Row Shifting as a Puzzle Mechanic in Generalized Connect Four

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***Abstract*—Connect Four is a well-studied two-player game for fixed board sizes, however, the complexity of the generalized game is still open. Here, we look at a variant of Connect Four that allows for row shifting. Shift-Tac-Toe is a two-player game similar to Connect Four with the goal of getting 3-in-a-row to win. What makes the game unique is that each row is connected and can be shifted left or right, which causes pieces to fall into a neighboring column or to be removed from the board. Here, we show that the standard 3×3 game is a first player win and provide a perfect game-tree AI, and then we look at a generalized version of the game. We show that as a one-player puzzle, knowing whether n-in-a-row can be achieved with only shift moves is NP-complete. We also provide an implementation of the game allowing for arbitrary board size, shift size, and number of players.**

***Index Terms*—connect 4, shift-tac-toe, board game, game complexity, algorithmic game theory, puzzle mechanics**

I. Introduction

Shift-Tac-Toe is a game played between two players where the goal is to get three pieces in a row [20][1] . Although the name implies a variant of Tic-Tac-Toe (Noughts & Crosses), the game is actually a variation of Connect Four [7]. What makes the game unique is that players can either place a piece or they can shift a row left or right. When a row is shifted, pieces either fall in the new column, or are removed from the game (See Figure 2a). Thus, unlike Connect Four, the game is unbounded (or loopy), so play could last indefinitely.

The standard Connect Four game (7×6 board) was solved in 1988 by both Allen [1] and Allis [2] and proven to be a first player win. Several other board sizes have been investigated as well [25], [26]. Looking at the complexity of a generalized version, any game is within PSPACE since the number of moves and the board is bounded. However, little else has been proven. Given a board configuration of pieces, it is known to be NP-hard to determine if that is a legal configuration, i.e., whether the configuration can be reached through standard play [27]. In [14], they show how Connect Four positions can be encoded in Quantified Boolean Formulas, and it was shown that infinite cylindrical Connect Four is a draw [30]–[32], but there has been no significant progress on the general complexity. Several other variants of Connect Four and Tic-Tac-Toe have been explored, but none have investigated this variant or a row shifting mechanic.

Sliding Tile Mechanics. The concept of shifting or sliding tiles in a puzzle is one of the oldest puzzle mechanics, and related to many geometric games and models of computation. Many games employing a three-match style of play, such as Bejeweled [13] and Candy Crush [17], have been around for decades. This style of game is also hard from a complexity standpoint [15]. However, there seem to be few related games that have allowed row shifting in this capacity, with a notable one being Yoshi's Cookie [24], although the shifting wrapped the tiles around rather than removing them.

The game is also tangentially related to the tilt model of self-assembly and robot motion planning, which has several variations with the two most-studied being the full-tilt [4] and single-step [5] models. The tilt model gets its name from the classic Labyrinth tilting marble mazes [8]. Shift-Tac-Toe can be viewed as full-tilt in one direction (south) and single-step in any east/west movement along the rows. These mechanics are the basis of several board games such as Ricochet Robots [22], Lunar Lockout [33], TILT [23], and several video/mobile games such as Atomix [18], Mega Maze [19], Jelly No Puzzle [21], Snakebird [12], and Tomb of the Mask [16]. These types of puzzles even appear in Pokémon levels as the character sliding on ice to reach a destination. Knowing whether a single block can reach a destination space is known to be hard (PSPACE-complete) in both models [4], [9], and is hard (NP-complete) in the single-step model even without obstacles [10].

Algorithmic Game Theory. AGT is motivated by understanding how difficult a game, or some aspect of it, is from a theoretical standpoint. This often has very real-world consequences to the mechanics that might be incorporated into the game itself. Certain items or skills are often updated in a game to balance unforeseen advantages, techniques, or hacks that they allow in a game. With complexity results focused on a specific mechanic, the inherent difficulty can often be better incorporated as part of the world to offer more strategy-based play. An example of this is the incorporation of moving block puzzles into the landscape of games such as Pokémon and Zelda, mazes and one-directional doors or warps in open world games, or code deciphering through gathered logic clues. Each

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[1]Although no longer sold, there are files to 3D print one [3].

