Filterless Fixed-Time Extremum Seeking for Scalar Quadratic Maps

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*Abstract***— In this paper, we study a novel fixed-time extremum-seeking algorithm that eliminates the need for filters to obtain an appropriate estimation of the gradient of a static map for optimization problems where the cost function is available only via measurements or evaluations. Previous research leveraged these filters to facilitate the application of averaging theory in analyzing the stability properties of the system. Specifically, they were employed to separate, using multi-time scale techniques, the non-smooth terms of the algorithms from the rapidly fluctuating oscillatory terms associated with periodic dithers. This separation was achieved through a singular perturbation argument, where the filter acted as boundary layer system with a sufficiently fast transient. However, since in many practical applications such transient cannot be made arbitrarily fast, and since classic extremum-seeking algorithms are also known to be stable even in the absence of filters, it is natural to ask whether the fixed-time extremum-seeking dynamics can also be simplified by removing the filters while achieving semi-global practical fixed-time convergence properties. This paper addresses this question for scalar quadratic cost functions, providing positive and negative answers depending on the structure of the cost. Additionally, we demonstrate that removing the filters results in average dynamics distinct from the conventional fixed-time gradient flow dynamics found in existing literature. Furthermore, we provide numerical examples to illustrate our findings.**

I. Introduction

We study a class of extremum-seeking (ES) dynamics aimed at solving model-free optimization problems in static maps. Standard ES algorithms, grounded in classic averaging theory, are recognized for approximating gradient-flows (on average) as the frequency of the dither increases, thereby inheriting the stability and convergence properties of these gradient flows. In recent years, efforts have been made to enhance the convergence rate of ES algorithms, particularly when the cost function exhibits certain qualitative properties such as being quadratic, strongly convex, invex, etc. For instance, in the case of smooth strongly convex functions, the basic ES scheme studied in [1], [2] can achieve exponential rates of convergence of order $\mathcal{O}(e^{-\kappa t})$, where $\kappa > 0$ denotes the strong convexity constant. In [3], a Newton-based ESC method was introduced to eliminate the dependency on κ in the convergence rate. In [4], momentum-based methods and hybrid techniques were used to achieve rates of convergence of order $\mathcal{O}(e^{-\sqrt{\kappa}t})$ and $\mathcal{O}(\frac{1}{t^2})$. Furthermore, prescribed-time results were recently derived in [5] using chirpy probing signals whose frequency increases over time. Similarly, fixed-time results (of semi-global practical nature) were investigated in [6], [7] employing non-smooth dynamics that blend sub-linear and super-linear feedback, inspired by the constructions of [8] and [9].

The findings of [6] relied on a low-pass filter to decouple the oscillatory terms and the non-smooth terms of the algorithm, thereby facilitating analysis through averaging and singular perturbation theory. However, this approach relies on analyzing the filter as a "parasitic" dynamics via a sufficiently large time-scale separation between the low-pass filter and the main state dynamics, which may be challenging to achieve in many practical applications. Based on this, it is natural to ask whether the filters can be eliminated from the fixed-time scheme proposed in [6], while retaining, to some extent, the semi-global practical fixed-time properties. Establishing this property for non-smooth ES algorithms is challenging, primarily due to the complexity of explicitly computing several integrals that emerge in the average non-smooth dynamics associated with the system.

In this paper, we advance the study of fixed-time ES dynamics by exploring approaches that do not rely on filters. Naturally, this investigation entails a detailed computation and analysis of the average dynamics of the proposed non-smooth algorithm, and establishing the connection between the stability properties of these dynamics and those of the actual algorithm. In this paper we primarily focus on quadratic scalar cost functions and demonstrate that even for such "simple" cost functions, the average dynamics without filters differ from the target fixed-time gradient flows examined in [6], [8]. We also establish that despite the disparity between the average dynamics obtained without filters and the flows of [6], [8], the resulting average dynamics still exhibit the same type of fixed-time stability property provided the cost has no offset. In other words, we establish that for such cost functions, achieving fixed-time extremum seeking (in a semi-global practical sense) is feasible without resorting to low-pass filters. The analysis is complemented by a judicious application of Lyapunov-based arguments and an averaging result for non-Lipschitz systems, where the average dynamics are only semi-globally practically asymptotically stable with respect to certain parameters. We finish the paper by presenting numerical examples that serve to illustrate the main theoretical results.

The rest of this paper is organized as follows: Section II presents preliminaries on notation and stability properties. Section III presents the problem statement. Section IV

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presents the main results. Section V presents a numerical example, and Section VI ends with the conclusions.

