

A FAMILY OF KÄHLER FLYING WING STEADY RICCI SOLITONS

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ABSTRACT. In 1996, H.-D. Cao constructed a $U(n)$ -invariant steady gradient Kähler-Ricci soliton on $\mathbb{C}^n$ and asked whether every steady gradient Kähler-Ricci soliton of positive curvature on $\mathbb{C}^n$ is necessarily $U(n)$ -invariant (and hence unique up to scaling). Recently, Apostolov-Cifarelli answered this question in the negative for $n = 2$. Here, we construct a family of $U(1) \times U(n-1)$ -invariant, but not $U(n)$ -invariant, complete steady gradient Kähler-Ricci solitons with strictly positive curvature operator on real $(1, 1)$ -forms (in particular, with strictly positive sectional curvature) on $\mathbb{C}^n$ for $n \geq 3$, thereby answering Cao's question in the negative for $n \geq 3$. This family of steady Ricci solitons interpolates between Cao's $U(n)$ -invariant steady Kähler-Ricci soliton and the product of the cigar soliton and Cao's $U(n-1)$ -invariant steady Kähler-Ricci soliton. This provides the Kähler analog of the Riemannian flying wings construction of Lai. In the process of the proof, we also demonstrate that the almost diameter rigidity of $\mathbb{P}^n$ endowed with the Fubini-Study metric does not hold even if the curvature operator is bounded below by 2 on real $(1, 1)$ -forms.

1. INTRODUCTION

1.1. Overview. Ricci solitons are self-similar solutions of the Ricci flow that serve as generalizations of Einstein manifolds. They play an important role in the singularity analysis of the Ricci flow. Specifically, a *Ricci soliton* is a triple (M, g, X) , where M is a Riemannian manifold endowed with a complete Riemannian metric g and a complete vector field X , such that

$$(1.1) \quad \text{Ric}_g + \frac{1}{2}\mathcal{L}_X g = \lambda g$$

for some $\lambda \in \mathbb{R}$. A Ricci soliton is called *steady* if $\lambda = 0$, *expanding* if $\lambda < 0$, and *shrinking* if $\lambda > 0$. If $X = \nabla^g f$ for some smooth real-valued function f on M , then we say that (M, g, X) (or (M, g, f)) is a *gradient Ricci soliton*, and we call f the (*soliton*) *potential function*. In this case, the soliton equation (1.1) becomes

$$\text{Ric}_g + \nabla^2 f = \lambda g.$$

If the potential function moreover has a critical point $p \in M$, then we denote the soliton by (M, g, f, p) .

In (1.1), if g is Kähler and X is real holomorphic, then we say that (M, g, X) is a *Kähler-Ricci soliton*. Let ω denote the Kähler form of g . If (M, g, X) is in addition gradient, then (1.1) may be rewritten as

$$\rho_\omega + i\partial\bar{\partial}f = \lambda\omega,$$

where ρ_ω is the Ricci form of ω and f is the potential. In this article, we are concerned with complete steady gradient Kähler-Ricci solitons.

In real dimension 2, the only example of a non-flat steady gradient Ricci soliton is Hamilton's cigar soliton [51] which is rotationally symmetric. For real dimension $n \geq 3$, the only n -dimensional non-flat rotationally symmetric (i.e., $O(n)$ -symmetric) steady gradient Ricci soliton is the Bryant soliton [12]. More recently, the third-named author constructed a family of $\mathbb{Z}_2 \times O(n-1)$ -symmetric, non-rotationally symmetric, steady gradient Ricci solitons on $\mathbb{R}^n$ for $n \geq 3$. These solitons are collapsed for $n = 3$ and are called “flying wings”, reflecting the fact that they are asymptotic to two-dimensional sectors at infinity. For $n \geq 4$, they are non-collapsed. Moreover, in even dimensions $n \geq 4$ they are not Kähler, since the curvature operator is strictly positive everywhere; cf. Section 2 or [43].

In the Kähler world, for any complex dimension $n \geq 1$, Cao [16] constructed a $U(n)$ -invariant steady gradient Kähler-Ricci soliton on $\mathbb{C}^n$, and on the canonical bundle $K_{\mathbb{P}^n}$ of complex projective space $\mathbb{P}^n$, and in doing so generalized Hamilton's cigar soliton to higher dimensions. Further generalizations were subsequently obtained by Dancer-Wang [41], Yang [89], Biquard-Macbeth [8], the second-named author and Deruelle [36], and more recently by Schäfer [75, 76]. Cao-Hamilton [20] showed that the underlying complex manifold of a complete steady gradient Kähler-Ricci soliton with strictly positive Ricci curvature admitting a critical point of the scalar curvature must be diffeomorphic to $\mathbb{R}^{2n}$. Bryant [13] and Chau-Tam [24] independently improved this result by showing that the underlying complex manifold of such a steady gradient Kähler-Ricci soliton must in fact be biholomorphic to $\mathbb{C}^n$. As it turns out, Cao's steady gradient Kähler-Ricci soliton on $\mathbb{C}^n$ has strictly positive sectional curvature. In light of this fact, he conjectured the following.

Conjecture 1.1 (Cao's Conjecture [16]). *A complete steady gradient Kähler-Ricci soliton with positive curvature on $\mathbb{C}^n$ must be isometric (up to scaling) to Cao's $U(n)$ -invariant steady gradient Kähler-Ricci soliton on $\mathbb{C}^n$.*

As pointed out by Cao [16], this conjecture is true when $n = 1$ as the real Killing vector field $J\nabla f$ provides the $U(1)$ -symmetry. In this case, the soliton is the cigar soliton. In higher dimensions, it is natural to investigate the local version of this uniqueness problem, namely whether or not Cao's soliton can be perturbed to another steady gradient Kähler-Ricci soliton nearby. To this end, Chau-Schnürer [23] demonstrated that Cao's soliton on $\mathbb{C}^n$ is dynamically stable under sufficiently small perturbations of the Kähler potential which have fast decay at infinity. Uniqueness of Cao's soliton under suitable C^1 asymptotic conditions at spatial infinity was proved by Cui [39] via Brendle's Killing vector field method [10, 11] (see also [29, 36, 76]).

In the more rigid *shrinking* soliton case, Ni [69] showed that $\mathbb{P}^n$ endowed with the $U(n+1)$ -invariant Fubini-Study metric is the unique shrinking gradient Kähler-Ricci soliton with strictly positive bisectional curvature. This condition is strictly weaker than strictly positive sectional curvature and strictly positive curvature operator on real $(1, 1)$ -forms. In the more flexible *expanding* soliton case, there is a one-parameter family of $U(n)$ -invariant expanding gradient Kähler-Ricci solitons with positive curvature on $\mathbb{C}^n$ constructed by Cao [17]. Furthermore, the second-named author and Deruelle [37] proved that there exist continuous families of asymptotically conical expanding gradient Kähler-Ricci solitons with positive curvature on $\mathbb{C}^n$. Heuristically speaking, steady Kähler-Ricci solitons exhibit behavior residing on the cusp of shrinking and expanding solitons. They are particularly delicate due to volume collapsing phenomenon [43]. It is therefore tempting to understand the general uniqueness of positively curved steady Kähler-Ricci solitons.

To this end, Apostolov-Cifarelli [2] recently constructed counterexamples to Conjecture 1.1 for $n = 2$. More precisely, they used Hamiltonian two-forms and toric geometry to construct a one-parameter family of $U(1) \times U(1)$ -invariant positively curved steady gradient Kähler-Ricci solitons on $\mathbb{C}^2$. They also constructed complete steady gradient Kähler-Ricci solitons on $\mathbb{C}^n$ with more general symmetries for $n \geq 3$. However, it is unclear whether or not their examples exhibit positive curvature in these dimensions, hence Conjecture 1.1 remains open for $n \geq 3$.

1.2. Main results.

1.2.1. Existence of steady Kähler-Ricci solitons. In our first result, we construct complete steady gradient Kähler-Ricci solitons on $\mathbb{C}^n$ with strictly positive curvature operator on $(1, 1)$ -forms that are not $U(n)$ -invariant. The condition of $\text{Rm} > 0$ on real $(1, 1)$ -forms in particular implies strictly positive sectional curvature. Thus, these steady solitons provide counterexamples to Conjecture 1.1 in any dimension $n \geq 2$.

Theorem A. *Let $n \geq 2$ and set $c_n = \frac{1}{2n(n+1)}$. Then for all $\alpha \in [0, c_n]$, there exists a $U(1) \times U(n-1)$ -invariant complete steady gradient Kähler-Ricci soliton (M, g, f, p) on $\mathbb{C}^n$ with strictly positive curvature operator on real $(1, 1)$ -forms such that $R(p) = 1$ and the lowest sectional curvature at p is equal to α .*

The family of steady solitons in this theorem interpolates between the product of the cigar soliton and Cao's $U(n-1)$ -invariant steady Kähler-Ricci soliton ($\alpha = 0$) and Cao's $U(n)$ -invariant steady Kähler-Ricci soliton ($\alpha = c_n$). These solitons serve as the Kähler analog of the $\mathbb{Z}_2 \times O(n-1)$ -symmetric n -dimensional Riemannian steady Ricci solitons from [61, 60], and so we call them Kähler flying wings. As demonstrated in Appendix B (cf. Corollary B.6), by comparing the respective soliton vector fields, we show that the steady solitons of Theorem A are not isometric to those of Apostolov-Cifarelli [2]. In light of the results from [26, 43, 69], the steady gradient Kähler-Ricci solitons of Theorem A are all collapsed, have zero asymptotic volume ratio, and have volume growth rate bounded below by r^n , where n is the complex dimension of the underlying manifold.

More generally, constructing explicit examples of complete non-compact Kähler manifolds with strictly positive sectional curvature in higher dimensions has long been an important problem in Kähler geometry. Surprisingly, not much progress was made until the mid-90s. In earlier work [55], Klembeck constructed $U(n)$ -invariant complete non-compact Kähler manifolds on $\mathbb{C}^n$ with strictly positive bisectional curvature $BK > 0$. When $n = 1$, the same example was also constructed by Hamilton [51] independently and is known as the cigar soliton. However, for $n \geq 2$, these examples do not have nonnegative sectional curvature [88, Example 1]. To the best of our knowledge, Cao's $U(n)$ -invariant expanding and steady gradient Kähler-Ricci solitons [16, 17] are the first examples of complete non-compact Kähler manifolds in higher dimensions with strictly positive sectional curvature in the literature (see [88]). Wu-Zheng [88] systematically study $U(n)$ -invariant Kähler manifolds with strictly positive sectional curvature and provided more non-trivial examples with $U(n)$ -symmetry, exotic volume growth, and scalar curvature decay. We refer the reader to [88, 90] for a more detailed historical account in this direction. Since positive curvature is an open condition, one may easily generate other Kähler manifolds with strictly positive sectional curvature via a small compact perturbation of the Kähler potential of the aforementioned metrics.

Theorem A generalizes previous examples in [2, 37] by giving positively curved sub- $U(n)$ -symmetric examples in higher dimensions that cannot be obtained by small perturbations of the Kähler potential of Cao's examples.

1.2.2. Non-almost diameter rigidity of $\mathbb{P}^n$. In the course of the proof of Theorem A, we obtain a non-almost diameter rigidity result for $\mathbb{P}^n$. The classical Myers theorem states that a complete Riemannian manifold (M, g) with $\text{Ric} \geq (n-1)g$ must have diameter $\text{diam}(M, g) \leq \pi$. Cheng [27] proved that $\text{diam}(M, g) = \pi$ if and only if M is isometric to the standard sphere of radius 1. It is natural to ask whether or not “almost diameter” rigidity holds, meaning whether or not a Riemannian manifold (M, g) with $\text{Ric} \geq (n-1)g$ having diameter close to π is also close to the standard sphere of radius 1 in a certain sense. However, the almost diameter rigidity does not hold even topologically. Indeed, there are metrics g with $\text{Ric}_g \geq (n-1)g$ and $\text{diam}(g) \geq \pi - \varepsilon$ for arbitrarily small $\varepsilon > 0$ on $\mathbb{P}^n$ for $n \geq 2$ by Anderson [1], and $S^k \times S^{n-k}$ for any $k \geq 2$ and $n-k \geq 3$ by Otsu [70]. On the other hand, the almost diameter rigidity does hold under additional conditions: With a suitable negative sectional curvature lower bound, Perelman proved that it is homeomorphic to S^n [72]. Very recently, Ren-Rong [74] showed that the manifold must be δ -bi-Hölder close to the standard sphere for any $\delta > 0$ if local universal covers are sufficiently close to the Euclidean ball under a uniform scale; see also [25, 34, 66] for further discussion of the almost rigidity assuming $\text{Ric}_g \geq (n-1)g$. Finally, the almost diameter rigidity does not hold under the even stronger curvature condition $\text{Rm} \geq 1$. The third-named author constructed examples with $\text{Rm} \geq 1$ that are arbitrarily close in the Gromov-Hausdorff sense to an interval of length π [61].

In the Kähler case, Li-Wang [63] proved that a complete Kähler manifold with bisectional curvature $BK \geq 2$ must have diameter bounded above by that of half of the Fubini-Study metric, that is, $\frac{\pi}{2}$. Recently, Datar-Seshadri [42] improved a rigidity result of Liu-Yuan [64] (see also [84]) by showing that if the diameter is equal to $\frac{\pi}{2}$, then the Kähler manifold is holomorphically isometric to $\mathbb{P}^n$. The next natural step then is to investigate the almost diameter rigidity in the Kähler case. In contrast to the Riemannian case, the Kähler structure imposes extra restrictions on the geometry of the manifold. By the results of Mori and Siu-Yau on the Frankel conjecture [68, 83], the positive bisectional curvature assumption already guarantees that the closed Kähler manifold is biholomorphic to $\mathbb{P}^n$. It is therefore interesting to ask if closeness to maximal diameter implies closeness to $\mathbb{P}^n$ in a more restrictive way, say the Gromov-Hausdorff sense, under the curvature condition. As a byproduct of our construction, we exhibit a degeneration of a family of Kähler metrics with bisectional curvature $BK \geq 2$ to a one-dimensional interval with optimal diameter equal to $\frac{\pi}{2}$. This demonstrates that the almost diameter rigidity of $\mathbb{P}^n$ in the Kähler case does not hold in general without further assumptions. More precisely, we show that almost diameter rigidity of $\mathbb{P}^n$ does not hold under an even stronger curvature condition.

Theorem B. *Let $n \geq 1$. Then for all $\varepsilon > 0$, there exists a $U(n)$ -invariant Kähler metric g on $\mathbb{P}^n$ having curvature operator Rm bounded below by 2 on real $(1, 1)$ -forms everywhere (in particular, the holomorphic bisectional curvature $BK \geq 2$) such that*

$$d_{GH}((\mathbb{P}^n, d_g), [0, \frac{\pi}{2}]) \leq \varepsilon.$$

In Theorem B, it is clear by volume comparison that the volume of g goes to 0 as $\varepsilon \rightarrow 0$. One may wonder if the almost diameter rigidity holds under an additional uniform volume

lower bound. Indeed as mentioned in [74], counterexamples in the Riemannian case with uniform volume lower bound were provided by Anderson [1] and Otsu [70] independently. In the Kähler case, we provide counterexamples in Corollary 4.3 by finding a sequence of $U(n)$ -invariant Kähler metrics on $\mathbb{P}^n$ with $\text{Rm} \geq 2$ on *real* $(1, 1)$ -forms with volumes uniformly bounded from below, whose diameters converge to $\frac{\pi}{2}$, but whose metrics are not close to half of the Fubini-Study metric in the Gromov-Hausdorff sense.

The precise definition of curvature operator $\text{Rm} \geq 2$ on real $(1, 1)$ -forms is given below in Definition 2.22. In particular, we show that this curvature condition implies that the holomorphic bisectional curvature $BK \geq 2$. The curvature condition $\text{Rm} > 0$ on complex-valued $(1, 1)$ -forms is also known as *strictly positive complex curvature operator* and was studied in [19, 26, 82, 88]. Here, we study the curvature condition $\text{Rm} > 2$ on *real* $(1, 1)$ -forms, and show that it is equivalent to the condition that the Kähler cone over the corresponding Sasaki manifold has $\text{Rm} > 0$ on real $(1, 1)$ -forms in the *transverse* directions. Moreover, these Kähler cones can be smoothed out by expanding Kähler-Ricci solitons with $\text{Rm} > 0$ on real $(1, 1)$ -forms by a result of the second-named author and Deruelle [37]. Theorem B therefore implies

Corollary C. *Let $n \geq 2$. Then for all $\varepsilon > 0$, there exists a $U(1) \times U(n-1)$ -invariant expanding gradient Kähler-Ricci soliton on $\mathbb{C}^n$ with $\text{Rm} > 0$ on real $(1, 1)$ -forms whose link (S^{2n-1}, h) satisfies*

$$d_{GH}((S^{2n-1}, d_h), [0, \frac{\pi}{2}]) \leq \varepsilon.$$

1.3. Outline of proofs. We begin by recalling some classical methods that have been used to construct Ricci solitons and explain their relevance to our construction.

ODE methods: If the metric is assumed to satisfy certain symmetries, then both the Einstein and Ricci soliton equation reduce to a family of ODEs. In the Riemannian case, assuming rotational symmetry, Hamilton constructed the two-dimensional cigar soliton [51] and Bryant constructed the n -dimensional Bryant soliton for all $n \geq 3$. Appleton [3] constructed four-dimensional $U(2)$ -invariant, non-collapsed, non-Kähler steady solitons on the line bundles $\mathcal{O}_{\mathbb{P}^1}(k)$, $k > 2$, over $\mathbb{P}^1$.

In the Kähler case, assuming $U(n)$ -symmetry, Cao [16] and Koiso [57] found shrinking Kähler-Ricci solitons on twisted projective line bundles over $\mathbb{P}^{n-1}$ for $n \geq 2$. Cao [17] also constructed a one-parameter family of complete expanding Kähler-Ricci solitons on $\mathbb{C}^n$, and steady Kähler-Ricci solitons on $\mathbb{C}^n$ and on the blow-up of $\mathbb{C}^n/\mathbb{Z}_n$ at the origin. Feldman-Ilmanen-Knopf [48] then constructed the corresponding blow-down shrinking Kähler-Ricci soliton; see [15, 18, 40, 41, 47, 50, 53, 71, 87] for more examples using ODEs.

Continuity methods: In [79], Siepmann used the continuity method to construct expanding Kähler-Ricci solitons coming out of Ricci-flat Kähler cones. Deruelle [44] extended the continuity method to the Riemannian case by constructing expanding gradient Ricci solitons coming out of positively curved cones. In [37], the second-named author and Deruelle extended the aforesaid work of Siepmann by using the continuity method to construct expanding Kähler-Ricci solitons emanating from Kähler cones with $\text{Rm} > 0$ on real $(1, 1)$ -forms. The key ingredient in the continuity method is to show that a deformation of the cone metric can be lifted to a deformation of the expanding gradient Ricci soliton. This relies on the invertibility of the linearized operator of the expanding soliton equation, which is true under the assumption of suitable positive curvature conditions. Recently,

Bamler-Chen [6] developed a new continuity method using degree theory that only requires nonnegative scalar curvature. In this situation, the linearized operator is not necessarily invertible. We also refer the reader to the following related works for expanding solitons: [21, 22, 28, 29, 38, 46, 62, 65, 77, 79].

Collapsing methods: In [61], the third-named author observed an intricate relationship between expanding and steady gradient Ricci solitons. Namely, for any sequence of expanding Ricci solitons with collapsing asymptotic volume ratio, zooming in on the points with the highest curvature, a steady Ricci soliton can be observed. A major step is to find a sequence of cone metrics with collapsing asymptotic volume ratios which themselves can be lifted to expanding gradient Ricci solitons by the continuity method. Finding a collapsing sequence of cone metrics is equivalent to finding a sequence of metrics on the link of the cone with collapsing volumes. Depending on the curvature conditions of the expanding solitons, the collapsing links should also satisfy certain curvature conditions.

In the Riemannian case [61], the links need to satisfy $Rm > 1$ for the cones over them to be lifted to expanding Ricci solitons with $Rm > 0$ [44]. Here, we require the link to be a Sasaki metric on S^{2n-1} over a Kähler metric on $\mathbb{P}^{n-1}$ with $Rm > 2$ on real $(1, 1)$ -forms. We show that this is equivalent to the corresponding cone having $Rm > 0$ on real $(1, 1)$ -forms. These cones can therefore be lifted to expanding gradient Kähler-Ricci solitons with $Rm > 0$ on real $(1, 1)$ -forms by work of the second-named author and Deruelle [36]. In the following, we explain the construction of the desired metrics on $\mathbb{P}^{n-1}$ and outline the proof.

In Section 2, we include some preliminaries on Kähler and Sasaki geometry, as well as the $U(1) \times U(n-1)$ -invariant Kähler cone metric on $\mathbb{C}^n$ induced by a $U(n-1)$ -invariant metric on $\mathbb{P}^{n-1}$. The details of the curvature computations are contained in Appendix A.

In Section 3, we construct a sequence of smooth Kähler metrics on $\mathbb{P}^{n-1}$ with $Rm \geq 2$ on real $(1, 1)$ -forms that converge to the interval $[0, \frac{\pi}{2}]$ in the Gromov-Hausdorff sense. We do this by first writing the Fubini-Study metric on $\mathbb{P}^{n-1}$ as a doubly warped product over the interval $[0, \frac{\pi}{2}]$. Then, by scaling down the two warping functions, we obtain a sequence of singular metrics collapsing to the interval $[0, \frac{\pi}{2}]$ with singularities at 0 and $\frac{\pi}{2}$, and satisfying $Rm \geq 2$ on real $(1, 1)$ -forms on the smooth part. By cutting off the conical singularity at 0 on arbitrarily small scales and gluing back a portion of a suitable Cao's expanding soliton, we can approximate the singular metrics by smooth Kähler metrics on $\mathbb{P}^{n-1}$ with $Rm \geq 2$ outside of an arbitrarily small neighborhood of 0, and with $Rm > 0$ everywhere.

In Section 4, we take a limit of Ricci flows starting from these approximating metrics, and obtain a Ricci flow starting from each singular metric which smooths out the singularities. We show that the Ricci flow satisfies $Rm \geq 2$ everywhere at all positive times. This yields the almost non-rigidity result of Theorem B.

In Section 5, we lift the sequence of Kähler metrics on $\mathbb{P}^{n-1}$ in Theorem B to a sequence of expanding Kähler-Ricci solitons with $Rm > 0$ on real $(1, 1)$ -forms. We will show that they converge to a steady soliton asymptotic to a two-dimensional sector of angle $\frac{\pi}{2}$, and the soliton splits off a cigar factor. Then, using the same interpolation construction of the third-named author from [61], we obtain the family of solitons of Theorem A. In Appendix B, we show that these solitons are not isometric to those of [2].

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2. PRELIMINARIES

2.1. Kähler cones and Sasaki metrics. In this subsection, we recall several useful notions and definitions, in particular that of a Kähler cone and Sasaki metric. For basics in Kähler geometry, we refer the reader to Huybrechts [52]. For a more comprehensive reference for Sasakian geometry, we refer the reader to Boyer-Galicki [9].

We begin with

Definition 2.1 (Riemannian cone). Let (S, g) be a compact connected Riemannian manifold. The *Riemannian cone* C_0 with *link* S is defined to be $\mathbb{R}_+ \times S$ with metric $g_0 = dr^2 \oplus r^2 g$ up to isometry. The radius function r is then characterized intrinsically as the distance from the tip in the metric completion. We normally identify S with the level set $\{r = 1\}$.

Then we have

Definition 2.2 (Kähler cone). A *Kähler cone* is a Riemannian cone (C_0, g_0) such that g_0 is Kähler, together with a choice of g_0 -parallel complex structure J_0 . This will in fact often be unique up to sign. We then have a Kähler form $\omega_0(X, Y) = g_0(J_0 X, Y)$, and $\omega_0 = \frac{i}{2} \partial \bar{\partial} r^2$ with respect to J_0 .

The vector field $r\partial_r$ on a Kähler cone is real holomorphic and $\xi := J_0 r\partial_r$ is real holomorphic and Killing. This latter vector field is known as the *Reeb field*. The closure of its flow in the isometry group of the link of the cone generates the holomorphic isometric action of a real torus on C_0 that fixes the tip of the cone. We call a Kähler cone “quasiregular” if this action is an S^1 -action (and, in particular, “regular” if this S^1 -action is free), and “irregular” if the action generated is that of a real torus of rank > 1 .

Given a Kähler cone $(C_0, \omega_0 = \frac{i}{2} \partial \bar{\partial} r^2)$ with radius function r , it is true that

$$(2.1) \quad \omega_0 = r dr \wedge \eta + \frac{1}{2} r^2 d\eta,$$

where

$$(2.2) \quad \eta = i(\bar{\partial} - \partial) \log r = d^c \log(r)$$

restricts to a contact form on the link $\{r = 1\}$ of C_0 and we define $d^c := i(\bar{\partial} - \partial)$. Clearly, any Kähler cone metric on $L^\times$, the contraction of the zero section of a negative line bundle L over a projective manifold, with some positive multiple of the radial vector field equal to the Euler vector field on $L \setminus \{0\}$, is regular. In fact, as the following theorem states, this property characterises all regular Kähler cones.

Theorem 2.3 ([9, Theorem 7.5.1]). *Let (C_0, ω_0) be a regular Kähler cone with Kähler cone metric $\omega_0 = \frac{i}{2}\partial\bar{\partial}r^2$, radial function r , and radial vector field $r\partial_r$. Then:*

- (i) *C_0 is biholomorphic to the blowdown $L^\times$ of the zero section of a negative line bundle L over a projective manifold D , with $a \cdot r\partial_r$ equal to the Euler field on $L \setminus \{0\}$ for some $a > 0$.*
- (ii) *Let $p : L \rightarrow D$ denote the projection. Then, writing ω_0 as in (2.1), we have that $\frac{1}{2}d\eta = p^*\omega^T$ for some Kähler form ω^T on D with $[\omega^T] = 2\pi a \cdot c_1(L^*)$.*

Evidently, the flow generated by the vector fields $\{r\partial_r, \xi\}$ produces the standard $\mathbb{C}^*$ -action on the fibres of L . We give the Kähler form ω^T a special name.

Definition 2.4. The Kähler form ω^T on D from Theorem 2.3(ii) is called the *transverse Kähler form* of ω_0 on D , with the corresponding Kähler metric g^T called the *transverse Kähler metric*.

In light of (2.1), the Kähler cone metric g_0 associated to ω_0 takes the form

$$(2.3) \quad g_0 = dr^2 + r^2(\eta^2 + p^*g^T).$$

As the next example demonstrates, Theorem 2.3 is reversible in that one can always endow $L^\times$ with the structure of a regular Kähler cone metric.

Example 2.5 ([9, Theorem 7.5.2]). Let L be a negative line bundle over a projective manifold D and let $p : L \rightarrow D$ denote the projection. Then L has a hermitian metric h with negative curvature. Locally, h is defined by a smooth nonnegative real-valued function which, by abuse of notation, we also write as h . This is just the norm with respect to h of the unit section in a local trivialisation of L . The real $(1, 1)$ -form $i\partial\bar{\partial}\log h$ then defines a (global) Kähler form on D .

