A note on best n-term approximation for generalized Wiener classes

Ronald DeVore, Guergana Petrova, Przemysław Wojtaszczyk

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Abstract

We determine the best n-term approximation of generalized Wiener model classes in a Hilbert space H. This theory is then applied to several special cases.

1 Introduction

One of the main themes in approximation theory is to prove theorems on how well functions can be approximated in a Banach space norm $\|\cdot\|_X$ by methods of linear or nonlinear approximation. The present paper is exclusively concerned with approximation in a separable Hilbert space H equipped with norm $\|\cdot\|$, induced by a scalar product $\langle\cdot,\cdot\rangle$. Let $\mathcal{D} := \{\phi_i,\, i \in \mathbb{N}\}$ be an orthonormal basis for H. This means that any function $f \in H$ has the unique representation

$$f = \sum_{j=1}^{\infty} f_j \phi_j, \quad \text{where} \quad \|f\|^2 = \sum_{j=1}^{\infty} |f_j|^2.$$

We are concerned with n term approximation of the elements $f \in H$. We denote by $\Sigma_n := \{S = \sum_{j\in\Lambda} c_j e_j : \Lambda \subset \mathbb{N},\, |\Lambda| = n\}$ and let

$$\sigma_n(f) := \inf_{S\in\Sigma_n} \|f - S\|, \quad n \geq 1,$$

be the *error of n-term approximation* of f. Given any $f \in H$, a best n-term approximation S_n of f is given by

$$S_n = S_n(f) := \sum_{j\in\Lambda_n} f_j \phi_j,$$

where $\Lambda_n := \Lambda_n(f)$ is a set of n indices j for which $|f_j| \geq |f_i|$ whenever $j \in \Lambda$ and $i \notin \Lambda$. Even though the set $\Lambda_n(f)$ is not uniquely defined because of possible ties in terms of the size of the absolute values of the coefficients f_j, the error of approximation $\sigma_n(f)$ is uniquely defined.

We are interested in model classes $K \subset H$ that are given by imposing conditions on the coefficients (f_j) of f. For such sets K, we define

$$\sigma_n(K) := \sup_{f\in K} \sigma_n(f) \tag{1.1}$$

and we are interested in the asymptotic decay of $\sigma_n(K) \to 0$ as $n \to \infty$.

Since we are only considering the approximation to take place in H, in going further it is sufficient to consider only the case

$$H = \ell_2 := \left\{ \mathbf{x} = (x_1, x_2, \dots,) : \|x\|_{\ell_2}^2 := \sum_{j=1}^{\infty} |x_j|^2 < \infty \right\}.$$

A classical result in this case is the following. Let $0 < p < 2$, and consider the unit ball in ℓ_p,

$$K = U(\ell_p) := \left\{ \mathbf{x} = (x_1, x_2, \dots,) : \|x\|_{\ell_p}^p := \sum_{j=1}^{\infty} |x_j|^p \le 1 \right\} \subset \ell_2.$$

It is known in this case that

$$\sigma_n(U(\ell_p)) \asymp n^{-1/p+1/2}, \quad n \to \infty, \tag{1.2}$$

with absolute constants in the equivalency[1] . This result is attributed to Stechkin [9].

Other results of the above type have been frequently obtained in the literature. To describe a general setting, let

$$\mathbf{w} := (w_j)_{j \in \mathbb{N}}, \quad 1 \le w_1 \le w_2 \le \dots,$$

be a monotonically nondecreasing sequence of positive weights. We consider the weighted ℓ_p space $\ell_p(\mathbf{w})$ defined as the set of all real valued sequences $\mathbf{x} \in \ell_2$ such that

$$\ell_p(\mathbf{w}) := \{\mathbf{x} \in \ell_2 : \|\mathbf{x}\|_{\ell_p(\mathbf{w})} < \infty\}, \quad 0 < p \le \infty,$$

where

$$\|\mathbf{x}\|_{\ell_p(\mathbf{w})} := \begin{cases} \left[\sum_{j=1}^{\infty} |w_j x_j|^p\right]^{1/p}, & 0 < p < \infty, \\ \\ \sup_j |w_j x_j|, & p = \infty. \end{cases} \tag{1.3}$$