II. Preliminaries

In this section we introduce some basic notation and stability definitions.

A. Notation and Auxiliary Lemmas

We use $\mathbb{R}_{>0}$ to denote the set of positive real numbers, $\mathbb{Z}_+$ to denote the set of positive integers, $\mathbb{N} := \mathbb{Z}_+ \cup \{0\}$, and $\mathbb{S}^1$ to denote the unit circle in $\mathbb{R}^2$. We denote the *floor* function $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ as the largest integer that does not exceed its argument. We also define the *sign* function, $\text{sgn}(\cdot) : \mathbb{R} \to \{-1, 1\}$, to be -1 when its argument is negative and 1 otherwise. A continuous function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it satisfies $\rho(0) = 0$ and is strictly increasing. It is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and, additionally, it grows unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if it is of class $\mathcal{K}$ in its first argument and for each $r > 0$, $\beta(r, \cdot)$ is non-increasing and $\lim_{t\to\infty} \beta(r, t) = 0$. Furthermore, $\beta \in \mathcal{KL}_\mathcal{T}$ (generalized $\mathcal{KL}$) if $\beta(\cdot, 0) \in \mathcal{K}$ and for each fixed $r \geq 0$, $\beta(r, \cdot)$ is continuous, non-increasing and there exists some $T(r) \in [0, \infty)$ such that $\beta(r, t) = 0$ for all $t \geq T(r)$. The mapping T is called the *settling time function*.

B. Fixed-Time Stability

Consider a system of the form

$$\dot{x} = f(x), \quad x(0) = x_0, \tag{1}$$

where $f : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function, $x \in \mathbb{R}^N$ is the state of the system, and $x_0 \in \mathbb{R}^N$ is the initial condition. In general, we shall not assume that f is differentiable or even Lipschitz. The following definition will be useful to study the properties of system (1).

Definition 1: System (1) is said to render the origin $x = 0$ *uniformly globally finite-time stable* if there exists a class $\mathcal{KL}_\mathcal{T}$ function β such that every solution of (1) satisfies:

$$|x(t)| \leq \beta(|x(0)|, t), \quad \forall\ t \geq 0, \tag{2}$$

where the settling time function T is continuous. System (1) is said to render the origin $x = 0$ *uniformly globally fixed-time stable* if, additionally, $\sup_{x_0 \in \mathbb{R}^N} T(x_0) < \infty$. □

The following Lemma corresponds to [10, Lemma 1]. The converse result is also established in [11, Thm.2] when the settling time T is continuous.

Lemma 1: Suppose there exists a smooth function $V : \mathbb{R}^N \to \mathbb{R}$ that is positive definite, radially unbounded, and satisfies:

$$\dot{V} := \langle \nabla V(x), f(x) \rangle \leq -c_1 V(x)^{p_1} - c_2 V(x)^{p_2}, \ \forall\, x \in \mathbb{R}^N,$$

for some $c_1, c_2 > 0$, $p_1 \in (0, 1)$ and $p_2 > 1$. Then, the origin $x = 0$ is globally fixed-time stable for the dynamics (1), and the settling time function satisfies the bound

$$T(x_0) \leq \frac{1}{c_1(1 - p_1)} + \frac{1}{c_2(p_2 - 1)}, \tag{3}$$

for all $x_0 \in \mathbb{R}^N$. Moreover, if $p_1 = 1 - \frac{1}{2\alpha}$, $p_2 = 1 + \frac{1}{2\alpha}$, and $\alpha > 1$, then the settling time function satisfies the bound

$$T(x(0)) \leq \frac{\alpha\pi}{\sqrt{c_1 c_2}}. \tag{4}$$

for all $x(0) \in \mathbb{R}^n$. □

Finally, in this paper we will also make use of the following practical stability properties.

Definition 2: Consider a parameterized system $\dot{x} = f_\varepsilon(x)$, with unique equilibrium point x^*, where $\varepsilon > 0$ is a tunable parameter. We say that x^* is semi-globally practically asymptotically stable if there exists $\beta \in \mathcal{KL}$ such that for each $\Delta > 0$ and $v > 0$ there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$ all solutions of the system satisfying $|x(0) - x^*| \leq \Delta$ also satisfy

$$|x(t) - x^*| \leq \beta(|x(0) - x^*|, t) + \nu, \ \ \forall\ \ t \geq 0. \tag{5}$$

If $\beta \in \mathcal{KL}_\mathcal{T}$ and the settling time function of β satisfies $\sup_{x_0 \in \mathbb{R}^N} T(r) < \infty$, then the system is said to render x^* semi-globally practically fixed-time stable. □