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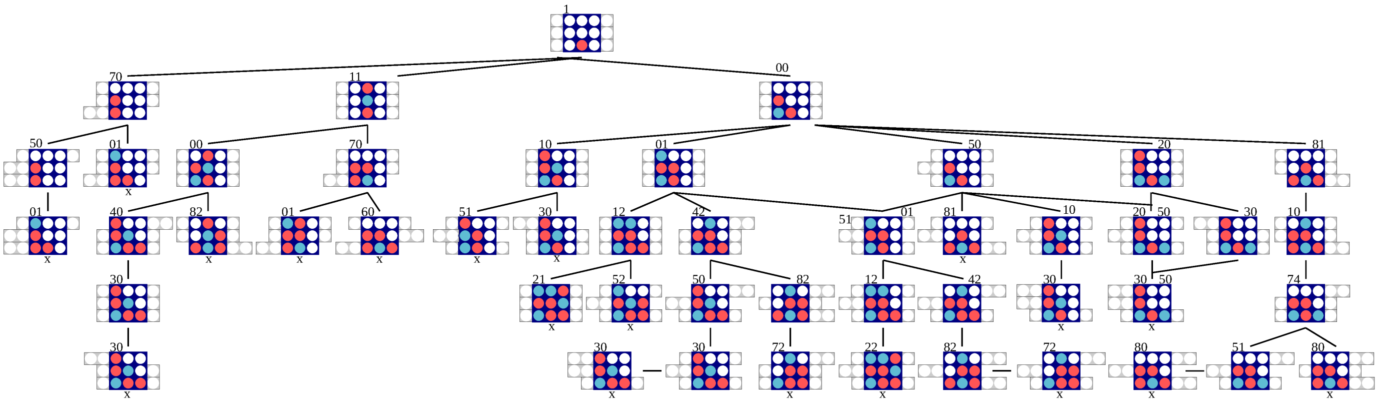


Fig. 1: Game tree starting with a move in the center. Each node represents a move by player two and an optimal counter move. The leaves are positions where any move by player two will lose. The numbers represent the moves made: 0) add column 0, 1) add column 1, 2) add column 2, 3) shift top row left, 4) shift top row right, 5) shift middle row left, 6) shift middle row right, 7) shift bottom row left, and 8) shift bottom row right. The maximum depth to win is 8 moves by Player 1.

of these mechanics (or minigames) alone are computationally hard, and thus the developer has the ability to tweak that aspect of the game, or the game altogether, to be as easy or difficult as desired, or to get harder as the game progresses.

In [6], they explore the many different mechanics used in The Legend of Zelda, and prove which ones are computationally hard, and which are not. This is interesting in demonstrating how complex the game is, and how much breadth and depth the developers gave themselves. They can create levels designed to exploit the complexity of a single mechanic or feature that in other levels is useless or nonexistent. Each one can make the gameplay entirely unique.

These results demonstrate the interplay between AGT and development as researchers seek to understand which mechanics make a game computationally hard and why that is the case, and developers seeking to make the gameplay novel with enough depth to build levels around, yet approachable to use.

A simple example to highlight why AGT matters in adding strategy to larger games is Tic-Tac-Toe. Several games include a Tic-Tac-Toe component to another skill such as archery, item throwing, racing, or even item collection where you are competing to get three-in-a-row before your opponent. This does not add any additional skill or strategy to the game since the strategy for Tic-Tac-Toe is simple. This merely adds the guise of increased difficulty or is just for fun, but is really only testing their accuracy (which may be intentional in some cases). If the same basic game mechanic were added but based on a game such as Hex, Go, or even Connect Four, then you not only have to be good at the skill, but at the subgame as well. The one caveat is if a player is unsure of their base skill or it is timed, they may alter the optimal Tic-Tac-Toe strategy to ensure a mark on the board, regardless of where.

Contributions. This paper makes a few contributions to the area of algorithmic game theory and general game understanding and design. First, we show that the standard commercial version of Shift-Tac-Toe is a first player win, and that the win only takes 8 moves. We then outline the general strategy. Following, we generalize the game, and show that even as a one-player game, it is computationally hard to know if some number of pieces can be placed in-a-row with only row shifting. This is an interesting puzzle in itself. We provide a simulator to setup and play these puzzles or to have multiplayer arbitrary-sized board games. We outline additional puzzle variants of interest and discuss other open problems that might give additional insight into the algorithmic complexity of Connect Four.

II. Shift-Tac-Toe is a First-Player Win

The shifting mechanic drastically changes the game compared to Connect Four. A 3×3 board of Connect Four (three) is an easy combinatorial game with only 869 possible positions [26]. However, with shifting, there are 91,125 possible positions. This is still fewer than the standard 7×6 Connect Four board, so even though game play may be quite different and may repeat positions, we can analyze the game in a fairly straightforward manner with a game tree.

In a standard Shift-Tac-Toe board, there are nine spots and each row can be shifted to three positions. Since each spot can be empty or a player piece, the total number of board configurations is $3^{12} = 531,441$. However, this includes invalid positions without a gravity constraint, so the total is actually only $91,125$. Most of these positions can be reached on either player's turn. With the shift mechanic, it is possible for both players to get 3-in-a-row in the same move, which is the only draw condition. Since pieces can be removed, the game does not actually have a draw configuration that you can not leave. However, players may repeat moves indefinitely triggering a draw through repetition.