We denote by $U(\ell_p(\mathbf{w}))$ the unit ball of this class

$$U(\ell_p(\mathbf{w}))) := \{\mathbf{x} \in \ell_p(\mathbf{w}) : \|\mathbf{x}\|_{\ell_p(\mathbf{w})} \le 1\}$$

and derive the rate of the error $\sigma_n(U_p(\mathbf{w}))$, see Theorem 2.2. It can happen that $\sigma_n(U_p(\mathbf{w}))$ is infinite for all n. Also, note that the class $\ell_p(\mathbf{w})$ is different from the classical weighted sequence spaces since the weights in the $\ell_p(\mathbf{w})$ norm are raised to a power. The problem considered here has been investigated in the context of best n-term approximation of diagonal operators in the general case of approximation in ℓ_q. The works of Stepanets, see [6, 7, 8], determine the rate of $\sigma_n(U(\ell_p(\mathbf{w}))))_{\ell_q}$ for general $0 < q \le \infty$ under the restriction $\lim_{k\to\infty} w_k = \infty$. Recently, in [4], based on results from [1], the condition on the sequence $\mathbf{w}$ in the case $0 < p < q$ has been removed, see Theorem 2.1(i). The case $q < p < \infty$ has also been considered, but under additional assumptions on the weight sequence $\mathbf{w}$, see Theorem 2.1(ii). In this paper, we consider only the case $q = 2$ and provide a simple, different unified approach for finding the rate of $\sigma_n(U(\ell_p(\mathbf{w}))))_{\ell_2}$ for all $0 < p \le \infty$ with no restriction on the weight $\mathbf{w}$.

[1]We use the notation $A \asymp B$ when there are absolute constants$C_1, C_2 > 0$such that we have $C_1 B \le A \le C_2 B$

We then go on to apply our result in the case of several special weights

$$w_j = j^\alpha(1+\log j)^\beta, \quad j \in \mathbb{N},$$

provided $\alpha > 0$, $\beta \in \mathbb{R}$, or $\alpha = 0$, $\beta \geq 0$. We show in Corollary 3.1 that when $\alpha = 0$, $\beta \geq 0$,

$$\sigma_n(U(\ell_p(\mathbf{w})) \asymp n^{-(1/p-1/2)}[\log n]^{-\beta}, \quad n > 1, \quad 0 < p < 2.$$

In the case $\alpha = \beta = 0$, this recovers Stechkin's result (1.2). In Corollary 3.2, we consider the case $\alpha > 0$, $\beta \in \mathbb{R}$, $0 < p \leq \infty$ and show that

$$U(\ell_p(\mathbf{w}))) \asymp n^{-(\alpha+1/p-1/2)}[\log n]^{-\beta}, \quad n > 1, \quad \alpha + 1/p - 1/2 > 0.$$

We call $\ell_p(\mathbf{w})$ a *generalized Wiener class* in analogy with the definition of Wiener spaces in Fourier analysis when $H = L_2([0,1])$ and ϕ_j is the Fourier basis. Our results have some overlap with the study of Wiener classes in the Fourier setting, that is, when $\mathcal{D}$ is the Fourier basis $\mathcal{F}$. When considering the specific case of Fourier basis, our results, which are restricted to approximation in Hilbert spaces, are valid for L_2. Several results in the literature consider the approximation of Wiener classes in L_q when the basis is the Fourier basis $\mathcal{F}$. For example, the case $\alpha > 1/2$, $\beta = 0$, and $p = 1$ has been analyzed in [2] and upper bounds for the error in L_∞, have been obtained for the Wiener spaces with $\mathcal{D}$ being the d-dimensional Fourier basis $\mathcal{F}^d$, see Lemma 4.3(i). Recently, these results have been improved in [3], where matching up to logarithm lower and upper bounds for multidimensional Wiener spaces with $\mathcal{D} = \mathcal{F}^d$ are given for the case $\beta = 0$, $\alpha > 0$, $0 < p \leq 1$, see Corollary 4.3 in [3], and $\beta = 0$, $\alpha > 1 - 1/p$, $1 < p \leq q$, all when the error is measured in L_q, $2 \leq q \leq \infty$, see Theorem 4.5 in [3]. In particular, when the dimension $d = 1$ and the error is measured in the Hilbert space norm (i.e. $q = 2$), the results from [3] give the rate $\sigma_n(U_p(\mathbf{w}, \mathcal{F}))_{L_2} \asymp n^{-(\alpha+1/p-1/2)}$, provided

$$\alpha > 0, \quad 0 < p \leq 1, \quad \text{or} \quad \alpha > 1 - 1/p, \quad 1 < p \leq 2.$$

This latter result is a special case of our analysis.

