The Nature of an Online Work Group Reveals the Teaching Knowledge of Inquiry-Oriented Linear Algebra Instructors through Their Goals for Instruction

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Postsecondary instructors interested in inquiry-oriented instruction of linear algebra participated in a sequence of eight one-hour online work group meetings with other inquiry-oriented linear algebra instructors and facilitators. Recordings were analyzed for how two participants referenced goals for instruction in discussions of implementing a new instructional unit on subspaces. We identified four goals for the instruction of teaching subspaces. We discuss the intersections of several goals that exist due to the tension caused by real-world contexts and abstract mathematical concepts. The instructors presented resolutions to the tension by utilizing varying teaching knowledge. Based on the results, we make suggestions for those who want to transition to inquiry-oriented instructional approaches.

Keywords: teaching knowledge, goals for instruction, inquiry-oriented linear algebra, online work group

Inquiry-Oriented Linear Algebra (IOLA) is a reform-oriented instructional approach that derives from Realistic Mathematics Education (RME). This instructional approach encourages teachers to support students in their reinvention of mathematical concepts through inquiry (Freudenthal, 1991; Kelley & Johnson, 2022). Research has shown that authentic engagement with mathematics through this instructional approach, can benefit student achievement and possibly incite equitable outcomes among students (Burke et al., 2020; Freeman et al., 2014; Haak et al., 2011; Kogan & Larsen, 2014). This instructional practice is difficult to enact, however, because instructors may not fully utilize the teaching knowledge needed to inform their practice. This is especially true for novice instructors implementing inquiry-oriented approaches (Wagner et al., 2007). If long-lasting instructional change is needed for the desirable outcomes available by IOLA, then teachers need to shift their instructional approach (Cohen, 1990; Henderson et al., 2011) by growing teaching knowledge. Thus, researchers have declared "a need for professional development programs that foster the development of undergraduate mathematics instructors' pedagogical reasoning" (Andrews-Larson et al., 2019, p. 129).

This lays the groundwork for the following problems: what is being done to address the professional development gap, what teaching knowledge IOLA instructors possess, and how do we capture it. Thus, an Online Work Group (OWG) for postsecondary mathematics instructors is examined in this study. The OWG was part of the IOLA-X project and was initially created to provide IOLA instructors a chance to collaborate with other instructors who are interested in their continuous pursuit of enacting IOLA. These IOLA instructors are guided by facilitators who are researchers of IOLA-X. Instructors from various universities across the United States joined this OWG for eight sessions across the Spring 2022 semester. They worked on an IOLA task unit “subspace” and discussed their teaching practices with the other researchers, instructors, and facilitators of the OWG. The researchers of the IOLA-X project took their contributions to inform their continual effort to adjust their curriculum and to create teacher notes for other IOLA instructors. The instructors' contributions can be valuable to capture how the OWG makes way for discussion of teacher practice and to also collect teacher knowledge IOLA instructors possess. Here, our research question is “How does the OWG serve opportunities for instructors to exhibit knowledge for teaching in inquiry-oriented ways?”

Literature Review

Linear Algebra is a postsecondary mathematics course that is often a requirement for students in STEM-related majors. As a result, many students at some point enroll in this course. IOLA is one instructional approach to active learning (Freeman et al., 2014) and inquiry-based mathematics education (Laursen & Rasmussen, 2019). Laursen and Rasmussen (2019) discuss this approach to mathematics education as "student engagement in meaningful mathematics, students' collaboration for sensemaking, instructor inquiry into student thinking, and equitable instructional practice to include all in rigorous mathematical learning and mathematical identity-building" (p. 140).

As stated previously, inquiry-oriented instructional approaches are difficult to implement. Mostly because instructors at first may not possess the reasoning and knowledge necessary for enactment (Andrews-Larson et al., 2019). The knowledge we are referring to is mathematical subject matter knowledge (Ball et al., 2008; Hill et al., 2008) and pedagogical content knowledge (Ball et al., 2008; Hill et al., 2008; Shulman, 1986). Shulman in 1986 first introduced the idea of Pedagogical Content Knowledge (PCK) as the subject-matter knowledge for teaching. This includes at the time the unnamed domains of PCK which are knowledge of content and teaching (KCT) and knowledge of content and students (KCS). These domains capture knowledge for teaching such as an instructor's knowledge of the best representation to present to students or knowledge of what ideas or conceptions students will bring to the table. The domains are not restricted to the teaching of a specific content area.

