARN IN GEOMETRY COURSES FOR TEACHERS?

Patricio Herbst	Soobin Jeon	Amanda Brown
University of Michigan	University of Michigan	University of Michigan
pgherbst@umich.edu	sbjeon@umich.edu	amilewsk@umich.edu

We report partial analysis of a survey of instructors of undergraduate geometry courses for teachers, attending to how they described the nature of the mathematical work they engage students in and the opportunities to learn that students had. Analysis of latent construct correlations showed that engagement of students in inquiry into geometry was significantly associated with opportunity to learn about mathematical definitions and conjecturing and engagement of students in the study of geometry was significantly associated with opportunity to learn about axioms and about history of geometry. Latent variable means comparisons showed group differences in claimed opportunity to learn between instructors whose highest degree was in mathematics and those whose highest degree was in mathematics education.

Keywords: geometry, teacher knowledge, undergraduate instruction, inquiry, opportunity to learn, survey

Objectives

We report on an analysis of survey responses from instructors of geometry courses for teachers (GeT) focusing on the curricular choices of instructors. Herbst et al. (2024a) reported on two distinct sets of characteristics of the mathematical work undergraduate geometry students may be engaged in: inquiring into geometry and studying geometry. Here, we investigated whether the type of mathematical work promoted could predict the topics stressed in different classes by looking into correlations between the former and the latter sets of variables and whether the field of highest degree attained by instructors could predict those curricular choices.

Literature Review

The mathematical preparation of teachers is an important component of secondary mathematics teacher preparation. This is so not only because teachers need to know the subject matter they will teach but also because the work of teaching, particularly when teaching for understanding, includes organizing the mathematical environments in which their students will learn and making sense of how students demonstrate their understanding (Manouchehri, 1998). Among the mathematical knowledge teachers need is the capacity to organize and manage mathematical work (Kuzniak & Nechache, 2021).

What knowledge to aim for and what mathematical work to engage prospective teachers in mathematics courses for teachers are important decisions instructors need to make. Though classically secondary mathematics teachers took courses equivalent to the mathematics major, the value of this choice has been questioned (e.g., Proulx & Bednarz, 2008). For a while, mathematics education researchers, mathematicians, and mathematics teacher educators have taken an interest in improving the mathematical preparation of teachers (Bass, 1997; Martin et al., 2020; Wasserman et al., 2023). This interest has been demonstrated in several ways. The promotion of inquiry-based learning in undergraduate mathematics courses has included mathematics courses for teachers demonstrating some learning of mathematical knowledge for teaching (see Laursen et al., 2016; Yoshinobu & Jones, 2013). Also, design research approaches to the teaching of mathematics courses for teachers have endeavored to connect the content of advanced mathematics courses with occasions of use in school mathematics teaching (Buchbinder & McCrone, 2023; Wasserman, et al, 2022). And communities of mathematics teacher educators and mathematicians have worked together to develop shared ownership of the problem of mathematical preparation of teachers as well as create curricular resources (e.g., CBMS, 2001, 2012; Martin et al., 2020; Senk et al., 2004; Usiskin et al., 2003).

Of particular interest to our study is the work of GeT: A Pencil, a community of instructors of geometry courses for secondary teachers (including mathematicians and mathematics educators) who have been working together since 2018 to improve those courses (see getapencil.org; An et al., 2023, 2024). An outcome of the work this group has been a consensual set of 10 essential student learning objectives (SLOs) that are meant to be a common core that diverse curriculum materials and pedagogical strategies could aim to align with. These 10 student learning outcomes are presented in Figure 1.

SLO	Description	SLO	Description
1	Derive and explain geometric arguments and proofs.	2	Evaluate geometric arguments and approaches to solving problems.
3	Understand the ideas underlying current secondary geometry content standards.	4	Understand the relationships between axioms, theorems, and different geometric models in which they hold.
5	Understand the role of definitions in mathematical discourse.	6	Effectively use technologies to explore geometry and geometric relationships.
7	Demonstrate knowledge of Euclidean geometry, including its history.	8	Be able to carry out and justify basic Euclidean constructions.
9	Compare Euclidean geometry to other geometries such as hyperbolic or spherical.	10	Use transformations to explore definitions and theorems about congruence, similarity, and symmetry.