III. Problem Statement

We consider the problem of minimizing a scalar cost function $J : \mathbb{R} \to \mathbb{R}$ using only measurements or evaluations. In this way, it is assumed that the exact mathematical form of J and its derivatives is unknown. Such types of problems can be tackled using extremum-seeking (ES) algorithms [1], [2]. The simplest ES scheme that can tackle this problem is characterized by the following dynamics with state x:

$$\dot{x} = -\frac{2k}{a} \sin(\omega t) J(x + a \sin(\omega t)), \tag{6}$$

where a, ω, k are positive tunable parameters. System (6) can be analyzed using a multi-time scale approach. In particular, when k is sufficiently small compared to ω, and a is sufficiently small compared to k, the average dynamics of (6) can be shown to be:

$$\dot{\tilde{x}} = -\nabla J(\tilde{x}) + \mathcal{O}(a). \tag{7}$$

where $\tilde{x}$ denotes the average of state x. Under suitable assumptions on J, and for sufficiently small values of a, this system renders the set of minimizers of J semi-globally practically asymptotically stable as $a \to 0^+$. Indeed, as $a \to 0^+$, the trajectories of (7) approximate those of a standard gradient flow $\dot{y} = -\nabla J(y)$. Moreover, as $\omega \to \infty$, or, equivalently ($k \to 0$), the trajectories of (6) approximate those of (7) (on compact time domains and compact sets of initial conditions) [12], thus eventually establishing a connection between the stability properties of (6) and those of a standard gradient flow.

A. Fixed-Time ESC: A Filterless Approach

To improve over the asymptotic rates of convergence of smooth ES dynamics, a class of fixed-time ES algorithms was studied in [6]. Such algorithms seek to approximate fixed-time gradient flows [8] of the form:

$$\dot{x} = -k \frac{\nabla J(x)}{|\nabla J(x)|^{p_1}} - k \frac{\nabla J(x)}{|\nabla J(x)|^{p_2}}, \tag{8}$$

where $k > 0$, $p_1 \in (0,1)$ and $p_2 < 0$. Indeed, as shown in [6], [8], if J is strongly convex with a globally Lipschitz gradient, then the minimizer x^* of J is fixed-time stable under the dynamics (8). Based on this observation, and following the same ideas behind the study of (6) and (7), one can hypothesize that the following filterless ES algorithm could provide a model-free version of (8):

$$\dot{x} = -\frac{k_1\mu(t)J(x+a\mu(t))}{|\mu(t)J(x+a\mu(t))|^{p_1}} - \frac{k_2\mu(t)J(x+a\mu(t))}{|\mu(t)J(x+a\mu(t))|^{p_2}}, \quad (9)$$

where $\mu(t) = \sin(\omega t)$, $k > 0$, $p_1 \in (0,1)$ and $p_2 < 0$. Note that this system has a continuous right-hand side. The main goal of this paper is to study to what extent the above hypothesis is true, and whether the average dynamics of (9) are indeed given by (8). Since the dynamics described in equation (9) are highly non-smooth, which can rapidly render several computations intractable, we will concentrate on quadratic cost functions.

Assumption 1: The cost J has the form $J(x) = \frac{1}{2}Hx^2 + Bx + H_0$, where $H > 0$. □

While the class of quadratic cost functions is not as general as the class of strongly cost functions studied in [6], it will be shown in the next section that, whenever the filters are removed, the quadratic structure of J may induce some unexpected phenomena depending on the nature of the bias. Furthermore, scalar quadratic cost functions can provide suitable approximations for functions that emerge in different practical applications, see [13].

IV. Main Results

To study system (9) for scalar quadratic maps, without loss of generality we can take $\omega = 1$, since otherwise a suitable change of time scale can be used to obtain this condition. Then, system (9) takes the form

$$\dot{x} = -\frac{k_1 J(x+a\sin t)\sin t}{|J(x+a\sin t)\sin t|^{p_1}} - \frac{k_2 J(x+a\sin t)\sin t}{|J(x+a\sin t)\sin t|^{p_2}}. \quad (10)$$

By Assumption 1, $J(x+a\sin t)$ can be written as:

$$J(x+a\sin t) = \frac{1}{2}H\left(x + \frac{B}{H} + a\sin t\right)^2 + H_0 - \frac{B^2}{4H}.$$

Using the transformation $\bar{x} = x + \frac{B}{H}$, we can assume without loss of generality that $B = 0$. In this way, we can focus on studying the stability properties of the optimal point $x^* = 0$. We now divide our analysis into two different cases: $H_0 = 0$ and $H_0 > 0$.