Set $r^2 = (h|w|^2)^a > 0$ for any $a > 0$, with w the coordinate on the fibre. Then $\frac{r^2}{2}$ defines the Kähler potential of a Kähler cone metric on $L^\times$, the contraction of the zero section of L , with Kähler form $\omega_0 = \frac{i}{2}\partial\bar{\partial}r^2$ and radial vector field $r\partial_r$, a scaling of the Euler vector field on $L \setminus \{0\}$ by $\frac{1}{a}$. Finally, writing ω_0 as in (2.1), we have that $\frac{1}{2}d\eta = p^*\omega^T$, where $\omega^T = a \cdot i\partial\bar{\partial}\log h$ is the transverse Kähler form, a Kähler form on D with $[\omega^T] = 2\pi a \cdot c_1(L^*)$.

As a specific example, we have

Example 2.6. In Example 2.5, one can consider the holomorphic line bundle $\pi : \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \rightarrow \mathbb{P}^{n-1}$ endowed with the hermitian metric h whose curvature form is $-\omega_{FS}$, that is, negative the Fubini-Study metric on $\mathbb{P}^{n-1}$ [52, Example 4.3.12]. For any $a > 0$, consider the Kähler cone metric defined by $r^2 = (h|w|^2)^a$, with w the coordinate on the fibre. The corresponding Kähler cone via the usual identification of the blowdown of the zero section of $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ with $\mathbb{C}^n$ resulting from the construction is $\mathbb{C}^n$ endowed with the Kähler cone metric $\omega_0 = \frac{i}{2}\partial\bar{\partial}r^2 = \frac{i}{2}\partial\bar{\partial}(|z|^{2a})$, with the Reeb vector field ξ a scaling by $\frac{1}{a}$ of that whose flow rotates the Hopf fibres with period 2π . In this case, the transverse Kähler form is given by $\frac{a}{2} \cdot \omega_{FS}$. Clearly, when $a = 1$, we obtain the flat metric on $\mathbb{C}^n$.

One may deform a Kähler cone to generate more examples in the following way.

Definition 2.7 (Type II deformation). Let $(C_0, \omega_0 = \frac{i}{2}\partial\bar{\partial}r^2)$ be a Kähler cone with complex structure J_0 and let $\varphi : C_0 \rightarrow \mathbb{R}$ be a smooth real-valued function satisfying

$\mathcal{L}_{r\partial_r}\varphi = \mathcal{L}_{J_0r\partial_r}\varphi = 0$ with $\tilde{\omega}_0 = \frac{i}{2}\partial\bar{\partial}(r^2e^{2\varphi}) > 0$. Then $(C_0, \tilde{\omega}_0)$, a Kähler cone with radius function $\tilde{r} := re^\varphi$ and radial vector field $\tilde{r}\partial_{\tilde{r}} = r\partial_r$, is called a *deformation of type II* (of (C_0, ω_0)).

Let $\tilde{\eta} = i(\bar{\partial} - \partial)\log \tilde{r}$. Then by (2.1), $\tilde{\omega}_0$ may be written as

$$\tilde{\omega}_0 = \frac{i}{2}\partial\bar{\partial}(r^2e^{2\varphi}) = \tilde{r}d\tilde{r} \wedge \tilde{\eta} + \frac{1}{2}\tilde{r}^2d\tilde{\eta} = \tilde{r}d\tilde{r} \wedge \tilde{\eta} + \frac{1}{2}\tilde{r}^2(d\eta + i\partial\bar{\partial}\varphi).$$

We refer the reader to [9, Section 7.5.1] or [49, Proposition 4.2] for more details.

The link of a Kähler cone is called a “Sasaki” manifold.

Definition 2.8 (Sasaki manifolds). A compact (odd real dimensional) Riemannian manifold (S, g) is *Sasaki* if and only if the Riemannian cone over (S, g) is a Kähler cone.

A Sasaki manifold (S, g) is naturally a contact manifold with contact form η and Reeb vector field given by the restriction of (2.2) and ξ to $\{r = 1\} \cong S$, respectively. We will only consider *regular* Sasaki manifolds, i.e., those Sasaki manifolds for which the corresponding Kähler cone C_0 is regular. Then via Theorem 2.3(ii), we have a map $p : C_0 \rightarrow D$ onto a compact Kähler manifold (D, g^T) , where g^T is the transverse Kähler metric whose Kähler form ω^T satisfies $\frac{1}{2}d\eta = p^*\omega^T$. Moreover, restricting p to $S \cong \{r = 1\}$, we get a map $p : (S, g) \rightarrow (D, g^T)$ which is a Riemannian submersion, as $g = \eta^2 + p^*g^T$ thanks to (2.3). This realizes S as the total space of an S^1 -bundle over D , the fibres of which are precisely the orbits of the flow of ξ .

Example 2.9. In the Kähler cone described in Example 2.6, the corresponding Sasaki manifold (S, g) is the $(2n - 1)$ -sphere $S = S^{2n-1}$ with g the round metric of curvature 1, $p : (S^{2n-1}, g) \rightarrow (\mathbb{P}^{n-1}, g^T)$ is the Hopf fibration, and $g^T = \frac{1}{2}g_{FS}$ is the Fubini-Study metric on $\mathbb{P}^{n-1}$ normalised so that $\text{Ric}(g_{FS}) = ng_{FS}$. The contact form η in this case is the restriction of the one-form $d^c \log r$ to $S^{2n-1} \subset \mathbb{R}^{2n}$.

For more details on Sasaki manifolds, we refer the reader to [9].

2.2. Doubly warped product metrics. In this section, we introduce doubly warped product metrics and highlight the key features of such metrics that we need.

For $n \geq 2$, let (M, g, η, ξ) be a (possibly irregular) $(2n - 1)$ -dimensional Sasaki manifold as defined in Definition 2.8, with Sasaki metric g , contact one-form η , Reeb vector field ξ , and transverse Kähler metric g^T . For a given connected open interval $I = (0, L) \subset \mathbb{R}$, $L > 0$, we define on the real $2n$ -dimensional manifold $\widehat{M} := M \times I$ a doubly-warped product Riemannian metric $\hat{g}$ by

$$(2.4) \quad \hat{g} := ds^2 + a^2(s)\eta^2 + b^2(s)g^T,$$

where s is the coordinate on $(0, L)$. Without loss of generality, we assume that $a(s), b(s) > 0$. The archetypical example of this construction is the Kähler cone itself.

Example 2.10. Set $a(s) = b(s) = s$ and $L = \infty$. Then one obtains on $\widehat{M}$ the Kähler cone over the Sasaki manifold (M, g) .

We endow $\widehat{M}$ with a complex structure in the following way. Since (M, g) is Sasaki, we know that the cone $C_0 = M \times \mathbb{R}_+$ over M is Kähler, and is in particular a complex manifold

with complex structure we denote by J_0 . Let r denote the coordinate on the $\mathbb{R}_+$ -factor of C_0 . We define a map $\phi : (0, L) \rightarrow (0, \infty)$ as the unique solution to the ODE:

$$(2.5) \quad \begin{cases} a(s)\phi'(s) = \phi(s), \\ \phi\left(\frac{L}{2}\right) = 1. \end{cases}$$

This is given explicitly by $\phi(s) = e^{\int_{\frac{L}{2}}^s \frac{du}{a(u)}}$. Then since $\phi'(s) > 0$ for $s > 0$, ϕ defines a diffeomorphism onto its image. We define a map $\Phi : \widehat{M} \rightarrow C_0$ by $\Phi(x, s) = (x, \phi(s))$. This is also a diffeomorphism onto its image, and so we define a complex structure on $\widehat{M}$ by $\widehat{J} := \Phi^* J_0$. By construction $\Phi_*(a(s)\partial_s) = r\partial_r$, and so $\widehat{J}(a(s)\partial_s) = \xi$.

It turns out that $\hat{g}$ is hermitian with respect to $\widehat{J}$ with fundamental form given by

$$\widehat{\omega} = \hat{g}(\widehat{J}(\cdot), \cdot) = a(s)ds \wedge \eta + b(s)^2\omega^T.$$

We give necessary and sufficient conditions for when $(\widehat{M}, \hat{g}, \widehat{J})$ is Kähler.

Lemma 2.11. *$(\widehat{M}, \hat{g}, \widehat{J})$ is Kähler if and only if $a(s) = b(s)b'(s)$. If this is the case, then the Kähler form $\widehat{\omega}$ of $(\widehat{M}, \hat{g}, J_0)$ is given by*

$$\widehat{\omega} = dd^c \left(\frac{1}{2} \int_{\frac{L}{2}}^s \frac{b(u)^2}{a(u)} du \right),$$

where $d^c = i(\bar{\partial} - \partial)$.

Proof. The fundamental form $\widehat{\omega}$ of $\hat{g}$ is given by

$$(2.6) \quad \widehat{\omega} = a(s)ds \wedge \eta + b(s)^2\omega^T.$$

Clearly $d\widehat{\omega} = 0$ if and only if $a(s) = b(s)b'(s)$. Here we use the fact that $d\eta = 2\omega^T$.

Regarding the last statement, it is clear that from (2.6) that

$$\widehat{\omega} = d \left(\frac{1}{2} b(s)^2 \eta \right).$$

Now, in light of (2.5), we can write

$$\eta = d^c \log(s) = d^c \log(\phi(s)) = \left(\frac{\phi'(s)}{\phi(s)} \right) d^c s = \frac{d^c s}{a(s)}.$$

The desired expression follows. $\square$

This leads us on to our next examples. The first illustrates the fact that the Fubini-Study metric on $\mathbb{P}^n$ can be realized as a doubly warped product.

Example 2.12 ([73, p.17]). Let (M, g, η, ξ) be the round Sasaki structure on S^{2n-1} , set $a(s) = \sin(s)\cos(s) = \frac{1}{2}\sin(2s)$ and $b(s) = \sin(s)$, and let $I = (0, \frac{\pi}{2})$. Then the doubly warped product metric closes up at 0 by adding a point and at $\frac{\pi}{2}$ by adding a $\mathbb{P}^{n-1}$. The resulting metric is one half of the Fubini-Study metric on $\widehat{M} = \mathbb{P}^n$. In light of Lemma 2.11, the Kähler form ω_{FS} of the Fubini-Study metric may be written as

$$\frac{1}{2}\omega_{FS} = dd^c \left(\frac{1}{2} \int_{\frac{\pi}{4}}^s \tan(u) du \right) = \frac{1}{2} dd^c (\ln \sec(s)) \quad \text{on } M \times \left(0, \frac{\pi}{2}\right).$$

The next example yields complete doubly warped product Kähler metrics on $\mathbb{P}^n$.

Example 2.13. Again working with the round Sasaki structure on the $(2n - 1)$ -sphere (S^{2n-1}, g) , realized as an S^1 -bundle $p : (S^{2n-1}, g) \rightarrow (\mathbb{P}^{n-1}, g^T)$ over $\mathbb{P}^{n-1}$ via the Hopf fibration, with transverse metric $g^T = \frac{1}{2}g_{FS}$, one half of the Fubini-Study metric g_{FS} on $\mathbb{P}^{n-1}$ normalised so that $\text{Ric}(g_{FS}) = ng_{FS}$ (cf. Example 2.9), the doubly warped product construction yields Kähler metrics on $(0, L) \times S^{2n-1}$ which close up at 0 by adding a point and at L by adjoining a $\mathbb{P}^{n-1}$ to give a complete Kähler metric on $\mathbb{P}^n$ if and only if a and b can be extended smoothly to 0 and L such that

$$(2.7) \quad a^{(\text{even})}(0) = 0, \quad a'(0) = 1, \quad a^{(\text{even})}(L) = 0, \quad a'(L) = -1,$$

and

$$(2.8) \quad b^{(\text{even})}(0) = 0, \quad b'(0) = 1, \quad b^{(\text{odd})}(L) = 0, \quad b(L) > 0.$$

(The smoothness condition at $s = L$ was also considered by Tran [86, Lemma 3.3].) One can verify that one half of the Fubini-Study metric on $\mathbb{P}^n$, considered in Example 2.12 where $L = \frac{\pi}{2}$, $a(s) = \frac{1}{2}\sin(2s)$, and $b(s) = \sin(s)$, satisfies these conditions. These conditions have the following inferences for the functions a and b . Using [67, Lemma 2.1], we can write them in the following way:

(2.9)

$$\begin{aligned} a(x) &= xg_1(x) & \text{for } x \in [0, L), \quad g_1(0) = a'(0) = 1, \text{ and } g_1'(0) = \frac{1}{2}a''(0) = 0, \\ a(x) &= (L - x)g_2(L - x) & \text{for } x \in (0, L], \quad g_2(0) = -a'(L) = 1, \text{ and } g_2'(0) = \frac{1}{2}a''(L) = 0, \\ b(x) &= xg_3(x) & \text{for } x \in [0, L), \text{ and } g_3(0) = b'(0) = 1, \text{ and } g_3'(0) = \frac{1}{2}b''(0) = 0, \\ b(x) &= b(L) + (L - x)g_4(L - x) & \text{for } x \in (0, L], \quad g_4(0) = -b'(L) = 0, \\ & & \text{and } g_4'(0) = \frac{1}{2}b''(L) = 0, \end{aligned}$$

where $g_i(x)$, $i = 1, \dots, 4$, are smooth functions.

Assume that a and b are chosen so that $[0, L] \times S^{2n-1}$ is smooth at 0. Let $t_i \rightarrow 0$ be a sequence of positive numbers converging to zero. Then the rescaled metrics $ds^2 + (t_i^{-1}a(t_i s))^2 \eta \otimes \eta + (t_i^{-1}b(t_i s))^2 g^T$ converge smoothly locally to the Euclidean metric $ds^2 + s^2 \eta \otimes \eta + s^2 g^T$ on $\mathbb{C}^n$ (realized as a Riemannian cone over S^{2n-1} ; cf. (2.3)). The smooth convergence immediately implies that $a(0) = b(0) = 0$ and $a'(0) = b'(0) = 1$. In fact, by rewriting the metric under the Cartesian product as in [73, Chapter 1, Sections 3.4 and 4.3], one can see that the pair of conditions (2.7) and (2.8) are necessary and sufficient for smoothness at 0 and L .

2.3. $U(1) \times U(n - 1)$ -Kähler cone metrics on $\mathbb{C}^n$ coming from $U(n - 1)$ -invariant metrics on $\mathbb{P}^{n-1}$. In this section, we will demonstrate how to construct $U(1) \times U(n - 1)$ -Kähler cone metrics on $\mathbb{C}^n$ from the following data. The main result of this section is Proposition 2.17.

Let $n \geq 3$ and recall Example 2.13 in the case of the round $(2n - 3)$ -sphere (S^{2n-3}, g) realized as an S^1 -bundle $p : (S^{2n-3}, g) \rightarrow (\mathbb{P}^{n-2}, g^T)$ over $(\mathbb{P}^{n-2}, g^T)$ with $2g^T = g_{FS}$, the Fubini-Study metric on $\mathbb{P}^{n-2}$ normalised so that $\text{Ric}(g_{FS}) = (n - 1)g_{FS}$. We consider doubly warped product Kähler metrics on $S^{2n-3} \times (0, L)$, $L > 0$, of the form

$$(2.10) \quad \bar{g} = ds^2 + \bar{a}(s)^2 \eta^2 + \bar{b}(s)^2 g^T.$$

Here s is the coordinate on the interval $(0, L)$ and η is the contact form associated to the round Sasaki structure on S^{2n-3} (see Example 2.9). Write ω^T for the Kähler form associated to g^T . We have that $\bar{a}(s) = \bar{b}(s)\bar{b}'(s)$ by Lemma 2.11 because $\bar{g}$ is Kähler, and in addition that (2.7) and (2.8) hold true so that, as described in Example 2.13, $\bar{g}$ closes up smoothly at the endpoints of $(0, L)$ to give a warped product Kähler metric $\bar{g}$ on $\mathbb{P}^{n-1}$. Without loss of generality, we can assume that $\bar{a}(s) > 0$. Then, as written in Lemma 2.11, the Kähler form $\bar{\omega}$ of $\bar{g}$ can be written as

$$\bar{\omega} = \bar{a}(s)ds \wedge \eta + \bar{b}(s)^2\omega^T = dd^c \left(\frac{1}{2} \int_{\frac{L}{2}}^s \frac{\bar{b}(u)^2}{\bar{a}(u)} du \right),$$

and the corresponding volume form can be computed as

$$\bar{\omega}^{n-1} = (n-1) \cdot \bar{a}(s) \cdot \bar{b}(s)^{2n-4} ds \wedge \eta \wedge (\omega^T)^{n-2}.$$

In particular,

$$(2.11) \quad \int_{\mathbb{P}^{n-1}} \bar{\omega}^{n-1} = c(n) \int_0^L \bar{a}(s) \cdot \bar{b}(s)^{2n-4} ds = c(n) \int_0^L \bar{b}'(s) \cdot \bar{b}(s)^{2n-3} ds = \frac{c(n)\bar{b}(L)^{2n-2}}{2n-2} > 0,$$

where $c(n) > 0$ is a dimensional constant and where we have used (2.8) and the fact that $\bar{a}(s) = \bar{b}(s)\bar{b}'(s)$.

We begin by proving some preliminary lemmas before stating and proving the main result.

Lemma 2.14. *The map $\phi : (0, L) \rightarrow (0, \infty)$ defined by*

$$\phi(s) = e^{\int_{\frac{L}{2}}^s \frac{du}{\bar{a}(u)}}$$

is invertible.

Proof. Since $\bar{a}(s) > 0$ by assumption, we see directly that $\phi'(s) > 0$ for $s \in (0, L)$ so that $\phi(s)$ is strictly increasing. In addition, in light of (2.9), with $|g_2(L-x) - 1| \leq \frac{1}{2}$ for x in an interval of the form $(0, \varepsilon)$ for some $\varepsilon > 0$ sufficiently small, we see that for $s \in (L-\varepsilon, L)$,

$$\int_{\frac{L}{2}}^s \frac{du}{\bar{a}(u)} = \int_{\frac{L}{2}}^{L-\varepsilon} \frac{du}{\bar{a}(u)} + \int_{L-\varepsilon}^s \frac{du}{\bar{a}(u)} \geq C + \frac{2}{3} \int_{L-\varepsilon}^s \frac{du}{L-u}$$

which tends to $+\infty$ as $s \rightarrow L^-$. Similar behavior occurs as $s \rightarrow 0^+$, hence $\phi(s)$ is indeed a diffeomorphism. $\square$

Let $(w_1, \dots, w_n)$ henceforth denote holomorphic coordinates on $\mathbb{C}^n$.

Lemma 2.15. *Define a function $\hat{s} : \mathbb{C}^n \setminus \{0\} \rightarrow [0, L]$ by*

$$\hat{s}(w_1, \dots, w_n) = \begin{cases} 0 & \text{if } w_2 = \dots = w_n = 0 \text{ (and } w_1 \neq 0), \\ \phi^{-1} \left(\frac{\sqrt{\sum_{j=2}^n |w_j|^2}}{|w_1|} \right) & \text{if } w_1 \neq 0 \text{ and } w_j \neq 0 \text{ for some } j \geq 2, \\ L & \text{if } w_1 = 0. \end{cases}$$

Then $\hat{s}$ is continuous and invariant under the diagonal $\mathbb{C}^$ -action on $\mathbb{C}^n$.*

Proof. The function $\hat{s}$ is clearly invariant under the diagonal $\mathbb{C}^*$ -action on $\mathbb{C}^n$. As such, it descends to a function that we denote by $\tilde{s} : \mathbb{P}^{n-1} \rightarrow [0, L]$ on $\mathbb{P}^{n-1}$. To complete the proof of the lemma, it suffices to show that $\tilde{s}$ is continuous in the induced homogeneous coordinates $[w_1 : \dots : w_n]$ on $\mathbb{P}^{n-1}$.

To this end, since ϕ extends to a continuous function with $\phi(0) = 0$, $\tilde{s}$ is clearly continuous on the open subset $\{[w_1 : w_2 : \dots : w_n] \in \mathbb{P}^{n-1} \mid w_1 \neq 0\}$. Let $2 \leq j \leq n$ and consider the chart $\{[w_1 : w_2 : \dots : w_n] \in \mathbb{P}^{n-1} \mid w_j \neq 0\}$. In this chart, we see that for $w_1 \neq 0$,

$$\tilde{s}([w_1 : \dots : w_n]) = \phi^{-1} \left(\left| \frac{w_j}{w_1} \right| \sqrt{1 + \sum_{\substack{k=2 \\ k \neq j}}^n |w_k|^2} \right).$$

Since the quantity inside the parentheses tends to $+\infty$ as $w_1 \rightarrow 0$, we see that $\tilde{s}$ is continuous, as required. $\square$

Lemma 2.16. *Define a function $\hat{\varphi} : \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{R}$ by*

$$\hat{\varphi}(w) = 2 \int_{\frac{L}{2}}^{\hat{s}(w)} \frac{\bar{b}(u)^2}{\bar{a}(u)} du - \bar{b}(L)^2 \cdot \ln \left(1 + e^{2 \int_{\frac{L}{2}}^{\hat{s}(w)} \frac{du}{\bar{a}(u)}} \right),$$

where $\hat{s}$ is the function defined in Lemma 2.15. Then $\hat{\varphi}$ is smooth and invariant under the diagonal $\mathbb{C}^*$ -action on $\mathbb{C}^n$.

Proof. Since $\hat{\varphi}$ is a function of $\hat{s}$ only, and $\hat{s}$ is invariant under the diagonal $\mathbb{C}^*$ -action, $\hat{\varphi}$ has the same property. As such, it descends to a function $\tilde{\varphi} = \tilde{\varphi}(\tilde{s}) : \mathbb{P}^{n-1} \rightarrow \mathbb{R}$, where $\tilde{s}$ is as in the proof of the previous lemma. To complete the proof of the lemma, it suffices to show that $\tilde{\varphi}$ is smooth in the induced homogeneous coordinates $[w_1 : \dots : w_n]$ on $\mathbb{P}^{n-1}$.

To this end, with s still denoting the coordinate on the $(0, L)$ -factor, let $\tau : (0, L) \rightarrow (0, \frac{\pi}{2})$ be the unique solution of the ODE

$$\begin{cases} \bar{a}(s)\tau'(s) = \frac{1}{2} \sin(2\tau(s)), \\ \tau\left(\frac{L}{2}\right) = \frac{\pi}{4}. \end{cases}$$

Then

$$\tau(s) = \arctan \left(e^{\int_{\frac{L}{2}}^s \frac{du}{\bar{a}(u)}} \right) = \arctan(\phi(s))$$

and hence is a diffeomorphism.

Next recall from Example 2.12 one half of the Fubini-Study metric on $S^{2n-3} \times (0, \frac{\pi}{2})$, the Kähler form of which is given by

$$(2.12) \quad \frac{1}{2} \omega_{FS} = dd^c \left(\frac{1}{2} \int_{\frac{\pi}{4}}^t \tan(u) du \right) = \frac{1}{2} dd^c (\ln \sec(t))$$

with t the coordinate on the interval factor. $\tau(s)$ has the additional property that $\tau_*(\bar{a}(s)\partial_s) = \frac{1}{2} \sin(2t)\partial_t$ and so we have an induced biholomorphism

$$(2.13) \quad T : (S^{2n-3} \times (0, L), \bar{\omega}) \rightarrow \left(S^{2n-3} \times \left(0, \frac{\pi}{2}\right), \frac{1}{2} \omega_{FS} \right),$$

$$T(x, s) = (x, \tau(s)) = (x, \arctan(\phi(s))),$$

which extends over the endpoints in the obvious way to give an automorphism of $\mathbb{P}^{n-1}$:

$$\begin{aligned} T : ((S^{2n-3} \times (0, L)) \cup \{0\} \cup (\mathbb{P}^{n-2} \times \{L\}), \bar{\omega}) \\ \rightarrow \left(\left(S^{2n-3} \times \left(0, \frac{\pi}{2}\right) \right) \cup \{0\} \cup \left(\mathbb{P}^{n-2} \times \left\{ \frac{\pi}{2} \right\} \right), \frac{1}{2} \omega_{FS} \right). \end{aligned}$$

Notice that $T^*t = \tau(s) = \arctan(\phi(s))$ and that

$$\begin{aligned} T^* \left(\frac{1}{2} \omega_{FS} \right) &= dd^c \left(\frac{1}{2} \int_{\frac{\pi}{4}}^{\tau(s)} \tan(u) du \right) = \frac{1}{2} dd^c (\ln \sec(\tau(s))) \\ &= \frac{1}{4} dd^c \left(\ln \left(1 + \left(e^{\int_{\frac{\pi}{2}}^s \frac{du}{\bar{a}(u)}} \right)^2 \right) \right) = \frac{1}{4} dd^c (\ln [1 + (\phi(s))^2]). \end{aligned}$$

Now $\bar{\omega}$ lies in a positive multiple of the Kähler class of $T^* \left(\frac{1}{2} \omega_{FS} \right)$, and so there exists a smooth real-valued function $\varphi : \mathbb{P}^{n-1} \rightarrow \mathbb{R}$, defined up to addition of a constant, and $c > 0$ such that $\bar{\omega} - cT^* \left(\frac{1}{2} \omega_{FS} \right) = i\partial\bar{\partial}\varphi$. In light of (2.11), we see that $c = \bar{b}(L)^2$. On $S^{2n-3} \times (0, L)$, we can write

$$\begin{aligned} (2.14) \quad \frac{1}{2} dd^c \varphi &= i\partial\bar{\partial}\varphi = \bar{\omega} - \bar{b}(L)^2 T^* \left(\frac{1}{2} \omega_{FS} \right) \\ &= \frac{1}{2} dd^c \left(\int_{\frac{L}{2}}^s \frac{\bar{b}(u)^2}{\bar{a}(u)} du - \frac{\bar{b}(L)^2}{2} \ln \left(1 + e^{2 \int_{\frac{L}{2}}^s \frac{du}{\bar{a}(u)}} \right) \right). \end{aligned}$$

Using the expansions of (2.9), it is easy to show that the function inside the parentheses on the right-hand side has a C^2 -continuation over $s = 0$, i.e., the function and its first and second derivative with respect to s extend as continuous functions over $s = 0$. The function also admits a C^2 -continuation over $s = L$. To see this, again use the expansions of (2.9) and rewrite the function in the following way:

$$\begin{aligned} \int_{\frac{L}{2}}^s \frac{\bar{b}(u)^2}{\bar{a}(u)} du - \frac{\bar{b}(L)^2}{2} \ln \left(1 + e^{2 \int_{\frac{L}{2}}^s \frac{du}{\bar{a}(u)}} \right) \\ = \int_{\frac{L}{2}}^s \left(\frac{\bar{b}(u)^2 - \bar{b}(L)^2}{\bar{a}(u)} \right) du - \frac{\bar{b}(L)^2}{2} \ln \left(1 + e^{-2 \int_{\frac{L}{2}}^s \frac{du}{\bar{a}(u)}} \right). \end{aligned}$$

In addition, its first derivative with respect to s vanishes when $s = 0, L$, and so the gradient and Laplacian of this function both extend as continuous functions to the whole of $\mathbb{P}^{n-1}$. With the Laplacian of this function vanishing, an integration by parts argument now implies that

$$(2.15) \quad \varphi - 2 \int_{\frac{L}{2}}^s \frac{\bar{b}(u)^2}{\bar{a}(u)} du + \bar{b}(L)^2 \ln \left(1 + e^{2 \int_{\frac{L}{2}}^s \frac{du}{\bar{a}(u)}} \right) \quad \text{is constant.}$$

In particular,

$$2 \int_{\frac{L}{2}}^s \frac{\bar{b}(u)^2}{\bar{a}(u)} du - \bar{b}(L)^2 \ln \left(1 + e^{2 \int_{\frac{L}{2}}^s \frac{du}{\bar{a}(u)}} \right)$$

is smooth on $\mathbb{P}^{n-1}$ because φ is.