2 Best n term approximation for $U(\ell_p(\mathbf{w}))$

In going forward, we assume $H = \ell_2$ with its canonical basis e_j, $j \in \mathbb{N}$. Before presenting our main theorem, let us introduce the decreasing rearrangement

$$\mathbf{x}^* = (x_j^*)_{j\in\mathbb{N}}$$

of the absolute values of the coordinates x_j of a sequence $\mathbf{x} = (x_j)_{j\in\mathbb{N}}$ that is an element of the sequence space $\mathbf{c_0}$ (consisting of all sequences whose elements converge to 0). Namely, we have that x_1^* is the largest of the numbers $|x_j|$, $j \in \mathbb{N}$, then x_2^* is the next largest, and so on. It follows that

$$x_1^* \geq x_2^* \geq \dots,$$

and $\|\mathbf{x}\|_{\ell_p} = \|\mathbf{x}^*\|_{\ell_p}$ for all $0 < p \leq \infty$. For each $n \geq 1$ and $\mathbf{x} \in \ell_2$, we have that

$$\sigma_n(\mathbf{x}) = \left[\sum_{j>n}[x_j^*]^2\right]^{1/2}, \tag{2.1}$$

where $\sigma_n(\mathbf{x})$ is the error of n-term approximation of $\mathbf{x}$ in the ℓ_2 norm. In order to prove our main result, we will need the following lemma.

Lemma 2.1. *If* $\mathbf{x} \in \ell_p(\mathbf{w})$*, then* $\mathbf{x}^* \in \ell_p(\mathbf{w})$, $\sigma_n(\mathbf{x}) = \sigma_n(\mathbf{x}^*)$*, and*

$$\|\mathbf{x}^*\|_{\ell_p(\mathbf{w})} \leq \|\mathbf{x}\|_{\ell_p(\mathbf{w})}. \tag{2.2}$$

Proof. It follows directly from the definitions of σ_n and $\mathbf{x}^*$ that $\sigma_n(\mathbf{x}) = \sigma_n(\mathbf{x}^*)$. Let the sequence $\mathbf{x} = (x_1, x_2, \dots)$ be in $\ell_p(\mathbf{w})$. We can assume that all x_j are non-negative since changing the signs of its entries does not effect neither its rearrangement nor its $\ell_p(\mathbf{w})$ norm. We shall first construct sequences $\mathbf{y}^{(m)} = (y_1^{(m)}, y_2^{(m)}, \dots)$, $m = 0, 1, \dots$, such that each $\mathbf{y}^{(m+1)}$ is gotten by swapping the positions of two of the entries in $\mathbf{y}^{(m)}$ with the indices of these entries each larger than m. Also, the first m entries of $\mathbf{y}^{(m)}$ satisfy

$$y_j^{(m)} = x_j^*, \quad j = 1, \dots, m.$$

Indeed, we start with $\mathbf{y}^0 = \mathbf{x}$. Assuming that $\mathbf{y}^{(m)}$ has been defined, we let $j > m$ be the smallest index larger than m such that $y_j^{(m)}$ is the largest of the entries $y_k^{(m)}$, $k > m$. We swap the entries with positions $m+1$ and j to create the sequence $\mathbf{y}^{(m+1)}$ from $\mathbf{y}^{(m)}$. We have for $m = 0, 1, 2, \dots$,

$$\|\mathbf{y}^{(m+1)}\|_{\ell_p(\mathbf{w})} \leq \|\mathbf{y}^{(m)}\|_{\ell_p(\mathbf{w})} \leq \dots \leq \|\mathbf{y}^{(0)}\|_{\ell_p(\mathbf{w})} = \|\mathbf{x}\|_{\ell_p(\mathbf{w})},$$

because the weights in $\mathbf{w}$ are non-decreasing. Note that for every m

$$\sum_{j=1}^{m} [w_j x_j^*]^p = \sum_{j=1}^{m} [w_j y_j^{(m)}]^p \leq \|\mathbf{y}^{(m)}\|_{\ell_p(\mathbf{w})}^p \leq \|\mathbf{x}\|_{\ell_p(\mathbf{w})}^p,$$

and therefore

$$\|\mathbf{x}^*\|_{\ell_p(\mathbf{w})} \leq \|\mathbf{x}\|_{\ell_p(\mathbf{w})},$$

which completes the proof. ∎

Now we are ready to determine the rate of $\sigma_n(U(\ell_p(\mathbf{w}))$ for all n and all weight sequences $\mathbf{w}$. We fix $\mathbf{w}$, n, and $0 < p < \infty$. From the sequence $\mathbf{w}$, we define the numbers

$$W_m := [w_1^p + w_2^p + \ldots + w_m^p]^{1/p}, \quad m \geq 1. \tag{2.3}$$

Then the following theorem holds

Theorem 2.2. *For any* $0 < p < \infty$ *and* $\mathbf{w}$*, we have*

$$\max_{m \geq n} (m - n)[W_m]^{-2} \leq \sigma_n^2(U(\ell_p(\mathbf{w}))) \leq \max_{m \geq n} (m - n + 1)[W_m]^{-2}, \tag{2.4}$$

and for $p = \infty$ *we have that*

$$\sigma_n^2(U(\ell_\infty(\mathbf{w}))) \asymp \sum_{j=n+1}^{\infty} w_j^{-2}.$$