Figure 1. Student Learning Objectives (SLOs)

The pursuit of all of those kinds of improvements can be facilitated by the existence of background information that describes the specific courses which are to be improved. Descriptive studies, such as TEDS-M, have contributed information about the qualities of teacher preparation in mathematics (e.g., Tatto & Senk, 2011) based on surveys and knowledge assessments, aiming to characterize how different nations prepare mathematics teachers. Tatto and Bankov (2018) provided an account of the opportunity to learn mathematics for secondary teachers in the United States based on an analysis of syllabi, noting that a large majority of prospective secondary teacher education programs provided opportunities to learn Euclidean or axiomatic geometry. However, information about what those geometry courses include both topically and in terms of mathematical work was beyond the scope of that study.

As regards geometry courses for secondary mathematics teachers, only two surveys have been conducted in the past. Wong (1970) surveyed leaders of mathematics departments and teacher education programs asking for their level of satisfaction with the geometry preparation of teachers. Grover and Connor (2000) reported on a survey of 108 instructors and were able to describe broad strokes of curriculum choices (e.g., that more than half of the courses emphasized Euclidean geometry from a synthetic perspective and that more than half included lectures with some discussion, but all group work was done outside of class). Herbst et al. (in press) interview study of 32 instructors showed they recognize a tension between sourcing the GeT curriculum from synthetic geometry and from geometry knowledge needed for teaching. It seemed important to develop a new instructor survey not only to update the description from Grover and Connor (2000) after broader emphases on mathematical knowledge for teaching and inquiry-based learning, but also to make more fine-grained claims. Though the survey targets questions about a range of issues on instruction and curriculum, the present report is focused on describing the geometry topics and geometric work students have opportunities to learn and do (for reports on other aspects of the survey see Herbst et al., 2024a, 2024b).

Theoretical Framework

We frame this inquiry using Cohen et al.'s (2003) instructional triangle which considers instruction as a transaction of content among instructor and students. We elaborate the content vertex of the triangle (see Figure 2, lower right) by noting that content is manifest in instruction in two different ways. Content is, on the one hand, a set of instructional goals or knowledge items which are at stake; and content is, on the other hand, the mathematical work that students are asked to do, in the form of problems and other tasks. In particular, different types of mathematical work with the content may be present for the same content. Geometry courses include many theorems about geometric concepts and students may all be expected to know the definitions and be able to prove the theorems. Yet the manner in which they get to attain such learning (the work in which they engage) may vary: In some classes they might participate in constructing the definitions or be given a chance to conjecture the theorems, while in other classrooms the definitions and the statements of such theorems may be given to them. That difference in the kind of mathematical work is an important one to track in geometry. Brousseau's (1997) notion of didactical contract can help distinguish between those classrooms. In particular, the survey as a whole pursues characterizing a contract that we name *geometric inquiry* (inquiry, hereafter) and one that we name the *study of geometry* (study, hereafter).

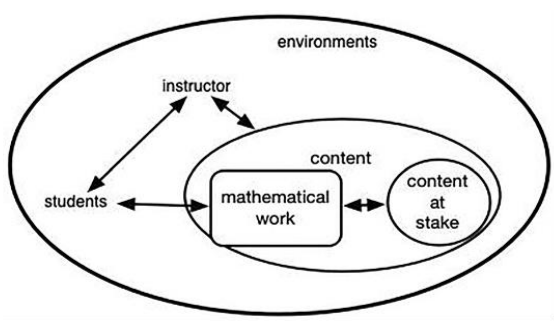

Figure 2. The instructional triangle adapted to include two manifestations of content