We next define a biholomorphism

$$\Psi : \left(S^{2n-3} \times \left(0, \frac{\pi}{2}\right) \right) \cup \{0\} \rightarrow \{[w_1 : w_2 : \dots : w_n] \in \mathbb{P}^{n-1} \mid w_1 \neq 0\}$$

in the following way. Choose an automorphism of the round Sasaki structures

$$\lambda : S^{2n-3} \rightarrow \{z = (z_1, \dots, z_{n-1}) \in \mathbb{C}^{n-1} \mid |z| = 1\}$$

and set

$$\begin{cases} \Psi(0) = [1 : 0 : \dots : 0], \\ \Psi(x, t) = [1 : z_1 : \dots : z_{n-1}] = [1 : \lambda(x) \cdot \tan(t)] \quad \text{for } t > 0 \text{ and } x \in S^{2n-3}. \end{cases}$$

Note that $\psi : (0, \frac{\pi}{2}) \rightarrow (0, \infty)$, $\psi(t) := \tan(t)$, is a diffeomorphism so that Ψ is one also. Moreover, $\psi(t)$ is the unique solution of the ODE

$$\begin{cases} \frac{1}{2} \sin(2t) \psi'(s) = \psi(t), \\ \psi(\frac{\pi}{4}) = 1, \end{cases}$$

hence Ψ in addition satisfies $\Psi_* (\frac{1}{2} \sin(2t) \partial_t) = r \partial_r$, where $r(z)^2 = \sum_{i=1}^{n-1} |z_i|^2$. Thus, Ψ is indeed a biholomorphism. Notice by construction that $\Psi^* r = \psi \circ t = \tan(t)$ and so in light of (2.12), Ψ extends in the obvious way to a holomorphic isometry

$$\Psi : \left(S^{2n-3} \times \left(0, \frac{\pi}{2} \right) \right) \cup \{0\} \cup \left(\mathbb{P}^{n-2} \times \left\{ \frac{\pi}{2} \right\} \right) \rightarrow \{[w_1 : \dots : w_n] \in \mathbb{P}^{n-1}\}$$

of $\mathbb{P}^{n-1}$ with respect to the Fubini-Study metric.

Recall that $\Psi^* r = \tan(t)$ and that $T^* t = \tau(s) = \arctan(\phi(s))$. Set

$$\Phi := \Phi = \Psi \circ T : (S^{2n-3} \times (0, L)) \cup \{0\} \cup (\mathbb{P}^{n-2} \times \{L\}) \rightarrow \{[w_1 : w_2 : \dots : w_n] \in \mathbb{P}^{n-1}\}.$$

Explicitly, Φ is given by

$$(2.16) \quad \begin{cases} \Phi(0) = [1 : 0 : \dots : 0], \\ \Phi(x, s) = \left[1 : \lambda(x) \cdot \phi(s) \right] = \left[1 : \lambda(x) \cdot e^{\int_{\frac{\pi}{4}}^s \frac{du}{a(u)}} \right] \quad \text{for } s > 0 \text{ and } x \in S^{2n-3}. \end{cases}$$

Since $\Phi^* r = \phi(s)$, we see directly that $\Phi^* \tilde{s} = s$. Thus, in light of (2.15), we find that

$$(2.17) \quad i\partial\bar{\partial}(\varphi - \Phi^* \tilde{\varphi}(\tilde{s})) = i\partial\bar{\partial}(\varphi - \tilde{\varphi}(s)) = 0,$$

and so $\tilde{\varphi}$ is smooth, as required. $\square$

The main result of this subsection is the following.

Proposition 2.17. *Let $n \geq 3$.*

(1) *Let $\bar{g}$ be a Kähler metric on $\mathbb{P}^{n-1}$ with Kähler form $\bar{\omega}$ invariant under the $U(n-1)$ -action defined by matrix multiplication on the last $(n-1)$ -homogeneous coordinates on $\mathbb{P}^{n-1}$. Then there exists $L > 0$, a smooth function $s : \mathbb{P}^{n-1} \rightarrow [0, L]$, and smooth functions $\bar{a}, \bar{b} : [0, L] \rightarrow [0, \infty)$ with $\bar{a}(s) = \bar{b}(s)\bar{b}'(s)$ satisfying (2.7) and (2.8) such that $\bar{g}$ can be written as a doubly warped product*

$$\bar{g} = ds^2 + \bar{a}^2(s)\eta^2 + \bar{b}^2(s)g^T$$

on $S^{2n-3} \times (0, L)$. Here, (S^{2n-3}, g^T, η) is the standard round Sasaki structure on S^{2n-3} as described in Example 2.9.

(2) *Let $\hat{\varphi} : \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{R}$ be as in Lemma 2.16 and let $\hat{g}$ be Kähler cone metric on $\mathbb{C}^n \setminus \{0\}$ defined by the radial function $\hat{r}(w) = |w|^{b(L)^2} e^{\hat{\varphi}(w)}$ (cf. Definition 2.7). Let $\hat{\omega}^T$ denote the tranverse Kähler form on $\mathbb{P}^{n-1}$. Then:*

(a) $\hat{g}$ is invariant under the $U(1) \times U(n-1)$ -action on $\mathbb{C}^n$ defined by

$$U(1) \times U(n-1) \ni (\alpha, A) \cdot (z, w) \in \mathbb{C} \times \mathbb{C}^{n-1} \mapsto (\alpha \cdot z, Aw).$$

(b) The induced Sasaki metric h on $\{\hat{r} = 1\} \cong S^{2n-1}$ satisfies

$$(2.18) \quad \text{diam}(\mathbb{P}^{n-1}, \hat{g}^T) \leq \text{diam}(S^{2n-1}, h) \leq \text{diam}(\mathbb{P}^{n-1}, \hat{g}^T) + \pi \bar{b}(L)^2$$

and

$$(2.19) \quad \text{vol}_h(S^{2n-1}) \leq 2\pi \bar{b}(L)^2 \text{vol}_{\hat{g}^T}(\mathbb{P}^{n-1}).$$

(c) $\hat{\omega}^T$ depends smoothly on $\bar{g}$ and there exists a biholomorphism Φ such that $\hat{\omega}^T = (\Phi^{-1})^* \bar{\omega}$.

Remark 2.18. Since $\hat{\varphi}$ is smooth and invariant under the diagonal $\mathbb{C}^*$ -action on $\mathbb{C}^n \setminus \{0\}$ by Lemma 2.16, it descends to a well-defined smooth real-valued function φ on projective space. The fact that $\hat{r}$ does indeed define the radial function of a Kähler cone metric on $\mathbb{C}^n \setminus \{0\}$ then follows from item (2c) of the proposition.

Remark 2.19. A version of Proposition 2.17 also holds when $n = 2$. In this case, the Kähler metric $\bar{g}$ is a warped product on $(0, L) \times S^1$ of the form

$$\bar{g} = ds^2 + \bar{a}^2(s)\eta^2$$

and $\bar{a}(s)$ satisfies (2.7). We define another function $\bar{b}(s) := \sqrt{2 \int_0^s a(\tau) d\tau}$. Then an induction argument shows that $\bar{b}(s)$ satisfies (2.8). With this, the proof of Lemmas 2.11, 2.14, 2.15, and 2.16, as well as that of Proposition 2.17, can be carried through. We leave the details to the interested reader.

Proof. (1) We begin by showing that $\bar{g}$ takes the form of a doubly warped product as stated. Many of the ideas we use come from [16].

First recall from Example 2.9 the round Sasaki structure on S^{2n-3} with contact one form η and transverse Kähler form $\omega^T = \frac{1}{2}\omega_{FS}$, one half of the Fubini-Study metric g_{FS} on $\mathbb{P}^{n-2}$ normalised so that $\text{Ric}(g_{FS}) = (n-1)g_{FS}$. Let $z = [z_1 : \dots : z_{n-1}]$ denote homogeneous coordinates on $\mathbb{P}^{n-1}$. The fact that $\bar{g}$ is invariant under the stipulated action means that the restriction of $\bar{g}$ to the open set $\{z_0 \neq 0\} \subseteq \mathbb{P}^{n-1}$ is invariant under the standard $U(n-1)$ -action on the holomorphic coordinates $(z_1, \dots, z_{n-1})$ induced from the standard coordinate chart covering the aforesaid open set. Let $|z| =: r : \mathbb{C}^{n-1} \rightarrow \mathbb{R}$ and write $r^2 = e^t$. Being $U(n-1)$ -invariant, we know that $\bar{\omega} = \frac{i}{2}\partial\bar{\partial}\Phi(t)$ for $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ a given smooth function. Thus, recalling that $d^c := i(\bar{\partial} - \partial)$ and $\eta = d^c \log(r) = \frac{1}{2}d^c t$, we can write

$$\bar{\omega} = \frac{i}{2}\partial\bar{\partial}\Phi(t) = \frac{1}{4}dd^c\Phi(t) = \Phi'(t)\frac{1}{4}dd^c t + \Phi''(t)\frac{dt}{2} \wedge \frac{d^c t}{2} = \mu(t)\omega^T + \mu'(t)\frac{dt}{2} \wedge \eta,$$

where $\mu(t) := \Phi'(t)$ is a smooth function on $(-\infty, \infty)$.

In defining a metric, we clearly have that both $\mu(t), \mu'(t) > 0$. We claim that $\bar{\omega}$ is a doubly warped product Kähler metric of the desired form with $L = \text{diam}(\mathbb{P}^{n-1}, \bar{g})$. To see this, set

$$s(t) := \frac{1}{2} \int_{-\infty}^t \sqrt{\mu'(\tau)} d\tau \quad \text{for all } t \in \mathbb{R}.$$

Then since ω is in particular rotationally symmetric, the Euclidean radial line emerging from the origin is also a minimizing geodesic. Hence $s(t)$ is well-defined as it is the distance from the origin to any given $z \in \mathbb{C}^{n-1}$ with $t = 2 \ln |z|$.

Next observe that $s'(t) = \frac{\sqrt{\mu'(t)}}{2} > 0$ and so t can be written as a function of s . Let

$$s_1 := \lim_{t \rightarrow -\infty} s(t) \quad \text{and} \quad s_2 := \lim_{t \rightarrow \infty} s(t).$$

Then $\bar{g}$ is a warped product metric over the interval (s_1, s_2) with

$$\bar{a}(s) := \sqrt{\mu'(t(s))} \quad \text{and} \quad \bar{b}(s) := \sqrt{\mu(t(s))}.$$

We claim that it is of the desired form. Indeed, for all $s \in (s_1, s_2)$, we have that

$$\bar{b}(s)\bar{b}'(s) = \frac{\mu'(t(s))}{2} \frac{dt}{ds} = \frac{\mu'(t(s))}{2} \frac{2}{\sqrt{\mu'(t(s))}} = \bar{a}(s)$$

and

$$\bar{\omega} = \mu(t)\omega^T + \mu'(t)\frac{dt}{2} \wedge \eta = \bar{b}(s)^2\omega^T + \bar{a}(s)ds \wedge \eta.$$

By the completeness of the metric, we see that

$$s_1 = \lim_{t \rightarrow -\infty} s(t) = 0 \quad \text{and} \quad s_2 = \lim_{t \rightarrow \infty} s(t) = \text{diam}(\mathbb{P}^{n-1}, \bar{g}) = L,$$

and by the arguments as laid out in Example 2.13 that (2.7) and (2.8) hold true, as required.

We now prove the remaining items of the proposition.

(2a) Recall the definition of $\hat{s}$ from Lemma 2.14. Since $\hat{\varphi}$ depends only on $\hat{s}$ which, by its very definition, is invariant under the action prescribed, it is clear that $\hat{\varphi}$, and hence $\hat{r}$, is also.

(2b) There exists a Riemannian submersion $p : (S^{2n-1}, h) \mapsto (\mathbb{P}^{n-1}, \hat{g}^T)$ which is an S^1 -bundle with S^1 -fiber parametrised by the flow of the Reeb vector field $\frac{\xi}{b(L)^2}$. Moreover, the Sasaki metric h on $\{\hat{r} = 1\}$ is given by

$$h = (\bar{b}(L)^2\eta + i(\bar{\partial} - \partial)\hat{\varphi}) \otimes (\bar{b}(L)^2\eta + i(\bar{\partial} - \partial)\hat{\varphi}) + p^*\hat{g}^T,$$

where η and ξ are the standard contact form and Reeb vector field of the flat Kähler cone $\mathbb{C}^n \setminus \{0\}$. Using the fact that

$$[i(\bar{\partial} - \partial)\hat{\varphi}](\xi) = d^c\hat{\varphi}(\xi) = -\mathcal{L}_{J\xi}\hat{\varphi} - d^c(\xi \cdot \hat{\varphi}) = 0$$

because $\hat{\varphi}$ is invariant under the diagonal $\mathbb{C}^*$ -action generated by ξ and $J\xi$, we see that the length of the S^1 -fiber of p is bounded above by $2\pi\bar{b}(L)^2$. (2.18) thus follows. We next apply the coarea formula to p to see that (2.19) holds.

(2c) We know that in homogeneous coordinates $[w_1 : \dots : w_n]$ on $\mathbb{P}^{n-1}$ induced from holomorphic coordinates $(w_1, \dots, w_n)$ on the ambient $\mathbb{C}^n$, $\hat{\omega}^T$ is given by

$$(2.20) \quad \hat{\omega}^T = \frac{1}{2}dd^c \log \hat{r} = \frac{\bar{b}(L)^2}{2}\omega_{FS} + i\partial\bar{\partial}\hat{\varphi},$$

where $\tilde{\varphi}$ is as in the proof of Lemma 2.16. As a holomorphic isometry, we then see that

$$\begin{aligned}\Psi^*\hat{\omega}^T &= \frac{\bar{b}(L)^2}{2}\omega_{FS} + i\partial\bar{\partial}(\Psi^*\tilde{\varphi}) \\ &= (T^{-1})^*\left(\bar{b}(L)^2T^*\left(\frac{1}{2}\omega_{FS}\right) + i\partial\bar{\partial}[(\Psi \circ T)^*\tilde{\varphi}]\right) \\ &= (T^{-1})^*\left(\bar{b}(L)^2T^*\left(\frac{1}{2}\omega_{FS}\right) + i\partial\bar{\partial}\varphi\right) = (T^{-1})^*\bar{\omega},\end{aligned}$$

where we have used (2.17) followed by (2.14) in the last two equalities. Hence $\hat{\omega}^T = (\Phi^{-1})^*\bar{\omega}$, where $\Phi = \Psi \circ T$ as in (2.16).

Finally, note that by (2.11), $\bar{b}(L)$ is determined by, and hence depends smoothly upon, $\text{vol}_{\bar{g}}(\mathbb{P}^{n-1})$. Hence in light of (2.20), showing that $\hat{\omega}^T$ depends smoothly on $\hat{g}$ comes down to showing that $\tilde{\varphi}$, or equivalently $\Psi^*\tilde{\varphi}$, depends smoothly on $\bar{g}$ (as a function on the interval $[0, \frac{\pi}{2}]$). To this end, observe from (2.14) that

$$i\partial\bar{\partial}(\Psi^*\tilde{\varphi}) = i\partial\bar{\partial}((T^{-1})^*\varphi) = (T^{-1})^*\bar{\omega} - \frac{\bar{b}(L)^2}{2}\omega_{FS}.$$

Contracting this equation with g_{FS} , the Fubini-Study metric, we see that, after imposing that $\varphi(\frac{L}{2}) = 0$, $\Psi^*\tilde{\varphi}$ is the unique solution of the following elliptic PDE:

$$(2.21) \quad \begin{cases} \Delta_{g_{FS}} u = \text{tr}_{g_{FS}}((T^{-1})^*\bar{\omega}) - \frac{n\bar{b}(L)^2}{2}, \\ u(\frac{\pi}{4}) = 0. \end{cases}$$

Now, from (2.13), we read that the map T^{-1} is determined by the map

$$\tau^{-1} = \phi^{-1} \circ \tan : \left(0, \frac{\pi}{2}\right) \rightarrow \mathbb{R}$$

and from its very definition we see that ϕ is defined in terms of data that is determined by $\bar{g}$, namely L which is equal to $\text{diam}(\mathbb{P}^{n-1}, \bar{g})$, s which is the distance from the point $[1 : 0 : \dots : 0] \in \mathbb{P}^{n-1}$, and $\bar{a}(s)^2$ which is the norm $[\bar{g}(\xi, \xi)](s)$ of the Reeb vector field ξ of the round Sasaki structure on S^{2n-3} . Thus, we deduce that ϕ^{-1} depends smoothly on $\bar{g}$. It is now clear that the right-hand side of (2.21), and as a result its unique solution u , depends smoothly on $\bar{g}$. This concludes the proof of item (2c). $\square$

2.4. Positive curvature conditions. In this subsection, we introduce several notions of positive curvature and their relation on Kähler manifolds. Throughout, we adopt the following convention of the curvature tensor (see also (A.1)):

$$\hat{R}(X, Y)Z = \hat{\nabla}_X(\hat{\nabla}_Y Z) - \hat{\nabla}_Y(\hat{\nabla}_X Z) - \hat{\nabla}_{[X, Y]}Z.$$

Let us recall the notion of the curvature operator and its lower bound on $(1, 1)$ -forms on a Kähler manifold (see [19, 26, 82]).

Definition 2.20. Fix any $p \in M$, we say that a Kähler manifold has curvature operator Rm strictly greater than (or bounded below by) $2\lambda \in \mathbb{R}$ on $(1, 1)$ -forms at p if for any nonzero $(1, 1)$ -form $i\xi_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}}$ at p ,

$$(2.22) \quad -\hat{R}_{\alpha\bar{\beta}\gamma\bar{\delta}}\xi^{\alpha\bar{\beta}}\overline{\xi^{\gamma\bar{\delta}}} > (\geq \text{ resp.}) 2\lambda(\hat{g}_{\alpha\bar{\beta}}\hat{g}_{\gamma\bar{\delta}} + \hat{g}_{\alpha\bar{\delta}}\hat{g}_{\gamma\bar{\beta}})\xi^{\alpha\bar{\beta}}\overline{\xi^{\gamma\bar{\delta}}},$$

where $\xi^{\alpha\bar{\beta}} = g^{\alpha\bar{\epsilon}} g^{\eta\bar{\beta}} \xi_{\eta\bar{\epsilon}}$.

Note that the negative sign before $\hat{R}$ in (2.22) (as well as (2.23) below) is due to the convention we adopt in (A.1). We may reformulate the condition above in a simpler manner in terms of curvature operator on *real* $(1, 1)$ -forms.

Definition 2.21 (Real $(1, 1)$ -form). Let (M, J) be a complex manifold of dimension n with complex structure J . A *real* $(1, 1)$ -form σ on M is a real two-form σ on M satisfying either one of the following two equivalent conditions:

$$(1) \sigma(J(\cdot), J(\cdot)) = \sigma(\cdot, \cdot);$$

(2) In local holomorphic coordinates $(z_1, \dots, z_n)$ on M , the complex bilinear extension $\sigma_{\mathbb{C}}$ of σ to $(TM \otimes \mathbb{C})^{\otimes 2}$ takes the form $\sigma_{\mathbb{C}} = i u_{\alpha\bar{\beta}} dz^{\alpha} \wedge dz^{\bar{\beta}}$, where $(u_{\alpha\bar{\beta}})_{\alpha\beta}$ is hermitian, i.e., $\overline{u_{\alpha\bar{\beta}}} = u_{\beta\bar{\alpha}}$ for all α and β .

By the symmetry of the curvature tensor, it induces a symmetric linear operator $\text{Rm} : \Lambda^2(M)_p \rightarrow \Lambda^2(M)_p$ such that for any $u \in \Lambda^2(M)_p$,

$$\text{Rm}(u)_{ij} = \hat{R}_{ijkl} g^{ka} g^{lb} u_{ab},$$

where $\Lambda^2(M)_p$ is the space of real 2-forms at p . We denote the space of real $(1, 1)$ -forms at p by $\Lambda_{\mathbb{R}}^{1,1}(M)_p \subseteq \Lambda^2(M)_p$. It can be seen from the Kählerity of the metric that $\text{Rm}(\Lambda^2(M)_p) \subseteq \Lambda_{\mathbb{R}}^{1,1}(M)_p$ and $\text{Rm} \equiv 0$ on the orthogonal complement of $\Lambda_{\mathbb{R}}^{1,1}(M)_p$ in $\Lambda^2(M)_p$ (see [19, 26]). In particular, Rm has non-trivial kernel in complex dimension $n \geq 2$.

Definition 2.22. A complex n -dimensional Kähler manifold is said to have *curvature operator* Rm *strictly greater than* (or *bounded below by*) $2\lambda \in \mathbb{R}$ on real $(1, 1)$ -forms at p if for any nonzero real $(1, 1)$ -form $i u_{\alpha\bar{\beta}} dz^{\alpha} \wedge dz^{\bar{\beta}}$ at p ,

$$(2.23) \quad -\hat{R}_{\alpha\bar{\beta}\gamma\bar{\delta}} u^{\alpha\bar{\beta}} u^{\gamma\bar{\delta}} > (\geq \text{ resp.}) 2\lambda(\hat{g}_{\alpha\bar{\beta}} \hat{g}_{\gamma\bar{\delta}} + \hat{g}_{\alpha\bar{\delta}} \hat{g}_{\gamma\bar{\beta}}) u^{\alpha\bar{\beta}} u^{\gamma\bar{\delta}},$$

where $u^{\alpha\bar{\beta}} = g^{\alpha\bar{\epsilon}} g^{\eta\bar{\beta}} u_{\eta\bar{\epsilon}}$.

It is not difficult to see that Condition (2.22) implies Condition (2.23). Moreover, by considering the following decomposition of $(1, 1)$ -forms

$$\xi_{\alpha\bar{\beta}} = \frac{\xi_{\alpha\bar{\beta}} + \overline{\xi_{\beta\bar{\alpha}}}}{2} + \frac{\xi_{\alpha\bar{\beta}} - \overline{\xi_{\beta\bar{\alpha}}}}{2} = a_{\alpha\bar{\beta}} + i b_{\alpha\bar{\beta}},$$

where $a_{\alpha\bar{\beta}} := \frac{1}{2}(\xi_{\alpha\bar{\beta}} + \overline{\xi_{\beta\bar{\alpha}}})$ and $b_{\alpha\bar{\beta}} := \frac{1}{2i}(\xi_{\alpha\bar{\beta}} - \overline{\xi_{\beta\bar{\alpha}}})$, we have $\overline{a_{\alpha\bar{\beta}}} = a_{\beta\bar{\alpha}}$, $\overline{b_{\alpha\bar{\beta}}} = b_{\beta\bar{\alpha}}$, and so Condition 2.23 also implies Condition 2.22, i.e., these two conditions are equivalent.

By abuse of notation, we still denote the optimal λ in (2.23) by λ , namely

$$(2.24) \quad \lambda(p) := \inf_{i u_{\alpha\bar{\beta}} dz^{\alpha} \wedge dz^{\bar{\beta}} \in \Lambda_{\mathbb{R}}^{1,1}(M)_p \setminus \{0\}} \frac{-\sum_{\alpha,\beta,\gamma,\eta=1}^n \hat{R}_{\alpha\bar{\beta}\gamma\bar{\eta}} u^{\alpha\bar{\beta}} u^{\gamma\bar{\eta}}}{\sum_{\alpha,\beta,\gamma,\eta=1}^n 2(\hat{g}_{\alpha\bar{\beta}} \hat{g}_{\gamma\bar{\eta}} + \hat{g}_{\alpha\bar{\eta}} \hat{g}_{\gamma\bar{\beta}}) u^{\alpha\bar{\beta}} u^{\gamma\bar{\eta}}}.$$

As $\text{Rm}(\Lambda^2(M)_p) \subseteq \Lambda_{\mathbb{R}}^{1,1}(M)_p$ and $\text{Rm} \equiv 0$ on the orthogonal complement of $\Lambda_{\mathbb{R}}^{1,1}(M)_p$, nonnegative curvature operator on $(1, 1)$ -forms implies nonnegative curvature operator. It is also well known that strictly positive curvature operator on $(1, 1)$ -forms implies strictly

positive sectional curvature [26, 82]. Indeed, let X, Y be two linearly independent real vectors in $T_p M$ and let v and w be the corresponding complex vectors in $T_p^{1,0} M$ given by

$$v = \frac{1}{2}(X - iJ_0 X) \quad \text{and} \quad w = \frac{1}{2}(Y - iJ_0 Y).$$

Then the sectional curvature $K(\sigma)$ of the 2-plane σ generated by X and Y is equal to (cf. [16, 82])

$$K(\sigma) = \frac{-\sum_{\alpha, \beta, \gamma, \delta=1}^n \hat{R}_{\alpha\bar{\beta}\gamma\bar{\delta}} \left(v^\alpha \bar{w}^\beta - w^\alpha \bar{v}^\beta \right) \left(w^\gamma \bar{v}^\delta - v^\gamma \bar{w}^\delta \right)}{\sum_{\alpha, \beta, \gamma, \delta=1}^n g_{\alpha\bar{\delta}} g_{\gamma\bar{\beta}} \left[\left(v^\alpha \bar{w}^\beta - w^\alpha \bar{v}^\beta \right) \left(w^\gamma \bar{v}^\delta - v^\gamma \bar{w}^\delta \right) - (v^\alpha w^\gamma - w^\alpha v^\gamma) \left(\bar{v}^\beta \bar{w}^\delta - \bar{w}^\beta \bar{v}^\delta \right) \right]}.$$

Hence by taking $u^{\alpha\bar{\beta}} = i \left(v^\alpha \bar{w}^\beta - w^\alpha \bar{v}^\beta \right)$ and $\lambda = 0$ in (2.23), strictly positive curvature operator on $(1, 1)$ -forms implies strictly positive sectional curvature and in turn strictly positive bisectional curvature. In the special case that M is $U(n)$ -invariant with fixed point p , then for any orthonormal pair X and Y in $T_p M$, the sectional curvature of $\sigma = \text{span}\{X, Y\}$ is given by

$$(2.25) \quad K(\sigma) = \lambda (1 + 3g(X, JY)^2),$$

where λ is as in (2.24). Recall from [63, 64, 84] that the bisectional curvature $BK \geq 2\lambda$ at $p \in M$ if for all holomorphic vectors $v, w \in T_p^{1,0} M$,

$$-\hat{R}_{\alpha\bar{\beta}\gamma\bar{\delta}} v^\alpha \bar{v}^\beta w^\gamma \bar{w}^\delta \geq 2\lambda (\hat{g}_{\alpha\bar{\beta}} \hat{g}_{\gamma\bar{\delta}} + \hat{g}_{\alpha\bar{\delta}} \hat{g}_{\gamma\bar{\beta}}) v^\alpha \bar{v}^\beta w^\gamma \bar{w}^\delta.$$

It is not difficult to see that $\text{Rm} \geq 2\lambda$ on $(1, 1)$ -forms implies that $BK \geq 2\lambda$. More precisely:

Lemma 2.23. *Let M be a Kähler manifold and let $p \in M$. Suppose that $\text{Rm} \geq 2\lambda$ on $(1, 1)$ -forms at p , where $\lambda \in \mathbb{R}$. Then the holomorphic bisectional curvature is bounded below by 2λ at p .*

Remark 2.24. The converse of this lemma is not true in general [88, Theorem 4], i.e., a bisectional curvature lower bound is a weaker condition than a lower bound on the curvature operator on $(1, 1)$ -forms.