Proof. We fix, $n, p, \mathbf{w}$. For every sequence $\mathbf{x} \in U(\ell_p(\mathbf{w}))$, we consider its decreasing rearrangement $\mathbf{x}^*$, which according to Lemma 2.1 is also an element of the unit ball and has the same n-term approximation. We next construct a new sequence $\tilde{\mathbf{x}}$ from $\mathbf{x}^*$ by making its first n entries equal to x_n^* and not touching the rest of the sequence, that is,

$$\tilde{x}_j = \begin{cases} x_n^*, & j = 1, \dots, n, \\ x_j^*, & j > n. \end{cases}$$

Note that because the sequence $\mathbf{x}^*$ is nonincreasing and the weights are nondecreasing, we have

$$\|\tilde{\mathbf{x}}\|_{\ell_p(\mathbf{w})} \le \|\mathbf{x}^*\|_{\ell_p(\mathbf{w})} \le 1, \quad \text{and} \quad \sigma_n(\tilde{\mathbf{x}}) = \sigma_n(\mathbf{x}^*) = \sigma_n(\mathbf{x}).$$

Let us denote by $b := x_n^*$ and notice that $\sum_{j=1}^n w_j^p \tilde{x}_j^p = W_n^p b^p$. We have

$$\sigma_n(\mathbf{x}) = \sigma_n(\tilde{\mathbf{x}}) = \sum_{j=n+1}^{\infty} [\tilde{x}_j]^2 = \sum_{j=n+1}^{\infty} [x_j^*]^2.$$

We are now interested in how to change the tail of $\tilde{\mathbf{x}}$, that is, how to change the x_j^*'s with $j > n$ to new quantities y_j, $j > n$ so that we maximize the n-term approximation $\sigma_n^2(\mathbf{y}) = \sum_{j=n+1}^{\infty} [y_j]^2$, under the restrictions $b = x_n^* \ge y_{n+1} \ge \cdots$ and

$$\sum_{j=n+1}^{\infty} w_j^p y_j^p \le 1 - W_n^p b^p =: S. \tag{2.5}$$

Notice that an investment of $w_j^p y_j^p$ towards S gives a return y_j^2 at coordinate j. Since the w_j are non-decreasing, to maximize $\sigma_n(\mathbf{y})$, it is best to invest as much as we can for $j = n+1$, $j = n+2$, and so on. So, the sequence which will maximize $\sigma_n^2(\mathbf{y})$ has to have $y_j = b$, $j = n+1, \dots$ until we have used up our capital S. In other words, given that our sequence $\mathbf{y}$ has first n coordinates b, then to maximize $\sigma_n(\mathbf{y})$ we should take $\mathbf{y} = (b, b, \dots, b, c, 0, 0, \dots) \in U(\ell_p(\mathbf{w}))$, where $0 \le c < b$. The membership in the unit ball of $\ell_p(\mathbf{w})$ requires that

$$b^p \sum_{j=1}^{m} w_j^p + c^p w_{m+1}^p \le 1 \quad \Rightarrow \quad b^p [W_m]^p \le 1$$

and

$$\sigma_n^2(\mathbf{y}) = (m-n)b^2 + c^2 < (m-n+1)b^2 \le (m-n+1)[W_m]^{-2}.$$

Therefore, we have

$$\sigma_n^2(U(\ell_p(\mathbf{w}))) \le \sup_{m \ge n} (m-n+1) W_m^{-2}.$$

Next, consider the special sequence $\mathbf{s}^{(m)}$ with entries $\mathbf{s}_j^{(m)}$ given by

$$\mathbf{s}_j^{(m)} = \begin{cases} W_m^{-1}, & j = 1, \dots, m, \\ 0, & j > m. \end{cases}$$

Clearly $\mathbf{s}^{(m)} \in U(\ell_p(\mathbf{w}))$ for all m and

$$\sigma_n^2(\mathbf{s}^{(m)}) = (m-n)[W_m]^{-2}, \quad m > n.$$

Such sequences provides the lower bound in the case $1 < p < \infty$, and thus (2.4) is proven.