Ball and colleagues (2008) expanded on the work of Shulman in their framework of Mathematical Knowledge for Teaching (MKT). Their framework includes PCK as half of their domains of MKT. The other half is subject-matter knowledge, in other words, mathematical knowledge that is unrelated to the practice of teaching. Teaching knowledge for mathematics instructors has been studied for decades as either declarative or “knowledge-in-use" (Andrews et al., 2022). These studies capture how experienced teachers approach their instruction. Although there is little evidence that more experience means more teaching knowledge (Andrews et al., 2022), there are studies that point to the differences in teaching knowledge between experts and novices (Auerbach et al., 2018). Thus, analyzing the teaching knowledge of experienced IOLA instructors can prove to be worthwhile as they highlight areas of instruction novice IOLA instructors may not consider.

Theoretical Framework

Wagner and colleagues (2007) studied the MKT and PCK regarding the challenges of a novice instructor teaching an inquiry-oriented differential equations course. As a result, Wagner et al. (2007) identified four types of instructional goals in the context of inquiry-oriented instruction at the undergraduate level: classroom orchestration goals, cognitive/learning goals, assessment goals, and content goals. These goals encompass their framework called *goals for instruction.* Each goal is summarized as follows:

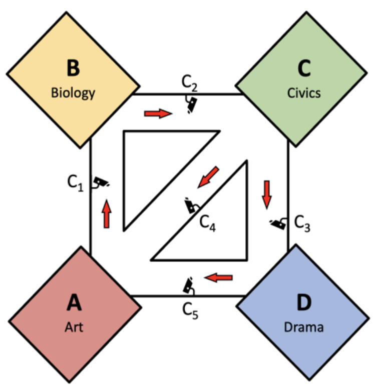
1. Classroom orchestration goals: How instructors orchestrate, intervene, and redirect the discussions and negotiate an agenda with emerging ideas.
2. Cognitive/Learning goals: What student understanding, questions, and activities look like.
3. Assessment goals: How to assess student learning, what the evidence of understanding is, and how to design a pace or curriculum.

4. Content goals: What and how specifically mathematical concepts should be learned.

Using the work from Wagner et al. (2007) as a priori scheme, this proposal identifies how instructional goals were discussed by experienced IOLA instructors in the OWG.

Study Context: Inquiry-Oriented Linear Algebra and Online Work Group

The IOLA-X project focuses on developing student materials composed of challenging and coherent task sequences that facilitate an inquiry-oriented approach to the teaching and learning of linear algebra (Wawro et al., 2013). There are five main phases in the design research spiral: *Task Design*, *Paired Teaching Experiment* (*PTE*), *Classroom Teaching Experiment* (*CTE*), *Online Work Group* (*OWG*), and *Web* (Wawro et al., 2023). The participants of our study come from the OWG in the fourth phase of the research project. The main purpose of the OWG for the IOLA research team is to learn from instructors how IOLA is implemented in various classrooms with various student populations and to gain insights to develop instructor notes and revise tasks (Wawro et al., 2023). At the center of the OWG for this study was the discussion of a unit of tasks about subspaces and Table 1 illustrates the overview of the subspace unit. In this unit, the tasks were contextualized in a problem about students walking in one-way hallways past cameras monitoring their traffic (See Figure 1). To draw out the feedback from the instructors, the facilitators managed mathematical discussions about the tasks as well as facilitated discourse about the preparation and implementation of the tasks. Through examining discussion and input from the experienced undergraduate instructors participating in the OWG, questions, and thoughts about the goals for instruction and challenges with implementation naturally arose.



The hallways in one wing of Ida B. Wells High School were changed to one-way corridors to promote social distancing during a pandemic (red arrows in the diagram). These hallways connect classrooms A-D (the Art room, Biology lab, Civics room, and Drama room) as shown in the diagram. Each hall has a security camera that allows Principal McDaniel to monitor student movement through the hallways (cameras 1-5, as shown in the diagram). As a further precaution, each wing is isolated from the rest, so the students in a wing stay within that wing and no students from any other part of the school will enter the west wing.