Proof of Lemma 2.23. For any $v, w \in T_p^{1,0} M$, let $u^{\alpha\bar{\beta}} = i \left(v^\alpha \bar{w}^\beta - w^\alpha \bar{v}^\beta \right)$. Then by (2.23), we have that

$$\begin{aligned} -2\hat{R}(v, \bar{w}, w, \bar{v}) + \hat{R}(v, \bar{w}, v, \bar{w}) + \hat{R}(\bar{v}, w, \bar{v}, w) \geq \\ -2\lambda (\hat{g}_{\alpha\bar{\beta}} \hat{g}_{\gamma\bar{\delta}} + \hat{g}_{\alpha\bar{\delta}} \hat{g}_{\gamma\bar{\beta}}) (v^\alpha \bar{w}^\beta v^\gamma \bar{w}^\delta + w^\alpha \bar{v}^\beta w^\gamma \bar{v}^\delta - v^\alpha \bar{w}^\beta w^\gamma \bar{v}^\delta - w^\alpha \bar{v}^\beta v^\gamma \bar{w}^\delta). \end{aligned}$$

Similarly, replacing w by iw , substituting $u^{\alpha\bar{\beta}} = i \left(v^\alpha \bar{iw}^\beta - iw^\alpha \bar{v}^\beta \right)$ into (2.23), and adding the resulting expression to the inequality above, we derive that

$$-4\hat{R}(v, \bar{v}, w, \bar{w}) = -4\hat{R}(v, \bar{w}, w, \bar{v}) \geq 8\lambda (\hat{g}_{\alpha\bar{\beta}} \hat{g}_{\gamma\bar{\delta}} + \hat{g}_{\alpha\bar{\delta}} \hat{g}_{\gamma\bar{\beta}}) v^\alpha \bar{v}^\beta w^\gamma \bar{w}^\delta.$$

Here, we have used the symmetry $\hat{R}_{\alpha\bar{\delta}\gamma\bar{\beta}} = \hat{R}_{\alpha\bar{\beta}\gamma\bar{\delta}}$ of a Kähler curvature operator (see for instance [30, Lemma 2.10]). $\square$

Henceforth, we consider a complex $(n-1)$ -dimensional Kähler manifold of the form (2.10) and fix a special real orthonormal frame $\{\hat{e}_k\}_{k=1}^{2n-2}$ such that $\hat{e}_{2n-3} = \partial_r$ and for all $k = 1, \dots, n-1$, $J_0(\hat{e}_{2k-1}) = \hat{e}_{2k}$. We also let $\{\hat{E}_\alpha\}_{\alpha=1}^{n-1}$ and $\{\hat{E}_{\bar{\alpha}}\}_{\alpha=1}^{n-1}$ denote the corresponding holomorphic and anti-holomorphic frames, namely

$$\hat{E}_\alpha = \frac{1}{2}(\hat{e}_{2\alpha-1} - i\hat{e}_{2\alpha}), \quad \hat{E}_{\bar{\alpha}} = \frac{1}{2}(\hat{e}_{2\alpha-1} + i\hat{e}_{2\alpha}), \quad \alpha = 1, \dots, n-1.$$

We extend the curvature tensor complex linearly. Lengthy computations using Cartan's formulas (see (A.2) in Appendix A) give us that

$$\begin{aligned} \hat{R}_{\alpha\bar{\beta}\gamma\bar{\delta}} &= \frac{\tilde{R}_{\alpha\bar{\beta}\gamma\bar{\delta}}}{b(r)^2} + \frac{1}{2} \left(\frac{b'(r)}{b(r)} \right)^2 \delta_{\alpha\delta} \delta_{\beta\gamma} + \left(\frac{a(r)^2}{2b(r)^4} \right) \delta_{\alpha\beta} \delta_{\gamma\delta} \quad \text{if } 1 \leq \alpha, \beta, \gamma, \delta \leq n-2, \\ \hat{R}_{\alpha\bar{\beta}\gamma\bar{\delta}} &= \left(\frac{b''(r)}{2b(r)} \right) \delta_{\gamma, n-1} \delta_{\delta\alpha} \quad \text{if } 1 \leq \alpha \leq n-2, \beta = n-1, \\ \hat{R}_{\alpha\bar{\beta}\gamma\bar{\delta}} &= \frac{1}{2} \left[\left(\frac{b''(r)}{b(r)} \right) (\delta_{\gamma\delta} - \delta_{\gamma, n-1} \delta_{\delta, n-1}) + \left(\frac{a''(r)}{2a(r)} \right) \delta_{\gamma, n-1} \delta_{\delta, n-1} \right] \quad \text{if } \alpha = n-1, \beta = n-1, \end{aligned}$$

where $\tilde{R}$ is the curvature tensor of the tranverse metric g^T . With the above formulas, we study the curvature lower bound of $\hat{g}$.

Lemma 2.25. *In the above situation, for any nonnegative real number $\lambda \geq 0$ and $r \in (0, L)$, $\hat{g}$ has curvature operator strictly greater than 2λ on $(1, 1)$ -forms at $(r, w_0) \in (0, L) \times S^{2n-3}$ if and only if all of the following conditions are satisfied:*

- (1) $(1 - \lambda b^2(r) - (b'(r))^2) > 0$,
- (2) $-a''(r) > 4\lambda a(r)$,
- (3) $-b''(r) > \lambda b(r)$,
- (4) $\frac{2(n-1)(1 - \lambda b^2(r) - (b'(r))^2)}{(n-2)b^2(r)} \left(-\frac{a''(r)}{4a(r)} - \lambda \right) > \left(-\frac{b''(r)}{b(r)} - \lambda \right)^2$.

Proof. Let $(r, w_0) \in (0, L) \times S^{2n-3}$. By Lemma A.2, we have curvature operator strictly greater than 2λ on $(1, 1)$ -forms if and only if for any nonzero hermitian matrix $u^{\alpha\bar{\beta}}$,

$$\begin{aligned} 0 &< \frac{2(1 - \lambda b^2 - (b')^2)}{b^2} \left(\frac{1}{4} \left(\sum_{\alpha=1}^{n-2} u^{\alpha\bar{\alpha}} \right)^2 + \frac{1}{4} \sum_{\alpha, \beta=1}^{n-2} |u^{\alpha\bar{\beta}}|^2 \right) + \left(-\frac{a''}{4a} - \lambda \right) |u^{n-1\bar{n-1}}|^2 \\ &\quad + \left(-\frac{b''}{b} - \lambda \right) \sum_{\alpha=1}^{n-2} |u^{\alpha\bar{n-1}}|^2 + \left(-\frac{b''}{b} - \lambda \right) u^{n-1\bar{n-1}} \sum_{\alpha=1}^{n-2} u^{\alpha\bar{\alpha}}. \end{aligned}$$

Thus, by choosing u suitably, we see that the curvature lower bound implies Conditions (1)-(3). Furthermore, setting u to be the diagonal matrix $u^{\alpha\bar{\alpha}} = \frac{2}{n-2}$, $u^{\alpha\bar{n-1}} = 0 = u^{n-1\bar{\alpha}} = u^{\alpha\bar{\beta}}$ for $\alpha, \beta = 1, \dots, n-2$ with $\alpha \neq \beta$, and

$$u^{n-1\bar{n-1}} = -\frac{\left(\frac{b''}{b} + \lambda \right)}{\left(\frac{a''}{4a} + \lambda \right)},$$

gives us Condition (4).

Conversely, if Conditions (1)–(4) hold, then by the Cauchy-Schwarz inequality and completing the square, for any nonzero hermitian u , we have that

$$\begin{aligned}
0 &\leq \left[\frac{2(n-1)(1-\lambda b^2-(b')^2)}{(n-2)b^2} - \frac{\left(-\frac{b''(r)}{b(r)} - \lambda\right)^2}{\left(-\frac{a''}{4a} - \lambda\right)} \right] \frac{1}{4} \left(\sum_{\alpha=1}^{n-2} u^{\alpha\bar{\alpha}} \right)^2 \\
&\quad + \left(-\frac{a''}{4a} - \lambda \right) \left[u^{n-1\overline{n-1}} + \frac{\left(-\frac{b''(r)}{b(r)} - \lambda\right)}{2\left(-\frac{a''}{4a} - \lambda\right)} \sum_{\alpha=1}^{n-2} u^{\alpha\bar{\alpha}} \right]^2 + \left(-\frac{b''}{b} - \lambda \right) \sum_{\alpha=1}^{n-2} |u^{\alpha\overline{n-1}}|^2 \\
&\leq \frac{2(1-\lambda b^2-(b')^2)}{b^2} \left(\frac{1}{4} \left(\sum_{\alpha=1}^{n-2} u^{\alpha\bar{\alpha}} \right)^2 + \frac{1}{4} \sum_{\alpha,\beta=1}^{n-2} |u^{\alpha\bar{\beta}}|^2 \right) + \left(-\frac{a''}{4a} - \lambda \right) |u^{n-1\overline{n-1}}|^2 \\
&\quad + \left(-\frac{b''}{b} - \lambda \right) \sum_{\alpha=1}^{n-2} |u^{\alpha\overline{n-1}}|^2 + \left(-\frac{b''}{b} - \lambda \right) u^{n-1\overline{n-1}} \sum_{\alpha=1}^{n-2} u^{\alpha\bar{\alpha}}.
\end{aligned}$$

The expression on the right-hand side of this inequality is positive unless u is equal to 0. This completes the proof of the lemma. $\square$

We will only consider asymptotically conical expanding gradient Kähler-Ricci solitons [37, 44, 45] satisfying $\text{Rm} \geq 0$ in this article. This latter condition is equivalent to the link of the asymptotic cone satisfying $\text{Rm} \geq 1$. In this case, the potential function is a strictly convex function with quadratic growth, the underlying manifold is diffeomorphic to $\mathbb{R}^n$, and the level sets of the soliton potential function are diffeomorphic to S^{n-1} . We will consider asymptotically conical expanding gradient Kähler-Ricci solitons resulting from the following existence result by the second-named author and Deruelle.

Theorem 2.26 ([37, Theorems A & E]). *Let g_0 be a Kähler cone metric on $\mathbb{C}^n$ with radial function r such that $a \cdot r \partial_r$ is the Euler vector field, where $a \in (0, 1)$. Then there exists a unique expanding gradient Kähler-Ricci soliton g with soliton vector field $r \partial_r$ such that*

$$|\nabla_{g_0}^k (g - g_0)|_{g_0} = O(r^{-2-k}) \quad \text{for all } k \geq 0.$$

Moreover, if the induced metric on the complex space $\mathbb{P}^{n-1}$ has curvature operator on $(1, 1)$ -forms strictly greater than 2, then g has strictly positive curvature operator on $(1, 1)$ -forms.

Remark 2.27. The lower bound of the induced metric on the base space $\mathbb{P}^{n-1}$ should have curvature operator on $(1, 1)$ -forms strictly greater than 2 in the statement of [37, Theorem E] as written here. To account for this change, the constant a in [37, p.38, Proof of Theorem E] should be a function of t as in the proof of Theorem A.

Remark 2.28. In the standard complex coordinates $z_k = x_k + iy_k$ on $\mathbb{C}^n$, the expanding gradient Kähler-Ricci soliton from of Theorem 2.26 has soliton vector field given by $X = cx_\alpha \frac{\partial}{\partial x_\alpha} + cy_\alpha \frac{\partial}{\partial y_\alpha}$ for some $c < 0$. This has a unique critical point at $0 \in \mathbb{C}^n$. Moreover at 0, the Ricci curvature Ric_g of g has only one distinct eigenvalue, i.e., $\text{Ric}_g = \mu g$ for some $\mu \in \mathbb{R}$. To see this, we compute the Lie derivative $\mathcal{L}_X g$. Indeed, for $\alpha, \beta = 1, 2, \dots, n$, we

have that

$$\begin{aligned}
\mathcal{L}_X g \left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta} \right) &= g \left(\nabla_{\frac{\partial}{\partial x_\alpha}} X, \frac{\partial}{\partial x_\beta} \right) + g \left(\frac{\partial}{\partial x_\alpha}, \nabla_{\frac{\partial}{\partial x_\beta}} X \right) \\
&= cg \left(\delta_{\alpha\gamma} \frac{\partial}{\partial x_\gamma} + x_\gamma \nabla_{\frac{\partial}{\partial x_\alpha}} \frac{\partial}{\partial x_\gamma} + y_\gamma \nabla_{\frac{\partial}{\partial x_\alpha}} \frac{\partial}{\partial y_\gamma}, \frac{\partial}{\partial x_\beta} \right) \\
&\quad + cg \left(\frac{\partial}{\partial x_\alpha}, \delta_{\beta\gamma} \frac{\partial}{\partial x_\gamma} + x_\gamma \nabla_{\frac{\partial}{\partial x_\beta}} \frac{\partial}{\partial x_\gamma} + y_\gamma \nabla_{\frac{\partial}{\partial x_\beta}} \frac{\partial}{\partial y_\gamma} \right) \\
&= 2cg \left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta} \right)
\end{aligned}$$

when evaluated at 0, and so $\mathcal{L}_X g = 2cg$. Thus, by (1.1), we find that

$$\text{Ric}_g = -\frac{1}{2}\mathcal{L}_X g - \lambda g = -(c + \lambda)g \quad \text{at } 0.$$

3. DEFORMATION OF COMPLEX PROJECTIVE SPACE

The main goal of this section is to prove Proposition 3.3 which produces for each $n \geq 3$ a sequence of smooth $U(n-1)$ -invariant Kähler metrics on $\mathbb{P}^{n-1}$ with $\text{Rm} \geq 0$ on real $(1, 1)$ -forms everywhere, and $\text{Rm} \geq 2$ on $(1, 1)$ -forms outside of arbitrarily small subsets, which collapse to $[0, \frac{\pi}{2}]$ in the Gromov-Hausdorff sense. In the next section, we will improve this result to obtain a sequence of metrics that everywhere satisfy $\text{Rm} \geq 2$ on real $(1, 1)$ -forms. The existence of these metrics will imply the almost non-rigidity of diameter in Theorem B and serve in the proof of Theorem A as the complex base of the asymptotic cone of the expanding gradient Kähler-Ricci solitons that degenerate to our desired steady Kähler-Ricci solitons.

To motivate and sketch the ideas of the construction in Proposition 3.3, we demonstrate a singular example; a variation of the Fubini-Study metric. Upon substituting $L = \frac{\pi}{2}$, $a(r) = \frac{\sin(2r)}{2k}$, $b(r) = \frac{\sin(r)}{\sqrt{k}}$ into (2.4) to obtain the metric

$$(3.1) \quad h_k := dr^2 + \frac{\sin^2(2r)}{4k^2} \eta \otimes \eta + \frac{\sin^2(r)}{k} g^T \quad \text{on } \left(0, \frac{\pi}{2}\right) \times S^{2n-3},$$

where g^T is the transverse metric as in Example 2.13 with n replaced by $(n-1)$, it is clear that h_k collapses to $[0, \frac{\pi}{2}]$ as $k \rightarrow \infty$ in the Gromov-Hausdorff sense and satisfies $\text{Rm} \geq 2$ on real $(1, 1)$ -forms except for $r = 0$ and $r = \frac{\pi}{2}$, where a and b don't fulfill the smooth boundary conditions for any $k > 1$ (see Example 2.13). However, we will build our desired metrics by gluing these singular metrics with suitable expanding solitons near $r = 0$ to smooth out the conical singularities.

In preparation for the gluing construction, we recall some basic facts about expanding gradient Kähler-Ricci solitons, and in particular Cao's examples of such solitons. To this end, let (M^n, g, f, p) be an expanding gradient Ricci soliton with p a critical point of f . Then (M^n, g, f, p) satisfies

$$\text{Ric} + \lambda g = \nabla^2 f$$

for some $\lambda > 0$ and generates a canonical Ricci flow of expanding Ricci solitons defined by $g(t) := (2\lambda t)\phi_{t-\frac{1}{2\lambda}}^* g$, $f_t = \phi_{t-\frac{1}{2\lambda}}^* f$, $t \in (0, \infty)$, where $\{\phi_s\}_{s \in (-\frac{1}{2\lambda}, \infty)}$ is the one-parameter

family of diffeomorphisms generated by the time-dependent vector field $\frac{-1}{1+2\lambda s}\nabla f$ with ϕ_0 the identity, and

$$\text{Ric}(g(t)) + \frac{1}{2t}g(t) = \nabla^2 f_t.$$

For any $\alpha > 1$, Cao proved that there exists a unique smooth $U(n-1)$ -invariant expanding gradient Kähler-Ricci soliton $(\mathbb{C}^{n-1}, g_\alpha, f_\alpha, p_\alpha)$ with positive curvature operator on real $(1, 1)$ -forms and with $R(p_\alpha) = \max R = 1$ that is asymptotic to the following cone metric [16, 26]:

$$(3.2) \quad g_{\text{cone}, \alpha} = dr^2 + \frac{r^2}{\alpha^2} \eta \otimes \eta + \frac{r^2}{\alpha} g^T.$$

Under the radial coordinates, we can write the expanding soliton metric g_α as

$$(3.3) \quad g_\alpha = dr^2 + a_\alpha^2(r) \eta \otimes \eta + b_\alpha^2(r) g^T.$$

Let $g_\alpha(t)$ be the canonical Ricci flow associated to g_α . Since the expanding Ricci soliton is asymptotically conical, we know that the flow $g_\alpha(t)$ converges smoothly locally to the cone metric $g_{\text{cone}, \alpha}$ away from the tip $*$ of the cone as $t \rightarrow 0$ [79, c.f. Theorem 4.3.1]. In the following lemma, we show that this local convergence is uniform for all α in a compact subset $I \subset (1, \infty)$.

Lemma 3.1 (Uniform convergence to asymptotic cones). *For any compact subset $I \subset (1, \infty)$, $k \in \mathbb{N}$, $\varepsilon > 0$, and $D > 1$, there exists $t_0 > 0$ such that for all $0 < t < t_0$ and all $\alpha \in I$,*

$$\|\nabla^k(g_\alpha(t) - g_{\text{cone}, \alpha})\| \leq \varepsilon$$

on $B(, D) \setminus B(*, D^{-1})$, where the derivatives, norms, and metric balls are measured with respect to the cone metric $g_{\text{cone}, \alpha}$.*

Proof. Suppose that this was not the case. Then there exist $\varepsilon_0 > 0$, $D_0 > 0$, $k_0 \in \mathbb{N}$, and sequences $t_i \rightarrow 0$ and $\alpha_i \in I$ such that for the expanding Kähler-Ricci soliton $(\mathbb{C}^{n-1}, g_{\alpha_i}, f_i, p_i)$ and the canonical Ricci flow $g_{\alpha_i}(t)$, we have that

$$(3.4) \quad \|\nabla^{k_0}(g_{\alpha_i}(t_i) - g_{\text{cone}, \alpha_i})\| \geq \varepsilon_0$$

on $B(*, D_0) \setminus B(*, D_0^{-1})$. Recall that $R_{g_{\alpha_i}}(p_i) = 1$.

Assume that $g_{\alpha_i} = g_{\alpha_i}(C_i)$ for some positive constant $C_i > 0$. We claim that there exists a positive constant C such that $C_i < C$ and for all $t > 0$,

$$|\text{Rm}_{g_{\alpha_i}(t)}| \leq \frac{C}{t}.$$

To prove this, we assume the contrary. Then there is a sequence $C_i \rightarrow \infty$ such that $|\text{Rm}_{g_{\alpha_i}(t)}| \leq \frac{C_i}{t}$ and $R(p_i, C_i) = 1$. By the same limiting argument as in the proof of [61, Lemma 2.3], we can take a smooth sequential limit pointed at p_i of the flows $g_{\alpha_i}(t + C_i)$, $t \in (-C_i, \infty)$, to obtain a smooth Ricci flow for $t \in (-\infty, \infty)$ which is associated to a smooth steady Kähler-Ricci soliton $(\mathbb{C}^{n-1}, g_\infty, f_\infty, p_\infty)$. Since $I \subset (0, \infty)$ is compact, these solitons are uniformly non-collapsed in the sense that there is some $\kappa > 0$ such that $B_{g_{\alpha_i}}(p_i, r) \geq \kappa r^n$ for all $r > 0$, hence the limit steady Ricci soliton and the asymptotic cone are also κ -non-collapsed, and $R(p_\infty, 0) = 1$. By Perelman's curvature estimate for Ricci flows with nonnegative curvature operator (see for example [54, Corollary 45.1(b)] and also [69, Theorem 2] for the Kähler case), this implies that the steady Ricci soliton is flat. But

this contradicts the fact that $R(p_\infty, 0) = 1$ and so we have the claim. By Shi's estimates [78], it follows that

$$(3.5) \quad |\nabla^k \text{Rm}_{g_{\alpha_i}(t)}| \leq \frac{C_k}{t^{\frac{k+2}{2}}}$$

for some $C_k > 0$ depending k and uniform for all i .

By the curvature bounds (3.5), we may assume, after taking a subsequence if necessary, that $\alpha_i \rightarrow \alpha_\infty \in I$ and $g_{\alpha_i}(t)$ converges smoothly locally on $(0, \infty)$ to a flow $g_\infty(t)$ on $(0, \infty)$. Moreover, if $\alpha_i \rightarrow \alpha_\infty$ as $i \rightarrow \infty$, then by the same argument as in [61, Lemma 2.3], one can show that $g_\infty(t) = g_{\alpha_\infty}(t)$ is the canonical Ricci flow of the expanding Ricci soliton g_{α_∞} asymptotic to the cone metric $g_{\text{cone}, \alpha_\infty}$. Since I is compact in $(1, \infty)$, for each $k \in \mathbb{N}$, we can find $A_k > 0$, uniform over all $i = 1, \dots, \infty$, such that on each smooth Riemannian cone $g_{\text{cone}, \alpha_i}$, we have that

$$(3.6) \quad \sup_{x \in \mathbb{C}^{n-1}} r_{\text{cone}, \alpha_i}^{2+k}(x) |\nabla^k \text{Rm}|(x) = \sup_{x \in \partial B(*, 1)} |\nabla^k \text{Rm}|(x) = A_k < \infty,$$

where $r_{\text{cone}, \alpha_i}(x) = d_{g_{\text{cone}, \alpha_i}}(x, *)$ and the curvature and derivatives are with respect to $g_{\text{cone}, \alpha_\infty}$. Since the Ricci flows $g_{\alpha_i}(t)$ all converge smoothly locally to their asymptotic cones $g_{\text{cone}, \alpha_i}$ as $t \rightarrow 0$ on $\mathbb{C}^{n-1} \setminus \{*\}$, $g_{\alpha_i}(t)$ can be extended smoothly to $t = 0$ on $\mathbb{C}^{n-1} \setminus \{*\}$. Moreover, by (3.5), (3.6), and Shi's local derivative estimates [31], we can deduce the following uniform curvature estimates for all $i = 1, \dots, \infty$: for all $k \in \mathbb{N}$ and $D > 0$, there exists $A_k(D) > 0$ such that $|\nabla^k \text{Rm}_{g_{\alpha_i}(t)}| \leq A_k$ on $B(*, D) \setminus B(*, D^{-1})$ for all $t > 0$. Therefore, after passing to a subsequence if necessary, we may assume the Ricci flows $g_{\alpha_i}(t)$ converge smoothly to $g_{\alpha_\infty}(t)$ on any compact subset of $\mathbb{C}^{n-1} \setminus \{*\}$, uniformly on any $[0, \infty)$. Thus, we have that

$$\begin{aligned} \|\nabla^k(g_{\alpha_i}(t_i) - g_{\text{cone}, \alpha_i})\| &\leq \|\nabla^k(g_{\alpha_i}(t_i) - g_{\alpha_\infty}(t_i))\| + \|\nabla^k(g_{\text{cone}, \alpha_\infty} - g_{\alpha_\infty}(t_i))\| \\ &\quad + \|\nabla^k(g_{\text{cone}, \alpha_\infty} - g_{\text{cone}, \alpha_i})\| \\ &< \frac{\varepsilon_0}{2} \end{aligned}$$

for all sufficiently large i on $B(*, D) \setminus B(*, D^{-1})$, where the norms and derivatives are taken with respect to $g_{\text{cone}, \alpha_\infty}$. Choosing $D > D_0$, this contradicts (3.4) and we are done. $\square$

In the next lemma, we show that for any singular metric from (3.1), there exists a expanding Ricci soliton from Cao's family such that we can glue a compact $U(n-1)$ -invariant subset of it with a $U(n-1)$ -invariant subset of the singular metric to obtain a metric on $\mathbb{P}^{n-1}$ which is C^1 -smooth near $r = 0$.