Game Tree Pruning. Given the size of the game tree (9^n for n moves), several rules were employed to prune unnecessary plays and subtrees. We employ symmetry pruning since the board has horizontal symmetry. Shifting empty rows leads to immediate losses, so we ignore these moves. We count repeated positions as a draw through repetition. If we identify one guaranteed winning path for a player, we prune the other options. For instance, the first player wins if they move in the center or a side column, so the AI just picks one and prunes the other subtree. If no winning branch exists, a draw move is chosen, and if one does not exist, it picks a losing branch at random.

Theorem 1. *The standard Shift-Tac-Toe board of size 3×3 with a shift of 2 (centered initially) is a first-player win.*

Proof. This is a proof through exhaustive search on the game tree. Within 8 moves (by the first player), the first player can always win. A game tree analysis and optimal AI are available at [11], [29] and the game tree is given in Figure 1.

Strategy. The 8-move strategy is fairly straightforward and it does not matter where the first piece is placed, but for simplicity, we assume a move in the center. Figure 1 shows the full game tree for this start with each node representing two moves (a move from player two, and the response). The numbers say what the two moves were (given in the caption). Any options not given are moves where player one would immediately win. The leaves are positions where P1 wins regardless of the move P2 makes. □

III. Shifting Gravity Puzzles are NP-Complete

Another interesting game stems from using the board as the basis for different types of puzzles. Here, we look at generalized Shift-Tac-Toe, and show that it is NP-complete to know if you can make n-in-a-row on an $n \times n$ board with only shift moves. We show this via a reduction from Directed Hamiltonian Path by setting up a board where a shift move corresponds to selecting the next vertex to visit.

Definition 1 (Generalized Shift-Tac-Toe). *A Shift-Tac-Toe board B is given as an $m \times n$ board with r-shift on each row ($r \leq n$). A given configuration consists of the number each row is shifted ($\leq r$) and the pieces on the board. The goal is to get some $k \leq n$ pieces in a row.*

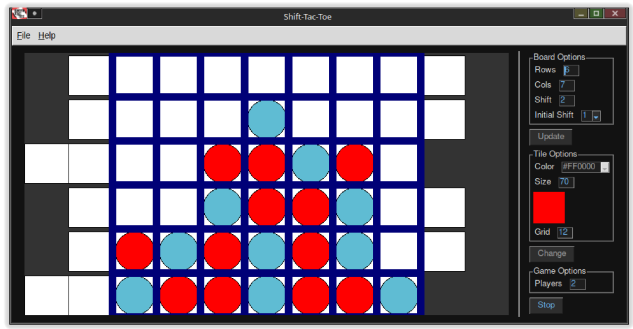
Path Gadget. For a Hamiltonian path on n vertices, $\langle p_0, p_1, \ldots, p_{n-1} \rangle$ where $p_0 = s$ and $p_{n-1} = t$, we define path gadgets. For our reduction, the only row that could create n-in-a-row is row $k = \lceil n/2 \rceil$. Each path gadget consists of two columns: the vertex choice column and the path selector column (p_4 gadget highlighted in Figure 3). Each vertex column has $k-1$ blue pieces and one red piece on row k. Every row above k is associated with a vertex. For p_i, the pieces above row k are red if that vertex could be visited as the i^{th} node in the path (determined by looking at vertices from p_{i-1}), and blue otherwise.

For the path selector column in gadget p_i, there are i empty spots starting at row k. Thus, those columns have $n-k-i$ pieces in the column (all blue but the top one). The one red piece is so n-in-a-row is not possible with blue pieces.

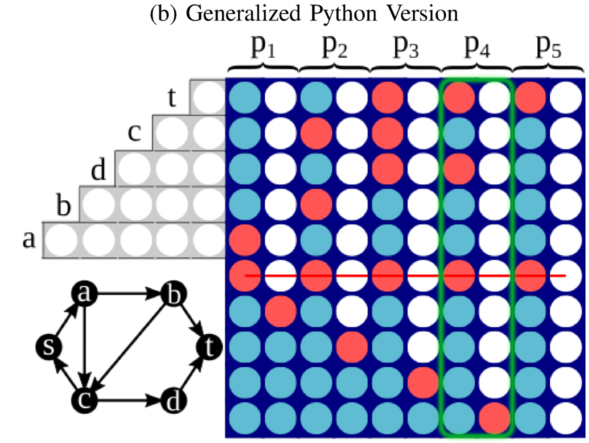
Theorem 2. *Given an $n \times n$ Shift-Tac-Toe configuration with a shift of $\lceil n/2 \rceil$, determining whether a k-in-a-row configuration ($k \leq n$ and $k = \mathcal{O}(n)$) is reachable with $\mathcal{O}(n)$ shift moves is NP-complete.*