SCENARIO ONE: TRACKING MOVEMENT

Before the school year starts, Principal McDaniel goes into the school when it is empty to spend a day learning how to use data from the camera system. Her daughter Ella comes with her, and she asks Ella to help her test the system by walking between rooms – so long as she stays in the building and follows the one-way hallway rules. Ella decides to start in the Art room, passes Camera 1 (C_1) as she walks from the Art room toward the biology lab, and then continues walking past Camera 2 (C_2) as she walks from the biology lab to the Civics room. Afterwards, Principal McDaniel sees that the camera system has recorded the number of people who walked by each camera with the vector $p = \begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}$.

Figure 1. The hallway scenario of Tasks 1-2 in the IOLA subspace unit

Methods

Our primary data source was the recorded videos of the OWG meetings—held and recorded via Zoom, and group artifacts such as Google Slides and Jamboards that served as secondary data sources. In this OWG, there were six members: F1 and F2 (facilitators), R1 and R2 (IOLA researchers), and I1 and I2 ("pure" participants who are experienced inquiry-based instructors but not IOLA researchers). This study focuses on the pure participants, so the participation and contributions of I1 and I2 serve as the main data of our study. I1 is an associate professor at a small private college in the Northwestern United States and I2 is a senior instructor at a large public university in the Northeastern United States and they all taught linear algebra at their universities at the time of the OWG sessions. Other than pure participants, this team involves one graduate student (F2), two associate professors (F1 and R1), and one full professor (R2).

Table 1. The IOLA subspace unit overview

Task	Driving Question	Mathematical Formalism
Hallway Task 1: Finding Paths	What are the possible paths from room A to room C and from room C to room C?	Closure (of "inputs")
Hallway Task 2: Managing Populations	What are the possible paths that induce a specific change in room populations? What are the possible paths that leave the room populations unchanged?	Correspondence between ("input" and "output") vectors
Hallway Task 3: New Wings	What are the possible paths for a different wing of Ida B Wells High School (defined by a matrix) that will leave populations unchanged?	Null spaces as a type of closed "input" spaces and column spaces as a type of closed "output" spaces

Each OWG meeting was approximately one hour and there were eight meetings throughout the Spring 2022 semester. Thus, a total of around eight hours of OWG meetings were conducted and recorded over several days. In the first four videos, the OWG members discussed the IOLA subspace tasks, either as if they were students or teachers, so they shared the mathematical progression of an IOLA subspace unit comprised of three tasks. The subspaces unit focused primarily on notions of closure of sets of vectors under vector addition and scalar multiplication, as well as null and column spaces. So, in the first four meetings, participants worked on the mathematical problems as a group and then discussed mathematical goals, approaches, and links to other ideas and topics. The remainder of the meetings took place throughout the participants' implementation of the sequence, with each participant reporting on how the implementation went, what they liked, how their students reasoned about tasks, what they would change or what they would do differently.

We first analyzed all eight videos in terms of goals for instruction. To do so, videos were transcribed by Otter, an online artificially intelligent transcription application. Both authors separately coded all transcripts using Nvivo software. In this analysis, the four goals of instruction were the code schemes. We coded for all the participants of the OWG—even though this report focuses on two pure participants. While coding, we assigned four codes, which mean four goals, at the level of a single turn of talk. Then, we compared codes to reach agreements to build inter-rater reliability. We identified common themes within each code and found out that there were many intersections between two or more goals. We decided to dig into the intersections more precisely—and so analyzed what kinds of pieces of knowledge of IOLA instructors were discussed, considering the goals for instructions and main themes of OWG meetings. For this report, we selected several examples of what I1 and I2 shared and contributed.

Results

Generally, in the OWG meetings, the pure participants discuss how to manage discussions of contextualized tasks about closures and subspaces (classroom orchestration goal), what kinds of discussion topics and communication emerged in engaging in IOLA tasks (cognitive/learning goal), curricular trajectories and mathematical content relevant to subspaces reorganized by instructors (content goal), and pacing, timing, testing, and grading of inquiry-based teaching (assessment goal). Within the findings, the main notable pattern in this OWG is there are many intersections of two or more goals for instruction, except for assessment goals. Also, it is interesting these intersections are rooted in some tensions between RME-based context and abstract mathematical concepts in implementing IOLA tasks. The following examples address those intersections.