A. Case I: $H_0 = 0$

When $H_0 = 0$, the optimal value of the cost satisfies $J^* = 0$, and the dynamics (10) can be written as:

$$\dot{x} = -\Psi_1(x) - \Psi_2(x), \quad (11)$$

where

$$\Psi_i(x) = k_i\left(\frac{H}{2}\right)^{1-p_i}\frac{\sin t}{|\sin t|^{p_i}}|x+a\sin t|^{2-2p_i}, \quad (12)$$

for $i \in \{1,2\}$. To analyze this system, we will consider two different conditions on $x : |x| > a$ and $|x| \le a$.

1) Suppose that $|x| > a$: By applying the generalized binomial theorem to $|x+a\sin t|^{2-2p_i}$, and using the observation that $|x+a\sin t| = |x| + \mathrm{sgn}(x)a\sin t$ we obtain

$$(|x| + \mathrm{sgn}(x)a\sin t)^{2-2p_i} = |x|^{2-2p_i} + a(2-2p_i)\frac{x}{|x|^{2p_i}}\sin t + \sum_{k=2}^{\infty} C_{k,i}\frac{|x|\,\mathrm{sgn}(x)^k}{|x|^{2p_i+k-1}}(a\sin t)^k, \quad (13)$$

for all $|x| > a$, where

$$C_{k,i} = \frac{1}{k!}\prod_{j=1}^{k}(3-2p_i-j). \quad (14)$$

Using a Stirling approximation, it can be shown that $\sum_{k=1}^{\infty}|C_{k,i}| < \infty$ for $i \in \{1,2\}$. Substituting (13) into (11), we obtain the following alternative expression for (10) whenever $|x| > a$:

$$\dot{x} = -\sum_{i=1}^{2} k_i\left(\frac{H}{2}\right)^{1-p_i}\frac{\sin t}{|\sin t|^{p_i}}\Bigg(|x|^{2-2p_i} + a(2-2p_i)\frac{x}{|x|^{2p_i}}\sin t + \sum_{k=2}^{\infty} C_{k,i}\frac{|x|\,\mathrm{sgn}(x)^k}{|x|^{2p_i+k-1}}(a\sin t)^k\Bigg) \quad (15)$$

System (15) is continuous and can be studied via averaging theory [14], [15]. By setting $k_i = \frac{1}{a}$ for $i = 1,2$ and computing the average vector field, we obtain:

$$\dot{x} = -\sum_{i=1}^{2}\Bigg(q_i 2^{p_i-1}H^{p_i}\frac{Hx}{|Hx|^{2p_i}} + \sum_{k=1}^{\infty} a^{2k}C^*_{k,i}\frac{x}{|x|^{2p_i+2k}}\Bigg) \quad (16)$$

where q_i and $C^*_{k,i}$ are given by:

$$q_i = \frac{2-2p_i}{2\pi}\int_{-\pi}^{\pi}|\sin t|^{2-p_i}dt \quad (17)$$

$$C^*_{k,i} = \left(\frac{H}{2}\right)^{1-p_i}\frac{C_{2k+1,i}}{2\pi}\int_{-\pi}^{\pi}|\sin t|^{2k+2-p_i}dt. \quad (18)$$

To arrive at (16), we used the following facts:

$$\int_{-\pi}^{\pi}\frac{(\sin t)^{k+1}}{|\sin t|^{p_i}} = 0 \quad \forall\ k \quad \text{even}$$
$$|x|\,\mathrm{sgn}(x)^k = x \quad \forall\ k \quad \text{odd},$$

which follow immediately from the fact that an odd periodic function has average 0. To study the stability properties of the average dynamics (16), we consider the Lyapunov function $V(x) = \frac{1}{2}x^2$, whose time derivative along the trajectories of the average system satisfies:

$$\dot{V} = -\sum_{i=1}^{2}\Bigg(q_i\left(\frac{H}{2}\right)^{1-p_i}|x|^{2-2p_i} + \sum_{k=1}^{\infty} C^*_{k,i}a^{2k}|x|^{2-2p_i-2k}\Bigg) \quad (19)$$

for all x where $|x| > a$. To proceed, it is helpful to state the following lemma, as it will be instrumental for our simplification of the expression in (19).