Our survey improved upon Grover and Connor's (2000) by taking into consideration the findings from Shultz (2022) about inquiry-oriented instruction (IOI). Shultz (2022) found that instructors' IOI practices were not homogeneous: Whereas some instructors who affirmed using inquiry involved students in conjecturing and defining, others who also affirmed teaching through inquiry interpreted it mostly in terms of student-teacher interaction (e.g., discussions, group work). Our survey built on Shultz's and also included questions that helped gauge the incidence of non-inquiry practices in teacher-student interaction and in the nature of the mathematical work. Our items were designed to answer separately questions about the incidence of different hypothesized factors associated with inquiry and study. Specifically, our survey included items that indicated constructs associated with inquiry and study regarding how instructors interact with students and how students interact with each other, and constructs associated with inquiry and with study concerning the nature of the mathematical work students do, including whether they participate in defining new concepts or rather receive the definitions from instructors. Herbst et al. (2024b) examined the distinctions among study and inquiry contracts in regard to the instructor-student interactions depicted on the left arrow in Figure 2. In this paper, we focus on the horizontal arrow shown in Figure 2 (how students interacted with content, which we call students' mathematical work) and some aspects denoted by the right arrow in Figure 2 (specifically the content that instructors recognized to be at stake).

Methods

To investigate the relationship between the mathematical work students were engaged in (i.e., study or inquiry) and the geometric ideas instructors recognized students had the opportunity to learn about, a survey was designed and distributed among instructors of geometry courses for secondary teachers in the United States. We targeted mathematics departments in all US universities where an undergraduate geometry course is regularly taught and required for students seeking certification to teach secondary mathematics. The survey was sent to all mathematics departments whose website included mention of such a geometry course (n=670). Emails were sent to department heads asking them to forward a link to the survey to the instructor who had taught the course last. We recognize that surveys provide only an approximation of teaching practice (Kennedy, 1999), and that more robust conclusions often need richer data collection. At the same time, a survey affords to see general trends in practice at low cost.

Figure 3 provides a list of the items used to describe students' mathematical work in relation to the constructs study and inquiry. We also used the SLOs (Figure 1) to operationalize what instructors might recognize among the opportunities to learn provided to their students in their courses, creating items that indicated each of the SLOs (see Appendix for some SLO-related items). Items shown in Figures 3 and items associated with the SLOs (see Appendix) were included in a larger survey administered through Qualtrics, which also asked questions about instructor demographics, prior preparation, and experience. The analysis focuses on a portion of the survey about mathematical work assigned to students and students' opportunity to learn the

SLOs, particularly looking at associations between the kind of students' mathematical work and the mathematical content recognized by the instructor to be at stake.

List of items that indicate Study (6-point Likert, from strongly disagree to strongly agree)	
821104	For the theorems whose proofs they had to learn, the proof was fully provided to them.
821105	The corollaries (i.e., consequences) of theorems students were supposed to use were explicitly stated for them.
821106	The constructions students were expected to learn were presented step by step to them.
List of items that indicate Inquiry (same scale as above)	
821204	Students were assigned to write (or improve) definitions.
821205	Students were asked to critique definitions given by either you [the instructor] or the textbook.
821219	Students were asked to critique construction procedures.

Figure 3. Items that indicate study and inquiry

Sample

About a third of the targeted departments had instructors return surveys. Our effective sample consisted of 140 GeT instructors who completed all survey items, including the GeT Instructor survey and a background questionnaire. Our sample participants confirmed they had taught a geometry course required for secondary mathematics teachers in the previous ten years. Approximately 69% had their highest degree in mathematics, while 28% had their highest degree in mathematics education (in both cases, highest degree is a Ph.D. or a Masters); also 35% had prior teaching experience in high school geometry. A sizeable 83% of participants held either tenured or tenure-track faculty positions, while 15% occupied non-tenure roles including lecturers and graduate students.

Results

The consistency of the inquiry and study scales for forms of mathematical work was reported elsewhere (Herbst et al., 2024a). To estimate a measurement model for opportunity to learn, we performed confirmatory factor analysis (CFA) on the items associated with the 10 hypothesized SLO constructs (see Figure 1). Given the limitations of that measurement model, we conducted exploratory factor analysis (EFA) on those same items to construct a new model. We felt the need to go beyond confirming our hypothesized model because some constructs had only two items; inter-item correlations within items indicating some constructs, namely SLO 1 and SLO 3, were too low suggesting poor internal consistency (Furr, 2017), and the item loadings under some of the constructs were low or cross-loaded to other constructs (Worthington et al., 2006).