I1: Yeah, my only hesitancy in all of this is the fact that you know, in the other task sequences, we have this clear and direct parallel between the intuitive contextual setting and the formal definitions. And this one, we're a little loosey-goosey in a couple of places, and I just, I don't know how that's gonna translate. Like, are they like, are they going to internalize what has been their tendency to think only about scalars that are, you know, positive whole numbers in the first place, right? And so, is this going to somehow reinforce that? Um.

Here, I1 expresses her concern of how students will take up subspaces according to the "loosey-goosey" definition in the task. That it may be difficult to align the abstract with what students would develop intuitively as the formal definition of subspace. This excerpt also is an intersection between classroom orchestration, cognitive/learning, and content goals. This intersection illustrates challenges for the instructor to discern what content ought to be taught (content goal), how those jive with the class activities (classroom orchestration goal) and concerns that students may "think only about scalars that are...positive whole numbers" in the context of the problem and if the task will continue to "somehow reinforce that" knowledge (cognitive/learning goal).

Similarly, another instructor participant, I2 also talks about the transition from the task activity to the introduction to the abstract version of subspace definition.

I2: So, I do want to say like, so it seems like we, the intent or how people have been talking about this is that we're going to use these like non-negative integers for the exploration stuff, but then tell the students to use real numbers for the actual subspace definition.

This is where I2 has the intersection between the classroom orchestration goal and a content goal. In terms of classroom orchestration, I2 anticipates how the task will be used for "exploration stuff" and also plans when there will possibly be direct instruction to then "just tell the students" to use real numbers for the actual subspace definition. Also, in terms of the content goal, I2 discusses what mathematical concept—the actual subspace definition, should be brought up during instruction using the IOLA task. Like above, throughout the overall OWG meetings, I2 expresses some tension in the negotiation between the real-world context of the task and the abstract mathematical concepts.

The intersections between classroom orchestration, cognitive/learning, and content goals stand out in OWG discussions of Task 3, where the concept of subspace is introduced. Task 3 starts from a new 5×7 matrix that represents a new scenario of camera record in another school wing, and the set U is all the possible camera data vectors which leaves the number of students—in each room unchanged. In the last part of Task 3 (See Figure 2), the set U is meant to be connected to the concept of subspace, and then null space. Here, I1 and I2 communicate with each other by discussing their anticipation of implementing Task 3.

I2: Yeah. So, in the previous prompt, they have to, you know, U as defined as, you know, actual students and actual cameras, right? So that means the entries on U are the entries in the vectors and U, I think, have to be non-negative integers. It's then, closed under scalar multiplication for those scalars for non-negative integer scalars. And then we change the scalars in the box definition, but we don't change U. So, our students going to be confused about, I think, "no" is a reasonable answer to C. Right? I think they might say "no", because it is not closed under vector scalar multiplication because I can't scale by a fraction, or a non-negative number, and I'll get something that is not in U because we didn't change U. So, either. So do either. So, I feel like the changes have to be synced, right? Like the change to U. And the change to the scalars needs to be synced, otherwise C turns into false.

I1: I assume that's what we want. Is, is that not what we wanted? like to just point out that like, well, U is closed to being a subspace, but because the scalars need to be any real numbers. It's not? Maybe I misunderstood? The…

I2: I don't know. That was one I thought. I kind of wanted there to be like a thing you found was a subspace, punch line, students like that. I mean, maybe there's used to it.

Recall: Definitions for closure of a set under vector addition and scalar multiplication:

- A set of vectors S is called **closed under vector addition** when $v, u \in S \Rightarrow v + u \in S$.
- A set of vectors S is called **closed under scalar multiplication** when $v \in S \Rightarrow k{*}v \in S$ for any scalar k.

New Definition: Subspaces

- A non-empty set of vectors in R^n is called a **subspace of** R^n if it is closed under vector addition and scalar multiplication (where, for our purposes, we assume scalars come from the real numbers R).