Lemma 3.2. *For all large $k \in \mathbb{N}$, there exists a sequence $\{(\mathbb{C}^{n-1}, g_i, r_i, s_i)\}_{i=1}^\infty$, where $r_i, s_i \rightarrow 0$ as $i \rightarrow \infty$, and $(\mathbb{C}^{n-1}, g_i)$ is a $U(n-1)$ -invariant expanding gradient Kähler-Ricci soliton which takes the form*

$$g_i = dr^2 + a_i^2(r)\eta \otimes \eta + b_i^2(r)g^T$$

for some positive smooth functions $a_i, b_i : (0, \infty) \rightarrow (0, \infty)$ satisfying

$$(3.7) \quad b_i^{(m)}(s_i) = \frac{\sin^{(m)}(r_i)}{\sqrt{k}} \quad \text{for } m = 0, 1, 2.$$

Proof. Let $100 \leq k \in \mathbb{N}$. By Lemma 3.1, for each $i \in \mathbb{N}$ we can find a large $R_i > 0$ such that the rescaled expanding soliton $R_i^{-2}g_\alpha$ (where g_α is from (3.3)) is $\frac{1}{100i^2}$ -close to the cone metric $g_{\text{cone}, \alpha}$ on near radius 1 for all $\alpha \in [k-1, k+1]$. In particular, this implies that

$$-\frac{R_i^2 b''_\alpha(R_i)}{b_\alpha(R_i)} + |b'_\alpha(R_i) - \frac{1}{\sqrt{\alpha}}| \leq \frac{1}{i^2},$$

noting that these two quantities vanish on $g_{\text{cone}, \alpha}$. In particular, we see that

$$0 < \lambda_{i, \alpha} := -\frac{b''_\alpha(R_i)}{b_\alpha(R_i)} \leq \frac{1}{R_i^2 i^2} \quad \text{and} \quad b'_\alpha(R_i) \in \left[\frac{1}{\sqrt{\alpha + \frac{1}{2}}}, \frac{1}{\sqrt{\alpha - \frac{1}{2}}} \right].$$

Furthermore, by the smoothness conditions $b_\alpha(0) = 0$, $b'_\alpha(0) = 1$, and the concavity of b_α , we know that $b_\alpha(R_i) \leq R_i$. Consider the further rescaled expanding soliton $h_\alpha := \lambda_{i, \alpha} g_\alpha$ which has the form

$$h_\alpha = ds^2 + \bar{a}_\alpha^2(s) \eta \otimes \eta + \bar{b}_\alpha^2(s) g^T.$$

Then we have

$$-\frac{\bar{b}''_\alpha(s_{i, \alpha})}{\bar{b}_\alpha(s_{i, \alpha})} = 1, \quad \bar{b}_\alpha(s_{i, \alpha}) \leq s_{i, \alpha} \leq \frac{1}{i},$$

where $s_{i, \alpha} = \sqrt{\lambda_{i, \alpha}} R_i \leq \frac{1}{i}$, and

$$(3.8) \quad \bar{b}'_\alpha(s_{i, \alpha}) = b'_\alpha(R_i) \in \left[\frac{1}{\sqrt{\alpha + \frac{1}{2}}}, \frac{1}{\sqrt{\alpha - \frac{1}{2}}} \right].$$

We can assume $i > k$ is sufficiently large. Then there is a unique $r_{i, \alpha} \in (0, \frac{\pi}{2})$ such that

$$(3.9) \quad -\bar{b}''_\alpha(s_{i, \alpha}) = \bar{b}_\alpha(s_{i, \alpha}) = \frac{\sin(r_{i, \alpha})}{\sqrt{k}} \leq \frac{1}{i},$$

and we have $r_{i, \alpha} \leq \frac{2\sqrt{k}}{i} \leq \frac{2}{\sqrt{i}}$ and also

$$(3.10) \quad \frac{1}{\sqrt{k+1/2}} < \frac{\cos(r_{i, \alpha})}{\sqrt{k}} < \frac{1}{\sqrt{k-1/2}}.$$

In particular, (3.9) implies that the assertion (3.7) holds for $m = 0, 2$ with our choice of $r_{i, \alpha}$, $s_{i, \alpha}$ and the expanding soliton h_α for any $\alpha \in [k-1, k+1]$. We will further determine a value of α so that (3.7) also holds for $m = 1$. To do this, we first observe that by combining (3.8) and (3.10), that $\bar{b}'_\alpha(s_{i, \alpha}) > \frac{\cos(r_{i, \alpha})}{\sqrt{k}}$ for $\alpha = k-1$ and $\bar{b}'_\alpha(s_{i, \alpha}) < \frac{\cos(r_{i, \alpha})}{\sqrt{k}}$ for $\alpha = k+1$. It is easy to see that $r_{i, \alpha}$ and $s_{i, \alpha}$ are both continuous in $\alpha \in [k-1, k+1]$, so by the intermediate value theorem there exists an $\alpha = \alpha(i, k) \in (k-1, k+1)$ such that $b'_\alpha(s_{i, \alpha}) = \frac{\cos(r_{i, \alpha})}{\sqrt{k}}$. This implies that (3.7) also holds for $m = 1$. $\square$

For each singular metric $h_k = dr^2 + \frac{\sin^2(2r)}{4k^2} \eta \otimes \eta + \frac{\sin^2(r)}{k} g^T$, $r \in [0, \frac{\pi}{2}]$ from (3.1), by gluing h_k with the expanding solitons obtained in Lemma 3.2 at arbitrarily small scales near $r = 0$, we can obtain a sequence of metrics that are C^1 -smooth near $r = 0$ and converge to h_k . In the next proposition, we will further modify these metrics to obtain a sequence of smooth Kähler metrics which converge to h_k , and satisfy $\text{Rm} \geq 0$ on real $(1, 1)$ -forms everywhere, and $\text{Rm} \geq 2$ on $(1, 1)$ -forms outside of arbitrarily small subsets.

Proposition 3.3 (Smoothing out cone points by gluing expanding solitons). *Let $n \geq 3$ and let $k \in \mathbb{N}$ be any fixed large number. Then there exists a sequence of positive numbers $\delta_i \rightarrow 0$ as $i \rightarrow \infty$, and a sequence of smooth $U(n-1)$ -invariant Kähler manifolds $(\mathbb{P}^{n-1}, g_i)$ which can be written as $g_i = dr^2 + a_i^2(r)\eta \otimes \eta + b_i^2(r)g^T$, $r \in [0, \frac{\pi}{2} - \delta_i]$, satisfying the following:*

- (a) $\text{Rm}_{g_i} > 0$ on real $(1, 1)$ -forms everywhere;
- (b) $\text{Rm}_{g_i} \geq 2$ on real $(1, 1)$ -forms on $r^{-1}([\delta_i, \frac{\pi}{2} - \delta_i])$;
- (c) For every $\varepsilon > 0$, g_i smoothly converges to h_k on $r^{-1}([\varepsilon, \frac{\pi}{2} - \varepsilon])$ as $i \rightarrow \infty$.

Proof. Fix k large, for each i let $r_i, s_i > 0$ be the sequences from Lemma 3.2 converging to zero, and let $b_1 : [0, \infty) \rightarrow [0, \infty)$ be the warping function of the corresponding expanding Ricci soliton, where we omit the dependence of b_1 (and all the following functions a_1, a_2, a_3, a_4 and b, b_2, b_3, b_4) in i . Let $b : [0, \frac{\pi}{2} + s_i - r_i] \rightarrow [0, \infty)$ be defined by

$$b(r) = \begin{cases} b_1(r) & \text{for } r \in [0, s_i], \\ b_2(r) := \frac{1}{\sqrt{k}} \sin(r - s_i + r_i) & \text{for } r \in [s_i, \frac{\pi}{2} + s_i - r_i]. \end{cases}$$

Then b is C^2 by Lemma 3.2. We also write $a_2 := b_2 b_2' = \frac{1}{2k} \sin(2(r - s_i + r_i))$.

Let $\varphi : (-\infty, \infty) \rightarrow [0, \infty)$ be a non-decreasing smooth function such that $\varphi(r) = 0$ on $(-\infty, 0]$ and $\varphi(r) = 1$ on $[t, \infty)$, where $t \in (0, \min\{s_i, r_i\})$ is some sufficiently small number whose value we will choose later. Define

$$B_i(r) := \varphi(s_i - r)b_1'''(r) + (1 - \varphi(s_i - r))b_2'''(r) \quad \text{for } r \in [0, \frac{\pi}{2} + s_i - r_i].$$

Then B_i is smooth. Let $b_3 : [0, \frac{\pi}{2} + s_i - r_i] \rightarrow \mathbb{R}$ be the unique smooth function solving

$$(3.11) \quad \begin{cases} b_3(0) = 0, & b_3'(0) = 1, & b_3''(0) = 0, \\ b_3'''(r) = B_i(r). \end{cases}$$

Because $B_i \equiv b_1'''$ on $[0, s_i - t]$ and $B_i \equiv b_2'''$ on $[s_i, \frac{\pi}{2} + s_i - r_i]$, we know that $b_3 \equiv b_1$ on $[0, s_i - t]$ and that b_3 satisfies the smoothness condition (2.8) at 0. Henceforth, for fixed r_i, s_i , we shall write $o(1)$ to denote all constants that go to zero as $t \rightarrow 0$. By the C^2 -regularity of b , it is then clear that

$$(3.12) \quad \|b_3 - b\|_{C^2[0, \frac{\pi}{2} + s_i - r_i]} \leq o(1).$$

Moreover, since $b_3''' \equiv b_2'''$ on $[s_i, \frac{\pi}{2} + s_i - r_i]$, by letting $a_3 = b_3 b_3'$, we also see that

$$(3.13) \quad \|a_3 - a_2\|_{C^2[s_i, \frac{\pi}{2} + s_i - r_i]} \leq o(1).$$

For $\alpha > 0$ to be chosen later, define a smooth function $A_{i,\alpha} : \mathbb{R} \rightarrow \mathbb{R}$ by

$$A_{i,\alpha}(r) = \varphi\left(\frac{\pi}{2} - r_i - r\right) \frac{-a_3''}{a_3} + \left(1 - \varphi\left(\frac{\pi}{2} - r_i - r\right)\right) \alpha^2$$

and let $a_4 : [0, \infty) \rightarrow \mathbb{R}$ be the unique smooth function, omitting its dependence on α , with $a_4 = a_3$ on $[0, \frac{\pi}{2} - r_i - t]$ and solving

$$\begin{cases} a_4\left(\frac{\pi}{2} - 2r_i\right) = a_3\left(\frac{\pi}{2} - 2r_i\right), & a_4'\left(\frac{\pi}{2} - 2r_i\right) = a_3'\left(\frac{\pi}{2} - 2r_i\right), \\ a_4'' = -a_4 \cdot A_{i,\alpha}. \end{cases}$$

Set $b_4(r) := \sqrt{\int_0^r 2a_4(\tau) d\tau}$. Then $a_4 = b_4 b_4'$, and (3.13) implies

$$(3.14) \quad \|a_4 - a_2\|_{C^2[s_i, \frac{\pi}{2} - r_i - t]} \leq o(1).$$

In the following we will show that there exists some $\alpha = \alpha_\#$ such that a_4 and b_4 satisfy the smoothness conditions (2.7) and (2.8), where $a_4 = 0$. Let $v_i(\alpha)$, $c_i(\alpha) > 0$ be two smooth functions such that $a_4 \geq 0$ on $[0, \frac{\pi}{2} - r_i + v_i(\alpha)]$ and

$$(3.15) \quad a_4(\frac{\pi}{2} - r_i + v_i(\alpha)) = 0, \quad a'_4(\frac{\pi}{2} - r_i + v_i(\alpha)) = -c_i(\alpha).$$

Then it suffices to find $\alpha_\#$ such that $c_i(\alpha_\#) = 1$. To do this, we consider the function

$$w_{i,\alpha}(r) := c_{0,i}(\alpha) \cdot \alpha^{-1} \sin(\alpha(\frac{\pi}{2} - r_i + v_{0,i}(\alpha) - r)),$$

where $c_{0,i}(\alpha), v_{0,i}(\alpha) > 0$ are two smooth functions in α such that $w_{i,\alpha} \geq 0$ on $[\frac{\pi}{2} - r_i, \frac{\pi}{2} - r_i + v_{0,i}(\alpha)]$ and $w_{i,\alpha}$ solves the following IVP:

$$\begin{cases} w(\frac{\pi}{2} - r_i) = a_2(\frac{\pi}{2} - r_i), & w'(\frac{\pi}{2} - r_i) = a'_2(\frac{\pi}{2} - r_i), \\ w'' = -w \cdot \alpha^2. \end{cases}$$

In particular, for

$$\alpha_0 = \sqrt{\frac{4k^2(1 - \frac{1}{k^2}\cos^2(2s_i))}{\sin^2(2s_i)}}, \quad v_0 = \frac{\arccos(\frac{1}{k}(\cos(2s_i)))}{\alpha_0},$$

direct computation shows that $c_{0,i}(\alpha_0) = 1$, and so

$$w_{i,\alpha_0}(\frac{\pi}{2} - r_i + v_0) = 0, \quad w'_{i,\alpha_0}(\frac{\pi}{2} - r_i + v_0) = -c_{0,i}(\alpha_0) = -1.$$

It is also easy to see that $c_{0,i}(\alpha_0 - 1) < 1$ and $c_{0,i}(\alpha_0 + 1) > 1$. By (3.13), in letting $t \rightarrow 0$, a_4 converges in the C^1 -sense to a_2 on $[s_i, \frac{\pi}{2} - r_i]$ and to $w_{i,\alpha}$ on $[\frac{\pi}{2} - r_i, \frac{\pi}{2} - r_i + \max\{v_{0,i}(\alpha), v_i(\alpha)\}]$. In particular, this implies that

$$c_i(\alpha) = c_{0,i}(\alpha) + o(1) \quad \text{and} \quad v_i(\alpha) = v_{0,i}(\alpha) + o(1).$$

Thus, for t sufficiently small, we have that $c_i(\alpha_0 - 1) < 1$ and $c_i(\alpha_0 + 1) > 1$. Continuity and the intermediate value theorem now give us an $\alpha_\# \in (\alpha_0 - 1, \alpha_0 + 1)$ such that $c_i(\alpha_\#) = 1$.

Henceforth fixing this $\alpha_\#$, denote the corresponding $v_i(\alpha_\#)$ by v_i . Then $g_i := dr^2 + a_4^2(r)\eta \otimes \eta + b_4^2(r)g^T$ is a $U(n-1)$ -invariant smooth Kähler metric on $\mathbb{P}^{n-1}$. It is moreover clear from (3.12) and (3.13) that

$$\|b_4 - b_2\|_{C^2[2r_i, \frac{\pi}{2} - 2r_i]} + \|a_4 - a_2\|_{C^2[2r_i, \frac{\pi}{2} - 2r_i]} \leq o(1).$$

So g_i clearly satisfies assertion (c) of the proposition.

It now suffices to show g_i satisfies $\text{Rm} > 0$ on real $(1, 1)$ -forms on $[0, s_i]$, and $\text{Rm} \geq 2 + o(1)$ on real $(1, 1)$ -forms on $[s_i, \frac{\pi}{2} - r_i + v_i]$. If this is true, then there exists a sequence $\varepsilon_i \rightarrow 0$, such that the rescaled metrics $(1 - \varepsilon_i)g_i$ satisfy all assertions of the proposition. This is equivalent to checking that a_4, b_4 satisfy inequalities (1)–(4) of Lemma 2.25 with $\lambda > 0$ on $[0, s_i]$ and $\lambda = 1 + o(1)$ on $[s_i, \frac{\pi}{2} - r_i + v_i]$ (see (4.1) for the geometric meaning of λ).

First, we verify a_4, b_4 satisfy inequalities (1) and (3) with $\lambda > 0$ on $[0, s_i]$ and $\lambda = 1 + o(1)$ on $[s_i, \frac{\pi}{2} - r_i - t]$. Since the expanding Ricci soliton $dr^2 + a_1^2(r)\eta \otimes \eta + b_1^2(r)g^T$ satisfies $\text{Rm} > 0$ on all $(1, 1)$ -forms, there exists $\delta_i > 0$ which tends to 0 as $i \rightarrow \infty$, such that a_1, b_1 satisfy inequalities (1)–(4) of Lemma 2.25 on $[0, s_i]$. In particular, inequalities (1) and (3) only involve at most second order derivatives of $b_4 = b_3$, so by (3.12), we see that b_4 satisfies (1) and (3) with $\lambda = \delta_i$ on $[0, s_i]$. Similarly, since $h_k = dr^2 + a_2^2(r)\eta \otimes \eta + b_2^2(r)g^T$ satisfies $\text{Rm} \geq 2$ on $(1, 1)$ -forms except at $r = 0$ and $r = \frac{\pi}{2}$, it follows that a_2, b_2 satisfy (1)–(4) with $\lambda = 1$. So by (3.12) we see that a_4, b_4 satisfy (1) and (3) with $\lambda = 1 + o(1)$ on $[s_i, \frac{\pi}{2} - r_i - t]$.

Next, we show (2) and (4) on $[0, \frac{\pi}{2} - r_i - t]$. Since b'_2 is decreasing and $b'_2(\frac{\pi}{2} - r_i + s_i) = 0$, for sufficiently small t , it follows from (3.12) that $b'_3 \geq \frac{1}{2}b'_2(\frac{\pi}{2} - r_i) > 0$ on $[0, \frac{\pi}{2} - r_i]$. By (3.11) and (3.12), we therefore find that

$$-\frac{a''_3}{4a_3} = -\frac{3b''_3}{4b_3} - \frac{1b'''_3}{4b'_3} = -\frac{3b''}{4b} - \frac{1B_i}{4b'} + o(1) \quad \text{on } [0, \frac{\pi}{2} - r_i - t].$$

On $[s_i - t, s_i]$, as B_i takes values between $b'''_1(s_i) + o(1)$ and $b'''_2(s_i) + o(1)$ on $[s_i - t, s_i]$, we see that $-\frac{a''_4}{4a_4}$ is also bounded between $-\frac{a''_1}{4a_1}(s_i) + o(1) \geq \frac{\delta_i}{2}$ and $-\frac{a''_2}{4a_2}(s_i) + o(1) \geq 1 + o(1)$, and hence $-\frac{a''_4}{4a_4} \geq \frac{\delta_i}{4}$, which implies (2) with $\lambda = \frac{\delta_i}{2}$ on $[0, s_i]$. On $[s_i, \frac{\pi}{2} - r_i - t]$, because $-\frac{a''_2}{4a_2} = 1$ and by (3.14), we have $-\frac{a''_4}{4a_4} \geq 1 + o(1)$, which implies (2) with $\lambda = 1 + o(1)$. Similarly, we can verify inequality (4).

Secondly, we verify (1)–(4) on $[\frac{\pi}{2} - r_i - t, \frac{\pi}{2} - r_i]$ with $\lambda = 1 + o(1)$. Since $-\frac{a''_3}{4a_3} \leq 1 + o(1) < \alpha_\#$ and $\alpha_\# \in (\alpha_0 - 1, \alpha_0 + 1)$, we see that

$$(3.16) \quad \|a_4 - a_3\|_{C^1[\frac{\pi}{2} - r_i - t, \frac{\pi}{2} - r_i]} = o(1),$$

and also

$$(3.17) \quad -\frac{a''_4}{4a_4} \geq -\frac{a''_3(\frac{\pi}{2} - r_i)}{4a_3(\frac{\pi}{2} - r_i)} + o(1).$$

(3.16) together with (3.13) implies

$$(3.18) \quad \|a_4 - a_2\|_{C^1[\frac{\pi}{2} - r_i - t, \frac{\pi}{2} - r_i]} + \|b_4 - b_2\|_{C^2[\frac{\pi}{2} - r_i - t, \frac{\pi}{2} - r_i]} = o(1),$$

which immediately implies (1) and (3) hold with $\lambda = 1 + o(1)$ on $[\frac{\pi}{2} - r_i - t, \frac{\pi}{2} - r_i]$. Moreover, together with (3.17), this implies (2) and (4).

Lastly, we verify (1)–(4) on $[\frac{\pi}{2} - r_i, \frac{\pi}{2} - r_i + v_i]$. First, (2) follows from

$$(3.19) \quad -\frac{a''_4}{4a_4} \geq \frac{(\alpha_0 - 1)^2}{4} \geq \frac{\alpha_0^2}{8} \geq \frac{k^2}{16r_i^2} \geq 1.$$

Let $w_i > 0$ denote all constants with $w_i \rightarrow 0$ as $r_i \rightarrow 0$. Then we have

$$(3.20) \quad b'_4(r) = \frac{a_4(r)}{b_4(r)} \leq \frac{a_4(\frac{\pi}{2} - r_i - t)}{b_4(\frac{\pi}{2} - r_i - t)} \leq w_i.$$

It is also easy to see that $|b_4^2 - b_4^2(\frac{\pi}{2} - r_i - t)| = w_i$, and so the bound $|b_4^2(\frac{\pi}{2} - r_i - t) - \frac{1}{k}| \leq w_i$ implies that $|b_4^2 - \frac{1}{k}| \leq w_i$. Using in addition (3.20), we derive inequality (1) by

$$(3.21) \quad \frac{1 - (b'_4)^2}{b_4^2} = \frac{1 - \left(\frac{a_4}{b_4}\right)^2}{b_4^2} \geq k - w_i \geq k - 1 \geq 1.$$

Next, by using the fact that $-a'_4$ is increasing, (3.18), and $b'_2(r) \leq w_i$, (3) follows from

$$-\frac{b''_4}{b_4} = \frac{-a'_4 + (b'_4)^2}{b_4^2} \geq \frac{-a'_2(\frac{\pi}{2} - r_i - t) + (b'_2)^2(\frac{\pi}{2} - r_i - t)}{b_2^2(\frac{\pi}{2} - r_i - t)} - w_i = 1 - w_i.$$

Finally, by using the inequalities $a'_4 \geq a'_4(\frac{\pi}{2} - r_i + v_i) = -1$, $|b_4^2 - \frac{1}{k}| \leq w_i$, and $0 \leq b'_4 \leq w_i$, we derive that

$$(3.22) \quad -\frac{b''_4}{b_4} = \frac{-a'_4 + (b'_4)^2}{b_4^2} \leq k + w_i \leq k + 1.$$

In conclusion, (3.21), (3.19), and (3.22) together give us that

$$\begin{aligned} \frac{2(n-1)(1-\lambda b_4^2(r)-(b_4'(r))^2)}{(n-2)b_4^2(r)} \left(-\frac{a_4''(r)}{4a_4(r)} - \lambda \right) &\geq \frac{2(n-1)}{(n-2)}(k-1-\lambda) \left(\frac{k^2}{16r_i^2} - \lambda \right) \\ &> (k+1-\lambda)^2 \geq \left(-\frac{b_4''(r)}{b_4(r)} - \lambda \right)^2, \end{aligned}$$

which implies (4) for all large k . This completes the proof. $\square$

4. RICCI FLOWS EMANATING FROM THE SINGULAR METRIC

Let $n \geq 3$. The goal of this section is to show that there is a Ricci flow that smooths out the singular metric (3.1), namely

$$h_k = dr^2 + \frac{\sin^2(2r)}{4k^2} \eta \otimes \eta + \frac{\sin^2(r)}{k} g^T,$$

where $r(\cdot) : \mathbb{P}^{n-1} \rightarrow [0, \frac{\pi}{2}]$ is the radial function, $r^{-1}(0) = \{p\}$, and $r^{-1}(\frac{\pi}{2})$ is biholomorphic to $\mathbb{P}^{n-2}$. We construct this Ricci flow by taking limits of the Ricci flows coming out of the metrics $\{g_k^i\}_{i=1}^\infty$ from Proposition 3.3 that converge to h_k . These Ricci flows are smooth, have uniform existence time, and satisfy a uniform curvature decay $|\text{Rm}| \leq \frac{C}{t}$. We will show that these smooth Ricci flows satisfy $\text{Rm} \geq 0$ everywhere, and $\text{Rm} \geq 2$ outside of arbitrarily small subsets, so eventually the Ricci flows converge to a limit flow that satisfies $\text{Rm} \geq 2$ on real $(1, 1)$ -forms *everywhere*.

The curvature condition of $\text{Rm} \geq 2$ on real $(1, 1)$ -forms is equivalent to $\lambda \geq 1$, where $\lambda \in \mathbb{R}$ is half of the infimum of Rm on real $(1, 1)$ -forms given by (2.24) in complex $(n-1)$ -dimensions, namely

$$(4.1) \quad \lambda := \frac{1}{2} \inf_{u \in \Lambda_{\mathbb{R}}^{1,1}(M) \setminus \{0\}} \frac{\text{Rm}(u, \bar{u})}{|u|_*^2},$$

where for $u = i u_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$, the norm $|u|_*$ is defined by

$$|u|_*^2 = \sum_{\alpha, \beta, \gamma, \eta=1}^{n-1} (\hat{g}_{\alpha\bar{\beta}} \hat{g}_{\gamma\bar{\eta}} + \hat{g}_{\alpha\bar{\eta}} \hat{g}_{\gamma\bar{\beta}}) u^{\alpha\bar{\beta}} \bar{u}^{\gamma\bar{\eta}},$$

and we have $\text{Rm}(u, \bar{u}) = -\sum_{\alpha, \beta, \gamma, \eta=1}^{n-1} \hat{R}_{\alpha\bar{\beta}\gamma\bar{\eta}} u^{\alpha\bar{\beta}} \bar{u}^{\gamma\bar{\eta}}$. Using the fact that $u^{\alpha\bar{\beta}} = g^{\alpha\bar{\epsilon}} g^{\eta\bar{\beta}} u_{\eta\bar{\epsilon}}$ and $\bar{u}_{\alpha\bar{\beta}} = u_{\beta\bar{\alpha}}$ for any α and β , we have

$$|u|_*^2 = \left| \sum_{\alpha, \beta=1}^{n-1} g_{\alpha\bar{\beta}} u^{\alpha\bar{\beta}} \right|^2 + u_{\alpha\bar{\beta}} g^{\alpha\bar{\gamma}} g^{\eta\bar{\beta}} \bar{u}_{\gamma\bar{\eta}} \geq u_{\alpha\bar{\beta}} g^{\alpha\bar{\gamma}} g^{\eta\bar{\beta}} \bar{u}_{\gamma\bar{\eta}} \geq 0$$

with equality if and only if $u = 0$.

We first observe that λ is a supersolution of the heat equation.