In the case $p = \infty$, we have that the entries x_j of a sequence $\mathbf{x} \in \ell_\infty(\mathbf{w})$ satisfy

$$|x_j| \leq \|\mathbf{x}\|_{\ell_\infty(\mathbf{w})} w_j^{-1} \quad \Rightarrow \quad \sigma_n^2(\mathbf{x}) \leq \left[\sum_{j=n+1}^{\infty} w_j^{-2} \right] \|\mathbf{x}\|_{\ell_\infty(\mathbf{w})}.$$

On the other hand, for the sequence

$$\mathbf{w}^{-1} := (w_1^{-1}, w_2^{-1}, \ldots) \in U(\ell_\infty(\mathbf{w})),$$

we have that $\sigma_n^2(\mathbf{w}^{-1}) = \sum_{j=n+1}^{\infty} w_j^{-2}$, and the proof is completed. ∎

3 Special cases of sequence spaces

In this section, we discuss several special cases of sequences $\mathbf{w}$ that are used in the definition of the classical Wiener spaces, see [3, 5] and the references therein.

Corollary 3.1. *Consider the classes $\ell_p(\mathbf{w})$ with $w_j := (1 + \log j)^\beta$, $\beta \geq 0$, $0 < p < 2$. Then we have*

$$\sigma_n(U(\ell_p(\mathbf{w}))) \asymp m^{-(1/p-1/2)}[\log m]^{-\beta}.$$

Proof. Let us start by calculating the W_m,

$$W_m^p = \sum_{j=1}^{m} w_j^p = \sum_{j=1}^{m} (1 + \log j)^{\beta p} \geq \frac{m}{2}(1 + \log(m/2))^{\beta p}.$$

But we also have $W_m^p \leq m(1 + \log m)^{\beta p}$, so for m sufficiently big we get

$$W_m \asymp m^{1/p}[\log m]^\beta. \tag{3.1}$$

Using this in Theorem 2.2 gives that

$$\sigma_n(U(\ell_p(\mathbf{w}))) \asymp n^{1/2-1/p}[\log n]^{-\beta},$$

provided $0 < p < 2$. ∎

Corollary 3.2. *Consider the classes $\ell_p(\mathbf{w})$, $0 < p \leq \infty$, with*

$$w_j := \max_{1 \leq i \leq j} i^\alpha \log(i+1)^\beta, \quad j \in \mathbb{N}.$$

with $\alpha > 0$, $\beta \in \mathbb{R}$. Then we have

$$\sigma_n(U(\ell_p(\mathbf{w}))) \asymp n^{-(\alpha+1/p-1/2)}[\log(n+1)]^{-\beta},$$

provided $\alpha + 1/p - 1/2 > 0$.

Proof. **Case** $0 < p < \infty$**:** Let us observe that for any $\delta > 0$ and $\gamma \in \mathbb{R}$, the function $\varphi(x) := x^\delta \log(x+1)^\gamma$, is an increasing function on $[c, \infty)$, if $c = c(\delta, \gamma)$ is sufficiently large. Therefore, we have that

$$W_m^p = \sum_{j=1}^{m} j^{\alpha p} \log(j+1)^{\beta p} \asymp m^{\alpha p+1} \log(m+1)^{\beta p}, \quad m \geq M,$$

provided M is sufficiently large. This gives

$$W_m \asymp m^{\alpha+1/p} \log(m+1)^\beta, \quad m \geq M. \tag{3.2}$$

Theorem 2.2 now gives that

$$\sigma_n(U(\ell_p(\mathbf{w}))) \asymp n^{1/2-\alpha-1/p}[\log(n+1)]^{-\beta},$$

for n sufficiently large as desired.
Case $p = \infty$**:** In this case we have the restriction that $2\alpha > 1$. According to Theorem 2.2, we have that

$$\sigma_n^2(U(\ell_\infty(\mathbf{w}))) \asymp \sum_{j=n+1}^{\infty} j^{-2\alpha}[\log(j+1)]^{-2\beta} \asymp n^{-2\alpha+1} \log(n+1)^{-2\beta}.$$

■

Remark 3.3. *Note that when $\beta = 0$, the ranges of α and p in Corollary 3.2 are*

$$\alpha > 0, \quad 0 < p \leq 2 \quad \textit{or} \quad \alpha > 1/2 - 1/p > 0, \quad 2 \leq p \leq \infty.$$

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