5. Several statements are written about U and F below. Circle ones that are true, and modify the ones that are false so that they become true statements.

a. U is a subset of R^7.	a. F is a subset of R^7.
b. U is closed under vector addition	b. F is closed under vector addition
c. U is closed under scalar multiplication (scalars $\in R$)	c. F is closed under scalar multiplication (scalars $\in R$)
d. U is a subspace of R^7	d. F is a subspace of R^7
e. U is all of R^7	e. F is all of R^7
f. <add your own statement here>	f. <add your own statement here>

Figure 2. The statements about subspaces in Task 3

I1 and I2 presented contrasting approaches to the IOLA instructions, especially in terms of the tensions between the RME context and formal concepts in the subspace. It seems I1 liked to engage her students in conversation and whole classroom discussions related to the tensions in the subspace tasks. From a previous OWG session, I1 remarked on her experience in a previous "stellar class" with their discussion of span. When her students discussed span, she "...thought it was like, superfluous, but it turned out not to be." As it turned out, there was a tension or confusion caused by the RME-based scenarios in IOLA tasks so students only used positive whole numbers as scalar multiples. She faciliated a classroom discussed where she "freaked out" her students by introducing other real numbers such as e and pi. That led to her students and she having "deeper and deeper" communicatively engaged conversations, and so "that meant content coverage was sacrificed a little. And I decided I was okay with that." On the other hand, I2 seemed to prefer focusing on what may confuse students so he wanted to reduce confusion beforehand. The second instructor wrestles with what content should he bring into discussions in his class between himself on the teacher side and the mathematician side. This wrestling particularly happened when he addressed more formal mathematical concepts such as closure under scalar multiplication, non-emptiness, and dimensions of a subspace—they usually have the potential to conflict with the context of the IOLA tasks. He stated, "I feel it makes me a bad mathematician. Probably a better math teacher to not show them that."

Discussion

The results of this analysis demonstrate several points. Firstly, the goals for the instruction framework can be used as a coding scheme that highlights teaching knowledge instructors utilize as they reflect on their instruction. Secondly, both participants of the study demonstrated often overlapping goals for their instruction. This is most likely due to the varying knowledge they rely on to inform their instruction. We saw from Participant I1 a demonstration of pedagogical knowledge being utilized as she anticipated ways to host a discussion with her students to untangle the difference in the contextualized and abstract definition of subspace. The other participant, I2, mostly relied on his mathematical subject-matter knowledge and his knowledge of students to hypothesize ways to deliver content in the most streamlined manner possible. These insights of teaching knowledge these two instructors possessed were made possible due to the semi-casual nature of the OWG. The instructors participated the most in the work group, yet the facilitators actively engaged in questioning instructors on their decision-making all the while. It is because of the structure a lot of varying teaching knowledge is revealed.

It was through discussion between the participants that revealed tension in implementing the subspace task due to its incomplete definition. The two participants presented different approaches to the IOLA instruction as it related to ironing out this wrinkle. Our two participants highlighted how experienced IOLA instructors will utilize various domains of teaching knowledge while balancing their knowledge of the principles of Realistic Mathematics Education to problem-solve. Thus, we think postsecondary instructors can have different avenues to becoming IOLA instructors so that their approaches to resolving tensions would be different. Therefore, it will be important for both novice and experienced IOLA instructors to have a professional development space—that may look like OWG—to unpack their speculations and approaches and then move forward. After implementation, reflection on instruction is also vital for developing teaching knowledge of oneself and others.

In terms of teacher knowledge—in addition to reflection on teaching practice, the OWG provided an opportunity for instructors to communicate with other instructors and also with the curriculum developers on the insights of the instructional design. This process of examining tasks and reflecting on their implementation is especially vital for IOLA instructors, so the OWG serves the place for them to analyze and reflect on the curriculum they implement in their classrooms. As the instructors examine instructional task designs after listening to what other instructors unpack from their implementation, their approaches can adjust based on their previous examination. As a result, they were able to build their teaching knowledge as it relates to adjusting curriculum to serve their student populations. That demonstrates the importance of reflection for IOLA instructors. Sharing approaches to task implementation and analysis is beneficial, yet it becomes more powerful for other instructors if it sparks reflection on practice.

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