Lemma 4.1. *Let $(M, g(t))$, $t \in [0, T]$, be a smooth Kähler-Ricci flow (not necessarily complete) with nonnegative curvature operator. Then*

$$\partial_t \lambda \geq \Delta \lambda$$

holds in the following barrier sense: for any $(q, \tau) \in M \times (0, T)$, there exists a neighborhood $U \subset M \times (0, T)$ of (q, τ) and a C^∞ (upper barrier) function $\phi : U \rightarrow \mathbb{R}$ such that $\phi \geq \lambda$ on U , with equality at (q, τ) and

$$\partial_t \phi \geq \Delta \phi \quad \text{at } (q, \tau).$$

Proof. Fix (p_0, t_0) and suppose that the infimum in (4.1) is attained at a nonzero $(1, 1)$ -form $w_0 \in \Lambda_{p_0}^{1,1}$. We may assume by scaling that $|w_0|_{*, g_{t_0}} = 1$. We first extend w_0 to a spacetime neighborhood $V \times (t_0 - \varepsilon, t_0 + \varepsilon)$ of (p_0, t_0) in the following way. Let $\{e_k\}_{k=1}^{2n-2}$ be an orthonormal basis of $(T_{p_0}M, g_{t_0}|_{T_{p_0}M})$ such that $J(e_{2\alpha-1}) = e_{2\alpha}$. By abusing notation and shrinking the neighborhood V of p_0 if necessary, we extend $\{e_k\}_{k=1}^{2n-2}$ to a smooth local orthonormal frame near p_0 via parallel translation. Then $J(e_{2\alpha-1}) = e_{2\alpha}$ on V and we have at (x_0, t_0) that

$$(4.2) \quad \nabla_{e_j} e_k = 0 \quad \text{and} \quad \nabla_X \nabla_X e_i = 0 \quad \text{for any } X \in T_{p_0}M \text{ at } p_0.$$

Let ∇_t denote the natural space-time extension of the Levi-Civita connection $\nabla^{g(t)}$ so that it is compatible with the metric; that is to say, $\partial_t |X|_{g(t)}^2 = 2\langle \nabla_t X, X \rangle_{g(t)}$ for any time-dependent vector field X . We extend $\{e_k\}_{k=1}^{2n-2}$ to a time-dependent local frame $\{e_k(t)\}_{k=1}^{2n-2}$ so that $e_k(x, t_0) = e_k(x)$ for all $x \in V$, and

$$(4.3) \quad \nabla_t e_k(x, t) = \partial_t(e_k(x, t)) - \text{Ric}(e_k(x, t)) = 0.$$

By ODE theory, after choosing a smaller V if necessary, we can solve the system on $(t_0 - \varepsilon, t_0 + \varepsilon)$ for some small $\varepsilon > 0$ with $e_k(x, t)$ smooth in $(x, t) \in V \times (t_0 - \varepsilon, t_0 + \varepsilon)$. It also follows from the fact that $J \circ \text{Ric} = \text{Ric} \circ J$ and the uniqueness of solutions of ODEs that $J(e_{2\alpha-1}(x, t)) = e_{2\alpha}(x, t)$. Let $\{\theta_k(t)\}_{k=1}^{2n-2}$ denote the dual frame. We introduce the corresponding holomorphic and antiholomorphic frames:

$$E_\alpha(x, t) = \frac{1}{2}(e_{2\alpha-1}(x, t) - i e_{2\alpha}(x, t)) \quad \text{and} \quad E_{\bar{\alpha}}(x, t) = \frac{1}{2}(e_{2\alpha-1}(x, t) + i e_{2\alpha}(x, t));$$

and their dual frames

$$\Theta_\alpha(x, t) = \theta_{2\alpha-1}(x, t) + i \theta_{2\alpha}(x, t) \quad \text{and} \quad \Theta_{\bar{\alpha}}(x, t) = \theta_{2\alpha-1}(x, t) - i \theta_{2\alpha}(x, t).$$

Thanks to (4.2) and (4.3), at (p_0, t_0) we have that

$$(4.4) \quad \nabla_t \Theta_\alpha = \nabla \Theta_\alpha = \nabla \Theta_{\bar{\alpha}} = \Delta \Theta_{\bar{\alpha}} = 0.$$

Since w_0 is a real $(1, 1)$ -form, we may assume $w_0 = i w_{\alpha\bar{\beta}} \Theta_\alpha(p_0, t_0) \wedge \Theta_{\bar{\beta}}(p_0, t_0)$ for some complex numbers $\overline{w_{\alpha\bar{\beta}}} = w_{\beta\bar{\alpha}}$, $\alpha, \beta = 1, \dots, n$. Let w_t be the 2-form given by

$$w_t = i w_{\alpha\bar{\beta}} \Theta_\alpha(x, t) \wedge \Theta_{\bar{\beta}}(x, t)$$

on $V \times (t_0 - \varepsilon, t_0 + \varepsilon)$. Then by the Kählerity of the metric, w_t is J -invariant, i.e., $w_t(J(\cdot), J(\cdot)) = w_t(\cdot, \cdot)$, and so it is of type $(1, 1)$. Moreover, by (4.4) at (p_0, t_0) , we have that

$$(4.5) \quad \nabla_t w_t = 0, \quad \nabla w_t = 0 \quad \text{and} \quad \Delta w_t = 0.$$

Next we define the smooth function φ by

$$\varphi(x, t) := \frac{1}{2} \frac{\text{Rm}(w_t, \overline{w}_t)}{|w_t|_*^2} = \frac{1}{2} \text{Rm}(w_t, \overline{w}_t),$$

where the curvature and norms are all with respect to $g(t)$, and we use that $|w_t|_* \equiv 1$. It can be seen from the definition that $\varphi \geq \lambda$ on $V \times (t_0 - \varepsilon, t_0 + \varepsilon)$ and equality holds at (p_0, t_0) . Recall that

$$\nabla_t \text{Rm} = \Delta \text{Rm} + Q(\text{Rm}) = \Delta \text{Rm} + \text{Rm}^2 + \text{Rm}^\sharp,$$

and both Rm^2 and $\text{Rm}^\sharp$ are nonnegative operators when the curvature operator Rm is nonnegative. Hence by (4.5), when evaluated at (p_0, t_0) , we have that

$$\begin{aligned} (\partial_t - \Delta)\varphi &= \frac{1}{2}(\partial_t - \Delta)(\text{Rm}(w_t, \bar{w}_t)) = ((\nabla_t - \Delta)\text{Rm})(w_t, \bar{w}_t) \\ &= \text{Rm}^2(w_t, \bar{w}_t) + \text{Rm}^\sharp(w_t, \bar{w}_t) \geq 0. \end{aligned}$$

This proves the lemma. $\square$

From Lemma 4.1, using the maximum principle, it is easy to see that the lower bound $\lambda \geq 1$ is preserved along any smooth Ricci flow. In Lemma 4.2 below, we will show this also holds for the Ricci flow coming out of the singular metric h_k , where $\lambda \geq 1$ only holds on the smooth part. This is achieved by a heat kernel estimate on the smooth Ricci flows that converge at positive times to the limit flow that smooths out h_k . The heat kernel estimate guarantees that the heat comes evenly from everywhere at $t = 0$ to positive times, and hence one can ignore the lack of enough curvature from the arbitrarily small regions. We first recall some basic facts about heat kernels.

Suppose that $(M, g(t))$ is a smooth complete Ricci flow for $t \in [a, b]$ with bounded curvature. Let $G(x, t; y, s)$ be the heat kernel of the heat equation $\partial_t u = \Delta u$, i.e.,

$$\begin{aligned} (\partial_t - \Delta_{x,t})G(\cdot, \cdot; y, s) &= 0 \quad \text{and} \quad \lim_{t \rightarrow s^+} G(\cdot, t; y, s) = \delta_y(\cdot), \\ (-\partial_s - \Delta_{y,s} + R)G(x, t; \cdot, \cdot) &= 0 \quad \text{and} \quad \lim_{s \rightarrow t^-} G(x, t; \cdot, s) = \delta_x(\cdot). \end{aligned}$$

In particular, we have $\int_M G(x, t; y, s) d_s y = 1$ for all $s < t$; see [32, Chapter 26] for more discussion of the heat kernel. We also recall the heat kernel upper bound given in terms of the lower bound on the pointed Nash entropy by Bamler in [4, Theorem 7.1]:

$$(4.6) \quad G(x, t; y, s) \leq \frac{C}{(t-s)^{\frac{n}{2}}} \exp(-\mathcal{N}_{(x,t)}(t-s)),$$

where $C > 0$ is a constant depending on $R_{\min}(t-s)$, and also the lower bound on the Nash entropy $\mathcal{N}_{(x,t)}(\cdot)$ in terms of the lower bound on the volume ratio in [4, Theorem 8.1]:

$$(4.7) \quad \frac{\text{vol}_{g(t)}(B_{g(t)}(x, r))}{r^n} \leq C \exp(\mathcal{N}_{(x,t)}(r^2)) \exp(C_0),$$

where $C_0 > 0$ is a dimensional constant and $C > 0$ is some constant depending on $R_{\min} r^2$.

Now we prove the main result of this section.

Lemma 4.2. *Let $n \geq 3$. Then for each fixed $k \in \mathbb{N}$, there exist $C_k, T_k > 0$ and a Ricci flow $(\mathbb{P}^{n-1}, g_k(t))$, $t \in [0, T_k]$, such that*

- (1) $|\text{Rm}| \leq \frac{C_k}{t}$ and $g_k(t) \rightarrow h_k$ smoothly on any compact subset of $r^{-1}((0, \frac{\pi}{2}))$ as $t \rightarrow 0$;
- (2) The metrics $d_{g_k(t)}$ converge to d_{h_k} as $t \rightarrow 0$;
- (3) $\lambda(x, t) \geq 1$ for all $(x, t) \in \mathbb{P}^{n-1} \times (0, T_k]$.

Proof. For each fixed $k \in \mathbb{N}$, let $\{(\mathbb{P}^{n-1}, g_k^i)\}_{i=1}^\infty$ be the sequence of metrics from Proposition 3.3. Then $(\mathbb{P}^{n-1}, g_k^i)$ converge smoothly to the singular metric $(\mathbb{P}^{n-1}, h_k)$ on any compact subset in $r^{-1}((0, \frac{\pi}{2}))$ as $i \rightarrow \infty$. Let $(\mathbb{P}^{n-1}, g_k^i(t))_{t \in [0, T_k^i]}$ be the Ricci flow starting from g_k^i , where $T_k^i > 0$ is the maximal existence time. It is clear that there exists $v_k > 0$ such that $\text{vol}_{h_k}(B_{h_k}(x, 1)) \geq v_k$ for all $x \in \mathbb{P}^{n-1}$, so that $\text{vol}_{g_k^i}(B_{g_k^i}(x, 1)) \geq v_k$ everywhere for all large i . Since $g_k^i(0)$ in addition satisfies $\text{Rm} \geq 0$, it follows from [5, 59], the existence theorem for Ricci flows under almost nonnegative curvature and non-collapsing conditions, that there exist $0 < T_k < 1$ and $C_k > 0$ such that $T_k^i \geq T_k$ for all i , and $|\text{Rm}| \leq \frac{C_k}{t}$ for all $t \in (0, T_k]$. Moreover, using the initial volume bound and the curvature bound $|\text{Rm}| \leq \frac{C_k}{t}$, we can argue exactly as in [80, Corollary 6.2] to get

$$(4.8) \quad \text{vol}_{g_k^i(t)}(B_{g_k^i(t)}(x, 1)) \geq \tilde{v}_k$$

for some $\tilde{v}_k > 0$ depending on v_k, C_k .

After passing to a subsequence, these flows converge smoothly to a Ricci flow $(\mathbb{P}^{n-1}, g_k(t))$, $t \in (0, T_k]$, satisfying $|\text{Rm}| \leq \frac{C_k}{t}$ for some $T_k, C_k > 0$. By Proposition 3.3(c), the curvature of the metrics g_k^i on any compact subset of $r^{-1}((0, \frac{\pi}{2}))$ has a uniform bound for all large i . By Perelman's pseudolocality theorem and Shi's derivative estimates (see for example [54]), this implies that the Ricci flows $\{g_k^i(t)\}$ uniformly smoothly converge on any compact subset of $r^{-1}((0, \frac{\pi}{2}))$ as $t \rightarrow 0$. The limit Ricci flow $g_k(t)$ therefore smoothly converges to h_k on any compact subset of $r^{-1}((0, \frac{\pi}{2}))$ as $t \rightarrow 0$. Finally, convergence of metric spaces $(\mathbb{P}^{n-1}, d_{g_k(t)})$ to $(\mathbb{P}^{n-1}, d_{h_k})$ as $t \rightarrow 0$ follows from standard distance distortion estimates; see for example [5, 59, 81].

Fix $(x, t) \in \mathbb{P}^{n-1} \times (0, T_k]$. It remains to verify that $\lambda(x, t) \geq 1$ for any $x \in \mathbb{P}^{n-1}$ and $t \in (0, T_k]$. To begin with, to estimate the heat kernel $G(x, t; y, s)$, by using (4.8) in (4.7) we obtain $\mathcal{N}_{(x,t)}(t-s) \geq C^{-1}$ for all $s \in [0, \frac{t}{2}]$, where here and below $C > 0$ denotes a generic constant that only depends on the point (x, t) . It follows from (4.6) that for all $(y, s) \in \mathbb{P}^{n-1} \times (0, \frac{t}{2}]$,

$$(4.9) \quad G(x, t; y, s) \leq C.$$

Next, since $\lambda \geq 1$ on $r^{-1}((0, \frac{\pi}{2})) \times \{0\}$ and λ is continuous on $r^{-1}((0, \frac{\pi}{2})) \times [0, T_k]$, for any $\varepsilon > 0$ we can find $\delta \in (0, \frac{t}{2})$ with $\delta \rightarrow 0$ as $\varepsilon \rightarrow 0$ such that for all $(y, s) \in r^{-1}((\delta, \frac{\pi}{2} - \delta)) \times [0, \delta]$, we have

$$(4.10) \quad \lambda(y, s) \geq 1 - \varepsilon.$$

By Lemma 4.1, λ is a super-solution to the heat equation, and so we have the lower bound

$$\begin{aligned} \lambda(x, t) &\geq \int_{\mathbb{P}^{n-1}} \lambda(y, \delta) G(x, t; y, \delta) d_\delta y \\ &\geq (1 - \varepsilon) \int_{r^{-1}(\delta, \frac{\pi}{2} - \delta)} G(x, t; y, \delta) d_\delta y \\ &\geq (1 - \varepsilon) \left(1 - C \cdot \text{vol}_\delta \left(r^{-1}([0, \delta] \cup [\frac{\pi}{2} - \delta, \frac{\pi}{2}])\right)\right), \end{aligned}$$

where in the second inequality we have used (4.10), and in the last inequality we have used $\int_{\mathbb{P}^{n-1}} G(x, t; y, s) d_s y = 1$ for all $s < t$, as well as (4.9). Since the Hausdorff measure is weakly continuous under the Gromov-Hausdorff convergence of $(\mathbb{P}^{n-1}, d_{g_k(t)})$ to $(\mathbb{P}^{n-1}, d_{h_k})$

as $t \rightarrow 0$ (cf. [14]), we deduce that $\text{vol}_\delta \left(r^{-1} \left([0, \delta] \cup \left[\frac{\pi}{2} - \delta, \frac{\pi}{2} \right] \right) \right) \rightarrow 0$ as $\delta \rightarrow 0$. Letting $\varepsilon \rightarrow 0$, this implies that $\lambda(x, t) \geq 1$. $\square$

We conclude this section with the following corollary which implies Theorem B.

Corollary 4.3. *Let $n \geq 2$. Then there exists a sequence of $U(n-1)$ -invariant metrics $\{g_k\}_{k=1}^\infty$ on $\mathbb{P}^{n-1}$ satisfying $\lambda \geq 1$ such that $(\mathbb{P}^{n-1}, d_{g_k})$ converges in the Gromov-Hausdorff sense to the interval $[0, \frac{\pi}{2}]$.*

Proof. For $n = 2$, we have that $U(1) = SO(2)$, and so the sequence of $SO(2)$ -invariant metrics is constructed in [61]. We may therefore assume that $n \geq 3$. Let $g_k(t)$ be the Ricci flow constructed in Lemma 4.2 that smooths out the singular metric h_k . Since h_k converges to $[0, \frac{\pi}{2}]$ and $g_k(t)$ converges to h_k as $t \rightarrow 0$ in the Gromov-Hausdorff sense, we can find a sequence $t_k \rightarrow 0$ so that $g_k(t_k)$ satisfies the assertion of the corollary. $\square$

5. CONSTRUCTION OF THE STEADY RICCI SOLITONS

In this section, we complete the construction of the family of $U(1) \times U(n-1)$ -invariant steady Kähler-Ricci solitons for $n \geq 2$ and prove Theorem A. We present the proof in higher dimensions and point out the essential differences for complex dimension 2. In the proof, we use some standard notions and facts from Alexandrov geometry and metric comparison. We refer the reader to [58, Section 2.4] for a more detailed exposition.

Proof of Theorem A. By Corollary 4.3, we have a sequence of $U(n-1)$ -invariant smooth Kähler metrics $(\mathbb{P}^{n-1}, h_i)$ satisfying the following conditions:

- (1) $\text{diam}(\mathbb{P}^{n-1}, h_i) \rightarrow \frac{\pi}{2}$ as $i \rightarrow \infty$;
- (2) $\text{Rm}(h_i) > 2$ on real $(1, 1)$ -forms (or equivalently, sectional curvature > 4 if $n = 2$);
- (3) $\lim_{i \rightarrow \infty} \text{vol}_{h_i}(\mathbb{P}^{n-1}) = 0$.

In particular, by (2), h_i has positive holomorphic bisectional curvature for all i . For each i , we run a normalized Kähler-Ricci flow $h_i(t)$ on each $(\mathbb{P}^{n-1}, h_i)$. Then by the results of Collins-Székeleyhidi [35] and Tian-Zhang-Zhang-Zhang [85], the normalized flow exists for all time and converges smoothly to a Kähler-Einstein metric on $\mathbb{P}^{n-1}$, which is a positive multiple of the Fubini-Study metric g_{FS} giving the same volume as h_i , by the uniqueness of Kähler-Einstein metrics up to biholomorphism [7]. Since there is a positive lower bound for Rm on real $(1, 1)$ -forms for all times along the Kähler-Ricci flow, and $\text{Rm} > 2$ for $t = 0$, reparametrizing the flow $h_i(t)$ and rescaling it by a suitable time-dependent positive function yields a smooth family of $U(n-1)$ -invariant Kähler metrics $(h_{i,\mu})_{\mu \in [0,1]}$ on $\mathbb{P}^{n-1}$ with

- (1) $h_{i,1} = h_i$ and $h_{i,0} = c_i g_{FS}$ for some $c_i > 0$;
- (2) $\text{diam}(\mathbb{P}^{n-1}, h_{i,1}) = \text{diam}(\mathbb{P}^{n-1}, h_i) \rightarrow \frac{\pi}{2}$ as $i \rightarrow \infty$;
- (3) $\text{Rm}(h_{i,\mu}) > 2$ on real $(1, 1)$ -forms (or equivalently, sectional curvature > 4 if $n = 2$) for all $\mu \in [0, 1]$;
- (4) $\text{vol}_{h_{i,\mu}}(\mathbb{P}^{n-1}) \rightarrow 0$ as $i \rightarrow \infty$ uniformly for all $\mu \in [0, 1]$.

For each fixed i , since the metrics $h_{i,\mu}$ on $\mathbb{P}^{n-1}$ are $U(n-1)$ -invariant and vary smoothly in μ , we see from Proposition 2.17 that there is a $U(1) \times U(n-1)$ -invariant path of Kähler cone metrics $C_{i,\mu}$ on $\mathbb{C}^n \setminus \{0\}$ varying smoothly in μ . The links of these cones are Sasaki

metrics on S^{2n-1} over $h_{i,\mu}$. The above conditions (2) and (4) satisfied by $h_{i,\mu}$, together with (2.18) and (2.19) in Proposition 2.17, then imply that the length of the orbit of the flow of the Reeb vector field tends to 0 as $i \rightarrow \infty$, and the Sasaki metrics over $h_{i,\mu}$ also satisfy conditions (2) and (4). Moreover, by condition (3) and Theorem A.1, the Kähler cones $C_{i,\mu}$ have positive curvature operator on real $(1, 1)$ -forms in the transverse direction. Therefore, by the local uniqueness theorem [37, Theorem 5.3], there exists a unique deformation of $g_{i,0}$ on $\mathbb{C}^n$ by a smooth path of $U(1) \times U(n-1)$ -invariant expanding gradient Kähler-Ricci solitons $(M_{i,\mu}, g_{i,\mu}, p_{i,\mu})$ with positive curvature operator on real $(1, 1)$ -forms asymptotic to the Kähler cone over $h_{i,\mu}$. Here, $p_{i,\mu}$ denotes the critical points of the soliton potential function and $R(p_{i,\mu}) = 1$.

Since the Sasaki metrics over $h_{i,\mu}$ satisfy (4), it follows that the asymptotic volume ratio of the expanding solitons $g_{i,\mu}$ decreases to zero uniformly for all μ as $i \rightarrow \infty$, and so by the same argument as in [61, Lemma 2.3], we can show that for any sequence $\mu_i \in [0, 1]$, the $U(1) \times U(n-1)$ -invariant expanding gradient Kähler-Ricci solitons $(M_{i,\mu_i}, g_{i,\mu_i}, p_{i,\mu_i})$ converge smoothly in the Cheeger-Gromov sense to a $U(1) \times U(n-1)$ -invariant steady gradient Kähler-Ricci soliton. In particular, on one hand, for $\mu_i \equiv 0$, since the Kähler cone is $U(n)$ -invariant, the expanding solitons $g_{i,0}$ are also $U(n)$ -symmetric by the uniqueness theorem in [29], therefore subsequentially converge to a limit steady soliton $(M_{\infty,0}, g_{\infty,0}, p_{\infty,0})$. By Cao's uniqueness result [16, Proposition 2.1], this limit has to be Cao's positively curved $U(n)$ -invariant steady soliton constructed in [16]. On the other hand, for $\mu_i \equiv 1$, we may assume that the expanding solitons $(M_{i,1}, g_{i,1}, p_{i,1})$ converge to the steady soliton $(M_{\infty,1}, g_{\infty,1}, p_{\infty,1})$. Moreover by Remark 2.28, Ric has only one positive distinct eigenvalue at the critical point of the steady soliton. Since the sequence of Sasaki metrics over $h_{i,1}$ also converge to the interval $[0, \frac{\pi}{2}]$ in the Gromov-Hausdorff sense, it follows that the asymptotic Kähler cones of the expanding solitons converge to the positive quadrant in the Euclidean plane $\mathbb{R}^2$.

We now show that the steady Kähler-Ricci soliton $(M_{\infty,1}, g_{\infty,1}, p_{\infty,1})$ must split as a product of a $(2n-2)$ -dimensional $U(n-1)$ -invariant steady Kähler-Ricci soliton, and a two-dimensional $U(1)$ -invariant steady Kähler-Ricci soliton. To this end, first note that since it exhibits $U(1) \times U(n-1)$ symmetry, there exists a $(2n-2)$ -dimensional totally geodesic submanifold N_1 fixed by the $U(1)$ -isometry, and a two-dimensional totally geodesic submanifold N_2 fixed by the $U(n-1)$ -isometry. Similarly, we write $N_{1,i}, N_{2,i}$ for the fixed point sets in the expanding soliton $(M_{i,1}, g_{i,1}, p_{i,1})$ of the $U(1)$ - and $U(n-1)$ -action, respectively. Note that we can find a $U(1)$ -invariant point and a $U(n-1)$ -invariant point in the Sasaki metric over $h_{i,1}$ such that their distance is greater than $\frac{\pi}{2} - \varepsilon_i$, where here and below ε_i denotes a general sequence such that $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$. So arguing as [61, Lemma 4.2], for any $x_i \in N_{1,i}, y_i \in N_{2,i}$ with $d_{g_{i,1}}(x_i, p_{i,1}) = d_{g_{i,1}}(y_i, p_{i,1})$, the comparison angle satisfies $\tilde{\angle} x_i p_{i,1} y_i \geq \frac{\pi}{2} - \varepsilon_i$. In particular, let $\{C_k\}_{k=0}^\infty$ be a sequence of positive numbers going to infinity as $k \rightarrow \infty$. Then for any points $x_{k,i} \in N_{1,i}, y_{k,i} \in N_{2,i}$ with $d_{g_{i,1}}(x_{k,i}, p_{i,1}) = d_{g_{i,1}}(y_{k,i}, p_{i,1}) = C_k$, we have

$$(5.1) \quad \tilde{\angle} x_{k,i} p_{i,1} y_{k,i} \geq \frac{\pi}{2} - \varepsilon_i.$$

On $(M_{\infty,1}, g_{\infty,1}, p_{\infty,1})$, on the one hand, for any points $x_{k,\infty} \in N_1, y_{k,\infty} \in N_2$ with

$$d_{g_{\infty,1}}(x_{k,\infty}, p_{\infty,1}) = d_{g_{\infty,1}}(y_{k,\infty}, p_{\infty,1}) = C_k,$$

let $\gamma_{1,k}, \gamma_{2,k} : [0, C_k] \rightarrow M_{\infty,1}$, be the unit speed minimizing geodesics connecting $p_{\infty,1}, x_{k,\infty}$ and $p_{\infty,1}, y_{k,\infty}$, and let $\gamma_1, \gamma_2 : [0, \infty) \rightarrow M_{\infty,1}$, be two unit speed geodesic rays starting from $p_{\infty,1}$ obtained as any subsequential limit of $\gamma_{1,k}, \gamma_{2,k}$. Then (5.1) implies that the asymptotic angle between γ_1, γ_2 satisfies

$$(5.2) \quad \angle \gamma_1 \gamma_2 = \lim_{k \rightarrow \infty} \tilde{\angle} x_{k,\infty} p_{\infty,1} y_{k,\infty} \geq \frac{\pi}{2}.$$

On the other hand, since the $U(1) \times U(n-1)$ -action induces an isometric action on the tangent space at the unique fixed point $p_{\infty,1}$ and induces an orthogonal decomposition $T_{p_{\infty,1}} M_{\infty,1} = V_1 \perp V_2$, where V_1, V_2 are the $(2n-2)$ -dimensional and 2-dimensional subspaces fixed by the $U(1)$ - and $U(n-1)$ -actions respectively, and $U(n-1)$ acts on V_1 transitively and $U(1)$ acts transitively on V_2 , suppose that $\gamma'_{1,k}(0) = v_{1,k} + v_{2,k}$ and $\gamma'_{2,k}(0) = w_{1,k} + w_{2,k}$, where $v_{1,k}, w_{1,k} \in V_1$ and $v_{2,k}, w_{2,k} \in V_2$. Then

$$(5.3) \quad \langle \gamma'_{1,k}(0), \gamma'_{2,k}(0) \rangle = \langle v_{1,k}, w_{1,k} \rangle + \langle v_{2,k}, w_{2,k} \rangle.$$

By replacing $\gamma_{1,k}$ with its image under a suitable isometry in $U(n-1)$, we can keep invariant $v_{2,k}$ and replace $v_{1,k}$ by any vector of the same norm in V_1 , and thus assume that $\langle v_{1,k}, w_{1,k} \rangle \geq 0$ and equality is achieved if and only if $v_{1,k} = 0$ or $w_{1,k} = 0$. Similarly, by replacing $\gamma_{2,k}$ with its image under a suitable isometry in $U(1)$ we can keep invariant $w_{1,k}$ and replace $w_{2,k}$ by any vector in V_2 , and thus assume that $\langle v_{2,k}, w_{2,k} \rangle \geq 0$ and equality is achieved if and only if $v_{2,k} = 0$ or $w_{2,k} = 0$.

We claim that, after replacing $x_{k,\infty}, y_{k,\infty}$ with their images under suitable $U(n-1)$ - and $U(1)$ -actions respectively, we can assume that $\langle v_{i,k}, w_{i,k} \rangle \geq 0$ for $i = 1, 2$, and thus the angle formed by $\gamma_{1,k}, \gamma_{2,k}$ at $p_{\infty,1}$ satisfies

$$(5.4) \quad \angle(\gamma'_{1,k}(0), \gamma'_{2,k}(0)) \leq \frac{\pi}{2}.$$

Indeed, by replacing $\gamma_{1,k}$ with its image under the $U(n-1)$ -action mapping $v_{1,k}$ to $-v_{1,k}$ if necessary, we have that $\langle v_{1,k}, w_{1,k} \rangle \geq 0$. Then by replacing $\gamma_{2,k}$ with its image under the $U(1)$ -action mapping $w_{2,k}$ to $-w_{2,k}$ if necessary, we have that $\langle v_{2,k}, w_{2,k} \rangle \geq 0$, so we obtain (5.4). The monotonicity of comparison angles then gives us that

$$(5.5) \quad \angle \gamma_1 \gamma_2 = \lim_{k \rightarrow \infty} \tilde{\angle} x_{k,\infty} p_{\infty,1} y_{k,\infty} \leq \limsup_{k \rightarrow \infty} \angle(\gamma'_{1,k}(0), \gamma'_{2,k}(0)) \leq \frac{\pi}{2}.$$

Combining (5.2) and (5.5), we find that $\angle \gamma_1 \gamma_2 = \frac{\pi}{2}$, and so by (5.4) and the monotonicity of comparison angles again, we arrive at the fact that

$$\tilde{\angle} \gamma_1(s) p \gamma_2(s) \equiv \frac{\pi}{2} \quad \text{for all } s \geq 0.$$

We claim that the sectional curvature of the 2-plane σ spanned by $\gamma'_1(0)$ and $\gamma'_2(0)$ at $p_{\infty,1}$ vanishes. Suppose the contrary. Then the positivity of the sectional curvature of σ implies that there is a small neighborhood in the two-dimensional submanifold $\Sigma := \exp_{p_{\infty,1}}(\sigma)$ in which the Gauss curvature is positive with respect to restricted metric g_Σ . However, this would imply that for some small $s > 0$, the g_Σ -minimizing geodesic in Σ connecting $\gamma_1(s)$ and $\gamma_2(s)$ has length strictly smaller than $\sqrt{2}s$, which implies that $\tilde{\angle} \gamma_1(s) p_{\infty,1} \gamma_2(s) < \frac{\pi}{2}$. This is a contradiction. The vanishing of the sectional curvature of σ implies that (5.3) vanishes, and

$$(5.6) \quad \langle v_{1,k}, w_{1,k} \rangle = \langle v_{2,k}, w_{2,k} \rangle = 0.$$

Consequently, we have $w_{1,k} = v_{2,k} = 0$ or $w_{2,k} = v_{1,k} = 0$. This implies that $\gamma'_1(0) \in N_1, \gamma'_2(0) \in N_2$ or $\gamma'_2(0) \in N_1, \gamma'_1(0) \in N_2$. In each case, we obtain

$$(5.7) \quad \text{Rm}(v_1, v_2, v_2, v_1) = 0 \quad \text{for any } v_1 \in V_1 \text{ and } v_2 \in V_2.$$

Since $\text{Rm} \geq 0$ is a symmetric linear operator acting on $\bigwedge^2 T_{p_{\infty,1}} M_{\infty,1}$, this implies that

$$(5.8) \quad \text{Rm}(v_1 \wedge v_2, \cdot) = 0 \quad \text{for any } v_1 \in V_1 \text{ and } v_2 \in V_2.$$

Now we show that the steady Ricci soliton splits isometrically as $M_{\infty,1} = M_1 \times M_2$ for some submanifolds M_1, M_2 of nonzero dimension. Indeed, note that, as a smooth limit of manifolds diffeomorphic to $\mathbb{C}^n$, $M_{\infty,1}$ is diffeomorphic to $\mathbb{C}^n$ and by the strong maximum principle we may assume that $\text{Ric} > 0$ on $M_{\infty,1}$. The results of Bryant [13] and Chau-Tam [24] subsequently infer that $M_{\infty,1}$ is in fact biholomorphic to $\mathbb{C}^n$. We may then argue as in [26, Lemma 4.1] or [33, Theorem 7.34] to deduce that if $M_{\infty,1}$ is irreducible, then $M_{\infty,1}$ has either positive curvature operator on real $(1, 1)$ -forms or is symmetric or is Einstein. The latter two cases are impossible as they would imply constant scalar curvature on $M_{\infty,1}$, contradicting the fact that $\Delta R + \langle \nabla f, \nabla R \rangle = -2|\text{Ric}|^2$ and $\text{Ric} > 0$. $M_{\infty,1}$ must therefore have positive curvature operator on real $(1, 1)$ -forms and positive sectional curvature (see Section 2.3), which is again absurd due to the fact that the sectional curvature vanishes on a certain 2-plane. Thus, we conclude that $M_{\infty,1}$ is indeed reducible and that by the de Rham decomposition theorem, splits isometrically and holomorphically as a product of lower dimensional steady gradient Kähler-Ricci solitons [56].