We initially conducted an Exploratory Factor Analysis (EFA) with a smaller sample (n=118) and subsequently validated our model with additional samples (n=140). Following the identification and removal of items with low loadings, cross-loadings, or loading onto a two-item or one-item construct, we used eigenvalues and Kaiser's criterion to analyze and determine the number of constructs. This process resulted in a new model comprising five constructs with improved inter-item correlations.

Table 1: Correlation between constructs (in each model) and the mathematical work students were engaged in (student and inquiry)

	Study	Inquiry		Study	Inquiry
SLO 1	0.045	0.081*	Axiom	0.263*	0.085
SLO 2	0.039	0.165	Definition	-0.120	0.765***
SLO 3	-0.023	0.083	DGS	-0.252	0.412*
SLO 4	0.266*	0.094	History	0.644**	0.151
SLO 5	-0.105	0.906**	Conjecturing	-0.026	0.284*
SLO 6	-0.245	0.450*	**p<0.05, **p<0.01, ***p<0.001*		
SLO 7	0.372**	0.124			
SLO 8	-0.196	0.241*			
SLO 9	0.270	0.155			
SLO 10	0.019	0.432*			

**p<0.05, **p<0.01, ***p<0.001*

In comparison to the fit indices of the hypothesized model (CFI = 0.842, TLI = 0.814, RMSEA = 0.079, SRMR = 0.096), the new model demonstrated significant improvement (CFI = 0.947, TLI = 0.935, RMSEA = 0.062, SRMR = 0.072) (Hu & Bentler, 1999). Examination of individual items under constructs in the new model compared to those in the hypothesized model revealed that SLO 4 (referred to as Axiom in the new model) and SLO 5 (referred to as Definition in the new model) remained unchanged. SLO 6, SLO 7, and SLO 10 differed by only one item from the constructs DGS, History, and Conjecturing, respectively, in the new model. This similarity between the models supports the confirmation that some constructs in the hypothesized model were robust, while others were not.

Given the robustness of these constructs concerning student opportunity to learn, we focused on exploring the relationships between these constructs (in each model) and the mathematical work students were engaged in (i.e., study or inquiry). An item covariance between an item in the Inquiry construct and an item in the SLO 5 construct (or Definition construct in the new model) was added to the model, as suggested by the highest modification index. Adjusting these parameters not only improved the overall model fit but also brought the correlations to standardized estimates (see Table 1). We found a significant association between engaging students in geometric inquiry and giving students opportunity to learn about geometric

Kosko, K. W., Caniglia, J., Courtney, S., Zolfaghari, M., & Morris, G. A., (2024). *Proceedings of the forty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Kent State University.

transformations, digital geometry environments, and the role of definitions in mathematics (SLOs 5, 6, and 10). We also found a significant association between engaging students in the study of geometry and giving students opportunity to learn about Euclid's Elements, and the role of axiomatic systems (SLOs 4, and 7). Conversely, the distinction between study and inquiry did not significantly impact the extent to which instructors claimed their students learned about proof, evaluating arguments, the content of high school geometry, non-Euclidean geometry, or constructions.

Table 2: Latent Variable Mean difference between demographic groups (ME or M)

Received Highest Degree in Field of Mathematics Education (ME) (N=39) or Mathematics (M) (N=96)			
Constructs in hypothesized model	LVM in ME after setting M to 0	Constructs in new model	LVM in ME after setting M to 0
SLO_1	-0.166*		
SLO_2	-0.406*	Argument	-0.284*
SLO_3	0.064		
SLO_4	-0.372*	Axiom	-0.368*
SLO_5	0.336	Definition	0.330
SLO_6	0.418	DGS	0.405
SLO_7	-0.188	History	-0.229
SLO_8	0.116		
SLO_9	-0.535*		
SLO_10	-0.075	Conjecturing	-0.023

* p-value < 0.05

We also conducted a comparative analysis of latent variable means among instructors holding the highest degree in either mathematics (M) or mathematics education (ME) to explore whether their preparation could serve as a predictor for the likelihood of offering opportunities for students to learn content associated with various SLOs. To assess the size of between-group differences per construct, we set the latent variable means in the group with the highest degree in mathematics to zero and estimated the means in the group with the highest degree in mathematics education. Across both models, it became apparent that instructors with highest degrees in mathematics were more likely to engage students in learning axioms (SLO 4), and in learning geometric arguments, such as understanding proofs (SLO 1) and evaluating arguments (SLO 2). Notably, the SLO construct related to non-Euclidean geometry (SLO 9)—whose items indicated the construct named History in the second model—appeared to be associated with instructors holding the highest degree in mathematics.