Lemma 2: Let $S_i = \{\lfloor 1-p_i \rfloor + 1, \lfloor 1-p_i \rfloor + 2, ...\}$. For $k_i \in \mathbb{Z}_+ \setminus S_i$ and $i = 1,2$, we have $\text{sgn}(C^*_{k_i,i}) = 1$. Otherwise, if $k_i \in S_i$ then:

$$\text{sgn}(C^*_{k_1,1}) = \begin{cases} -1 & \text{if } p_1 \in (0, \frac{1}{2}) \\ 1 & \text{if } p_1 \in [\frac{1}{2}, 1) \end{cases}$$

$$\text{sgn}(C^*_{k_2,2}) = \begin{cases} -1 & \text{if } p_2 \in (-n-1, -n-\frac{1}{2}),\ n \in \mathbb{N} \\ 1 & \text{else} \end{cases}$$

Proof: This follows immediately from the observation that $\text{sgn}(C^*_{k,i}) = \text{sgn}(C_{2k+1,i})$ and by inspecting (14). ■

With Lemma 2 at hand, we can observe that for $p_1 \in [\frac{1}{2}, 1)$ and $p_2 \notin (-n-1, -n-\frac{1}{2}) \quad \forall n \in \mathbb{N}$, the inner summation in (19) is negative. Hence, for p_i satisfying these properties, we have that:

$$\begin{aligned} \dot{V} &\leq -q_1 2^{p_1-1} H^{1-p_1} |x|^{2-2p_1} - q_2 2^{p_2-1} H^{1-p_2} |x|^{2-2p_2} \\ &\leq -q_1 H^{1-p_1} V^{1-p_1} - q_2 H^{1-p_2} V^{1-p_2}, \quad \forall\ |x| > a. \end{aligned} \tag{20}$$

Now, suppose $p_1 \in [\frac{1}{2}, 1)$ and $p_2 \in (-n-1, -n-\frac{1}{2})$ for some $n \in \mathbb{N}$. Then we can proceed from (19) as follows:

$$\begin{aligned} \dot{V} &= -q_1 \left(\frac{H}{2}\right)^{1-p_1} |x|^{2-2p_1} - q_2 \left(\frac{H}{2}\right)^{1-p_2} |x|^{2-2p_2} \\ &\quad - \sum_{k=1}^{\infty} C^*_{k,1} a^{2k} |x|^{2-2p_1-2k} - \sum_{j=1}^{\infty} C^*_{j,2} a^{2j} |x|^{2-2p_2-2j} \\ &\leq -\frac{q_1}{2} \left(\frac{H}{2}\right)^{1-p_1} |x|^{2-2p_1} - q_2 \left(\frac{H}{2}\right)^{1-p_2} |x|^{2-2p_2} \\ &\quad + \left(a^{2-2p_2} \sum_{j=\lfloor 1-p_2 \rfloor + 1}^{\infty} |C^*_{j,2}| - \frac{q_1}{2} \left(\frac{H}{2}\right)^{1-p_1} a^{2-2p_1} \right). \end{aligned} \tag{21}$$

We choose $a > 0$ sufficiently small such that

$$a^{2-2p_2} \sum_{j=\lfloor 1-p_2 \rfloor + 1}^{\infty} |C^*_{j,2}| - \frac{q_1}{2} \left(\frac{H}{2}\right)^{1-p_1} a^{2-2p_1} < 0. \tag{22}$$

Thus, we can proceed from (21) and obtain

$$\begin{aligned} \dot{V} &\leq -\frac{q_1}{2} \left(\frac{H}{2}\right)^{1-p_1} |x|^{2-2p_1} - q_2 \left(\frac{H}{2}\right)^{1-p_2} |x|^{2-2p_2} \\ &\leq -H^{1-p_1} \frac{q_1}{2} V^{1-p_1} - H^{1-p_2} q_2 V^{1-p_2}, \quad \forall\ |x| > a. \end{aligned}$$

This last inequality directly implies by [16, Thm. 4] that the optimal point x^* is globally practically fixed-time stable as $a \to 0^+$ for the average system (18). Moreover, the settling time can be upper-bounded as:

$$T(x_0) \leq \frac{2}{H^{1-p_1} q_1 p_1} - \frac{1}{H^{1-p_2} q_2 p_2}. \tag{23}$$

Therefore, we have verified that as long as $p_1 \in [\frac{1}{2}, 1)$ and $|x| > a$, the average system of (11) satisfies the Lyapunov conditions for practical fixed time stability. If $p_2 \notin (-n - 1, -n - \frac{1}{2}) \quad \forall n \in \mathbb{N}$, the result holds even if a does not satisfy (22).