We now show that the decomposition of the tangent space $T_{p_{\infty,1}} M_{\infty,1} = V_1 \perp V_2$ induces a splitting of the manifold as $M_{\infty,1} = N_1 \times N_2$. First, we claim that the sectional curvature is positive on each N_i at $p_{\infty,1}$ for $i = 1, 2$. Otherwise, suppose a 2-plane of V_1 has zero sectional curvature. Then we would have that the sectional curvature is zero for all 2-planes of V_1 . Indeed, the curvature of the induced Kähler metric on N_1 at the fixed point $p_{\infty,1}$ of $U(n-1)$ has the form

$$-R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \kappa(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}})$$

for some constant $\kappa \geq 0$. For $\kappa > 0$, the ratio between the smallest sectional curvature and the scalar curvature of the induced Kähler metric on N_1 at $p_{\infty,1}$ is a strictly positive dimensional constant (see (2.25)), hence zero sectional curvature at a 2-plane implies zero curvature operator and thus zero sectional curvature for all 2-plane in $T_{p_{\infty,1}} N_1$. This implies that $\text{Ric}(v_i, \cdot) = 0$ for some non-zero $v_i \in V_1$, contradicting Remark 2.28. Next, since $M_{\infty,1} = M_1 \times M_2$, it follows that for any arbitrary non-zero vectors $v_1 + v_2 \in M_1$ and $w_1 + w_2 \in M_2$, where $v_1, w_1 \in V_1$ and $v_2, w_2 \in V_2$, we have that

$$(5.9) \quad R(v_1 + v_2, w_1 + w_2, w_1 + w_2, v_1 + v_2) = 0.$$

By (5.8) and the first Bianchi identity we deduce that $R(v_1, w_1, w_2, v_2) = 0$, which together with (5.8) and (5.9) implies that

$$(5.10) \quad R(v_1, w_1, w_1, v_1) = R(v_2, w_2, w_2, v_2) = 0.$$

Since the sectional curvatures are positive on V_1, V_2 , this implies that $v_1 \wedge w_1 = v_2 \wedge w_2 = 0$. If $v_1, v_2 \neq 0$, then this implies that $w_1 = av_1, w_2 = bv_2$ and $w_1 + w_2 = av_1 + bv_2$ for some $a, b \in \mathbb{R}$. Hence $T_{p_{\infty,1}} M_2 \subset \text{Span}(v_1, v_2)$. Since the dimension of M_2 is at least 2, we have that $T_{p_{\infty,1}} M_2 = \text{Span}(v_1, v_2)$, and in particular $v_1 + v_2 \in T_{p_{\infty,1}} M_2 \cap T_{p_{\infty,1}} M_1$, which is a contradiction. So we have that $v_2 = 0$ or $v_1 = 0$, which implies that $T_{p_{\infty,1}} M_1 \subset V_1$ or $T_{p_{\infty,1}} M_1 \subset V_2$. Without loss of generality we may assume that $T_{p_{\infty,1}} M_1 \subset V_1$. Similarly, we

can deduce either w_1 or w_2 vanishes and thus $T_{p_{\infty,1}}M_2 \subset V_2$ or $T_{p_{\infty,1}}M_2 \subset V_1$. The latter case cannot happen and so we must have $T_{p_{\infty,1}}M_2 \subset V_2$. Therefore, we obtain $T_{p_{\infty,1}}M_i = V_i$ for both $i = 1, 2$. Since N_1, N_2 are the totally geodesic subspaces, it follows that $M_i = N_i$ for $i = 1, 2$, and M_1 (resp. M_2) is a $U(n-1)$ -invariant (resp. $U(1)$ -invariant) steady gradient Kähler Ricci soliton. In particular, they are scalings of Cao's steady gradient Kähler Ricci solitons.

For a 2-plane of $T_{p_{i,\mu}}M_{i,\mu}$ spanned by a vector in the $U(1)$ -fixed subspace and another vector in the $U(n-1)$ -fixed subspace, we can identify it with the 2-plane σ of $T_{p_{\infty,1}}M_{\infty,1}$. Then the sectional curvature $K_{i,\mu} := K_{g_{i,\mu}}(\sigma)$ varies smoothly in μ for each fixed i . We have just demonstrated that $K_{\infty,1} = 0$. It is also clear that $K_{\infty,0} > 0$ in the non-flat $U(n)$ -invariant steady Ricci soliton $(M_{\infty,0}, g_{\infty,0}, p_{\infty,0})$. Therefore, for any given $\alpha \in (0, K_{\infty,0})$, we have that $K_{i,0} < \alpha < K_{i,1}$ for all sufficiently large i , and hence by the intermediate value theorem there exists $\mu_i \in (0, 1)$ such that $K_{i,\mu_i} = \alpha$. A subsequence of the expanding gradient Ricci solitons $(M_{i,\mu_i}, g_{i,\mu_i}, p_{i,\mu_i})$ therefore converges to a steady Kähler-Ricci soliton $(M_{\infty}, g_{\infty}, p_{\infty})$ with $K_{g_{\infty}}(\sigma) = \alpha$ and $R(p_{\infty}) = 1$. This proves Theorem A. $\square$

APPENDIX A. CURVATURE OF DOUBLY-WARPED PRODUCT KÄHLER METRICS

In this appendix, we present the curvature computations for the Riemannian curvature tensor of a doubly-warped product Kähler metric $\hat{g}$ on $\widehat{M} = M \times I$ of the form

$$\hat{g} := ds^2 + a^2(s)\eta \otimes \eta + b^2(s)g^T,$$

where M is a Sasaki manifold of real dimension $(2n-1)$ with $n \geq 2$ with contact one-form η and transverse metric g^T as described in Subsection 2.2, $I \subseteq \mathbb{R}$ is an interval, and $a, b : I \rightarrow [0, \infty)$ are smooth functions. We assume that $a(s) = b(s)b'(s)$ so that $(\widehat{M}, \hat{g})$ does indeed define a Kähler manifold (cf. Lemma 2.11) and we write $\hat{J}$ for the complex structure of $\widehat{M}$ (see Subsection 2.2 for more details). We compute using Cartan's structure equations for the connection one-forms and corresponding curvature two-forms.

A.1. Connection one-forms. We denote the Levi-Civita connection of $\hat{g}$ by $\hat{\nabla}$. We compute in this subsection the connection one-forms of $\hat{\nabla}$. Recall that a differential form σ on a Sasaki manifold is *basic* if $\xi \lrcorner \sigma = 0$ and $\mathcal{L}_{\xi}\sigma = 0$, where ξ is the Reeb vector field.

To this end, let $\theta_1, \dots, \theta_{2n-2}$ be a local basic orthonormal coframe of g^T satisfying $\theta_i \circ \hat{J} = -\theta_{i+1}$ for i odd so that $d\eta = 2\omega^T = 2 \sum_{j=1}^{n-1} \theta_{2j-1} \wedge \theta_{2j}$, and let $(\omega_{ij})_{1 \leq i, j \leq 2n-2}$ denote the matrix of connection one-forms of g^T . Then (ω_{ij}) solves the Cartan structure equations

$$\begin{cases} d\theta_i = \sum_{j=1}^{2n-2} \omega_{ji} \wedge \theta_j, \\ \omega_{ij} + \omega_{ji} = 0. \end{cases}$$

Next set

$$\hat{\theta}_i := b(s)\theta_i \quad \text{for } i = 1, \dots, 2n-2, \quad \hat{\theta}_{2n-1} := ds, \quad \text{and} \quad \hat{\theta}_{2n} := a(s)\eta.$$

Then the matrix of connection one-forms $(\hat{\omega}_{ij})_{1 \leq i, j \leq 2n}$ of $\hat{g}$ with respect to this coframe is given by

$$\hat{\omega}_{ij} = \omega_{ij} - \delta_{j, i+1} \frac{a(s)}{b(s)^2} \hat{\theta}_{2n} \quad 1 \leq i \leq 2n-2 \text{ odd}, \quad 1 \leq j \leq 2n-2.$$

$$\hat{\omega}_{ij} = \omega_{ij} + \delta_{j, i-1} \frac{a(s)}{b(s)^2} \hat{\theta}_{2n} \quad 1 \leq i \leq 2n-2 \text{ even}, \quad 1 \leq j \leq 2n-2.$$

$$\begin{aligned} \hat{\omega}_{2n-1, i} &= \left(-\frac{b'(s)}{b(s)} \right) \hat{\theta}_i = -b'(s) \theta_i, \quad 1 \leq i \leq 2n-2, \\ \hat{\omega}_{2n-1, 2n} &= \left(-\frac{a'(s)}{a(s)} \right) \hat{\theta}_{2n} = -a'(s) \eta, \\ \hat{\omega}_{2n, i} &= \begin{cases} \frac{a(s)}{b(s)^2} \hat{\theta}_{i+1} = \frac{a(s)}{b(s)} \theta_{i+1}, & 1 \leq i \leq 2n-2 \text{ odd}, \\ -\frac{a(s)}{b(s)^2} \hat{\theta}_{i-1} = -\frac{a(s)}{b(s)} \theta_{i-1}, & 1 \leq i \leq 2n-2 \text{ even}. \end{cases} \end{aligned}$$

This translates to the following for $i = 1, 2, \dots, n-1$:

$$\begin{aligned} \hat{\nabla} \hat{\theta}_{2i-1} &= \sum_{k=1}^{2n} \hat{\omega}_{k, 2i-1} \otimes \hat{\theta}_k = \sum_{k=1}^{2n-2} \left(\omega_{k, 2i-1} + \delta_{k, 2i} \left(\frac{a(s)}{b(s)^2} \right) \hat{\theta}_{2n} \right) \otimes \hat{\theta}_k - \frac{b'(s)}{b(s)} \hat{\theta}_{2i-1} \otimes \hat{\theta}_{2n-1} \\ &\quad + \frac{a(s)}{b(s)^2} \hat{\theta}_{2i} \otimes \hat{\theta}_{2n}, \\ \hat{\nabla} \hat{\theta}_{2i} &= \sum_{k=1}^{2n} \hat{\omega}_{k, 2i} \otimes \hat{\theta}_k = \sum_{k=1}^{2n-2} \left(\omega_{k, 2i} - \delta_{k, 2i-1} \left(\frac{a(s)}{b(s)^2} \right) \hat{\theta}_{2n} \right) \otimes \hat{\theta}_k - \frac{b'(s)}{b(s)} \hat{\theta}_{2i} \otimes \hat{\theta}_{2n-1} \\ &\quad - \frac{a(s)}{b(s)^2} \hat{\theta}_{2i-1} \otimes \hat{\theta}_{2n}, \\ \hat{\nabla} \hat{\theta}_{2n-1} &= \sum_{k=1}^{2n} \hat{\omega}_{k, 2n-1} \otimes \hat{\theta}_k = \sum_{k=1}^{2n-2} \frac{b'(s)}{b(s)} \hat{\theta}_k \otimes \hat{\theta}_k + \frac{a'(s)}{a(s)} \hat{\theta}_{2n} \otimes \hat{\theta}_{2n}, \\ \hat{\nabla} \hat{\theta}_{2n} &= \sum_{k=1}^{2n} \hat{\omega}_{k, 2n} \otimes \hat{\theta}_k = \sum_{k=1}^{n-1} -\frac{a(s)}{b(s)^2} \hat{\theta}_{2k} \otimes \hat{\theta}_{2k-1} + \sum_{k=1}^{n-1} \frac{a(s)}{b(s)^2} \hat{\theta}_{2k-1} \otimes \hat{\theta}_{2k} \\ &\quad - \frac{a'(s)}{a(s)} \hat{\theta}_{2n} \otimes \hat{\theta}_{2n-1}. \end{aligned}$$

A.2. Curvature two-forms. Recall that $a(s) = b(s)b'(s)$ so that $(\widehat{M}, \hat{g}, \widehat{J})$ is Kähler. We define the curvature tensor $\hat{R}(\cdot, \cdot)(\cdot)$ of $\hat{g}$ by

$$\hat{R}(X, Y)Z = \hat{\nabla}_X \hat{\nabla}_Y Z - \hat{\nabla}_Y \hat{\nabla}_X Z - \hat{\nabla}_{[X, Y]} Z \quad \text{for } X, Y, Z \in \Gamma(T\widehat{M}).$$

With this, we realize $\hat{R}$ as a $(0, 4)$ -tensor via

$$\hat{R}(X, Y, Z, W) = \langle \hat{R}(X, Y)W, Z \rangle_{\hat{g}} \quad \text{for } X, Y, Z, W \in \Gamma(T\widehat{M}),$$

and write

$$\begin{aligned}\hat{R} &= \hat{R}(\hat{e}_i, \hat{e}_j, \hat{e}_k, \hat{e}_l) \hat{\theta}_i \otimes \hat{\theta}_j \otimes \hat{\theta}_k \otimes \hat{\theta}_l \\ &= \langle \hat{R}(\hat{e}_i, \hat{e}_j) \hat{e}_l, \hat{e}_k \rangle_{\hat{g}} \hat{\theta}_i \otimes \hat{\theta}_j \otimes \hat{\theta}_k \otimes \hat{\theta}_l \\ &= \hat{R}_{ijkl} \hat{\theta}_i \otimes \hat{\theta}_j \otimes \hat{\theta}_k \otimes \hat{\theta}_l,\end{aligned}$$

where

$$(A.1) \quad \hat{R}_{ijkl} = \langle \hat{R}(\hat{e}_i, \hat{e}_j) \hat{e}_l, \hat{e}_k \rangle_{\hat{g}}.$$

We define the curvature two-forms $\hat{\Omega}_{ij}$, $1 \leq i, j \leq 2n$, of $\hat{R}$ by

$$\hat{R} = \hat{\theta}_i \otimes \hat{\theta}_j \otimes \hat{\Omega}_{ij}$$

so that

$$\hat{\Omega}_{ij} = \hat{R}_{ijkl} \hat{\theta}_k \otimes \hat{\theta}_l = \frac{1}{2} \hat{R}_{ijkl} \hat{\theta}_k \wedge \hat{\theta}_l.$$

Then

$$\hat{R}_{ijkl} = \langle \hat{R}(\hat{e}_i, \hat{e}_j) \hat{e}_l, \hat{e}_k \rangle_{\hat{g}} = \hat{\Omega}_{ij}(\hat{e}_k, \hat{e}_l)$$

and $\hat{\Omega}_{ij}$ satisfies the Cartan structure equations

$$\hat{\Omega}_{ij} = d\hat{\omega}_{ij} + \hat{\omega}_{ik} \wedge \hat{\omega}_{kj}.$$

By the symmetries of the curvature tensor, we also see that the two-forms $\hat{\omega}_{ij}$ are real $(1, 1)$ -forms. In this subsection, we compute the curvature two-forms $(\hat{\Omega})_{1 \leq i, j \leq n}$ of $\hat{g}$.

Let Ω_{ij} , $1 \leq i, j \leq 2n - 2$, denote the curvature two-forms of g^T . These satisfy

$$\Omega_{ij} = d\omega_{ij} + \sum_{k=1}^{2n-2} \omega_{ik} \wedge \omega_{kj} \quad \text{for } 1 \leq i, j \leq 2n - 2.$$

We have the freedom to choose the local basic orthonormal coframe $\{\theta_i\}_{i=1}^{2n-2}$ of g^T to be parallel at a point so that $(\omega_{ij})_{1 \leq i, j \leq 2n-2} = 0$ at this point. This we do. With this simplification, a computation shows that at this point, the two-forms Ω_{ij} , $1 \leq i, j \leq 2n - 2$,

are given by

$$\begin{aligned}
\widehat{\Omega}_{ij} &= \Omega_{ij} - \frac{b'(s)^2}{b(s)^2} \left(\hat{\theta}_i \wedge \hat{\theta}_j + \hat{\theta}_{i+1} \wedge \hat{\theta}_{j+1} \right), \quad 1 \leq i, j \leq 2n-2 \text{ odd}, \\
\widehat{\Omega}_{ij} &= \Omega_{ij} - \frac{b'(s)^2}{b(s)^2} \left(\hat{\theta}_i \wedge \hat{\theta}_j + \hat{\theta}_{i-1} \wedge \hat{\theta}_{j-1} \right), \quad 1 \leq i, j \leq 2n-2 \text{ even}, \\
\widehat{\Omega}_{ij} &= \Omega_{ij} - \frac{b'(s)^2}{b(s)^2} \left(\hat{\theta}_i \wedge \hat{\theta}_j - \hat{\theta}_{i+1} \wedge \hat{\theta}_{j-1} \right) - 2\delta_{j,i+1} \left[\left(\frac{a'(s)}{b(s)^2} - \frac{a(s)b'(s)}{b(s)^3} \right) \hat{\theta}_{2n-1} \wedge \hat{\theta}_{2n} \right. \\
&\quad \left. + \left(\frac{a(s)^2}{b(s)^2} \right) \omega^T \right], \quad 1 \leq i \leq 2n-2 \text{ odd}, 1 \leq j \leq 2n-2 \text{ even}, \\
\widehat{\Omega}_{i, 2n-1} &= \left(\frac{b''(s)}{b(s)} \right) \left(\hat{\theta}_{2n-1} \wedge \hat{\theta}_i - \hat{\theta}_{i+1} \wedge \hat{\theta}_{2n} \right), \quad i = 1, \dots, 2n-2, i \text{ odd}, \\
\widehat{\Omega}_{i, 2n-1} &= \left(\frac{b''(s)}{b(s)} \right) \left(\hat{\theta}_{2n-1} \wedge \hat{\theta}_i + \hat{\theta}_{i-1} \wedge \hat{\theta}_{2n} \right), \quad i = 1, \dots, 2n-2, i \text{ even}, \\
\widehat{\Omega}_{2n, i} &= \left(\frac{a'(s)}{b(s)^2} - \frac{a(s)b'(s)}{b(s)^3} \right) \left(\hat{\theta}_{2n-1} \wedge \hat{\theta}_{i+1} + \hat{\theta}_i \wedge \hat{\theta}_{2n} \right), \quad i = 1, \dots, 2n-2, i \text{ odd}, \\
\widehat{\Omega}_{2n, i} &= - \left(\frac{a'(s)}{b(s)^2} - \frac{a(s)b'(s)}{b(s)^3} \right) \left(\hat{\theta}_{2n-1} \wedge \hat{\theta}_{i-1} - \hat{\theta}_i \wedge \hat{\theta}_{2n} \right), \quad i = 1, \dots, 2n-2, i \text{ even}, \\
\widehat{\Omega}_{2n, 2n-1} &= 2 \left(a'(s) - \frac{a(s)b'(s)}{b(s)} \right) \omega^T + \frac{a''(s)}{a(s)} \hat{\theta}_{2n-1} \wedge \hat{\theta}_{2n}.
\end{aligned}$$

Henceforth working at this point at which the basis $\{\theta_i\}_{i=1}^{2n-2}$ of g^T is parallel, we have that $\widehat{J}\hat{e}_{2k-1} = \hat{e}_{2k}$ because $\hat{\theta}_k$ is dual to $\hat{e}_k$. We now define for $k = 1, \dots, n$,

$$\begin{aligned}
\hat{E}_k &= \hat{\theta}_{2k-1} + i \hat{\theta}_{2k}, & \hat{E}_{\bar{k}} &= \hat{\theta}_{2k-1} - i \hat{\theta}_{2k}, \\
\hat{E}_k &= \frac{1}{2} (\hat{e}_{2k-1} - i \hat{e}_{2k}), & \hat{E}_{\bar{k}} &= \frac{1}{2} (\hat{e}_{2k-1} + i \hat{e}_{2k}).
\end{aligned}$$

Similarly, for $k = 1, \dots, n-1$, we define the corresponding frame for g^T :

$$E_k = \frac{1}{2} (e_{2k-1} - i e_{2k}), \quad E_{\bar{k}} = \frac{1}{2} (e_{2k-1} + i e_{2k}).$$

In particular, observe that $\hat{E}_n = \frac{1}{2} \left(\partial_r - i \left(\frac{\xi}{a(s)} \right) \right)$ because by definition, $J_0 \partial_r = \frac{\xi}{a(s)}$. We extend the $(0, 4)$ -tensor $\hat{R}$ by complex multi-linearity and write the corresponding components as

$$\hat{R}_{\alpha\bar{\beta}\gamma\bar{\delta}} = \hat{R}(\hat{E}_\alpha, \hat{E}_{\bar{\beta}}, \hat{E}_\gamma, \hat{E}_{\bar{\delta}}).$$

Then

$$\hat{R} = \left(\hat{\Theta}_\alpha \wedge \hat{\Theta}_{\bar{\beta}} \right) \otimes \widehat{\Omega}_{\alpha\bar{\beta}},$$

where the complex-valued $(1, 1)$ -forms $\widehat{\Omega}_{\alpha\bar{\beta}}$, $1 \leq \alpha, \beta \leq n$, satisfy

$$\widehat{\Omega}_{\alpha\bar{\beta}}(\hat{E}_\gamma, \hat{E}_{\bar{\delta}}) = \hat{R}_{\alpha\bar{\beta}\gamma\bar{\delta}},$$

and are related to the real $(1, 1)$ -forms $\widehat{\Omega}_{ij}$, $1 \leq i, j \leq 2n$, by

$$\widehat{\Omega}_{\alpha\bar{\beta}} = \frac{1}{2} \left(\widehat{\Omega}_{2\alpha-1, 2\beta-1} + i \widehat{\Omega}_{2\alpha-1, 2\beta} \right).$$

For $1 \leq \alpha, \beta \leq n-1$, we have

$$\widehat{\Omega}_{2\alpha-1,2\beta-1} = \Omega_{2\alpha-1,2\beta-1} - \frac{1}{2} \left(\frac{b'(s)}{b(s)} \right)^2 \left[\hat{\Theta}_\alpha \wedge \hat{\Theta}_{\bar{\beta}} + \hat{\Theta}_{\bar{\alpha}} \wedge \hat{\Theta}_\beta \right]$$

and

$$\begin{aligned} \widehat{\Omega}_{2\alpha-1,2\beta} &= \Omega_{2\alpha-1,2\beta} - \frac{i}{2} \left(\frac{b'(s)}{b(s)} \right)^2 \left[\hat{\Theta}_\alpha \wedge \hat{\Theta}_{\bar{\beta}} - \hat{\Theta}_{\bar{\alpha}} \wedge \hat{\Theta}_\beta \right] \\ &\quad - \delta_{\alpha\beta} \left[i \left(\frac{b''(s)}{b(s)} \right) \hat{\Theta}_n \wedge \hat{\Theta}_{\bar{n}} + i \left(\frac{a(s)^2}{b(s)^4} \right) \sum_{l=1}^{n-1} \hat{\Theta}_l \wedge \hat{\Theta}_{\bar{l}} \right]. \end{aligned}$$

It therefore follows that

$$\widehat{\Omega}_{\alpha\bar{\beta}} = \frac{\Omega_{\alpha\bar{\beta}}}{b(s)^2} - \frac{1}{2} \left(\frac{b'(s)}{b(s)} \right)^2 \hat{\Theta}_{\bar{\alpha}} \wedge \hat{\Theta}_\beta + \frac{\delta_{\alpha\beta}}{2} \left[\left(\frac{b''(s)}{b(s)} \right) \hat{\Theta}_n \wedge \hat{\Theta}_{\bar{n}} + \left(\frac{a(s)^2}{b(s)^4} \right) \sum_{l=1}^{n-1} \hat{\Theta}_l \wedge \hat{\Theta}_{\bar{l}} \right]$$

for $1 \leq \alpha, \beta \leq n-1$.