Conclusion

The results shared provide a glimpse of how instructor claims about the opportunity to learn geometry they provide to their students relates to the kind of mathematical work they organize for them. In turn these results help see a baseline of implementation of the SLOs. Some differences in this implementation are associated with the field in which instructors were prepared. We notice that extending the consensus over the 10 SLOs may require more conversations across the differences among instructors, one of which seems to be their academic preparation. We also notice the need to better measure opportunity to learn; notably, engagement of students in proof (SLO 1), which Ion et al. (2023) showed to be something most instructors agree should be an objective in geometry courses for teachers could not be measured robustly.

References

An, T., Berzina-Pitcher, I., Bigelow, V., Buchbinder, O., Herbst, P., Hetrick, C., Milewski, A., Miller, N., Prasad, P., Pyzdrowski, L., St Goar, J., Sears, R., Szydlik, S., & Vestal, S. (2023). A cross-institutional faculty online learning community: Community-guided faculty development in teaching college geometry for teachers. In S. Linder, C. Lee, S. Stefl, and K. High (Eds.), *Handbook of STEM Faculty Development* (pp. 325-336). IAP.

An, T., Boyce, S., Brown, A., Buchbinder, O., Cohen, S., Dumitrascu, D., Escuadro, H., Herbst, P., Ion, M., Krupa, E., Miller, N. Pyzdrowski, L., Sears, R., St. Goar, J., Szydlik, S., & Vestal, S. (2024). (Toward) Essential student learning objectives for teaching geometry to pre-service secondary teachers. In B. Benken (Ed.), *The AMTE Handbook of Mathematics Teacher Education: Reflection on Past, Present and Future – Paving the Way for the Future of Mathematics Teacher Education*. IAP.

Bass, H. (1997). Mathematicians as educators. *Notices of the AMS, 44*(1), 18-21.

Buchbinder, O., & McCrone, S. (2023). Preparing prospective secondary teachers to teach mathematical reasoning and proof: the case of the role of examples in proving. *ZDM–Mathematics Education,55*, 779-792.

Conference Board of the Mathematical Sciences (CBMS) (2001). *The mathematical education of teachers*. American Mathematical Society and Mathematical Association of America.

Conference Board of the Mathematical Sciences (CMSB). (2012). *The mathematical education of teachers II*. American Mathematical Society and Mathematical Association of America.

Furr, R. M. (2017). *Psychometrics: An introduction*. Sage.

Grover, B. W., & Connor, J. (2000). Characteristics of the college geometry course for preservice secondary teachers. *Journal of Mathematics Teacher Education, 3*, 47-67.

Herbst, P., Brown, A. M., Ion, M., & Margolis, C. (in press, online first). Teaching geometry for secondary teachers: What are the tensions instructors need to manage? *International Journal of Research in Undergraduate Mathematics Education,* https://doi.org/10.1007/s40753-023-00216-0.

Herbst, P., Jeon, S., & Brown, A. (2024a). How do geometry courses for teachers support learning to teach? Results from an instructor survey. *Paper presented at the Annual Meeting of AERA.* Philadelphia, PA.

Herbst, P., Jeon, S., & Ko, I. (2024b). Instructor-Student Interactions in Geometry Courses for Secondary Teachers: Results from A National Survey. In press in *Proceedings of the 2024 Annual Conference on Research in Undergraduate Mathematics Education*. University of Nebraska Omaha.

Hu, L. & Bentler, P.M. (1999). Cutoff criteria for fit indexes in covariance structure analysis: Conventional criteria versus new alternatives. *Structural Equation Modeling, 6*(1), 1-55.