2) Suppose that $|x| \leq a$: In this case, to the best of the authors' knowledge, there is no well-defined method for approximating the average of (11). However, we can still analyze the qualitative properties of the average dynamics by considering the integral:

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{2} \frac{-k_i \left(\frac{H}{2}\right)^{1-p_i}}{2\pi} \int_{-\pi}^{\pi} \frac{\sin t}{|\sin t|^{p_i}} |x + a \sin t|^{2-2p_i} dt \\ &= \sum_{i=1}^{2} \frac{-k_i \left(\frac{H}{2}\right)^{1-p_i}}{2\pi} \int_{0}^{\pi} |\sin t|^{1-p_i} \Big(|x + a \sin t|^{2-2p_i} \\ &\qquad - |x - a \sin t|^{2-2p_i} \Big) dt. \end{aligned} \tag{24}$$

In particular, note that:

$$\text{sgn}(|x + a \sin t|^{2-2p_i} - |x - a \sin t|^{2-2p_i}) = \text{sgn}(x),$$

for all $t \in [0, \pi]$, and since (24) is continuous in x and vanishes at $x = 0$, we can conclude that the *average* converges to zero once it enters the neighborhood $[-a, a]$. In particular, using the same Lyapunov function $V = \frac{1}{2}x^2$, we have that $\dot{V} = x\dot{x}$ and since x and $\dot{x}$ always have different sign, then $\dot{V} \leq 0$. Moreover, the product $x\dot{x}$ vanishes only at $x = 0$. This analysis reveals that in the $\mathcal{O}(a)$ neighborhood of the origin, the average dynamics are also well-defined.

By combining the above results, we can obtain the following key technical result:

Proposition 1: Consider the average dynamics (16) and suppose that $p_1 \in [\frac{1}{2}, 1), p_2 < 0$. Then, the optimal point x^* is globally practically fixed-time stable as $a \to 0^+$. In particular, there exists $a^* > 0$, such that for all $a \in (0, a^*)$, all solutions of (16) satisfy

$$|x(t) - x^*| \leq \beta(|x(0) - x^*|, t) + \rho(a), \tag{25}$$

where $\rho(\cdot)$ is a positive definite function, and $\beta \in \mathcal{KL}_\mathcal{T}$ with settling time $T(\cdot)$ satisfying (23) for all $x(0) = x_0 \in \mathbb{R}$. □

Remark 1: The result of Proposition 1 establishes a key missing technical result in the literature of ESC. Namely, it shows that the average dynamics of the filterless ESC algorithm (10) do have (in a practical sense) the fixed-time stability property. However, the result also highlights an important fact: the average dynamics of system (10) are not exactly equal to the fixed-time gradient flow (8), even if $\mathcal{O}(a)$ terms are neglected outside an a-neighborhood of the origin. In particular, using (16), the average dynamics can be written as

$$\dot{\tilde{x}} = -\tilde{k}_1 \frac{\nabla J(\tilde{x})}{|\nabla J(\tilde{x})|^{2p_1}} - \tilde{k}_2 \frac{\nabla J(\tilde{x})}{|\nabla J(\tilde{x})|^{2p_2}} + \tilde{\mathcal{O}}(a), \tag{26}$$

where $\tilde{\mathcal{O}}(a)$ denotes terms bounded for all $|x| > a$, and the constants $\tilde{k}_1, \tilde{k}_2 > 0$ depend on H. This observation already highlights an important distinction compared to the results of [6], where a smooth filter was used to obtain a quasi-steady state approximation of (8). □

Remark 2: The closed-form expression presented in (16) also highlights an interesting behavior of the system whenever $p_1 = 1$. In this case, using (17), we obtain $q_1 = 0$,

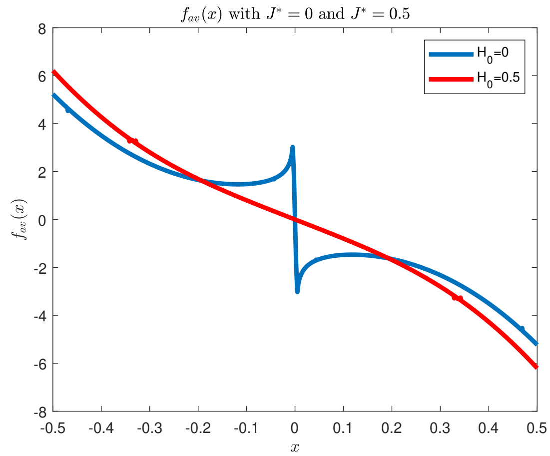

Fig. 1: Vector field of the average system of (11) with $a = 0.005, p_1 = 0.62, p_2 = -0.7$, and $H = 9.84$. For H_0 we use $H_0 = 0$ and $H_0 = 0.5$