Proof. Given an instance of directed Hamiltonian Path $H = \langle G, s, t \rangle$ s.t. $G = (V, E)$, where $s, t \in V$, and $\forall (a, b) \in E$,

(a) Game Boxes



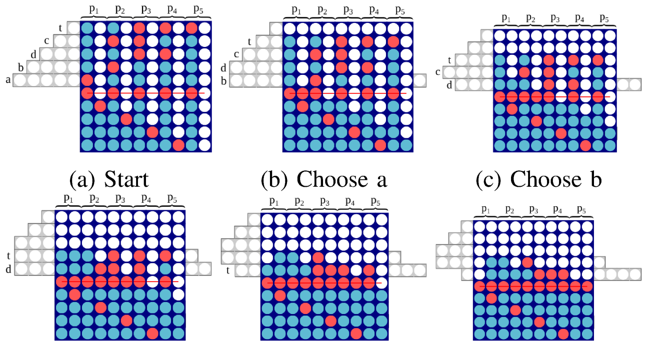
(b) Generalized Python Version



(c) Reduction Example

Fig. 2: (a) Original boxes of the English and French editions [20]. (b) Simulator that allows for arbitrary board size, shift size, and number of players [28]. (c) An example reduction from Directed Hamiltonian Path. Note that every row has a shift of 5, but for clarity, it only shows the amount that is possible to use (or the line is immediately impossible). The columns represent which vertex is chosen along the path. The rows represent the vertices to choose from. Note that when a vertex is chosen, its row is removed so the vertex rows above it move down one (See Figure 3).

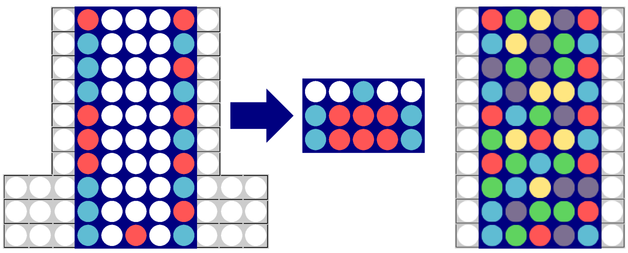
$a, b \in V$, we construct a Shift-Tac-Toe instance as described above. For convenience, let $n = |V| - 1$. Create a $2n \times 2n$ board B. Assign the vertices $V \setminus \{s\}$ to the top n rows. Then build the path gadgets as described for p_1 through p_n. From this configuration, a $2n$-in-a-row position is reachable with n shifts if and only if G has a Hamiltonian path from s to t.



(a) Start (b) Choose a (c) Choose b

(d) Choose c (e) Choose d (f) Choose t

Fig. 3: The steps to walk a Hamiltonian path in the Example of Figure 2c. Vertex s is p_0. Choose vertex a for p_1, b for p_2, c for p_3, d for p_4, and t for p_5.



(a) Pattern Puzzle (b) Match-Three Puzzle

Fig. 4: (a) An example easy shifting puzzle with a starting configuration and ending pattern to make. (b) A standard tile-matching puzzle based on shifting the rows where three-in-a-row causes the tiles to disappear and the tiles above to fall.

The proof of correctness is omitted due to space constraints.

Membership. This problem is in NP since this version does not allow adding pieces to the board, and is bounded to a polynomial number of shifts. □

IV. Conclusion

Here, we have taken some steps towards understanding how additional movement affects Connect Four, but there are many algorithmic and complexity questions that remain, and other areas to explore related to shifting puzzles in general.

One-player questions. For better understanding the generalized game, is the current problem still NP-hard with a constant shift size and a constant k-in-a-row? Figure 4a shows an example puzzle to make a pattern using this basic mechanic. How interesting are these types of puzzles? Is pattern making also NP-hard? In a related question, relocation of a piece is clearly easy, but is reconfiguration in k moves hard? Clearly, for a 1-player game, adding pieces to make a line is easy. Is some combination of shifts and adding pieces of interest for line building, pattern building, or reconfiguration in general?

Two-player questions. Is the Shift-Tac-Toe legality problem (retrograde Shift-Tac-Toe) hard? With a 2-player generalized version, the game is likely to be PSPACE-complete. For what size board does this occur? Is there a starting setup/condition of the 3×3 board that is a second-player win or draw? For other board sizes, is it always a first-player win?

Other questions. The complexity of Connect Four is still open. What other variants might give insight into the complexity of the game?

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