Ion, M., Herbst, P., Ko, I., and Hetrick, C. (2023). Agreeing on objectives of geometry for teachers' courses: Feedback from instructors on an initial list. *Proceedings of the 45th annual meeting of PMENA* (Vol. 2, pp. 362-369). University of Nevada, Reno.

Kennedy, M. M. (1999). Approximations to indicators of student outcomes. *Educational Evaluation and Policy Analysis*, *21*(4), 345-363.

Kuzniak, A., & Nechache, A. (2021). On forms of geometric work: A study with pre-service teachers based on the theory of Mathematical Working Spaces. *Educational Studies in Mathematics, 106*(2), 271-289.

Laursen, S. L., Hassi, M. L., & Hough, S. (2016). Implementation and outcomes of inquiry-based learning in mathematics content courses for pre-service teachers. *International Journal of Mathematical Education in Science and Technology, 47*(2), 256-275.
Manouchehri, A. (1998). Mathematics curriculum reform and teachers: What are the dilemmas? *Journal of Teacher Education*, *49*(4), 276-286.
Martin, W. G., Lawler, B. R., Lischka, A. E., & Smith, W. M. (Eds.). (2020). *The mathematics teacher education partnership: The power of a networked improvement community to transform secondary mathematics teacher preparation*. IAP.
Proulx, J., & Bednarz, N. (2008). Reconceptualizing the mathematical preparation of secondary school mathematics teachers. *Alberta Journal of Educational Research*, *54*(1), 124-127.
Senk, S. L., Keller, B., & Ferrini-Mundy, J. (2004). Teacher preparation: K-12 mathematics. *The Curriculum Foundations Project: Voices of the Partner Disciplines*, 145-155.
Shultz, M. C. (2022). The rationality of undergraduate mathematics instructors: The choice to use inquiry-oriented instructional practices. *Journal for Research in Mathematics Education*, *53*(3), 227-246.
Tatto, M. T., & Bankov, K. (2018). The intended, implemented, and achieved curriculum of mathematics teacher education in the United States. In *Exploring the Mathematical Education of Teachers Using TEDS-M Data* (pp. 69-133). Springer.
Tatto, M. T., & Senk, S. (2011). The mathematics education of future primary and secondary teachers: Methods and findings from the teacher education and development study in mathematics. *Journal of Teacher Education*, 62(2), 121-137.
Usiskin, Z., Peressini, A., Marchisotto, E. A., & Stanley, D. (2003). *Mathematics for high school teachers: An advanced perspective*. Pearson.
Wasserman, N. H., Buchbinder, O., & Buchholtz, N. (2023). Making university mathematics matter for secondary teacher preparation. *ZDM–Mathematics Education, 55*, 719-736.
Wasserman, N. H., Fukawa-Connelly, T., Weber, K., Ramos, J. P. M., & Abbott, S. (2022). *Understanding analysis and its connections to secondary mathematics teaching*. Springer Nature.
Wong, R. E. (1970). Geometry preparation for high school mathematics teachers. *The American Mathematical Monthly*, 77(1), 70-78.
Worthington, R. L., & Whittaker, T. A. (2006). Scale development research: A content analysis and recommendations for best practices. *The Counseling Psychologist*, 34(6), 806-838.
Yoshinobu, S. and Jones, M. (2013). An Overview of Inquiry-Based Learning in Mathematics. In Wiley *Encyclopedia of Operations Research and Management Science*

Appendix: Sample items indicating opportunity to learn

SLO	Item	Item statement: Students had the opportunity to …
1	4101	… learn to write geometric arguments (e.g., proofs)
2	4105	… check whether proofs were valid
3	4118	…analyze properties of different two-dimensional geometric shapes
4	4106	…work with different axiomatic systems
5	4132	…write definitions
6	4108	…use dynamic geometry software to explore figures.
7	4110	…learn about Euclid's Elements
8	4114	… perform basic Euclidean straightedge and compass constructions
9	4117	… learn differences between Euclidean geometry and other geometries
10	4126	… apply transformations to analyze mathematical situations