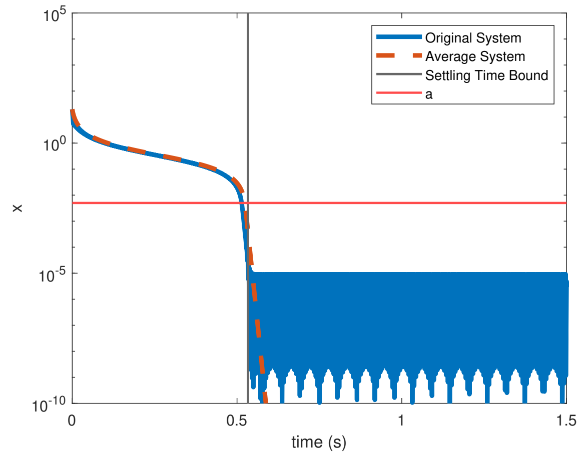

Fig. 2: Trajectories of the example system with $H = 6.74$ compared with its average system (30) We can see that the trajectories reach the set $(a, -a)$ within our computed settling time bound. For all simulations we use $a = 0.005$

which in turn implies that $\tilde{k}_1 = 0$ in (26) thus effectively removing any "finite-time" behavior from the average dynamics. Again, this is in contrast to the behavior obtained in the filter-based approach of [6], where $p_1 = 1$ recovers a finite-time gradient flow. □

By leveraging Proposition 1, we can now state the second main result of this paper, which studies the stability properties of the filterless ESC algorithm (10) when J has no offset.

Theorem 1: Consider the filterless ESC algorithm (10), and suppose that Assumption 1 holds, $p_1 \in [\frac{1}{2}, 1)$, $p_2 < 0$, and $H_0 = 0$. Then, the global minimizer x^* is semi-globally practically fixed-time stable as $(\frac{1}{\omega}, a) \to 0^+$.

Proof: The result follows by a direct application (in the correct time scale) of averaging results for non-Lipschitz systems whose average dynamics are only semi-globally practical stable, i.e., [15, Thm. 7]. Specifically, we consider the following system:

$$\dot{x} = -\frac{1}{a}\frac{\mu J(x + a\mu)}{|\mu J(x + a\mu)|^{p_1}} - \frac{1}{a}\frac{\mu J(x + a\mu)}{|\mu J(x + a\mu)|^{p_2}}, \tag{27a}$$

$$\frac{1}{\omega}\begin{bmatrix}\dot{\mu}\\ \dot{\xi}\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\mu\\ \xi\end{bmatrix}, \quad (\mu, \xi) \in \mathbb{S}^1. \tag{27b}$$

where the exosystem (27b) is used to generate the sinusoid signal $t \mapsto \mu(t)$. In particular, every solution to the linear oscillator (27b) satisfies $\mu(t) = \sin(\omega t + \phi_0)$, where $\phi_0 \in [0, 2\pi)$. Since the phase-shift is inconsequential due to the periodicity of the dither, by Proposition 1 the resulting dynamics (27a) have a well-defined average system satisfying (25) as $a \to 0^+$. Then, by [15, Thm. 7], the original system (27a) will retain the same $\mathcal{KL}$ bound, provided ω is sufficiently large and a is sufficiently small. This establishes the result. ■

According to Theorem 1, the initial hypothesis on the connections between the stability properties of (8) and (10) turns out to be true for the case when $H_0 = 0$, i.e., when the optimizer of the cost function is zero. However, in the next section we show that the inclusion of a non-zero offset in the cost function can significantly change the nature of the stability properties of the non-smooth filterless ES dynamics.

B. Case II: $H_0 > 0$

When the cost function J has a positive offset, the term $H_0 > 0$ essentially "smooths" out the filterless ESC dynamics, resulting in the loss of the fixed-time stability properties. To see this phenomena, note that in this case the dynamics can be written as:

$$\dot{x} = -\tilde{f}_1(x, t) - \tilde{f}_2(x, t), \tag{28}$$

where

$$\tilde{f}_i(x, t) = k_i\left(\frac{H}{2}\right)^{1-p_i}\frac{\sin t}{|\sin t|^{p_i}}\left((x + a\sin t)^2 + \frac{2H_0}{H}\right)^{1-p_i}.$$

Let $f_{av}(x)$ denote the time average of (28). Since $\frac{H_0}{H} > 0$, $\frac{\partial \tilde{f}_i(x,t)}{\partial x}$ is continuous for all t, x. Hence, we can differentiate $f_{av}(x)$ to obtain:

$$\frac{\partial f_{av}(x)}{\partial x} = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial}{\partial x}\left(\tilde{f}_1(x, t) + \tilde{f}_2(x, t)\right)dt, \tag{29}$$

which is clearly bounded if we restrict x on any compact set even if $a = 0$. Hence, the finite-time convergence property is not preserved. For a better visualization of the smoothing effect of having $J^* > 0$, please refer to Figure 1.