Next, for $1 \leq \alpha \leq n-1, \beta = n$, we have

$$\widehat{\Omega}_{2\alpha-1,2n-1} = \frac{b''(s)}{2b(s)} \left(\hat{\Theta}_n \wedge \hat{\Theta}_{\bar{\alpha}} + \hat{\Theta}_{\bar{n}} \wedge \hat{\Theta}_\alpha \right)$$

and

$$\widehat{\Omega}_{2\alpha-1,2n} = -\frac{i}{2} \left(\frac{b''(s)}{b(s)} \right) \left(\hat{\Theta}_n \wedge \hat{\Theta}_{\bar{\alpha}} - \hat{\Theta}_{\bar{n}} \wedge \hat{\Theta}_\alpha \right).$$

Hence in this case we see that

$$\widehat{\Omega}_{\alpha\bar{n}} = \frac{1}{2} \left(\widehat{\Omega}_{2\alpha-1,2n-1} + i \widehat{\Omega}_{2\alpha-1,2n} \right) = \frac{b''(s)}{2b(s)} \hat{\Theta}_n \wedge \hat{\Theta}_{\bar{\alpha}}.$$

Finally, for $\alpha = n, \beta = n$, we have that

$$\begin{aligned} \widehat{\Omega}_{n\bar{n}} &= \frac{1}{2} \left(\widehat{\Omega}_{2n-1,2n-1} + i \widehat{\Omega}_{2n-1,2n} \right) \\ &= \frac{i}{2} \widehat{\Omega}_{2n-1,2n} \\ &= \frac{1}{2} \left[\left(\frac{b''(s)}{b(s)} \right) \sum_{l=1}^{n-1} \hat{\Theta}_l \wedge \hat{\Theta}_{\bar{l}} + \frac{a''(s)}{2a(s)} \hat{\Theta}_n \wedge \hat{\Theta}_{\bar{n}} \right]. \end{aligned}$$

In summary, $\widehat{\Omega}_{\alpha\bar{\beta}}$ is equal to

$$\begin{cases} \frac{\Omega_{\alpha\bar{\beta}}}{b(s)^2} - \frac{1}{2} \left(\frac{b'(s)}{b(s)} \right)^2 \hat{\Theta}_{\bar{\alpha}} \wedge \hat{\Theta}_\beta + \frac{\delta_{\alpha\beta}}{2} \left[\left(\frac{b''(s)}{b(s)} \right) \hat{\Theta}_n \wedge \hat{\Theta}_{\bar{n}} + \left(\frac{a(s)^2}{b(s)^4} \right) \sum_{l=1}^{n-1} \hat{\Theta}_l \wedge \hat{\Theta}_{\bar{l}} \right] & 1 \leq \alpha, \beta \leq n-1, \\ \left(\frac{b''(s)}{2b(s)} \right) \hat{\Theta}_n \wedge \hat{\Theta}_{\bar{\alpha}} & 1 \leq \alpha \leq n-1, \beta = n, \\ \frac{1}{2} \left[\left(\frac{b''(s)}{b(s)} \right) \sum_{l=1}^{n-1} \hat{\Theta}_l \wedge \hat{\Theta}_{\bar{l}} + \frac{a''(s)}{2a(s)} \hat{\Theta}_n \wedge \hat{\Theta}_{\bar{n}} \right] & \alpha = \beta = n. \end{cases}$$

In particular, this yields that

$$(A.2) \quad \hat{R}_{\alpha\bar{\beta}\gamma\bar{\delta}} = \begin{cases} \frac{\tilde{R}_{\alpha\bar{\beta}\gamma\bar{\delta}}}{b(s)^2} + \frac{1}{2} \left(\frac{b'(s)}{b(s)} \right)^2 \delta_{\alpha\delta} \delta_{\beta\gamma} + \left(\frac{a(s)^2}{2b(s)^4} \right) \delta_{\alpha\beta} \delta_{\gamma\delta} & 1 \leq \alpha, \beta, \gamma, \delta \leq n-1, \\ \left(\frac{b''(s)}{2b(s)} \right) \delta_{\gamma n} \delta_{\delta\alpha} & 1 \leq \alpha \leq n-1, \beta = n, \\ \frac{1}{2} \left[\left(\frac{b''(s)}{b(s)} \right) (\delta_{\gamma\delta} - \delta_{\gamma n} \delta_{\delta n}) + \left(\frac{a''(s)}{2a(s)} \right) \delta_{\gamma n} \delta_{\delta n} \right] & \alpha = n, \beta = n, \end{cases}$$

where $\tilde{R}$ is the curvature tensor of g^T and $\tilde{R}_{\alpha\bar{\beta}\gamma\bar{\delta}} = \tilde{R}(E_\alpha, E_{\bar{\beta}}, E_\gamma, E_{\bar{\delta}})$.

A.3. Curvature of a Kähler cone. Recalling the construction of the n -dimensional Kähler cone from Example 2.10, we have that $a(s) = b(s) = s$, and so from the above we determine that for this example,

$$\hat{\Omega}_{\alpha\bar{\beta}} = \begin{cases} \frac{1}{s^2} \left(\Omega_{\alpha\bar{\beta}} - \frac{1}{2} \hat{\Theta}_{\bar{\alpha}} \wedge \hat{\Theta}_{\beta} + \frac{\delta_{\alpha\beta}}{2} \sum_{l=1}^{n-1} \hat{\Theta}_l \wedge \hat{\Theta}_{\bar{l}} \right) & \text{if } 1 \leq \alpha, \beta \leq n-1, \\ 0 & \text{if } 1 \leq \alpha \leq n, \beta = n. \end{cases}$$

It is well-known that a Riemannian cone has $\text{Rm} \geq 0$ if and only if the link has $\text{Rm} \geq 1$. The Kähler version of this fact is the following.

Theorem A.1 ([9]). *A Kähler cone has strictly positive curvature operator on real $(1, 1)$ -forms in the transverse directions if and only if the link is a Sasaki link whose base satisfies (2.23) with $\lambda = 1$.*

Proof. It suffices to show that for all nonzero real $(1, 1)$ -forms involving only transverse components, namely those of the form $\sum_{\alpha, \beta=1}^{n-1} i u_{\alpha\bar{\beta}} \hat{\Theta}_\alpha \wedge \hat{\Theta}_{\bar{\beta}}$, we have that

$$0 < -\hat{R}_{\alpha\bar{\beta}\gamma\bar{\eta}} u^{\alpha\bar{\beta}} u^{\gamma\bar{\eta}} = \frac{\left(-\tilde{R}_{\alpha\bar{\beta}\gamma\bar{\eta}} - \frac{1}{2} (\delta_{\alpha\beta} \delta_{\gamma\eta} + \delta_{\alpha\eta} \delta_{\beta\gamma}) \right)}{s^2} u^{\alpha\bar{\beta}} u^{\gamma\bar{\eta}},$$

where $u^{\alpha\bar{\beta}} = \hat{g}^{\alpha\bar{\gamma}} \hat{g}^{\eta\bar{\beta}} u_{\eta\bar{\gamma}}$. Since $g^T(E_\alpha, E_{\bar{\beta}}) = g_{\alpha\bar{\beta}}^T = \frac{\delta_{\alpha\beta}}{2}$, this happens precisely when the curvature tensor of the transverse metric satisfies

$$\begin{aligned} -\tilde{R}_{\alpha\bar{\beta}\gamma\bar{\eta}} u^{\alpha\bar{\beta}} u^{\gamma\bar{\eta}} &> \frac{1}{2} (\delta_{\alpha\beta} \delta_{\gamma\eta} + \delta_{\alpha\eta} \delta_{\beta\gamma}) u^{\alpha\bar{\beta}} u^{\gamma\bar{\eta}} \\ &= 2 \left(g_{\alpha\bar{\beta}}^T g_{\gamma\bar{\eta}}^T + g_{\alpha\bar{\eta}}^T g_{\gamma\bar{\beta}}^T \right) u^{\alpha\bar{\beta}} u^{\gamma\bar{\eta}}. \end{aligned}$$

Thus, the transverse metric has curvature operator strictly greater than 2 on real $(1, 1)$ -forms. This proves Theorem A.1. $\square$

In the next lemma, we consider a slightly more general situation than that of the Kähler cone.

Lemma A.2. *Let $\lambda \in \mathbb{R}$ and let $\hat{g}$ be a doubly-warped product Kähler metric on $(0, L) \times S^{2n-1}$, $L > 0$, of the form*

$$\hat{g} = ds^2 + a^2(s) \eta^2 + b^2(s) g^T,$$

where $a, b : (0, L) \rightarrow [0, \infty)$ are smooth functions and (S^{2n-1}, η, g^T) is the round Sasaki structure on S^{2n-1} as described in Example 2.9 with $g^T = \frac{1}{2} g_{FS}$, where g_{FS} is the Fubini-Study metric on $\mathbb{P}^{n-1}$ normalized so that $\text{Ric}(g_{FS}) = n g_{FS}$.

Let $p = (s, w_0) \in (0, L) \times S^{2n-1}$. Then $\hat{g}$ has curvature operator strictly greater than 2λ on $(1, 1)$ -forms at p if and only if for any nonzero hermitian matrix $u^{\alpha\bar{\beta}}$,

$$0 < \frac{2(1 - \lambda(b(s))^2 - (b'(s))^2)}{(b(s))^2} \left(\frac{1}{4} \left(\sum_{\alpha=1}^{n-1} u^{\alpha\bar{\alpha}} \right)^2 + \frac{1}{4} \sum_{\alpha,\beta=1}^{n-1} |u^{\alpha\bar{\beta}}|^2 \right) + \left(-\frac{a''(s)}{4a(s)} - \lambda \right) |u^{n\bar{n}}|^2 \\ + \left(-\frac{b''(s)}{b(s)} - \lambda \right) \sum_{\alpha=1}^{n-1} |u^{\alpha\bar{n}}|^2 + \left(-\frac{b''(s)}{b(s)} - \lambda \right) u^{n\bar{n}} \sum_{\alpha=1}^{n-1} u^{\alpha\bar{\alpha}}.$$

Proof. From the formulas in (A.2), it suffices to express the difference

$$-\hat{R}_{\alpha\bar{\beta}\gamma\bar{\eta}} - 2\lambda(\hat{g}_{\alpha\bar{\beta}}\hat{g}_{\gamma\bar{\eta}} + \hat{g}_{\alpha\bar{\eta}}\hat{g}_{\gamma\bar{\beta}})$$

in terms of the warping functions a and b . To this end, let $p = (s, w_0) \in (0, L) \times S^{2n-1}$ and for any nonzero real $(1, 1)$ -form $\sum_{\alpha,\beta=1}^n i u_{\alpha\bar{\beta}} \hat{\Theta}_\alpha \wedge \hat{\Theta}_{\bar{\beta}}$ at p , let $u^{\alpha\bar{\beta}} = \hat{g}^{\alpha\bar{\varepsilon}} \hat{g}^{\eta\bar{\beta}} u_{\eta\bar{\varepsilon}} = 4u_{\beta\bar{\alpha}}$. Then

(A.3)

$$2\lambda(\hat{g}_{\alpha\bar{\beta}}\hat{g}_{\gamma\bar{\eta}} + \hat{g}_{\alpha\bar{\eta}}\hat{g}_{\gamma\bar{\beta}})u^{\alpha\bar{\beta}}u^{\gamma\bar{\eta}} = \frac{\lambda}{2}(\delta_{\alpha\beta}\delta_{\gamma\eta} + \delta_{\alpha\eta}\delta_{\beta\gamma})u^{\alpha\bar{\beta}}u^{\gamma\bar{\eta}} \\ = \frac{\lambda}{2} \left[\left(u^{n\bar{n}} + \sum_{\alpha=1}^{n-1} u^{\alpha\bar{\alpha}} \right)^2 + \sum_{\alpha,\beta=1}^n |u^{\alpha\bar{\beta}}|^2 \right] \\ = \frac{\lambda}{2} \left[2|u^{n\bar{n}}|^2 + 2u^{n\bar{n}} \sum_{\alpha=1}^{n-1} u^{\alpha\bar{\alpha}} + \left(\sum_{\alpha=1}^{n-1} u^{\alpha\bar{\alpha}} \right)^2 + \sum_{\alpha,\beta=1}^{n-1} |u^{\alpha\bar{\beta}}|^2 + 2 \sum_{\alpha=1}^{n-1} |u^{\alpha\bar{n}}|^2 \right].$$

Here we have used the fact that $\overline{u^{\alpha\bar{n}}} = u^{n\bar{\alpha}}$ in the last equality. By the symmetries of the curvature tensor, we moreover have that

$$-\sum_{\alpha,\beta,\gamma,\eta=1}^n \hat{R}_{\alpha\bar{\beta}\gamma\bar{\eta}} u^{\alpha\bar{\beta}} u^{\gamma\bar{\eta}} = \sum_{\alpha,\beta,\gamma,\eta=1}^{n-1} -\hat{R}_{\alpha\bar{\beta}\gamma\bar{\eta}} u^{\alpha\bar{\beta}} u^{\gamma\bar{\eta}} - \hat{R}_{n\bar{n}n\bar{n}} u^{n\bar{n}} u^{n\bar{n}} \\ - 2 \sum_{\alpha=1}^{n-1} u^{\alpha\bar{n}} u^{n\bar{n}} \hat{R}_{\alpha\bar{n}n\bar{n}} - 2 \sum_{\alpha=1}^{n-1} u^{n\bar{\alpha}} u^{n\bar{n}} \hat{R}_{n\bar{\alpha}n\bar{n}} \\ - 2 \sum_{\alpha,\beta=1}^{n-1} u^{\alpha\bar{\beta}} u^{n\bar{n}} \hat{R}_{\alpha\bar{\beta}n\bar{n}} - 2 \sum_{\alpha,\beta=1}^{n-1} u^{\alpha\bar{n}} u^{n\bar{\beta}} \hat{R}_{\alpha\bar{n}n\bar{\beta}} \\ - \sum_{\alpha,\beta=1}^{n-1} u^{n\bar{\alpha}} u^{n\bar{\beta}} \hat{R}_{n\bar{\alpha}n\bar{\beta}} - \sum_{\alpha,\beta=1}^{n-1} u^{\alpha\bar{n}} u^{\beta\bar{n}} \hat{R}_{\alpha\bar{n}\beta\bar{n}} \\ - 2 \sum_{\alpha,\beta,\gamma=1}^{n-1} u^{\alpha\bar{\beta}} u^{n\bar{\gamma}} \hat{R}_{\alpha\bar{\beta}n\bar{\gamma}} - 2 \sum_{\alpha,\beta,\gamma=1}^{n-1} u^{\alpha\bar{\beta}} u^{\gamma\bar{n}} \hat{R}_{\alpha\bar{\beta}\gamma\bar{n}}.$$

This expression can be simplified using the fact that

$$\hat{R}_{\alpha\bar{\beta}n\bar{\gamma}} = \hat{R}_{\alpha\bar{\beta}\gamma\bar{n}} = \hat{R}_{\alpha\bar{n}n\bar{n}} = \hat{R}_{n\bar{\alpha}n\bar{n}} = \hat{R}_{\alpha\bar{n}\beta\bar{n}} = \hat{R}_{n\bar{\alpha}n\bar{\beta}} = 0.$$

Since the the curvature tensor of $\frac{1}{2}g_{FS}$ is given by $-\frac{1}{2}(\delta_{\alpha\beta}\delta_{\gamma\eta} + \delta_{\alpha\eta}\delta_{\beta\gamma})$, we see that

$$\begin{aligned} \sum_{\alpha,\beta,\gamma,\eta=1}^{n-1} -\hat{R}_{\alpha\bar{\beta}\gamma\bar{\eta}} u^{\alpha\bar{\beta}} u^{\gamma\bar{\eta}} &= \sum_{\alpha,\beta,\gamma,\eta=1}^{n-1} \left[\frac{(\delta_{\alpha\beta}\delta_{\gamma\eta} + \delta_{\alpha\eta}\delta_{\beta\gamma})}{2b^2} - \frac{(b')^2(\delta_{\alpha\beta}\delta_{\gamma\eta} + \delta_{\alpha\eta}\delta_{\beta\gamma})}{2b^2} \right] u^{\alpha\bar{\beta}} u^{\gamma\bar{\eta}} \\ &= \frac{1-(b')^2}{2b^2} \left[\left(\sum_{\alpha=1}^{n-1} u^{\alpha\bar{\alpha}} \right)^2 + \sum_{\alpha,\beta=1}^{n-1} |u^{\alpha\bar{\beta}}|^2 \right]. \end{aligned}$$

By the curvature formulas given in (A.2), we find that

$$-\hat{R}_{n\bar{n}n\bar{n}} u^{n\bar{n}} u^{n\bar{n}} = -\frac{a''}{4a} |u^{n\bar{n}}|^2,$$

$$\begin{aligned} -2 \sum_{\alpha,\beta=1}^{n-1} u^{\alpha\bar{\beta}} u^{n\bar{n}} \hat{R}_{\alpha\bar{\beta}n\bar{n}} &= -2 \sum_{\alpha,\beta=1}^{n-1} u^{\alpha\bar{\beta}} u^{n\bar{n}} \frac{b''}{2b} \delta_{\alpha\beta} = -\frac{b''}{b} u^{n\bar{n}} \sum_{\alpha=1}^{n-1} u^{\alpha\bar{\alpha}}, \\ -2 \sum_{\alpha,\beta=1}^{n-1} u^{\alpha\bar{n}} u^{n\bar{\beta}} \hat{R}_{\alpha\bar{n}n\bar{\beta}} &= -\sum_{\alpha,\beta=1}^{n-1} u^{\alpha\bar{n}} u^{n\bar{\beta}} \frac{b''}{b} \delta_{\alpha\beta} = -\frac{b''}{b} \sum_{\alpha=1}^{n-1} |u^{\alpha\bar{n}}|^2, \end{aligned}$$

and so

$$\begin{aligned} - \sum_{\alpha,\beta,\gamma,\eta=1}^n \hat{R}_{\alpha\bar{\beta}\gamma\bar{\eta}} u^{\alpha\bar{\beta}} u^{\gamma\bar{\eta}} &= \frac{1-(b')^2}{2b^2} \left[\left(\sum_{\alpha=1}^{n-1} u^{\alpha\bar{\alpha}} \right)^2 + \sum_{\alpha,\beta=1}^{n-1} |u^{\alpha\bar{\beta}}|^2 \right] - \frac{a''}{4a} |u^{n\bar{n}}|^2 \\ (A.4) \quad &\quad -\frac{b''}{b} u^{n\bar{n}} \sum_{\alpha=1}^{n-1} u^{\alpha\bar{\alpha}} - \frac{b''}{b} \sum_{\alpha=1}^{n-1} |u^{\alpha\bar{n}}|^2. \end{aligned}$$

The lemma now follows by combining (A.3) and (A.4), together with the fact that $u^{\alpha\bar{\beta}}$ is an arbitrary nonzero real $(1, 1)$ -vector. $\square$

APPENDIX B. COMPARISON WITH THE STEADY SOLITONS OF APOSTOLOV-CIFARELLI

Let $(z_1, \dots, z_n)$ denote coordinates on $\mathbb{C}^n$ with $z_k = x_k + iy_k$ and suppose that $(\mathbb{C}^n, g, X)$ is a steady Kähler-Ricci soliton with $\text{Ric}_g = \frac{1}{2}\mathcal{L}_X g$, where

$$(B.1) \quad X = 2 \operatorname{Re} \left(\sum_{\alpha=1}^n a_\alpha z_\alpha \partial_{z_\alpha} \right) = \sum_{\alpha=1}^n a_\alpha (x_\alpha \partial_{x_\alpha} + y_\alpha \partial_{y_\alpha}) \quad \text{with } a_\alpha \in \mathbb{R}.$$

We begin with a general lemma.

Lemma B.1. *If $\text{Ric}_g > 0$ at $0 \in \mathbb{C}^n$, then $a_\alpha > 0$ for all $\alpha = 1, \dots, n$.*

Proof. Without loss of generality, suppose that $a_1 \leq 0$ at $0 \in \mathbb{C}^n$. Let ∇ denote the Levi-Civita connection of g . Then

$$\begin{aligned}
0 < \text{Ric}_g(\partial_{x_1}, \partial_{x_1}) &= \frac{1}{2} \mathcal{L}_X g(\partial_{x_1}, \partial_{x_1}) \\
&= g\left(\nabla_{\partial_{x_1}} X, \partial_{x_1}\right) \\
&= \sum_{\alpha=1}^n a_\alpha g\left(\nabla_{\partial_{x_1}} (x_\alpha \partial_{x_\alpha} + y_\alpha \partial_{y_\alpha}), \partial_{x_1}\right) \\
&= \sum_{\alpha=1}^n a_\alpha g\left(\delta_{\alpha 1} \partial_{x_\alpha} + x_\alpha \nabla_{\partial_{x_1}} \partial_{x_\alpha}, \partial_{x_1}\right) + \sum_{\alpha=1}^n a_\alpha g\left(y_\alpha \nabla_{\partial_{x_1}} \partial_{y_\alpha}, \partial_{x_1}\right) \\
&= a_1 g(\partial_{x_1}, \partial_{x_1}) \quad \text{at } 0 \in \mathbb{C}^n.
\end{aligned}$$

This is a contradiction. $\square$

Next, we show that there exist holomorphic coordinates that linearise the soliton vector field of the Kähler flying wing steady solitons.

Lemma B.2. *For $(\mathbb{C}^n, g, X)$ the Kähler flying wing steady solitons, holomorphic coordinates exist so that X takes the form given in (B.1).*

Proof. The expanding Kähler-Ricci solitons in [37] are asymptotically conical, hence the scalar curvature has a maximum value in the interior of the manifold. Since these expanding solitons have positive Ricci curvature, it follows from the soliton identities that the maximum value of the scalar curvature is achieved at a zero of the soliton vector field. By construction, this is a single point. We denote this point by $0 \in \mathbb{C}^n$.

Now, the Kähler flying wing steady solitons are obtained as the pointed Cheeger-Gromov limits of the aforementioned expanding solitons based at the point 0, after rescaling the scalar curvature at 0 to be equal to 1. It follows that the scalar curvature of the limiting steady soliton g also has a maximum value of 1 at its base point. Since $\text{Ric}_g > 0$, by the soliton identities X vanishes at this point. The fact that the limiting soliton potential is convex because $\text{Ric}_g > 0$ then implies that this zero of X is unique.

Finally, by [13, Proposition 6], we can choose local holomorphic coordinates on $\mathbb{C}^n$ centred at the zero of X so that X takes the form (B.1). Since the zero of X is unique, these coordinates can be extended globally by [38, Proposition 2.28] on $\mathbb{C}^n$ so that the representation (B.1) of X is global. $\square$

Finally, we show that the soliton vector field of our Kähler flying wing steady solitons is a multiple of the Euler vector field on $\mathbb{C}^n$.

Proposition B.3. *For $(\mathbb{C}^n, g, X)$ the Kähler flying wing steady solitons in Theorem A, we have $a_1 = \dots = a_n > 0$ in (B.1).*

Proof. By Remark 2.27, the Ricci tensor of the expanding Kähler-Ricci solitons constructed in [37] is diagonal at $0 \in \mathbb{C}^n$, the unique critical point of the soliton vector field. It follows that $\text{Ric}_g = \mu g$ for some $\mu \in \mathbb{R}$ at $0 \in \mathbb{C}^n$ for the Kähler flying wing steady solitons. Since

$\text{Ric}_g > 0$, we see from Lemma B.1 above that $\mu > 0$. For any index $\alpha = 1, \dots, n$, we then have that

$$\begin{aligned}
\mu g(\partial_{x_\alpha}, \partial_{x_\alpha}) &= \text{Ric}_g(\partial_{x_\alpha}, \partial_{x_\alpha}) = \frac{1}{2} \mathcal{L}_X g(\partial_{x_\alpha}, \partial_{x_\alpha}) \\
&= g(\nabla_{\partial_{x_\alpha}} X, \partial_{x_\alpha}) \\
&= \sum_{\beta=1}^n a_{\beta} g(\nabla_{\partial_{x_\alpha}} (x_{\beta} \partial_{x_{\beta}} + y_{\beta} \partial_{y_{\beta}}), \partial_{x_\alpha}) \\
&= \sum_{\beta=1}^n a_{\beta} g(\delta_{\alpha\beta} \partial_{x_{\beta}} + x_{\beta} \nabla_{\partial_{x_\alpha}} \partial_{x_{\beta}}, \partial_{x_\alpha}) + \sum_{\beta=1}^n a_{\beta} g(y_{\beta} \nabla_{\partial_{x_\alpha}} \partial_{y_{\beta}}, \partial_{x_\alpha}) \\
&= a_{\alpha} g(\partial_{x_\alpha}, \partial_{x_\alpha}) \quad \text{at } 0 \in \mathbb{C}^n,
\end{aligned}$$

i.e., $a_{\alpha} = \mu > 0$ for all $\alpha = 1, \dots, n$. □

Next recall the steady solitons from [2, Theorem 1.2]. We consider this theorem with $\ell = 2$, $d_1 = 0$, and $d_2 = n - 2$. In this case, this theorem yields complete $U(1) \times U(n - 1)$ -invariant steady solitons on $\mathbb{C}^n$. By [2, Lemma 5.1], the soliton vector field X is given by

$$X = -\frac{a}{(\alpha_2 - \alpha_1)} \left(-\frac{q(\alpha_1)}{2} X_1 + \frac{q(\alpha_2)}{2(n-1)} X_2 \right),$$

where X_1 is the Euler vector field on the first $\mathbb{C}$ -factor and X_2 is the Euler vector field on the second $\mathbb{C}^{n-1}$ -factor of $\mathbb{C}^n$, where $q(t)$ is a polynomial of degree one, and $-\infty < \alpha_1 < \alpha_2 < \infty$ (cf. [2, Section 5.1]). Write $q(t) = q_0 + q_1 t$ and note that we can always normalise by translation and scaling so that $\alpha_1 = 0$ and $\alpha_2 = 1$. Then it is clear that

Lemma B.4. *X coincides with a multiple of the Euler vector field on $\mathbb{C}^n$ if and only if $q_1 = -nq_0$.*

The next proposition shows that this can never be the case.

Proposition B.5. *The vector field X of the steady solitons of [2, Theorem 1.2] with $U(1) \times U(n - 1)$ -symmetry can never be a multiple of the Euler vector field on $\mathbb{C}^n$.*

Proof. In this case we have that $p_c(t) = (t - 1)^{n-2}$ (cf. [2, p.14]). Suppose that the proposition is false. Then by Lemma B.4 we have that $q(t) = 1 - nt$ up to a non-zero scalar multiple. By construction, the solution F of the ODE satisfies [2, equation (5.1)]:

$$0 = F(1) = e^{-2a} \int_0^1 e^{2ax} q(x) p_c(x) dx,$$

which by definition of $p_c(t)$ and $q(t)$ equates to

$$0 = \int_0^1 e^{2ax} (1 - nx)(x - 1)^n dx.$$

Here $a > 0$ is a real constant. Then

$$\begin{aligned}
0 &= \int_0^1 (x(x-1)^{n-1}e^{2ax})' dx \\
&= \int_0^1 (x-1)^{n-1}e^{2ax} dx + (n-1) \int_0^1 x(x-1)^{n-2}e^{2ax} dx + 2a \int_0^1 x(x-1)^{n-1}e^{2ax} dx \\
&= \int_0^1 ((x-1) + (n-1)x)(x-1)^{n-2}e^{2ax} dx + 2a \int_0^1 x(x-1)^{n-1}e^{2ax} dx \\
&= - \underbrace{\int_0^1 (1-nx)(x-1)^{n-2}e^{2ax} dx}_{=0} + 2a \int_0^1 x(x-1)^{n-1}e^{2ax} dx.
\end{aligned}$$

This is a contradiction because $x(x-1)^{n-1}e^{2ax}$ has a sign on $(0, 1)$. $\square$

Finally, the steady solitons from [2, Theorem 1.4]. can never have a soliton vector field a scalar multiple of the Euler vector field on $\mathbb{C}^n$; cf. [2, Section 5.3]. As a result of these observations, we obtain

Corollary B.6. *The solitons of Theorem A are non-isometric to those of [2].*

Proof. If a steady soliton of [2] doesn't have positive sectional curvature or is not $U(1) \times U(n-1)$ -invariant, then the result is clear. So suppose that it has both of these properties. Then any isometry will map a critical point of the scalar curvature to a critical point of the scalar curvature. By the soliton identities, the fact that the Ricci curvature is positive implies that for each steady soliton, the critical points of the scalar curvature and the zero set of the soliton vector field coincide. Each soliton vector field has one zero at the origin and positive Ricci curvature implies that this zero is unique, hence any isometry will map the origin to the origin. At every point, the steady soliton equation implies that the eigenvalues of the Ricci tensor coincide with those of the Hessian of the potential function. In particular, at the origin, being a zero of the soliton vector field, the Hessian of the soliton potential is independent of the choice of metric. This yields a contradiction. Indeed, at the origin, Proposition B.3 implies that the Ricci tensor of the steady solitons of Theorem A is a multiple of the identity, whereas the other observations above imply that this is not the case for those steady solitons of [2] admitting $U(1) \times U(n-1)$ -symmetry. $\square$

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