V. Numerical Examples

In this section, we illustrate our theoretical results via a numerical example, where $p_1 = \frac{1}{2}$ and $p_2 = -\frac{1}{2}$. With this choice of p_i, we can obtain a more detailed characterization of the average dynamics of the system and also of the settling time. Similar to our previous analysis, we consider two cases. When $|x| > a$, we have that $|x + a\sin t| = \text{sgn}(x)(x +$

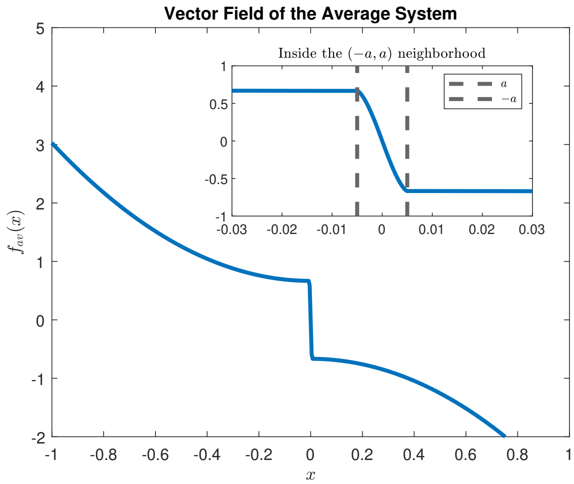

Fig. 3: Vector field of the average system of (30)

$a \sin t$). With this observation, we can obtain the average system as follows:

$$\dot{\tilde{x}} = -q_1 \left(\frac{H}{2}\right)^{\frac{1}{2}} \operatorname{sgn}(\tilde{x}) - q_2 2^{-\frac{3}{2}} H^{-\frac{1}{2}} \frac{H\tilde{x}}{|H\tilde{x}|^{-1}} - \frac{a^2 \left(\frac{H}{2}\right)^{\frac{3}{2}} \operatorname{sgn}(\tilde{x})}{2\pi} \int_{-\pi}^{\pi} |\sin t|^{\frac{9}{2}} dt, \tag{30}$$

where we set $k_1 = k_2 = \frac{1}{a}$. It can also be verified that q_1 and q_2 here are consistent with the formulas given in (17). We can also see that the first two terms in (30) are consistent with (16), since $\operatorname{sgn}(x) = \frac{x}{|x|^{2(\frac{1}{2})}}$ and $x|x| = \frac{x}{|x|^{2(-\frac{1}{2})}}$. Furthermore, since $C_{k,1} = 0$ for $k \geq 2$ and $C_{k,2} = 0$ for $k \geq 4$, it follows that only $C^*_{1,2}$ is nonzero. We can compare the last term in (30) with (18) and see that they are consistent. With the Lyapunov function $V(x) = \frac{1}{4}x^4$, we can compute its time derivative along the trajectories of (30) and obtain that for all $|x| > a$:

$$\dot{V} \leq -2q_1 H^{\frac{1}{2}} V^{\frac{3}{4}} - 2q_2 H^{\frac{3}{2}} V^{\frac{5}{4}}$$

By using (4) with $\alpha = 2$ we can obtain a settling time bound of $T(x_0) \leq \frac{3.594}{H}$. We simulate the trajectories of the system along with those generated by its average also for $H = 6.74$. The result is shown in Figure 2, and the computed settling time of $\frac{3.594}{H} = 0.533$. To better illustrate our results, we provide simulations for the vector field of the average system and the trajectories generated by the average system.

VI. Conclusions

In this work, we provide a thorough and comprehensive analysis of a proposed filterless ESC algorithm for the solution of fixed-time optimization problems in scalar quadratic maps that are only accessible via measurements. Compared to existing results, the approach simplifies the implementation by removing additional states and auxiliary filters. We provided a detailed analysis of the average dynamics of the system, uncovering the smoothing effect that non-zero cost minima can have on the algorithm, as well as conditions on the exponents under which the finite-time or fixed-time convergence properties are lost. Future research directions will extend the results to multi-variable and time-varying optimization problems.

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